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Towards Deriving the Standard Model Coupled to Gravity from Generalized Trace Dynamics via the Spectral Action Principle

Tejinder P. Singh

Tata Institute of Fundamental Research, Homi Bhabha Road, Mumbai 400005, India; tpsingh@tifr.res.in

Abstract

We present a spectral-action framework for connecting Generalized Trace Dynamics (GTD) to the structural form of the low-energy action of the observed universe. The fundamental single-STM-atom Lagrangian is decomposed exactly into a purely bosonic sector, boson–fermion cross terms, and bifermionic terms. This sectorwise decomposition furnishes a dictionary to almost-commutative spectral geometry: the bosonic sector supplies a quadratic GTD Dirac functional built from the six split-biquaternionic differential directions together with octonionic vector/gauge fluctuations; the cross sector supplies, under an explicit localization hypothesis, a sesquilinear fermionic pairing; and the bifermionic sector supplies the scalar/internal channel that is bosonized into the Higgs bridge field. We also record the principal-symbol link between the $SO(3,3)$ BF variables and the four-dimensional leafwise Dirac operator. Under stated assumptions — spontaneous localization, Euclidean continuation, six- to four-dimensional BF reduction, and a candidate observed-leaf finite geometry compatible with the $E_6/J_3(\mathbb{O}_{\mathbb{C}})$ inputs — the bosonic heat-kernel expansion yields the structural low-energy classes of terms: Einstein–Hilbert gravity, Yang–Mills kinetic terms, and scalar kinetic and potential terms. In addition, we provide a candidate finite spectral triple with explicit finite trace invariants, verify that the localization map respects the one-generation lepton/quark representation split, identify visible color-singlet scalar channels with electroweak quantum numbers $(\mathbf{1}, \mathbf{2}, \pm 1/2)$, and exhibit a smooth regulator family with explicit cutoff moments (f_0, f_2, f_4). Conversely, the assembled low-energy spectral action admits a natural inverse bilinear lift back to split bioctonionic trace dynamics. The result is a structural framework with conditional consistency checks and reductions, not a complete first-principles derivation.

Keywords: generalized trace dynamics; spectral action principle; noncommutative geometry; split bioctonions; exceptional Jordan algebra; Standard Model; quantum gravity; E_6 trinification; almost-commutative geometry; Higgs sector

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1. Introduction

1.1. Motivation

The search for a unified description of quantum theory and gravity remains one of the central challenges of fundamental physics. Any candidate framework must, in a suitable limit, reproduce the Standard Model of particle physics coupled to Einstein's general relativity, including the correct gauge-group structure, three fermion generations with their observed mass hierarchies, and the Higgs mechanism.

Generalized Trace Dynamics (GTD) [1,2] is a pre-quantum, pre-spacetime candidate theory that replaces the classical spacetime manifold with a non-commutative space labelled by quaternions and octonions, and replaces classical dynamical variables with matrix-valued operator degrees of freedom. Building on Adler's Trace Dynamics [3,4], GTD introduces matrix-valued gravitation through the spectral action principle of Chamseddine and Connes [5,6], and proposes a fundamental Lagrangian with $E_8 \times E_8$ symmetry [7,9]. From this pre-quantum dynamics, both quantum field theory and classical general relativity are expected to emerge in appropriate limits.

A central question for the GTD program is whether its fundamental action — which takes the simple form of a matrix-valued free-particle kinetic energy — can be organized into the ingredients needed for the low-energy Lagrangian after symmetry breaking and localization. The present paper addresses that question at the level of a structural framework: it sets up the sectorwise correspondence with spectral geometry, carries out several explicit consistency checks, and isolates the remaining steps that would still be needed for a complete derivation.

1.2. Strategy

The construction proceeds through the following chain:

1. **Full GTD expansion.** After substituting the bosonic–fermionic decomposition into the fundamental single-STM-atom Lagrangian, the action splits naturally into three pieces,
$$\mathcal{L}_{\rm GTD} = \mathcal{L}_{BB} + \mathcal{L}_{BF} + \mathcal{L}_{FF}, \tag{1}$$
where $\mathcal{L}_{BB}$ is purely bosonic, $\mathcal{L}_{BF}$ contains the boson–fermion cross terms, and $\mathcal{L}_{FF}$ is bifermionic.
2. **Bosonic spectral sector.** The bosonic piece $\mathcal{L}_{BB}$ is organized by the full normalized bosonic variable
$$\dot{Q}_B = \mathcal{D}_{\rm GTD} + \Phi_{B,L} + \Phi_{B,R}, \qquad \mathcal{D}_{\rm GTD} = \mathcal{D}_L + \omega\mathcal{D}_R, \tag{2}$$
with
$$\mathcal{D}_{L,R} = \dot{q}^{\rm curved}_{BL,R} + \frac{i\alpha}{L} q^{\rm vec}_{BL,R}, \qquad \Phi_{B,L} = \dot{q}_{B0L}, \qquad \Phi_{B,R} = \omega\dot{q}_{B0R}. \tag{3}$$
The assembled quaternionic leaf operators $\dot{q}^{\rm curved}_{BR} = D_{4\rm curved}$ and $\dot{q}^{\rm curved}_{BL} = D'_{4\rm curved}$ use only the nonzero imaginary quaternionic directions. The bosonic dotted zeroth quaternionic modes therefore supply the two branch-resolved scalar/Higgs seeds rather than part of the GTD Dirac operator itself. The quantity $\mathcal{D}_{\rm GTD}$ is dimensionless; the corresponding physical Dirac operator is $D^{\rm phys}_{\rm GTD} = \mathcal{D}_{\rm GTD}/L$, of dimension L^{-1}. The dimensionless cutoff used in the normalized spectral formulas is $\Lambda = L/L_{\rm P}$.
3. **Conditional reduction of the fermionic and scalar sectors.** Under an explicit localization hypothesis, the cross sector $\mathcal{L}_{BF}$ is rewritten as a sesquilinear pairing of operator-valued Dirac-mode coefficients. The bifermionic sector $\mathcal{L}_{FF}$ is rewritten as a bilinear seed which, after coarse-graining, generates an effective quartic scalar channel whose auxiliary-field completion yields the Higgs bridge field. These steps are consistency checks carried out under stated hypotheses, not full dynamical derivations.
4. **BF-to-Dirac reduction on the leaf.** The Wesley–Singh–Isidro SO$(3,3)$ BF mechanism is used at the principal-symbol level to relate the six-dimensional BF variables to a four-dimensional tetrad and spin connection on each leaf, and hence to the curved leafwise Dirac operator D_Σ entering the almost-commutative geometry.
5. **Low-energy action.** The bosonic spectral action $\mathrm{Tr}[f(\mathcal{D}_A/\Lambda)]$ is then expanded by the standard Seeley–DeWitt machinery, while the cross-sector reduction supplies the fermionic pairing $\langle\Psi, \mathcal{D}^{\rm phys}_A\Psi\rangle$. In this way gravity, gauge fields, the Higgs sector, and the fermionic action are traced back to distinct parts of the same GTD Lagrangian.
6. **Reverse-engineering consistency check.** Starting from the assembled low-energy spectral action, one can rewrite the theory in Dirac spectral variables, promote the bosonic spectral datum and the fermionic mode coefficients to matrix-valued pre-geometric variables in the spirit of trace dynamics, and lift the continuum point label to split bioctonionic STM coordinates. Under these assumptions the minimal bilinear inverse lift recovers the single-STM-atom GTD action as a natural inverse ansatz.

Conventions for normalized and physical operators.

Throughout the paper, symbols without the superscript "phys" denote normalized dimensionless GTD or spectral-action variables. Their physical counterparts are obtained by dividing by L, so that $\mathcal{D}_A^{\rm phys} = \mathcal{D}_A/L$, $D_{\rm GTD}^{\rm phys} = \mathcal{D}_{\rm GTD}/L$, and the corresponding eigenvalues have dimension L^{-1}. The normalized spectral action is written as $\mathrm{Tr}[f(\mathcal{D}_A/\Lambda)]$ with $\Lambda = L/L_{\rm P}$, whereas the equivalent physical description uses the dimensionful operator $\mathcal{D}_A^{\rm phys}$ and cutoff $\Lambda_{\rm phys} = 1/L_{\rm P}$.

We work in Euclidean signature throughout, Wick-rotating to Lorentzian signature at the end, following the standard practice in the non-commutative-geometry spectral-action program [5,6].

1.3. Key Assumptions and Status of the Construction

We begin by stating explicitly what is imported from prior work, what is newly derived in the present manuscript, and what enters as a working hypothesis at specific steps. Table 1 summarises the epistemic status of each ingredient. The reader is referred to this table whenever it becomes useful to know whether a given object is taken as input, derived here, or assumed at the level of a working hypothesis.

The fundamental assumptions of the construction are:

(A1) **Localization assumed.** Spontaneous localization of STM atoms has occurred, so we work with the classical (post-localization) action. The sum over many localized atoms reconstructs the spectral action $\mathrm{Tr}[f(D/\Lambda)]$.

(A2) **Euclidean signature.** We work in Euclidean signature on the six-dimensional base, then Wick-rotate the two emergent four-dimensional spacetimes.

(A3) **Six- to four-dimensional reduction.** The Wesley–Singh–Isidro $SO(3,3)$ BF mechanism [10] provides the dynamical reduction from the six-dimensional base to two overlapping four-dimensional leaves with Lorentz symmetry.

(A4) **Single atom as local building block.** The GTD action for a single STM atom is taken as the elementary local block. After spontaneous localization, summing over many STM atoms reconstructs the bosonic spectral trace, while the same sectorwise decomposition into BB, BF, and FF pieces is assumed to survive in the effective low-energy description.

Additional working hypotheses (H1)–(H6).

Beyond (A1)–(A4), several subsidiary inputs enter at specific steps of the construction, as catalogued in Table 1. These include: (H1) the explicit localization map that sends odd GTD variables to operator-valued Dirac-mode coefficients while preserving the lepton/quark representation split; (H2) the isolation of the visible scalar bridge channel by the e_0 projection of the bifermionic seed; (H3) the assumption that localization/coarse-graining generates an attractive quartic channel in that visible scalar sector; (H4) the use of a minimal observed-leaf finite geometry as an associative shadow of the full exceptional/nonassociative internal structure; (H5) the broken-phase support factors used when comparing the common spectral Yang–Mills coefficient with the visible low-energy couplings; and (H6) the standard auxiliary-field (Hubbard–Stratonovich) completion of the resulting attractive quartic channel. These ingredients are introduced where needed below; they are not presented as derived results of the present paper.

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Table 1. Status of the ingredients entering the construction. Items above the first horizontal rule are imported from prior work; items between the first and second rules are newly derived or constructed here; items between the second and third rules are working hypotheses (H1)–(H6) that enter specific steps; the final entry records a programmatic look-ahead recorded in Appendix D, which is not load-bearing for the main-text construction.

Ingredient	Status	Reference / Section
GTD action as starting point	Imported	[1,2]
$E_8 \times E_8$ symmetry of fundamental Lagrangian	Imported	[7,9]
Residual 288 as adjoint-lineage labels (not matter); two composite Higgs from $\mathbf{27} \otimes \mathbf{27}$	Imported	[8]; Sec. 7.2
Split bioctonionic scaffolding	Imported	[11]
Six- to four-dimensional BF reduction	Imported	[10]
Spectral action principle	Imported	[5,6]
Exceptional Jordan eigenvalue data and charged-fermion mass ratios	Imported	[12,14]
$BB/BF/FF$ sectorwise decomposition of $\mathcal{L}_{\text{GTD}}$	Derived here	Section 3.2
GTD Dirac operator $\mathcal{D}_{\text{GTD}} = \mathcal{D}_L + \omega \mathcal{D}_R$ identified with bosonic differential part	Derived here	Section 3.1
Sectorwise spectral assembly rule for $\mathcal{D}_A$	Derived here	Sections 3, 6, 7
Principal-symbol BF $\to$ leafwise Dirac link	Derived here	Section 5
Heat-kernel a_0, a_2, a_4 in present setup	Derived here (standard methodology)	Section 4
Candidate E_6-compatible finite spectral triple	Proposed here	Appendix A
Smooth regulator family with explicit (f_0, f_2, f_4)	Constructed here	Appendix B
Visible e_0-channel quantum-number check (color-singlet doublets, $Y = \pm 1/2$)	Derived here	Sections 3, 6
NJL mean-field critical-coupling formulas	Derived here	Section 6
Reverse-engineering bilinear lift to bioctonionic STM	Derived here	Section 10
(H1) Explicit localization map preserving lepton/quark representations	Working hypothesis	Section 3, used in 6
(H2) e_0 projection isolates the visible scalar bridge channel	Working hypothesis	Section 7
(H3) Localization/coarse-graining generates the attractive quartic channel	Working hypothesis	Section 7
(H4) Associative shadow suffices for the observed-leaf finite geometry	Working hypothesis	Appendix A; see also Section 1.4
(H5) Broken-phase support factors connect bare spectral coupling to visible couplings	Working hypothesis	Section 6.1
(H6) Auxiliary-field (Hubbard–Stratonovich) completion of the quartic channel	Standard technique applied	Section 7
Pre-breaking nonassociative scaffold	Programmatic look-ahead	Appendix D

Scope of the main text vs. Appendix D.

Throughout the body of the paper we work with the associative shadow of the internal geometry (hypothesis (H4)). The pre-breaking nonassociative, Jordan, and phase-space-like scaffold of Appendix D is recorded as a programmatic look-ahead: it suggests where the broken-phase associative geometry used here might sit inside a larger pre-breaking mathematical structure. Appendix D is not load-bearing for the main-text construction; the results in the body of the paper do not depend on the nonassociative structure recorded there.

1.4. Relation to Prior Work

This work builds on the $E_8 \times E_8$ unification proposal [7], the split-bioctonionic scaffolding construction [11], the bosonic Lagrangian and weak-mixing-angle derivation [9], the fermion mass ratios

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from the exceptional Jordan algebra [12,14,15], the gravi-weak unification via six-dimensional BF theory [10], and the comparison of GTD with causal fermion systems and non-commutative geometry [16]. It also draws on the spectral action principle of Chamseddine and Connes [5,6] and on Connes' earlier formulation [46] of the geometric foundations underlying the spectral action, on Landi–Rovelli's use of Dirac spectral variables in gravity [17], and on standard background references for the non-commutative-geometry program [18,19]. The mathematical structure of the exceptional Jordan algebra $J_3(\mathbb{O}_\mathbb{C})$ used throughout follows the standard treatments [47]. For the role of non-commutative algebraic structures and gauge groups in the broader trace-dynamics setting we refer to [48].

Within the broader octonionic and exceptional-unification landscape, the present program is complementary to other approaches that relate Standard-Model structure to complex octonions, triality, and Jordan geometry, including work by Furey, Boyle, Farnsworth, and Todorov–Drenska [20–24]. In particular, Boyle and Farnsworth's Jordan-geometric reformulation of Standard-Model and Pati–Salam structure, together with Farnsworth's later work on nonassociative spectral geometries with exceptional symmetry and charged scalar sectors, provides a useful neighbouring line of development to the present GTD-to-spectral-action program.

It is also useful to place the present discussion alongside the conceptual explanation of the finite Standard-Model geometry in non-commutative geometry, together with later extensions to non-associative and Lorentzian/twisted spectral settings, and alongside the standard continuum-limit reference for causal fermion systems [25–28].

The distinguishing aim in the present work is not merely to recover the representation-theoretic content of the low-energy fields, but to start from a trace-dynamical pre-spacetime action and connect it, via spectral-action methods, to the structural form of the low-energy Lagrangian.

1.5. Anticipated Spontaneous Localization Mechanism

The present paper assumes spontaneous localization as an input. The expected mechanism is motivated by the existing GTD / trace-dynamics literature and by related work on collapse and operator spacetime. In Adler's trace dynamics, quantum theory arises as an emergent thermodynamic approximation to an underlying matrix dynamics. GTD is expected to reduce, on Connes-time scales $\Delta\tau \gg \tau_{\mathrm{EW}}$ and in the regime where the anti-self-adjoint part of the trace Hamiltonian is negligible, to an equilibrium statistical thermodynamic description whose coarse-grained dynamics is precisely quantum theory (and, after localization of the background, quantum field theory) [1,16]. In this near-equilibrium regime, the Adler–Millard charge is equipartitioned and the canonical commutators, Heisenberg equations, and the rules of quantum field theory emerge as the effective description of the STM-atom ensemble.

When the anti-self-adjoint contribution to the trace Hamiltonian becomes significant, the system moves away from equilibrium and the quantum approximation breaks down. In GTD this happens when many STM atoms become sufficiently entangled. The resulting dynamics drives spontaneous localization: the matrix-valued state of an STM atom localises to one specific eigenvalue, selecting a definite classical configuration out of the pre-spacetime matrix dynamics [16,29]. Ref. [29] argues that in GTD the purely bosonic subsector admits a self-adjoint Hamiltonian, whereas the fermionic sector carries an intrinsic anti-self-adjoint contribution; since every term in the anti-self-adjoint part of the trace Hamiltonian contains fermionic variables, collapse channels are fermionic and bosonic fields classicalise only through the fermions they couple to. A more recent analysis [30] makes the reduction step explicit at the emergent Hilbert-space level: with an adjoint-preserving (imaginary) scalarization of the odd-grade Grassmann coefficient and normal ordering, the anti-self-adjoint fermionic Hamiltonian generates a norm-preserving nonlinear drift of continuous-spontaneous-localization (CSL) form, $-(\hat{G}_F - \langle \hat{G}_F \rangle)\psi$ with $\hat{G}_F$ a fermionic occupation operator, which drives reduction toward occupation (pointer) eigenstates. This is the GTD analogue of the spontaneous-collapse approach to the measurement problem [51–54]: in the present GTD setting, a fermionic STM atom localises to one of the available octonionic directions, selecting a definite spacetime-matter configuration and contributing to the emergence of a classical spacetime background [16].

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Two points must nonetheless be stated plainly [30]. First, this deterministic drift is *amplitude-blind*: on its own it selects the outcome of largest collapse coupling and discards the prepared weights $|c_k|^2$, so it constitutes genuine reduction but not yet a derivation of the Born rule. The Born weights reduce instead to the Itô (martingale) character of the effective stochastic equation that emerges when the unobserved many-atom/de Sitter sector is coarse-grained; whether the required collective (Adler–Millard fluctuation) response is nonzero is at present open, and the most natural inter-atom mechanism — a coassociative boson-exchange vertex — is provably Adler–Millard-neutral and hence collapse-inert. We therefore treat spontaneous localization as a well-motivated input whose deterministic reduction skeleton is established (modulo the stated scalarization and ordering choices) but whose Born-rule completion remains conditional.

The coarse-graining analysis of Ref. [31] shows that a deterministic, non-unitary, norm-preserving dynamics can, when observed at lower time resolution, appear stochastic — supplying exactly the genuine stochasticity that the Itô construction above requires — and supports the broader expectation that quantum linear superposition is an equilibrium phenomenon while spontaneous localization reflects fluctuations away from equilibrium. We do not derive the collapse mechanism from first principles for the full STM ensemble; we use the existing literature to motivate the working hypothesis that GTD implements this pattern in the split bioctonionic setting. Crucially for the present paper, the spectral-action route to the Standard Model is *insensitive* to the unresolved collapse/Born question: the gauge group, representations, generations, mass ratios, gauge couplings, and Newton's constant are read off at fixed Connes time from the algebra and from the self-adjoint Dirac spectrum entering $\mathrm{Tr}[f(\mathcal{D}_A/\Lambda)]$, and these are blind to the frequency structure of the anti-self-adjoint (collapse) sector [30]; the results of this paper therefore do not depend on how that question is ultimately settled. Assuming different localised STM atoms occupy distinct spectral slots, the single-STM-atom Lagrangian fixes the universal local contribution to the emergent classical Lagrangian, and the full classical action is obtained by summing this local structure over the localised STM ensemble.

1.6. *Broader Overview of the Octonionic $E_8 \times \omega E_8$ Program*

For context, the spectral-action analysis developed here sits inside a broader research program whose starting point is the claim that quantum theory should be reformulated without reliance on an external classical time parameter. Classical spacetime is available only when sufficiently many degrees of freedom have already classicalised; a more exact pre-spacetime description should therefore exist far below the Planck scale. Five mathematical ingredients play the central role in this broader construction: normed division algebras, Adler's trace dynamics, Connes' non-commutative geometry together with the spectral action principle, Clifford algebras, and the exceptional Lie and Jordan-algebraic structures. Each classical spacetime point is traded for a non-commutative split bioctonionic scaffold of sixteen real dimensions, rich enough to carry pre-spacetime directions, internal symmetry directions, and the algebraic data for three generations of chiral fermions [11,32,49].

At each bioctonionic point one overlays matrix-valued matter and field variables obeying the rules of trace dynamics, with one component per bioctonionic direction. The elementary building block is an atom of spacetime-matter (STM atom), whose action is the matrix-valued free-particle form written in Eq. (5). The evolution parameter is Connes time, arising from the Tomita–Takesaki theorem for von Neumann algebras [33]. Dirac eigenvalues serve as invariant spectral data for gravitation and Yang–Mills geometry, and GTD promotes these spectral variables to operator-valued degrees of freedom on the split bioctonionic scaffold: one original Dirac eigenvalue lifts to one copy of the generalized Dirac operator, i.e. to one STM atom. Spontaneous localization maps the matrix-valued variables back to definite eigenvalue configurations, yielding classical spacetime together with classical matter and gauge fields; STM atoms that do not undergo critical localization remain governed by the underlying trace dynamics or, to good approximation, by quantum field theory on the emergent classical background [16].

The universe is assumed to begin as a collection of very many STM atoms, each obeying an $E_8 \times \omega E_8$ symmetry. An inflation-like de Sitter phase persists until the electroweak scale, where the two E_8 branches are broken branchwise as

$$\begin{aligned} E_{8L} &\longrightarrow \mathrm{SU}(3)_L^{\mathrm{geom}} \times E_{6L} \longrightarrow \mathrm{SU}(3)_c \times \mathrm{SU}(3)_{F,L} \times \mathrm{SU}(3)_L \longrightarrow \mathrm{SU}(2)_L \times \mathrm{U}(1)_{Y_1} \longrightarrow \mathrm{U}(1)_Y, \\ E_{8R} &\longrightarrow \mathrm{SU}(3)_R^{\mathrm{geom}} \times E_{6R} \longrightarrow \mathrm{SU}(3)_{c'} \times \mathrm{SU}(3)_{F,R} \times \mathrm{SU}(3)_R \longrightarrow \mathrm{SU}(2)_R \times \mathrm{U}(1)_{Y_2} \longrightarrow \mathrm{U}(1)_{Y_{\mathrm{dem}}}. \end{aligned} \tag{4}$$

On the left branch, $\mathrm{SU}(3)_c$ is visible QCD, $\mathrm{SU}(3)_{F,L}$ is a global flavour symmetry, and $\mathrm{SU}(3)_L$ yields the visible electroweak sector. On the right branch, $\mathrm{SU}(3)_{c'}$ is not gauged in the minimal phenomenological limit, $\mathrm{SU}(3)_{F,R}$ organizes the right-handed family structure, and $\mathrm{SU}(3)_R$ breaks to $\mathrm{SU}(2)_R \times \mathrm{U}(1)_{Y_{\mathrm{dem}}}$. The broken $\mathrm{SU}(2)_R$ sector is intended to underlie gravitation, while the unbroken $\mathrm{U}(1)_{\mathrm{dem}}$ is the new long-range force of dark electromagnetism sourced by the square root of mass. The two geometric groups supply the split bioctonionic scaffold from which both spacetime and internal symmetry space emerge [7,10,11].

Triality in $\mathrm{Spin}(8) \subset E_6$ organizes the family sector. Complex split bioctonions generate the Clifford algebra $\mathrm{Cl}(7) \cong \mathrm{Cl}(6) \oplus \mathrm{Cl}(6)$; minimal left ideals of the two $\mathrm{Cl}(6)$ copies furnish one generation of chiral fermions, and triality replicates this structure across three generations. The resulting three-family data are encoded by the exceptional Jordan algebra; after flavour, left–right, and triality breaking, this gives the charged-fermion mass hierarchy, including the cross-sector relation $\sqrt{m_\tau/m_\mu} = \sqrt{m_s/m_d}$ [12,32].

A broader catalogue of the program's falsifiable consequences includes the second Higgs, three families of sterile right-handed neutrinos, dark electromagnetism, flavour-sector relations, and quantum-foundational signatures. The present paper addresses one specific part of that agenda: the spectral-action route from the GTD single-STM action to the structural form of the low-energy Lagrangian, clarifying how gravity, gauge sectors, fermionic pairing, and the effective Higgs data emerge from the same pre-spacetime matrix dynamics.

2. The Fundamental GTD Lagrangian

2.1. *Action for a Single STM Atom*

The fundamental action for a single atom of spacetime-matter is [2,7]

$$\frac{S}{\hbar} = \frac{1}{2} \int \frac{d\tau}{\tau_{\mathrm{Pl}}} \, \mathrm{Tr}\left[\frac{L_P^2}{L^2} \, \dot{Q}_1^\dagger \, \dot{Q}_2 \right]. \tag{5}$$

Here τ is Connes time, L_P is the Planck length, and L is the fundamental length scale of the STM atom (a two-brane). The two matrices Q_1 and Q_2 together describe the STM atom; their entries are Grassmann-number-valued. The action has the form of a matrix-valued free-particle kinetic energy and is assumed invariant under $E_8 \times E_8$ symmetry prior to symmetry breaking.

2.2. *Single-STM Connes-time action and the emergence of the classical spacetime action*

A conceptual distinction that should be kept explicit throughout is the distinction between the microscopic GTD Lagrangian of a *single* STM atom and the emergent classical spacetime action of the localized many-STM ensemble. In GTD, the primitive object is the single-STM-atom action

$$\frac{S_{\mathrm{STM}}}{\hbar} = \int \frac{d\tau}{\tau_{\mathrm{Pl}}} \, \mathcal{L}_{\mathrm{STM}}(\tau), \qquad \mathcal{L}_{\mathrm{STM}}(\tau) = \frac{1}{2} \, \mathrm{Tr}\left[\frac{L_P^2}{L^2} \, \dot{Q}_1^\dagger \dot{Q}_2 \right]. \tag{6}$$

Thus the traced quadratic quantity is, at the microscopic level, the *Connes-time Lagrangian* of one STM atom. It is not yet the full classical spacetime action, nor directly a spacetime Lagrangian density. The sectorwise decomposition carried out below should therefore first be read as a decomposition of the single-STM Connes-time Lagrangian into its bosonic, cross, and bifermionic parts.

By contrast, on the spectral-action side,

$$\mathrm{Tr}\big[f(D_A/\Lambda)\big]$$

is naturally an action functional of an emergent geometry. After the heat-kernel expansion, it becomes an integral of a local density over a four-dimensional classical spacetime. Accordingly, the GTD-to-spectral-action map necessarily involves a change of level:

$$\mathcal{L}^{(1)}_{\mathrm{GTD}}(\tau) \longrightarrow S^{(1)}_{\mathrm{GTD}} = \int \frac{d\tau}{\tau_{\mathrm{Pl}}}\,\mathcal{L}^{(1)}_{\mathrm{GTD}}(\tau) \longrightarrow S_{\mathrm{class}} = \sum_A S^{(A)}_{\mathrm{STM}} \sim \int d^4x\,\sqrt{g}\,\mathcal{L}_{\mathrm{eff}}(x). \tag{7}$$

In this sense, the single-STM-atom GTD Lagrangian fixes the universal local *action seed*, whereas the spacetime volume integral emerges only after summing over many localized STM atoms and passing to the corresponding coarse-grained classical limit.

This viewpoint also clarifies how the relation between Connes time and classical spacetime time should be understood. For each fixed value of Connes time τ, one expects an emergent classical configuration

$$\mathcal{G}(\tau) = \big(M_\tau, g_{\mu\nu}(\tau), A_\mu(\tau), \Psi(\tau), \Phi(\tau), \dots\big),$$

or, more cautiously, an emergent fiber-bundle-type classical structure. Within any one such classical configuration, one may then perform the usual $3+1$ split of general relativity, for example in ADM form,

$$ds^2_\tau = -N^2(\tau,t,x)\,dt^2 + h_{ij}(\tau,t,x)\,\big(dx^i + N^i dt\big)\big(dx^j + N^j dt\big). \tag{8}$$

Here t is the ordinary coordinate time internal to the emergent classical geometry at fixed τ, whereas τ orders the evolution of the emergent classical geometry itself. In this way, one does not replace the GR picture of spacelike hypersurfaces evolving in coordinate time; rather, one embeds it inside a deeper GTD/NCG description in which whole classical four-geometries arise and evolve along Connes time.

The same distinction clarifies the role of spontaneous localization. In the near-equilibrium regime, GTD is expected to reduce on large Connes-time scales to the emergent quantum description. When many STM atoms become sufficiently entangled, the anti-self-adjoint part of the trace Hamiltonian becomes significant, spontaneous localization sets in, and classical spacetime geometry emerges. In this passage, Connes time is not removed; rather, it remains the underlying evolution parameter along which the emergent classical four-geometry evolves.

The present paper does not yet derive the detailed map

$$\sum_A \int \frac{d\tau}{\tau_{\mathrm{Pl}}}\,\mathcal{L}_A(\tau) \longrightarrow \int d^4x\,\sqrt{g}\,\mathcal{L}_{\mathrm{eff}}(x),$$

nor does it prove rigorously that one localized STM atom should be identified with one point of a classical manifold. Those statements remain interpretive hypotheses. What is used here is the weaker but sufficient claim that the single-STM GTD action fixes the universal local building block whose repetition over the localized STM ensemble yields the classical low-energy action.

2.3. Bosonic–Fermionic Decomposition

After $E_8 \times E_8$ symmetry breaking (which coincides with electroweak symmetry breaking), the fundamental variables decompose as [7,9]

$$\dot{Q}_1^\dagger = \dot{Q}_B^\dagger + \frac{L_{\mathrm{P}}^2}{L^2}\,\beta_1\,\dot{Q}_F^\dagger, \qquad \dot{Q}_2 = \dot{Q}_B + \frac{L_{\mathrm{P}}^2}{L^2}\,\beta_2\,\dot{Q}_F, \tag{9}$$

$$\dot{Q}_B = \frac{1}{L}(i\alpha\, q_B + L\,\dot{q}_B), \qquad \dot{Q}_F = \frac{1}{L}(i\alpha\, q_F + L\,\dot{q}_F). \tag{10}$$

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Here β_1 and β_2 are unequal odd-grade Grassmann numbers, and α is the Yang–Mills coupling constant. The bosonic matrices (even-grade Grassmann entries) describe gauge fields and gravity; the fermionic matrices (odd-grade Grassmann entries) describe quarks and leptons. On the fermionic side, both the dotted piece $\dot{q}_F$ and the undotted piece q_F must be kept: the former belongs to the split-biquaternionic sector already present before the full octonionic extension, while the latter appears when the theory is enlarged to the full split bioctonionic space.

Dimensionally, we take the configuration variables Q_i, q_B, and q_F to have dimension of length, while differentiation with respect to Connes time gives dimensionless quantities $\dot{Q}_i$, $\dot{q}_B$, and $\dot{q}_F$. It follows that $\dot{Q}_B$ and $\dot{Q}_F$ are dimensionless normalized GTD Dirac variables. Whenever the comparison with the usual non-commutative-geometry Dirac operator is needed, the physical operator is obtained by dividing these normalized variables by L.

2.4. Decomposition on Split Bioctonionic Space

Following the GTD bookkeeping of the comparison paper [16], the dotted/undotted distinction should not be identified directly with the left/right distinction. Both dotted and undotted variables decompose into left and right sectors. For the bosonic variables, however, one must further separate the bosonic dotted zeroth quaternionic mode from the genuinely Dirac-type directions. The reason is simple: the split-biquaternionic Dirac gradient is built from the six *imaginary* directions of the split biquaternion, whereas the zeroth quaternionic bosonic mode is scalar and is therefore not part of the Dirac operator itself.

The bosonic dotted variable is written as

$$\dot{q}_B = \dot{q}_{B0L} + \dot{q}_{BL} + \omega\big(\dot{q}_{B0R} + \dot{q}_{BR}\big) = \dot{q}_{B0} + \dot{q}_{B,\nabla}, \tag{11}$$
$$\dot{q}_{B0} := \dot{q}_{B0L} + \omega\dot{q}_{B0R} = \Phi_{B,L} + \Phi_{B,R}, \qquad \dot{q}_{B,\nabla} := \dot{q}_{BL} + \omega\dot{q}_{BR}, \tag{12}$$

with

$$\Phi_{B,L} := \dot{q}_{B0L}, \qquad \Phi_{B,R} := \omega\dot{q}_{B0R}. \tag{13}$$

Here $\dot{q}_{BL}$ and $\dot{q}_{BR}$ denote the nonzero imaginary quaternionic directions on the two leaves. These are the directions that assemble into the leafwise Dirac operators. By contrast, $\Phi_{B,L}$ and $\Phi_{B,R}$ are the two bosonic zeroth-mode scalar components. In the $E_8 \times E_8$ interpretation the right seed $\Phi_{B,R}$ is the Standard-Model-like Higgs antecedent associated with the mass-giving right branch, whereas the left seed $\Phi_{B,L}$ is the antecedent of the second Higgs associated with the left branch [7].

For the undotted bosonic variable the split used in the comparison paper introduces only the additional internal octonionic directions:

$$q_B = q_{BL}^{\text{vec}} + \omega q_{BR}^{\text{vec}} =: q_{B,\text{vec}}, \tag{14}$$

where q_{BL}^{vec} and q_{BR}^{vec} carry the four internal octonionic directions. No additional undotted bosonic zeroth mode appears: the quaternionic zeroth direction is already carried by the dotted sector and is not duplicated in q_B.

The fermionic variables are written as

$$\dot{q}_F = \dot{q}_{F0L} + \omega\dot{q}_{F0R} + \dot{q}_{FL} + \omega\dot{q}_{FR} = (\dot{q}_{F0L} + \dot{q}_{FL}) + \omega(\dot{q}_{F0R} + \dot{q}_{FR}), \tag{15}$$
$$q_F = q_{FL} + \omega q_{FR}. \tag{16}$$

Thus the dotted fermionic sector remains entirely quaternionic, while the undotted fermionic sector supplies the additional octonionic directions. The crucial difference from the bosonic case is that we do *not* construct the spacetime Dirac operator directly from $\dot{q}_F$ or q_F. Instead, the fermionic variables are sent, after localization, to operator-valued mode coefficients and to the finite/internal data of the almost-commutative geometry. For this reason the special separation of the bosonic dotted zeroth mode does not induce an analogous subtraction from $\mathcal{D}_F$.

For later use, it is convenient to introduce the bosonic split

$$\dot{q}_B = \dot{q}_{B0} + \dot{q}_{B,\nabla}, \qquad q_B = q_{B,\text{vec}}, \tag{17}$$

and the fermionic quaternionic combinations

$$\dot{q}^{(\mathbb{H})}_{F,L} := \dot{q}_{F0L} + \dot{q}_{FL}, \qquad \dot{q}^{(\mathbb{H})}_{F,R} := \dot{q}_{F0R} + \dot{q}_{FR}. \tag{18}$$

The main point is that the bosonic dotted zeroth modes $(\Phi_{B,L}, \Phi_{B,R})$ are scalar data and therefore belong to the two-Higgs sector, whereas the genuinely Dirac-type bosonic directions are the six imaginary split-biquaternionic directions together with the non-scalar octonionic vector/gauge fluctuations carried by $q_B = q_{B,\text{vec}}$.

3. Expanding the GTD Lagrangian and assembling the spectral data

Compact operator dictionary.

The main symbols used in Sections 3–10 are the following: $\dot{Q}_B$ is the full normalized bosonic GTD variable; $\mathcal{D}_{\text{GTD}}$ is its genuine Dirac-type part built from the split-biquaternionic differential directions together with the non-scalar octonionic vector/gauge fluctuations; $(\Phi_{B,L}, \Phi_{B,R})$ are the branch-resolved bosonic scalar seeds; $\Psi_F = \Psi_q + \Psi_\ell$ is the unified normalized fermionic GTD variable; D^{phys}_Σ is the leafwise physical Dirac operator on a four-dimensional sheet; $\mathcal{D}_F$ is the finite/internal Dirac operator; and $\mathcal{D}_A$ denotes the assembled almost-commutative operator. Appendix C.2 records the fuller dictionary used in later sections.

3.1. The GTD Dirac Operator

The comparison paper distinguishes between the component-level split bioctonionic bookkeeping and the assembled curved leaf operators that enter the GTD action [16]. On the six-dimensional base of signature $(3, 3)$ one first writes the flat operator

$$D_6 = \hat{\imath}\,\partial_{t_1} + \hat{\jmath}\,\partial_{t_2} + \hat{k}\,\partial_{t_3} + \omega\big(\hat{\ell}\,\partial_{x_1} + \hat{m}\,\partial_{x_2} + \hat{n}\,\partial_{x_3}\big), \tag{19}$$

which decomposes into the two four-dimensional operators D_4 and D'_4 on the two leaves. At the curved level, the right-hand leaf operator is defined in Eq. (2.50) of the comparison paper as

$$\dot{q}^{\text{curved}}_{BR} \equiv D_{4\text{curved}} = \dot{q}_{Bt_1}\,\hat{\imath}\,\partial_{t_1} + \omega\Big(\dot{q}_{Bx_1}\,\hat{\ell}\,\partial_{x_1} + \dot{q}_{Bx_2}\,\hat{m}\,\partial_{x_2} + \dot{q}_{Bx_3}\,\hat{n}\,\partial_{x_3}\Big), \tag{20}$$

with an analogous left-hand operator $\dot{q}^{\text{curved}}_{BL} \equiv D'_{4\text{curved}}$ descending from D'_4. The crucial point is that these leafwise operators already use only the nonzero imaginary quaternionic directions. The bosonic dotted zeroth quaternionic mode is not part of the Dirac gradient.

Dimensionally, the coordinates Q_i, q_B, and q_F have dimension of length, whereas $\dot{Q}_i$, $\dot{q}_B$, and $\dot{q}_F$ are dimensionless Connes-time derivatives. It follows that the GTD object naturally appearing in the trace Lagrangian is a *dimensionless normalized* variable; the physical Dirac operator of non-commutative geometry is obtained only after dividing by L.

In the bosonic split of Eq. (17), the normalized leafwise GTD Dirac variables are therefore

$$\mathcal{D}_L := \dot{q}^{\text{curved}}_{BL} + \frac{\mathrm{i}\alpha}{L}\, q^{\text{vec}}_{BL}, \qquad \mathcal{D}_R := \dot{q}^{\text{curved}}_{BR} + \frac{\mathrm{i}\alpha}{L}\, q^{\text{vec}}_{BR}, \tag{21}$$

and the genuine normalized GTD Dirac variable is

$$\mathcal{D}_{\text{GTD}} := \mathcal{D}_L + \omega\mathcal{D}_R = \frac{1}{L}\big(\mathrm{i}\alpha\, q_{B,\text{vec}} + L\dot{q}_{B,\nabla}\big). \tag{22}$$

The bosonic dotted zeroth quaternionic modes instead define the normalized scalar seeds

$$\Phi_{B,L} := \dot{q}_{B0L}, \qquad \Phi_{B,R} := \omega\dot{q}_{B0R}, \qquad \Phi_{B,0} := \Phi_{B,L} + \Phi_{B,R}, \tag{23}$$

so that the full normalized bosonic variable in the GTD action is

$$\dot{Q}_B = \mathcal{D}_{\text{GTD}} + \Phi_{B,L} + \Phi_{B,R} = \mathcal{D}_{\text{GTD}} + \Phi_{B,0}. \tag{24}$$

This is the form of the bosonic GTD variable extracted from the action. The point of the split is that the dotted bosonic zeroth quaternionic modes are now treated as the two branch-resolved scalar/Higgs antecedents rather than as part of the Dirac operator. In the effective low-energy interpretation, $\Phi_{B,R}$ is the Standard-Model-like mass-giving Higgs antecedent, while $\Phi_{B,L}$ is the second Higgs antecedent associated with the left branch.

The corresponding physical operators are

$$D^{\text{phys}}_{\text{GTD}} := \frac{1}{L}\,\mathcal{D}_{\text{GTD}}, \qquad \Phi^{\text{phys}}_{B,L} := \frac{1}{L}\,\Phi_{B,L}, \qquad \Phi^{\text{phys}}_{B,R} := \frac{1}{L}\,\Phi_{B,R}, \tag{25}$$

with $[D^{\text{phys}}_{\text{GTD}}] = [\Phi^{\text{phys}}_{B,L}] = [\Phi^{\text{phys}}_{B,R}] = L^{-1}$. The same relation holds leafwise, $D^{\text{phys}}_{L,R} = \mathcal{D}_{L,R}/L$. The separation of the bosonic dotted zeroth mode is specific to the bosonic continuum variable used to build the Dirac gradient. It does not alter the role of the finite/internal Dirac operator $\mathcal{D}_F$, which is assembled separately from the fermionic/Jordan data discussed below.

Why the Dirac operator is naturally a split-biquaternionic gradient.

A useful way to motivate the GTD Dirac operator is to start from the observation that ordinary quaternions already package vector analysis: the quaternion product contains the usual dot and cross products, and a quaternionic differential operator packages a gradient in three directions. It is easily seen that when a 3D quaternionic gradient operator $D_3 = \hat{\imath}\partial_1 + \hat{\jmath}\partial_2 + \hat{k}\partial_3$ is squared, one finds $D_3^2 = -\nabla^2$. This points to a relation between division algebra and the Dirac operator as a gradient operator in the algebra. However, in four dimensions a naive quaternionic relativistic gradient $D_4 = i\partial_0 + \hat{\imath}\partial_1 + \hat{\jmath}\partial_2 + \hat{k}\partial_3$ is not fully satisfactory, because the commuting scalar imaginary unit $i = \sqrt{-1}$ does not provide a fourth *noncommuting* quaternionic direction. In this sense, a genuinely quaternionic Dirac gradient points beyond 4D.

Appendix C of [10] shows that the natural algebraic setting is instead the split biquaternion algebra

$$\text{Cl}(0,3) \simeq \mathbb{H} \oplus \omega\mathbb{H}, \qquad \omega^2 = 1,$$

whose two quaternionic copies provide two commuting triples of imaginary units,

$$(\hat{\imath}, \hat{\jmath}, \hat{k}), \qquad (\hat{\ell}, \hat{m}, \hat{n}) := (-\omega\hat{\imath}, -\omega\hat{\jmath}, -\omega\hat{k}).$$

These six imaginary directions define a natural 6D event variable and its associated gradient operator,

$$x_6 = t_1\hat{\imath} + t_2\hat{\jmath} + t_3\hat{k} + \omega(x_1\hat{\ell} + x_2\hat{m} + x_3\hat{n}),$$

$$D_6 = \hat{\imath}\,\partial_{01} + \hat{\jmath}\,\partial_{02} + \hat{k}\,\partial_{03} + \omega(\hat{\ell}\,\partial_1 + \hat{m}\,\partial_2 + \hat{n}\,\partial_3),$$

for which

$$D_6\widetilde{D}_6 = \partial_{01}^2 + \partial_{02}^2 + \partial_{03}^2 - \partial_1^2 - \partial_2^2 - \partial_3^2,$$

thus reproducing the Klein–Gordon operator with split signature $(3,3)$. In this sense, the six imaginary directions of the split biquaternion are not an ad hoc choice: they are precisely what is needed for a genuinely quaternionic Dirac gradient.

After symmetry breaking, this primitive 6D gradient admits two natural 4D Lorentzian reductions, corresponding to the two overlapping 4D leaves,

$$D_4^{(I)} = \hat{\imath}\,\partial_{01} + \omega(\hat{\ell}\,\partial_1 + \hat{m}\,\partial_2 + \hat{n}\,\partial_3),$$

$$D_4^{(II)} = \omega\hat{\ell}\,\partial_1 + \hat{\imath}\,\partial_{01} + \hat{\jmath}\,\partial_{02} + \hat{k}\,\partial_{03}.$$

These furnish the leafwise Dirac operators which, after curvature and gauge dressing, become the corresponding curved-space operators on the two 4D sectors.

Within the split bioctonionic extension used in GTD, the conceptual division is then natural: the split-biquaternionic directions supply the *derivative* or gradient part of the Dirac operator, whereas the remaining octonionic directions are taken to act *algebraically*, namely as internal non-derivative directions. They therefore enter the low-energy operator not as additional spacetime derivatives but as vector/gauge and finite/internal fluctuations. In this way one is led naturally to the fluctuated almost-commutative operator

$$D_A = D_\Sigma \otimes \mathbf{1} + A_\Sigma + \gamma^5 \otimes D_F + \gamma^5 \otimes \Phi,$$

with D_Σ descending from the split-biquaternionic gradient structure, and (A_Σ, D_F, Φ) encoding the algebraic fluctuations associated with the internal bioctonionic sector.

3.2. Full Sectorwise Expansion of the GTD Lagrangian

Introduce the dimensionless ratio

$$\eta := \frac{L_{\mathrm{P}}^2}{L^2}, \tag{26}$$

and define the shifted variables following Raj–Singh [9]

$$q_1^\dagger := q_B^\dagger + \eta\,\beta_1 q_F^\dagger, \qquad q_2 := q_B + \eta\,\beta_2 q_F. \tag{27}$$

Both the bosonic configuration variable q_B and the (matrix-valued) fermionic configuration variable q_F carry the same length dimension L in our conventions, so the additive combination in (27) is dimensionally consistent: the factor η is dimensionless and $\beta_{1,2}$ are dimensionless projectors on the fermionic sector. Then the fundamental GTD Lagrangian can be written as

$$\mathcal{L}_{\mathrm{GTD}} = \frac{L_{\mathrm{P}}^2}{2L^2}\,\mathrm{Tr}\Big[\dot{q}_1^\dagger \dot{q}_2 + \frac{\alpha^2}{L^2} q_1^\dagger q_2 + \frac{\mathrm{i}\alpha}{L}\left(\dot{q}_1^\dagger q_2 - q_1^\dagger \dot{q}_2\right)\Big]. \tag{28}$$

This is equivalent to $\mathcal{L}_{\mathrm{GTD}} = \frac{1}{2}\,\mathrm{Tr}[\eta\,\dot{Q}_1^\dagger \dot{Q}_2]$ once $\dot{Q}_j = \dot{q}_j + (\mathrm{i}\alpha/L)q_j$. The Connes-time Lagrangian is dimensionless, as required since the action $S/\hbar$ is dimensionless and the Connes-time integration measure $d\tau/\tau_{\mathrm{P}}$ is dimensionless. To see this term by term: with q of dimension L and $\dot{q}$ dimensionless (so that $\dot{Q}_F = (\mathrm{i}\alpha q_F + L\dot{q}_F)/L$ is dimensionless), each of $\dot{q}_1^\dagger \dot{q}_2$, $\alpha^2 q_1^\dagger q_2/L^2$, and $(\mathrm{i}\alpha/L)(\dot{q}_1^\dagger q_2 - q_1^\dagger \dot{q}_2)$ is dimensionless. The dimensionless prefactor $\eta/2 = L_{\mathrm{P}}^2/(2L^2)$ preserves this overall. Equivalently, in terms of the compact bosonic and fermionic *normalized* GTD variables,

$$\mathcal{D}_B := \dot{Q}_B = \mathcal{D}_{\mathrm{GTD}} + \Phi_{B,0}, \qquad \Psi_F := \dot{Q}_F = \frac{1}{L}(\mathrm{i}\alpha q_F + L\dot{q}_F), \tag{29}$$

with the useful fermionic decomposition

$$\Psi_F = \Psi_q + \Psi_\ell, \qquad \Psi_q := \frac{\mathrm{i}\alpha}{L}\,q_F, \qquad \Psi_\ell := \dot{q}_F. \tag{30}$$

Here Ψ_ℓ is the dotted split-biquaternionic contribution associated with leptonic modes, while Ψ_q is the undotted octonionic contribution associated with quark modes. On the bosonic side, by contrast, one

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must keep the split of Eq. (24): the genuine Dirac variable is $\mathcal{D}_{\text{GTD}}$, while $\Phi_{B,0}$ is the bosonic scalar seed.

This leptonic/quark split is not merely a matter of notation; it reflects the underlying division-algebraic Clifford structure. As discussed in [32] complex split biquaternions realize the algebra $Cl(3)$, which is already sufficient to describe one generation of chiral leptons on the split-signature $(3,3)$ spacetime scaffold. This is precisely the data carried by the dotted fermionic sector $\dot{q}_F$, which remains entirely split-biquaternionic. By contrast, the undotted sector q_F appears only after extending to complex split bioctonions, whose associated Clifford structure $Cl(7)$ incorporates the extra internal directions needed for the quark sector and its $\text{SU}(3)_{\text{color}}$ symmetry. In this sense, quaternions are already adequate for leptons, whereas quarks require the octonionic enlargement.

This viewpoint also clarifies two structural features of the model. First, leptons are naturally color singlets because the purely dotted split-biquaternionic sector does not yet carry the octonionic internal directions that generate $\text{SU}(3)_{\text{color}}$. Second, the weak and gravitational leafwise operators are universal for leptons and quarks alike because both sectors share the same quaternionic differential scaffold; what distinguishes quarks is the additional octonionic internal structure carried by the undotted variables.

One then has

$$\mathcal{L}_{\text{GTD}} = \frac{L_{\text{P}}^2}{2L^2}\,\text{Tr}\Big[\big(\mathcal{D}_B^\dagger + \eta\beta_1\Psi_F^\dagger\big)\big(\mathcal{D}_B + \eta\beta_2\Psi_F\big)\Big]. \tag{31}$$

Opening the brackets gives the full sector decomposition

$$\mathcal{L}_{\text{GTD}} = \mathcal{L}_{BB} + \mathcal{L}_{BF} + \mathcal{L}_{FF}, \tag{32}$$

with

$$\mathcal{L}_{BB} = \frac{L_{\text{P}}^2}{2L^2}\,\text{Tr}\Big(\mathcal{D}_B^\dagger\mathcal{D}_B\Big), \tag{33}$$

$$\mathcal{L}_{BF} = \frac{L_{\text{P}}^4}{2L^4}\,\text{Tr}\Big(\mathcal{D}_B^\dagger\beta_2\Psi_F + \beta_1\Psi_F^\dagger\mathcal{D}_B\Big), \tag{34}$$

$$\mathcal{L}_{FF} = \frac{L_{\text{P}}^6}{2L^6}\,\text{Tr}\Big(\beta_1\Psi_F^\dagger\beta_2\Psi_F\Big). \tag{35}$$

The bosonic piece expands to

$$\begin{aligned}\mathcal{L}_{BB} = \frac{L_{\text{P}}^2}{2L^4}\,\text{Tr}\Big[&\alpha^2 q_B^\dagger q_B + L^2\dot{q}_B^\dagger\dot{q}_B \\ &+ \text{i}\alpha L\big(\dot{q}_B^\dagger q_B - q_B^\dagger\dot{q}_B\big)\Big],\end{aligned} \tag{36}$$

while the cross and bifermionic pieces are

$$\begin{aligned}\mathcal{L}_{BF} = \frac{L_{\text{P}}^4}{2L^6}\,\text{Tr}\Big[&L^2\big(\dot{q}_B^\dagger\beta_2\dot{q}_F + \beta_1\dot{q}_F^\dagger\dot{q}_B\big) + \alpha^2\big(q_B^\dagger\beta_2 q_F + \beta_1 q_F^\dagger q_B\big) \\ &+ \text{i}\alpha L\big(\dot{q}_B^\dagger\beta_2 q_F + \beta_1\dot{q}_F^\dagger q_B - q_B^\dagger\beta_2\dot{q}_F - \beta_1 q_F^\dagger\dot{q}_B\big)\Big],\end{aligned} \tag{37}$$

$$\begin{aligned}\mathcal{L}_{FF} = \frac{L_{\text{P}}^6}{2L^8}\,\text{Tr}\Big[&L^2\beta_1\dot{q}_F^\dagger\beta_2\dot{q}_F + \alpha^2\beta_1 q_F^\dagger\beta_2 q_F \\ &+ \text{i}\alpha L\big(\beta_1\dot{q}_F^\dagger\beta_2 q_F - \beta_1 q_F^\dagger\beta_2\dot{q}_F\big)\Big].\end{aligned} \tag{38}$$

The bosonic sector itself admits a second, physically important decomposition. Using Eq. (24),

$$\mathcal{L}_{BB} = \mathcal{L}_{DD} + \mathcal{L}_{D\Phi} + \mathcal{L}_{\Phi\Phi}, \tag{39}$$

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where

$$\mathcal{L}_{DD} = \frac{L_{\mathrm{P}}^2}{2L^2}\,\mathrm{Tr}\Big(\mathcal{D}_{\mathrm{GTD}}^{\dagger}\mathcal{D}_{\mathrm{GTD}}\Big), \tag{40}$$

$$\mathcal{L}_{D\Phi} = \frac{L_{\mathrm{P}}^2}{2L^2}\,\mathrm{Tr}\Big(\mathcal{D}_{\mathrm{GTD}}^{\dagger}\Phi_{B,0} + \Phi_{B,0}^{\dagger}\mathcal{D}_{\mathrm{GTD}}\Big), \tag{41}$$

$$\mathcal{L}_{\Phi\Phi} = \frac{L_{\mathrm{P}}^2}{2L^2}\,\mathrm{Tr}\Big(\Phi_{B,0}^{\dagger}\Phi_{B,0}\Big). \tag{42}$$

Thus the full GTD expansion makes the bosonic scalar contribution explicit: $\mathcal{L}_{DD}$ is the genuine Dirac/vector precursor, while $\mathcal{L}_{\Phi\Phi}$ and $\mathcal{L}_{D\Phi}$ contain the two bosonic zeroth-mode Higgs seeds $\Phi_{B,L}$ and $\Phi_{B,R}$ together with their mixed couplings. The key point is that only $\mathcal{L}_{DD}$ is the direct Dirac precursor. The full bosonic sector $\mathcal{L}_{BB}$ already contains the two branch-resolved scalar antecedents before one ever turns to the bifermionic sector.

Equation (32) is the full expansion needed for the GTD-to-low-energy map. The cross sector $\mathcal{L}_{BF}$ and the bifermionic sector $\mathcal{L}_{FF}$ still carry information that must be used before the completed low-energy operator and action are assembled. In particular, the finite/internal operator $\mathcal{D}_F$ and the composite off-diagonal bridge are assembled from the fermionic sectors and are not modified by the bosonic zeroth-mode subtraction that defines $\mathcal{D}_{\mathrm{GTD}}$.

3.3. Bosonic Sector and the Quadratic Spectral Functional

The bosonic GTD dictionary has three pieces. The term $\mathcal{L}_{DD}$ is the natural GTD precursor of the differential-plus-vector part of the bosonic spectral functional. The term $\mathcal{L}_{\Phi\Phi}$ is the bosonic two-Higgs seed sector coming from the dotted bosonic zeroth quaternionic modes on the two branches. The mixed term $\mathcal{L}_{D\Phi}$ records the fact that, in the original GTD trace action, these bosonic Dirac and two-scalar channels appear inside the same normalized variable $\dot{Q}_B$.

In the low-energy almost-commutative description, the *physical* differential-plus-vector operator is

$$D_{\mathrm{vec}}^{\mathrm{phys}} := D_{\Sigma}^{\mathrm{phys}} \otimes \mathbf{1} + A_{\Sigma}^{\mathrm{phys}}, \tag{43}$$

with normalized dimensionless counterpart

$$D_{\mathrm{vec}} := L\,D_{\mathrm{vec}}^{\mathrm{phys}}. \tag{44}$$

Similarly, the physical cutoff Λ_{phys} and the normalized cutoff Λ are related by

$$\Lambda = L\,\Lambda_{\mathrm{phys}} = \frac{L}{L_{\mathrm{P}}}, \qquad \Lambda_{\mathrm{phys}} = \frac{1}{L_{\mathrm{P}}}. \tag{45}$$

At the bosonic level one has the algebraic correspondence

$$\mathcal{L}_{DD} \longleftrightarrow \frac{1}{2\Lambda^2}\,\mathrm{Tr}\Big(D_{\mathrm{vec}}^{\dagger}D_{\mathrm{vec}}\Big) = \frac{L_{\mathrm{P}}^2}{2}\,\mathrm{Tr}\Big[(D_{\mathrm{vec}}^{\mathrm{phys}})^{\dagger}D_{\mathrm{vec}}^{\mathrm{phys}}\Big]. \tag{46}$$

The bosonic dotted zeroth-mode seed is represented by the normalized scalar variable $\Phi_{B,0}$ and its physical counterpart

$$\Phi_{B,0}^{\mathrm{phys}} := \frac{1}{L}\Phi_{B,0}. \tag{47}$$

Accordingly, the full bosonic GTD precursor may be viewed algebraically as

$$\mathcal{L}_{BB} \longleftrightarrow \frac{1}{2\Lambda^2}\,\mathrm{Tr}\Big[(D_{\mathrm{vec}} + \Phi_{B,0})^{\dagger}(D_{\mathrm{vec}} + \Phi_{B,0})\Big], \tag{48}$$

with the important caveat that, in the completed almost-commutative geometry, $\Phi_{B,0}$ is inserted not as an extra spacetime derivative but as part of the scalar fluctuation of the finite/internal sector.

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Thus the formulas below are written in terms of normalized dimensionless Dirac variables, while the corresponding physical Dirac operators are obtained by dividing by L. The scope of the bosonic identification is as follows: $\mathcal{L}_{BB}$ already provides both the differential-plus-vector precursor and the bosonic scalar seed, but it does *not* yet provide the full chiral bridge/Yukawa structure. That additional off-diagonal information comes from the fermionic sectors discussed below.

3.4. Cross Terms and the Fermionic Pairing

The aim of this subsection is to identify the GTD cross sector $\mathcal{L}_{BF}$, given in compact form by Eq. (34) and in expanded form by Eq. (37), as the GTD-side counterpart of the emergent sesquilinear fermionic pairing $\langle\Psi, \mathcal{D}_A^{\mathrm{phys}}\Psi\rangle$ of the spectral action. Under the localization hypothesis stated below (Eq. (51)), the GTD-side normalized fermionic variable Ψ_F from Eq. (29) and the emergent-side physical fermion field $\Psi(x)$ from Eq. (49) are not distinct objects: they are the same physical fermionic content viewed before and after localization, and the lepton/quark decomposition $\Psi_F = \Psi_\ell + \Psi_q$ in Eq. (30) is the same decomposition as $\Psi(x) = \Psi_\ell(x) + \Psi_q(x)$ in Eq. (50), just on the two sides of the localization map. The conditional reduction performed in this subsection is therefore the explicit map

$$\mathcal{L}_{BF}[\mathcal{D}_B, \Psi_F] \xrightarrow{\text{localization}} \langle\Psi, \mathcal{D}_A^{\mathrm{phys}}\Psi\rangle,$$

relating the GTD cross sector of Eq. (34) to the emergent fermionic pairing.

The expanded form of $\mathcal{L}_{BF}$ in Eq. (37) contains three distinct structures: dotted–dotted couplings $\dot{q}_B^\dagger \dot{q}_F$, undotted–undotted couplings $q_B^\dagger q_F$, and mixed αL bridge terms that couple the dotted and undotted sectors. In the present interpretation these correspond, respectively, to the split-biquaternionic/leptonic channel, the octonionic/quark channel, and the channel that glues the two inside the unified GTD fermionic variable.

To connect this structure to the conventional fermionic action it is useful to follow Landi–Rovelli and expand the localized physical fermion field in an eigenspinor basis of the completed Dirac operator [17]. Let

$$\mathcal{D}_A^{\mathrm{phys}}\psi_n = \lambda_n^{\mathrm{phys}}\psi_n, \qquad \Psi(x) = \sum_n \hat{a}_n\,\psi_n(x), \tag{49}$$

where, in the spirit of trace dynamics, the coefficients $\hat{a}_n$ are odd operator-valued mode amplitudes before localization. (The lowercase ψ_n are the c-number eigenspinors of $\mathcal{D}_A^{\mathrm{phys}}$; the uppercase $\Psi(x)$ is the emergent-side fermion field, distinct in notation from the GTD-side configuration variable Ψ_F but connected to it by the localization map below.) To retain the GTD lepton/quark bookkeeping we also write

$$\Psi(x) = \Psi_\ell(x) + \Psi_q(x), \qquad \Psi_\ell(x) = \sum_n \hat{a}_n^{(\ell)}\psi_n(x), \qquad \Psi_q(x) = \sum_n \hat{a}_n^{(q)}\psi_n(x), \tag{50}$$

with $\hat{a}_n = \hat{a}_n^{(\ell)} + \hat{a}_n^{(q)}$. The localization hypothesis needed in the present paper is that the odd GTD variables descend to these coefficients,

$$\mathcal{L}_{\mathrm{loc}}^{(F)}: \qquad \dot{q}_F \mapsto \{\hat{a}_n^{(\ell)}\}, \qquad \frac{\mathrm{i}\alpha}{L}q_F \mapsto \{\hat{a}_n^{(q)}\}, \tag{51}$$

while the localized bosonic variable determines the matrix elements of the completed operator $\mathcal{D}_A^{\mathrm{phys}}$. Under this map the GTD-side leptonic component $\Psi_\ell = \dot{q}_F$ maps to the emergent leptonic mode expansion $\Psi_\ell(x)$, and the GTD-side quark component $\Psi_q = (\mathrm{i}\alpha/L)q_F$ maps to the emergent quark mode expansion $\Psi_q(x)$. The uppercase Ψ symbol carries the same physical content on both sides; the distinction is operator-valued (GTD side) versus mode-expanded (emergent side).

With this notation one has the exact sesquilinear identity

$$\langle\Psi, \mathcal{D}_A^{\mathrm{phys}}\Psi\rangle = \sum_{m,n}\hat{a}_m^\dagger\,K_{mn}\,\hat{a}_n, \qquad K_{mn} := \langle\psi_m, \mathcal{D}_A^{\mathrm{phys}}\psi_n\rangle, \tag{52}$$

and, in an eigenspinor basis of the completed operator,

$$K_{mn} = \lambda_n^{\mathrm{phys}}\,\delta_{mn}, \qquad \langle \Psi, \mathcal{D}_A^{\mathrm{phys}} \Psi\rangle = \sum_n \lambda_n^{\mathrm{phys}}\,\hat{a}_n^\dagger \hat{a}_n. \tag{53}$$

If one wishes to keep the lepton/quark channels separate before the final diagonalization, it is convenient to assemble the coefficients into a two-component vector

$$\hat{A}_n := \begin{pmatrix} \hat{a}_n^{(\ell)} \\ \hat{a}_n^{(q)} \end{pmatrix}, \qquad \mathbf{K}_{mn} := \begin{pmatrix} K_{mn}^{\ell\ell} & K_{mn}^{\ell q} \\ K_{mn}^{q\ell} & K_{mn}^{qq} \end{pmatrix}, \tag{54}$$

so that

$$\langle \Psi, \mathcal{D}_A^{\mathrm{phys}} \Psi\rangle = \sum_{m,n} \hat{A}_m^\dagger\,\mathbf{K}_{mn}\,\hat{A}_n. \tag{55}$$

Equation (37) has the same sesquilinear architecture as (55), but the physical interpretation needs one more representation-theoretic step. Before localization the mixed αL terms are off-diagonal in the dotted/undotted bookkeeping. After localization to a basis adapted to the finite algebra, however, the physical low-energy operator should be block-diagonal in the lepton/quark split because visible color is conserved.

For one generation, write

$$H_\ell^{(1)} := (L_L \oplus e_R \oplus \nu_R) \oplus (L_L^c \oplus e_R^c \oplus \nu_R^c), \qquad H_q^{(1)} := (Q_L \oplus u_R \oplus d_R) \oplus (Q_L^c \oplus u_R^c \oplus d_R^c), \tag{56}$$

with orthogonal projectors P_ℓ and P_q. The localization map is then required to preserve the representation content,

$$\mathcal{L}_{\mathrm{loc}}^{(F)}(\dot{q}_F) \subset H_\ell^{(1)}, \qquad \mathcal{L}_{\mathrm{loc}}^{(F)}\!\left(\frac{\mathrm{i}\alpha}{L} q_F\right) \subset H_q^{(1)}. \tag{57}$$

On the observed low-energy truncation the finite algebra $A_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$ acts trivially on $H_\ell^{(1)}$ and non-trivially on $H_q^{(1)}$ only through the color factor. Since D_Σ^{phys} is color-blind, A_Σ^{phys} is color-diagonal, and the finite operators (D_F, Φ) are color singlets, one has

$$[P_\ell, \mathcal{D}_A^{\mathrm{phys}}] = 0, \qquad [P_q, \mathcal{D}_A^{\mathrm{phys}}] = 0, \qquad P_\ell \mathcal{D}_A^{\mathrm{phys}} P_q = P_q \mathcal{D}_A^{\mathrm{phys}} P_\ell = 0 \tag{58}$$

in the physical basis. Consequently,

$$K_{mn}^{\ell q} = K_{mn}^{q\ell} = 0, \qquad \langle \Psi, \mathcal{D}_A^{\mathrm{phys}} \Psi\rangle = \sum_{m,n} (\hat{a}_m^{(\ell)})^\dagger K_{mn}^{\ell\ell} \hat{a}_n^{(\ell)} + \sum_{m,n} (\hat{a}_m^{(q)})^\dagger K_{mn}^{qq} \hat{a}_n^{(q)}. \tag{59}$$

The mixed αL terms in Eq. (37) should therefore be read as pre-localized bridge terms required by the unified GTD fermionic variable. After localization and projection onto irreducible gauge sectors, they collapse onto the physical diagonal leptonic and quark pairings rather than producing observable lepton–quark mixing.

Conditional reduction of the fermionic sector.

Assume that (i) the odd GTD matrices localize according to Eqs. (51) and (57), and (ii) the localized bosonic variable determines the matrix kernel of the completed operator $\mathcal{D}_A^{\mathrm{phys}}$. Then the GTD cross sector $\mathcal{L}_{BF}$ in Eq. (34), summed over the localized STM ensemble, reduces to the standard sesquilinear fermionic action on the emergent leaf:

$$\mathcal{L}_{BF}[\mathcal{D}_B, \Psi_F] \xrightarrow{\text{localization}} S_{BF}^{\mathrm{loc}} = \sum_{m,n} (\hat{a}_m^{(\ell)})^\dagger K_{mn}^{\ell\ell} \hat{a}_n^{(\ell)} + \sum_{m,n} (\hat{a}_m^{(q)})^\dagger K_{mn}^{qq} \hat{a}_n^{(q)} = \langle \Psi, \mathcal{D}_A^{\mathrm{phys}} \Psi\rangle, \tag{60}$$

which in an eigenbasis becomes (53). The left-hand side is the GTD cross sector of Eq. (34), viewed as a function of the GTD-side variables $\mathcal{D}_B$ and Ψ_F; the right-hand side is the standard sesquilinear fermionic pairing of the spectral action, viewed as a function of the emergent-side variable $\Psi(x)$. The intermediate sums make the algebraic content of the reduction explicit in the lepton/quark basis. Thus the GTD cross sector $\mathcal{L}_{BF}$ furnishes, under stated hypotheses, an explicit conditional reduction to the standard fermionic action $\langle \Psi, \mathcal{D}_A^{\text{phys}} \Psi \rangle$, with a representation-theoretic explanation of why physical lepton–quark mixing does not appear in the localized Dirac operator.

This does not yet amount to a full first-principles derivation from the GTD variational principle alone, because the localization map itself still has to be derived dynamically. But it is stronger than a purely formal analogy: the GTD cross sector and the low-energy fermionic action are now related by an explicit sesquilinear reduction.

3.5. Bifermionic Terms and the Scalar Bridge Field

The scalar dictionary has two bosonic/fermionic inputs. The bosonic dotted zeroth quaternionic modes provide a *pair* of scalar seeds, $\Phi_{B,L} = \dot{q}_{B0L}$ and $\Phi_{B,R} = \omega \dot{q}_{B0R}$, already inside $\mathcal{L}_{BB}$. The sector $\mathcal{L}_{FF}$ then supplies the additional finite/internal scalar data that are needed to turn these bosonic seeds into off-diagonal Higgs bridges with Yukawa couplings. This is consistent with the comparison paper, which states that terms of schematic type q_F^2 encode the Higgs-potential sector. In the bosonic split, the bosonic scalar seeds are carried by the dotted quaternionic zeroth modes rather than by a separate undotted bosonic scalar direction [16]. The important refinement is that the single-STM action supplies a *bilinear seed*, whereas a literal Hubbard–Stratonovich transformation should be applied only after an effective quartic scalar channel has been generated by localization and coarse-graining. In the $E_8 \times E_8$ interpretation, $\Phi_{B,R}$ is identified with the branch whose breaking is responsible for ordinary mass generation, while $\Phi_{B,L}$ supplies the second Higgs associated with the complementary branch and the acquisition of electric charge after left–right symmetry breaking [7,12].

A convenient composite scalar channel is the e_0 projection of the bifermionic bilinear,

$$\mathcal{O}_H := P_{e_0}\Big(\beta_1 \Psi_F^\dagger \beta_2 \Psi_F\Big), \tag{61}$$

with P_{e_0} denoting projection onto the scalar bridge direction. The bare single-STM bifermionic sector then provides the bilinear seed

$$\mathcal{L}_{FF} = \frac{L_P^6}{2L^6} \operatorname{Tr}\Big(\beta_1 \Psi_F^\dagger \beta_2 \Psi_F\Big) \quad \rightsquigarrow \quad \mathcal{O}_H, \tag{62}$$

which is to be combined with the bosonic scalar seed $\Phi_{B,0}$ rather than viewed as the entire Higgs sector by itself.

To check that this channel can really play the role of the Higgs bridge, it is useful to resolve the visible one-generation quantum numbers carried by the bilinear. Using the multiplets in Appendix A, the relevant color-singlet scalar channels arise from

$$\begin{aligned} \bar{d}_R Q_L &\sim (\bar{\mathbf{3}}, \mathbf{1}, +1/3) \otimes (\mathbf{3}, \mathbf{2}, +1/6) = (\mathbf{1} \oplus \mathbf{8}, \mathbf{2}, +1/2), \\ \bar{e}_R L_L &\sim (\mathbf{1}, \mathbf{1}, +1) \otimes (\mathbf{1}, \mathbf{2}, -1/2) = (\mathbf{1}, \mathbf{2}, +1/2), \\ \bar{u}_R Q_L &\sim (\bar{\mathbf{3}}, \mathbf{1}, -2/3) \otimes (\mathbf{3}, \mathbf{2}, +1/6) = (\mathbf{1} \oplus \mathbf{8}, \mathbf{2}, -1/2), \\ \bar{\nu}_R L_L &\sim (\mathbf{1}, \mathbf{1}, 0) \otimes (\mathbf{1}, \mathbf{2}, -1/2) = (\mathbf{1}, \mathbf{2}, -1/2). \end{aligned} \tag{63}$$

Thus the color-singlet part of the e_0 channel contains precisely two electroweak-doublet scalar channels,

$$\mathcal{O}_{H_d} \sim P_{1_c, e_0}(\bar{d}_R Q_L + \bar{e}_R L_L) \in (\mathbf{1}, \mathbf{2}, +1/2), \qquad \mathcal{O}_{H_u} \sim P_{1_c, e_0}(\bar{u}_R Q_L + \bar{\nu}_R L_L) \in (\mathbf{1}, \mathbf{2}, -1/2), \tag{64}$$

together with color-octet companions that are removed by the singlet projection and/or assumed heavy in the low-energy truncation. In other words, the GTD bifermionic seed is representation-theoretically compatible with the usual Higgs doublet and its conjugate. For notational compactness the discussion below continues to write a single symbol $\mathcal{O}_H$ and coupling G_H; when the two visible

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channels are resolved explicitly, one simply replaces them by $(\mathcal{O}_{H_u}, \mathcal{O}_{H_d})$ and channel-dependent couplings (G_u, G_d).

At the microscopic GTD level the visible e_0-projected channel need not by itself be self-adjoint. In view of the intrinsic anti-self-adjoint contribution of the fermionic GTD Hamiltonian [29], Eq. (65) should be read as the coarse-grained effective quartic channel for the *self-adjoint part* of the visible scalar projection. The explicit derivation of this self-adjoint reduction remains open and is recorded in Section 11.

The localized effective theory is assumed to generate an attractive quartic channel of Nambu–Jona-Lasinio type,

$$S_H^{(4F)} = -G_H \int d^4x \sqrt{g}\, \mathcal{O}_H^\dagger \mathcal{O}_H, \qquad G_H > 0, \tag{65}$$

through cumulants of the localized ensemble, integrating out fast Connes-time modes, or an equivalent coarse-graining procedure. Equation (65) is the appropriate starting point for an auxiliary-field completion. The auxiliary-field completion used here is in the spirit of the Nambu–Jona-Lasinio composite-scalar mechanism and later dynamical Higgs/top-condensation models [34,35].

Introduce an off-diagonal auxiliary field Φ_{LR} and define the Euclidean action

$$S_{\text{aux}}[\Phi_{LR}, \Psi_F] = \int d^4x \sqrt{g}\left[\frac{1}{G_H}\, \Phi_{LR}^\dagger \Phi_{LR} - \Phi_{LR}^\dagger \mathcal{O}_H - \mathcal{O}_H^\dagger \Phi_{LR}\right]. \tag{66}$$

Completing the square gives

$$S_{\text{aux}} = \int d^4x \sqrt{g}\left[\frac{1}{G_H}\, |\Phi_{LR} - G_H \mathcal{O}_H|^2 - G_H\, \mathcal{O}_H^\dagger \mathcal{O}_H\right], \tag{67}$$

so integrating out Φ_{LR} reproduces the quartic channel (65) up to a field-independent determinant. Conversely, keeping Φ_{LR} as the fundamental infrared variable yields the Yukawa-like bridge coupling

$$\mathcal{L}_{\Phi\Psi\Psi} = -\Phi_{LR}^\dagger \mathcal{O}_H - \mathcal{O}_H^\dagger \Phi_{LR} \quad \longrightarrow \quad -\bar{\Psi}_L\, \Phi_{LR}\, Y\, \Psi_R - \bar{\Psi}_R\, \Phi_{LR}^\dagger\, Y^\dagger\, \Psi_L, \tag{68}$$

and a bosonic mass term $G_H^{-1} \Phi_{LR}^\dagger \Phi_{LR}$.

At the next stage, integrating out fermionic fluctuations around the localized background produces the scalar effective potential,

$$V_{\text{eff}}(\Phi_{LR}) = m_\Phi^2\, \Phi_{LR}^\dagger \Phi_{LR} + \lambda_\Phi\, (\Phi_{LR}^\dagger \Phi_{LR})^2 + \cdots, \qquad m_\Phi^2 = \frac{1}{G_H} - \Pi_2(0), \tag{69}$$

where $\Pi_2(0)$ denotes the quadratic fermion loop contribution and the quartic term is induced similarly. For the visible one-generation channels of Eq. (64), a hard-cutoff NJL estimate with cutoff Λ_H gives

$$m_{u,d}^2 \simeq \frac{1}{G_{u,d}} - \frac{\Lambda_H^2}{8\pi^2}\, \Xi_{u,d}, \qquad \Xi_u := 3|y_u|^2 + |y_\nu|^2, \qquad \Xi_d := 3|y_d|^2 + |y_e|^2, \tag{70}$$

and

$$\lambda_{u,d} \simeq \frac{\ln(\Lambda_H^2/\mu^2)}{8\pi^2}\, \Sigma_{u,d}, \qquad \Sigma_u := 3|y_u|^4 + |y_\nu|^4, \qquad \Sigma_d := 3|y_d|^4 + |y_e|^4. \tag{71}$$

Accordingly, symmetry breaking occurs in a given channel when

$$G_{u,d} > G_{u,d}^{\text{crit}} := \frac{8\pi^2}{\Lambda_H^2\, \Xi_{u,d}}. \tag{72}$$

These formulas do not derive the couplings $G_{u,d}$ from first principles, but they do show that once the localized ensemble produces an attractive quartic channel, the resulting composite scalar has the correct electroweak quantum numbers and a standard NJL mean-field instability toward Higgs-bridge condensation. In the broken-phase parametrization used later one may rewrite Φ_{LR} in terms of visible and right-sector multiplets Φ_L and Φ_R; the mixed coupling λ_{LR} then arises either from mixed quartic channels already present in the localized effective action or from radiative communication between the two sectors.

This auxiliary-field completion also clarifies the double-counting issue in the scalar bookkeeping. The field Φ_{LR} introduced in (66) is *not* an extra Higgs on top of the bosonic zeroth modes. Rather, the effective low-energy scalar fluctuations should be viewed as being assembled branchwise from the bosonic scalar seeds and the composite bridge data,

$$\Phi_L^{\text{eff}} := \Phi_{B,L} + \eta_L\,\Phi_{LR}, \qquad \Phi_R^{\text{eff}} := \Phi_{B,R} + \eta_R\,\Phi_{LR}, \tag{73}$$

with real mixing coefficients η_L and η_R to be fixed by the detailed finite geometry and symmetry-breaking pattern. Thus $\Phi_{B,L}$ supplies the left-branch scalar antecedent, $\Phi_{B,R}$ supplies the right-branch scalar antecedent, and Φ_{LR} supplies the off-diagonal chiral/Yukawa bridge. In the GTD/NCG dictionary, the low-energy two-Higgs sector is assembled from two bosonic branch seeds and one composite bridge field; non-commutative geometry then explains how the resulting scalar doublets enter the fluctuated finite Dirac operator and the spectral action.

Accordingly, the correct chain is

$$\mathcal{L}_{BB}^{(0)} \longrightarrow (\Phi_{B,L}, \Phi_{B,R}), \qquad \mathcal{L}_{FF}^{\text{bare}} \longrightarrow \mathcal{O}_H \longrightarrow S_H^{(4F)} \longleftrightarrow S_{\text{aux}}[\Phi_{LR}, \Psi_F] \longrightarrow (\mathcal{D}_F, \Phi_{LR}), \tag{74}$$

which is stronger than a purely symbolic arrow, while still depending on the coarse-grained effective hypothesis that generates (65). The completed low-energy two-Higgs sector is obtained only after these bosonic and bifermionic inputs are assembled together.

3.6. From the full GTD expansion to the low-energy action

The low-energy action is therefore assembled *sectorwise*. The GTD decomposition (32) is mapped to the effective action by

$$S_{\text{eff}}[g, A, \Phi, \Psi] = \underbrace{\operatorname{Tr}[f(\mathcal{D}_A/\Lambda)]}_{\text{bosonic spectral action in normalized variables}} + \underbrace{\langle \Psi, \mathcal{D}_A^{\text{phys}} \Psi \rangle}_{\text{fermionic pairing}} + \underbrace{S_{\text{res}}^{(4F)}}_{\text{residual four-fermion terms, if any}} . \tag{75}$$

where $\mathcal{D}_A = L\mathcal{D}_A^{\text{phys}}$ is the normalized dimensionless Dirac operator, the bosonic sector $\mathcal{L}_{BB}$ fixes both the differential/vector part of $\mathcal{D}_A$ and the branch-resolved bosonic scalar seeds $(\Phi_{B,L}, \Phi_{B,R})$, the fermionic pairing is fixed by $\mathcal{L}_{BF}$, and the composite/off-diagonal scalar bridge data $(\mathcal{D}_F, \Phi_{LR})$ are fixed by $\mathcal{L}_{FF}$. If the bifermionic channel is fully bosonized, the residual term $S_{\text{res}}^{(4F)}$ vanishes and the standard non-commutative-geometry form $\operatorname{Tr}[f(\mathcal{D}_A/\Lambda)] + \langle \Psi, \mathcal{D}_A^{\text{phys}} \Psi \rangle$ is recovered, with the effective two-Higgs fluctuation assembled from $(\Phi_{B,L}, \Phi_{B,R})$ and Φ_{LR}.

4. Heat-Kernel Expansion of the Bosonic Spectral Action

4.1. Product Geometry and Dimensional Reduction

After the Wesley–Singh–Isidro six- to four-dimensional reduction [10], the six-dimensional operator is projected to a four-dimensional leaf Σ and coupled to a finite internal space F determined by the E_6 trinification. The completed almost-commutative operator entering the bosonic spectral action has a physical form and a normalized form. The physical operator is

$$\mathcal{D}_A^{\text{phys}} = D_\Sigma^{\text{phys}} \otimes \mathbf{1} + \gamma^5 \otimes \mathcal{D}_F^{\text{phys}} + A_\Sigma^{\text{phys}} + \gamma^5 \otimes \Phi^{\text{phys}}, \tag{76}$$

and the normalized dimensionless operator used in the GTD spectral formulas is

$$\mathcal{D}_A := L\,\mathcal{D}_A^{\rm phys}. \tag{77}$$

We follow the standard Connes–Chamseddine sign convention: γ^5 is the four-dimensional Euclidean chirality operator satisfying $(\gamma^5)^2 = +\mathbf{1}$ and $\{\gamma^5, D_\Sigma\} = 0$ in the Wick-rotated/Euclidean form used throughout. This convention matches the one used in the Seeley–DeWitt heat-kernel expansion of Section 4 and in [6]. Here $D_\Sigma^{\rm phys}$ descends from the leafwise imaginary quaternionic sector identified in $\mathcal{L}_{DD}$, $A_\Sigma^{\rm phys}$ is the vector/gauge connection built from the non-scalar octonionic sector $q_{B,{\rm vec}}$, and the scalar fluctuation $\Phi^{\rm phys}$ is assembled from the bosonic branchwise zeroth-mode seeds $(\Phi_{B,L}^{\rm phys}, \Phi_{B,R}^{\rm phys})$ together with the off-diagonal bridge data extracted from $\mathcal{L}_{FF}$. The finite operator $\mathcal{D}_F^{\rm phys}$ remains the separate fermionic/Jordan operator and is not altered by the bosonic zeroth-mode subtraction. Equivalently, one may write $\mathcal{D}_A = L\mathcal{D}_A^{\rm phys}$, $\mathcal{D}_F = L\mathcal{D}_F^{\rm phys}$, and $\Phi = L\Phi^{\rm phys}$. The bosonic spectral action $\mathrm{Tr}[f(\mathcal{D}_A/\Lambda)]$ is the normalized form of $\mathrm{Tr}[f(\mathcal{D}_A^{\rm phys}/\Lambda_{\rm phys})]$, while the fermionic term is $\langle\Psi, \mathcal{D}_A^{\rm phys}\Psi\rangle$.

To justify the appearance of D_Σ more concretely, it is useful to make explicit the principal-symbol chain behind Assumption (A3) listed in Section 1.3 above. The six-dimensional reduction is formulated in terms of SO(3,3) BF data,

$$S_{BF}^{(3,3)}[B,A] = \int_{M_6} B_{IJ} \wedge F^{IJ}(A) + \Phi_{IJKL}\, B^{IJ} \wedge B^{KL}, \tag{78}$$

with indices $I, J = 1, \dots, 6$. When the simplicity constraints and symmetry-breaking conditions select a rank-four distribution corresponding to a given leaf, the B field takes the simple form

$$B^{ab}\big|_\Sigma \sim \epsilon^{ab}{}_{cd}\, e^c \wedge e^d, \qquad A^{ab}\big|_\Sigma \sim \omega^{ab}, \tag{79}$$

with $a, b = 0, 1, 2, 3$ on the leaf. One then recovers the Palatini data (e, ω) and hence the leafwise gravitational action.

The corresponding four-dimensional physical Dirac operator is

$$D_\Sigma^{\rm phys} = \mathrm{i}\gamma^a e_a{}^\mu \left(\partial_\mu + \frac{1}{4}\omega_{\mu bc}\gamma^{bc} \right), \tag{80}$$

whose principal symbol is

$$\sigma\left(D_\Sigma^{\rm phys}\right)(x,\xi) = \mathrm{i}\gamma^a e_a{}^\mu(x)\,\xi_\mu. \tag{81}$$

Thus, even before the full BF reduction is completed in detail, any leafwise GTD operator descending from the reduced BF data must share the same first-order principal symbol. This is the precise sense in which the assembled quaternionic operators $\dot{q}_{BR}^{\rm curved} = D_{4\rm curved}$ and $\dot{q}_{BL}^{\rm curved} = D'_{4\rm curved}$ are identified with the curved Dirac operators on the two leaves.

BF-to-Dirac principal-symbol map.

Assume that the Wesley–Singh–Isidro mechanism yields on each localized leaf a tetrad $e_a{}^\mu$ and spin connection $\omega_{\mu bc}$ extracted from the reduced BF variables. Then the quaternionic leaf operator used in the GTD action has the same principal symbol as the curved Dirac operator (80). Consequently the use of D_Σ inside the almost-commutative operator (76) is justified at the level of the leading differential structure, even though the full BF reduction still remains to be worked out in detail.

The internal symmetry is organized by the E_6 trinification:

$$E_{6L} \to \mathrm{SU}(3)_c \times \mathrm{SU}(3)_{F,L} \times \mathrm{SU}(3)_L, \qquad \mathrm{SU}(3)_L \to \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y, \tag{82}$$

$$E_{6R} \to \mathrm{SU}(3)_{c'} \times \mathrm{SU}(3)_{F,R} \times \mathrm{SU}(3)_R, \qquad \mathrm{SU}(3)_R \to \mathrm{SU}(2)_R \times \mathrm{U}(1)_{Y_{\rm dem}}. \tag{83}$$

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In the right branch we use the GTD interpretation $SU(3)_{c'} \equiv SU(3)_{grav}$ as a working hypothesis. This identification is structurally motivated by the split-octonionic construction, but it should still be regarded as provisional until the explicit finite geometry is fully fixed. The internal Hilbert space for three generations of Standard Model fermions (including right-handed neutrinos) has dimension

$$\dim \mathcal{H}_F = 96. \tag{84}$$

4.2. The Seeley–DeWitt Expansion

For standard derivations and normalization conventions for the heat-kernel / Seeley–DeWitt coefficients used throughout, see [36,50].

The heat-kernel expansion used here is naturally *leafwise*. That is, once the six-dimensional GTD/BF data have been reduced to a four-dimensional Lorentzian leaf with its corresponding almost-commutative operator $D_{A,\Sigma}$, the standard four-dimensional Seeley–DeWitt expansion applies on that leaf exactly as in ordinary spectral-action geometry. If both four-dimensional leaves are retained explicitly, one has a sum of two leafwise spectral actions,

$$S_{\rm spec}^{(2\,\rm leaves)} = \mathrm{Tr}\big[f(D_{A,L}/\Lambda)\big] + \mathrm{Tr}\big[f(D_{A,R}/\Lambda)\big],$$

and the corresponding finite traces in Appendix A double in the left–right symmetric limit. In the present paper, however, the explicit coefficient extraction is written for the observed leaf and its associated minimal finite geometry.

This does *not* mean that the second leaf is interpreted as a second visible copy of the Standard Model weak interaction. Rather, the same leafwise heat-kernel structure is read with different branch labels: on the observed leaf one recovers the visible $SU(3)_c \times SU(2)_L \times U(1)_Y$ interpretation, whereas the complementary leaf carries the right-sector / pre-gravitational data $SU(3)_{grav} \times SU(2)_R \times U(1)_{Y_{dem}}$. From the viewpoint of the observed low-energy sector, the second leaf therefore contributes the corresponding right-branch gauge/scalar data rather than a second visible weak interaction. The formulas below are thus to be read as the one-leaf coefficient formulas, with the second leaf obtained, when desired, by the replacement $L \leftrightarrow R$ together with the corresponding replacement $Y \to Y_{dem}$ and $t_L^i \to t_R^i$.

For a four-dimensional manifold, the spectral action has the heat-kernel expansion [5,6]

$$\mathrm{Tr}[f(\mathcal{D}_A/\Lambda)] \approx 2f_4\Lambda^4 a_0 + 2f_2\Lambda^2 a_2 + f_0 a_4 + \mathcal{O}(\Lambda^{-2}), \tag{85}$$

where $\mathcal{D}_A = L\mathcal{D}_A^{\rm phys}$ and $\Lambda = L\Lambda_{\rm phys} = L/L_{\rm P}$ are the normalized dimensionless variables, while the equivalent physical form is $\mathrm{Tr}[f(\mathcal{D}_A^{\rm phys}/\Lambda_{\rm phys})]$ with $\Lambda_{\rm phys} = 1/L_{\rm P}$. The coefficients f_k are moments of the cutoff function f. Appendix A supplies a candidate E_6-compatible finite geometry for the observed sector, while Appendix B gives an explicit regulator family with computable moments.

We note that the leafwise spectral expansion is the intermediate bookkeeping device, whereas only the observed leaf is retained as explicit spacetime in the final effective action, with the complementary leaf contributing right-sector gauge/scalar/pre-gravitational data rather than an independent second cosmological term.

4.3. The a_0 Coefficient: Cosmological Constant

The zeroth coefficient is

$$a_0 = \frac{1}{16\pi^2}\int d^4x\,\sqrt{g}\,\mathrm{tr}_F(\mathbf{1}_F) = \frac{96}{16\pi^2}\,\mathrm{Vol}(\Sigma_R). \tag{86}$$

This gives the cosmological constant

$$\Lambda_{\rm cosm} = \frac{12 f_4}{\pi^2}\,\frac{L^4}{L_{\rm P}^4}\cdot 16\pi G. \tag{87}$$

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4.4. The a_2 Coefficient: Einstein–Hilbert Action and Higgs Mass

The second coefficient is

$$a_2 = \frac{1}{16\pi^2} \int \mathrm{d}^4x \sqrt{g}\, \mathrm{tr}_F\Big(-\frac{R}{6} \mathbf{1}_F + \mathcal{D}_F^2 \Big). \tag{88}$$

The first term gives the Einstein–Hilbert action with Newton's constant

$$\frac{1}{16\pi G} = \frac{2 f_2 L^2}{\pi^2 L_{\mathrm{P}}^2}. \tag{89}$$

In the GTD interpretation this relation should be read as a coefficient-matching condition for the localized four-dimensional Einstein term, not as the definition of a new fundamental gravitational scale. The microscopic constants are L_{P}, τ_{P}, and $\hbar$; with $c = L_{\mathrm{P}}/\tau_{\mathrm{P}}$ one may write

$$G = \frac{L_{\mathrm{P}}^5}{\hbar\, \tau_{\mathrm{P}}^3}. \tag{90}$$

Thus the electroweak/localization scale discussed later is an emergent transition scale at which the low-energy coefficient is matched, whereas Newton's constant is already fixed at the one-STM level [37].

For the scalar sector the GTD/NCG dictionary is slightly richer. The non-scalar bosonic variable $q_{B,\mathrm{vec}}$ supplies the gauge/vector fluctuation, the bosonic dotted zeroth mode $\dot{q}_{B0}$ supplies the bosonic scalar seed $\Phi_{B,0}$, and the effective finite operator $\mathcal{D}_F$ together with the bifermionic channel supplies the off-diagonal chirality-mixing bridge. A minimal block form is

$$\mathcal{D}_F = \begin{pmatrix} 0 & M^\dagger \\ M & 0 \end{pmatrix} \oplus \text{conjugate sector}, \qquad M = \mathrm{diag}(Y_\nu, Y_e, Y_u \otimes \mathbf{1}_3, Y_d \otimes \mathbf{1}_3), \tag{91}$$

with the Jordan ansatz $Y_f = y_f Y_J$ detailed in Appendix A. After inner fluctuation one has $\mathcal{D}_F \mapsto \mathcal{D}_F + \Phi + \Phi^\dagger$, where Φ is off-diagonal between the left- and right-chiral finite subspaces. In GTD variables this effective scalar is assembled from the bosonic dotted zeroth-mode seed $\Phi_{B,0}$ together with the bosonized e_0-channel of the bifermionic term $q_F^\dagger q_F$.

The a_2 coefficient therefore yields the scalar mass term through the Yukawa trace, but with a normalization that must be matched to the bosonic dotted zeroth-mode seed discussed above,

$$a_{\mathrm{eff}} = \mathrm{tr}\Big(Y_\nu^\dagger Y_\nu + Y_e^\dagger Y_e + 3 Y_u^\dagger Y_u + 3 Y_d^\dagger Y_d \Big), \tag{92}$$

with the structural scaling

$$\mu^2 \sim \frac{2 f_2 L^2}{\pi^2 L_{\mathrm{P}}^2}\, a_{\mathrm{eff}}. \tag{93}$$

The precise numerical coefficient depends on the normalization chosen for the effective scalar field and on whether one works with a single left–right bridge field or with the broken-phase parameterization (Φ_L, Φ_R) introduced in Section 7.

4.5. The a_4 coefficient: Yang–Mills, Higgs quartic, and gravitational corrections

The fourth coefficient is

$$a_4 = \frac{1}{16\pi^2} \int \mathrm{d}^4x \sqrt{g}\, \mathrm{tr}_F\Big(\frac{1}{12} \Omega_{\mu\nu}^2 + \frac{1}{2} E^2 + \frac{1}{6} \nabla^2 E + \cdots \Big). \tag{94}$$

The curvature of the spinor-gauge bundle gives the Yang–Mills kinetic terms. In the GTD dictionary the vector part of the connection is supplied by the non-scalar bosonic sector $q_{B,\mathrm{vec}}$, while the scalar fluctuation Φ is assembled from the bosonic dotted zeroth-mode seed $\Phi_{B,0}$ and the off-diagonal finite/internal bridge extracted from the bifermionic sector. After localization, the common spectral coefficient of the vector kinetic terms is

$$g = \frac{\alpha L_P}{L}. \tag{95}$$

At the level of the canonical heat-kernel normalization one therefore has the bare spectral relation

$$g_{3,\text{spec}}^2 = g_{2,\text{spec}}^2 = \frac{5}{3} g_{1,\text{spec}}^2 = g^2 = \frac{\alpha^2 L_P^2}{L^2}. \tag{96}$$

If, as argued in the octonionic construction, the $E_8 \times E_8$ breaking/localization transition is an electroweak-scale phenomenon, then Eq. (96) should be read as a boundary condition on the pre-matched spectral coefficients at a scale $\mu_* \sim m_Z$–TeV, not as a statement about a long desert running from a trans-Planckian unification scale [9]. The observed visible-leaf QCD coupling $g_3(\mu_*)$ can then differ from $g_{3,\text{spec}}$ because of broken-phase support effects, localization thresholds, and the leaf-dependent normalization of the gauge modes. Canonical normalization of the Yang–Mills kinetic terms requires

$$g^2 = \frac{\pi^2}{f_0}. \tag{97}$$

The endomorphism E gives the scalar kinetic term, quartic potential, and non-minimal coupling to gravity:

$$a_4 \supset \int d^4x\, \sqrt{g} \left[|D_\mu \Phi|^2 + \frac{\pi^2 b_{\text{eff}}}{2 f_0 a_{\text{eff}}^2} |\Phi|^4 + c_1 C_{\mu\nu\rho\sigma}^2 + c_2 R^* R^* \right], \tag{98}$$

where

$$b_{\text{eff}} = \text{tr}\Big((Y_\nu^\dagger Y_\nu)^2 + (Y_e^\dagger Y_e)^2 + 3(Y_u^\dagger Y_u)^2 + 3(Y_d^\dagger Y_d)^2 \Big) \tag{99}$$

and the effective quartic coupling scales as

$$\lambda_{\text{eff}} \sim \frac{\pi^2 b_{\text{eff}}}{2 f_0 a_{\text{eff}}^2}. \tag{100}$$

Again, the precise normalization depends on the finite scalar content used to represent the left–right bridge field.

5. The Two Four-Dimensional Spacetimes and New Forces

5.1. The Six-Dimensional Base and its Decomposition

The six-dimensional base M_6 has signature $(3,3)$ and is constructed from split bi-quaternions [11]:

$$M_6 = \Im\mathbb{H}_L \oplus \omega\, \Im\mathbb{H}_R. \tag{101}$$

Two overlapping four-dimensional Lorentzian spacetimes are embedded as [11]

$$\Sigma_R = \Im\mathbb{H}_R \oplus \text{span}\{t_L\}, \qquad \text{signature } (3,1) \text{ (gravity)}, \tag{102}$$

$$\Sigma_L = \Im\mathbb{H}_L \oplus \text{span}\{\omega t_R\}, \qquad \text{signature } (1,3) \text{ (weak-force geometry)}. \tag{103}$$

The gravi-weak interaction lives on the full six-dimensional spacetime prior to symmetry breaking. The Wesley–Singh–Isidro BF mechanism [10] implements the dynamical reduction $SO(3,3) \to SO(3,1) \times SO(2)$, with the self-dual two-forms on Σ_R satisfying simplicity constraints that reproduce Einstein gravity, and the anti-self-dual sector on Σ_L giving the weak-force geometry.

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5.2. Forces on Each Leaf

The bookkeeping separates, on each branch, a dotted quaternionic sector that belongs to the leafwise differential operator from an undotted octonionic sector that carries the internal gauge degrees of freedom.

Left branch / Σ_L.

The assembled left-leaf quaternionic operator $\dot{q}_{BL}^{\text{curved}}$ carries the leafwise $\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ electroweak structure. At the component level it is built from the *nonzero imaginary* quaternionic directions introduced in Section 2.3; the bosonic dotted zeroth quaternionic mode is excluded from the Dirac operator and instead contributes to the scalar/Higgs seed. The non-scalar undotted octonionic variable q_{BL}^{vec} carries the visible color sector $\mathrm{SU}(3)_c$, while the dotted bosonic mode $\dot{q}_{B0L}$ contributes to the scalar seed on the same leaf.

Right branch / Σ_R.

The assembled right-leaf quaternionic operator $\dot{q}_{BR}^{\text{curved}} = D_{4\text{curved}}$ carries the leafwise $\mathrm{SU}(2)_R \times \mathrm{U}(1)_{Y_{\text{dem}}}$ gravi-DEM structure. After the Wesley–Singh–Isidro reduction, the geometric content of this right quaternionic sector is what is re-expressed as the gravitational branch on the observed leaf. Again, this operator is built only from the nonzero imaginary quaternionic directions of Section 2.3; the bosonic dotted zeroth quaternionic mode belongs to the scalar/Higgs seed. The non-scalar undotted octonionic variable q_{BR}^{vec} carries the internal gauge sector $\mathrm{SU}(3)_{\text{grav}}$, while the dotted bosonic mode $\dot{q}_{B0R}$ contributes to the scalar seed on the right leaf.

At the level of the underlying GTD coefficients, the gravi-gluon coupling differs from the visible color coefficient by a factor of α:

$$g_{\text{grav}}^{(0)} = \frac{L_{\text{P}}}{L}, \qquad g_{3,\text{spec}} = \frac{\alpha L_{\text{P}}}{L}. \tag{104}$$

This relative normalization is tied in GTD to the duality between the dotted quaternionic and undotted octonionic sectors, as discussed in Section 5.3. In the low-energy covariant derivative (112), however, g_3 denotes the matched visible-leaf QCD coupling. The octonionic phenomenological target is [38]

$$\alpha_s(\mu_*) = \frac{g_3^2(\mu_*)}{4\pi} \approx 16\,\alpha_{\text{em}}(\mu_*), \tag{105}$$

The gauge-sector analysis of Ref. [38] isolates two ingredients behind this relation. First, before broken-phase support effects one has the standard visible charge-trace normalization

$$\frac{\alpha_s}{\alpha_{\text{em}}^{(0)}} = \frac{8}{3}, \tag{106}$$

coming from one generation of quark and lepton charges. Second, the broken visible sector is modeled on the six real octonionic ladder directions H_6. If the unbroken electromagnetic mode is the democratic trace direction on this six-dimensional support space, while each visible color mode is localized on one effective support sector, the electromagnetic coupling is diluted by an additional factor of six:

$$\frac{\alpha_s}{\alpha_{\text{em}}} = \frac{8}{3} \times 6 = 16, \qquad e = \frac{g}{4}. \tag{107}$$

A convenient effective summary of the same mechanism is to write the broken-phase visible Yang–Mills term as

$$\mathcal{L}_{\text{YM}}^{\text{vis}} = -\frac{1}{4}\sum_i \frac{N_i}{g^2}\, F_{i\,\mu\nu}F_i^{\mu\nu}, \qquad g_{i,\text{vis}} = \frac{g}{\sqrt{N_i}}, \tag{108}$$

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where g is the common pre-broken visible Yang–Mills coefficient and N_i counts the effective support multiplicity of the corresponding visible mode. Choosing

$$N_3 = 1, \qquad N_\gamma = 16, \tag{109}$$

gives

$$g_3 = g, \qquad e = \frac{g}{4}, \qquad \frac{\alpha_s}{\alpha_{\rm em}} = 16. \tag{110}$$

The present paper does not derive the democratic/localized support hypothesis microscopically; it uses Eqs. (106)–(110) only as an effective low-energy summary of the broken-phase support mechanism that connects the common spectral coefficient to the visible couplings.

From the mixed terms.

The terms proportional to $i\alpha L$ couple the quaternionic leaf sectors to the octonionic internal sectors. In this bookkeeping this means the non-scalar Dirac/gauge pieces $\dot{q}_{B,\nabla}$ and $q_{B,\rm vec}$. After symmetry breaking they furnish the bridge terms from which the familiar electroweak interactions and their right-sector analogues are recovered, but they do not by themselves generate Yukawa masses.

5.3. Gauge–Gravity Duality

The bosonic Lagrangian exhibits a structural duality between the non-scalar undotted octonionic sector $q_{B,\rm vec}$ and the non-scalar dotted quaternionic sector $\dot{q}_{B,\nabla}$ [9]:

$$\frac{\alpha^2 L_{\rm P}^2}{L^4}\,{\rm Tr}\Big(q_{B,\rm vec}^\dagger q_{B,\rm vec}\Big) \longleftrightarrow \frac{L_{\rm P}^2}{L^2}\,{\rm Tr}\Big(\dot{q}_{B,\nabla}^\dagger \dot{q}_{B,\nabla}\Big). \tag{111}$$

Under the mapping $(\alpha/L)q_{B,\rm vec} \leftrightarrow \dot{q}_{B,\nabla}$, the normalization of the internal SU(3) sectors is tied to that of the quaternionic leaf sector. The bosonic dotted zeroth quaternionic modes are spectators to this particular duality because they belong to the scalar/Higgs seed rather than to the gauge–gravity exchange itself. A more explicit microscopic derivation of how this duality is realized after BF reduction remains to be given, but algebraically it is the source of the coupling relation (104).

Conceptually, this duality is closely related to the separation between the split-biquaternionic and octonionic directions. The split-biquaternionic directions are precisely the ones that carry the first-order gradient operator and hence the leafwise notion of locality and geometry. The remaining octonionic directions enter algebraically, as internal non-derivative fluctuations. In this sense the former play the role of geometric or "outer" data, while the latter play the role of internal or "inner" data in the sense familiar from non-commutative geometry. The comparison paper emphasizes that the continuum structure to be recovered is a fiber bundle rather than bare spacetime [16]. The safest interpretation is therefore not a literal boundary/bulk duality, but a geometry–gauge exchange between leaf/base and internal/fiber sectors: what is read as geometry on one leaf can reappear as internal gauge structure when viewed from the complementary sector [9,10,37].

5.4. Symmetry Bookkeeping after Trinification

The Dirac-operator definition makes the role of the gauge factors more transparent. There are two distinct kinds of symmetry action in the GTD Lagrangian:

1. the *leafwise quaternionic* sectors carried by the assembled operators $\dot{q}_{BL}^{\rm curved}$ and $\dot{q}_{BR}^{\rm curved}$, associated with $SU(2)_L \times U(1)_Y$ and $SU(2)_R \times U(1)_{Y_{\rm dem}}$ respectively;
2. the *internal octonionic* sectors carried by q_{BL} and q_{BR}, associated with $SU(3)_c$ and $SU(3)_{\rm grav}$ respectively.

The compact normalized GTD Dirac variable $\mathcal{D}_{\rm GTD} = \mathcal{D}_L + \omega\mathcal{D}_R$ packages only the leaf/gauge pieces into one split bioctonionic object. The full bosonic normalized variable is $\dot{Q}_B = \mathcal{D}_{\rm GTD} + \Phi_{B,L} + \Phi_{B,R}$, with $(\Phi_{B,L}, \Phi_{B,R})$ carrying the two bosonic scalar/Higgs seeds. The same bookkeeping applies in the fermionic channel: the dotted variable $\dot{q}_F$ carries the quaternionic lepton sector on the two leaves, whereas the undotted variable q_F carries the octonionic quark sector. The left–right Yukawa mixing itself resides in the off-diagonal finite block $\mathcal{D}_F$ and in the assembled scalar fluctuation $\Phi \sim \Phi_{B,L} + \Phi_{B,R} + \Phi_{LR}$.

Table 2. How the symmetry factors act on the different pieces of the GTD Lagrangian.

GTD piece	Lagrangian role	Symmetry factor	Chiral action
$q_{BL}^\dagger q_{BL}$	internal octonionic gauge sector	$\mathrm{SU}(3)_c$	vector-like on quark color indices
$q_{BR}^\dagger q_{BR}$	internal octonionic gauge sector	$\mathrm{SU}(3)_{\mathrm{grav}}$	vector-like hidden/right sector
$(\dot{q}_{BL}^{\mathrm{curved}})^\dagger \dot{q}_{BL}^{\mathrm{curved}}$	left-leaf quaternionic sector	$\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$	acts on left-chiral electroweak multiplets
$(\dot{q}_{BR}^{\mathrm{curved}})^\dagger \dot{q}_{BR}^{\mathrm{curved}}$	right-leaf quaternionic sector	$\mathrm{SU}(2)_R \times \mathrm{U}(1)_{Y_{\mathrm{dem}}}$	acts on right sector before symmetry breaking; geometric branch gives gravity after BF reduction
$\dot{q}_{B0}$ or $\Phi_{B,0}$	bosonic dotted zeroth mode / scalar seed	Higgs/scalar sector	does not act as a spacetime derivative; feeds the effective scalar fluctuation
$\dot{q}_F$	dotted fermionic / leptonic sector	$\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ on the left branch, $\mathrm{SU}(2)_R \times \mathrm{U}(1)_{Y_{\mathrm{dem}}}$ on the right branch	acts on lepton multiplets with both left- and right-handed pieces present
$(\mathrm{i}\alpha/L)q_F$	undotted fermionic / quark sector	$\mathrm{SU}(3)_c$ on the left branch, $\mathrm{SU}(3)_{\mathrm{grav}}$ on the right branch	acts on quark multiplets with both left- and right-handed pieces present
$\dot{q}_B^\dagger q_B - q_B^\dagger \dot{q}_B$	mixed quaternionic–octonionic terms	gravi-weak / gauge bridges	does *not* by itself generate Yukawa masses
$\mathcal{D}_F + \Phi$ or bosonized $q_F^\dagger q_F\vert_{e_0}$	Higgs / Yukawa sector	off-diagonal under left–right finite blocks	mixes H_L and H_R; assembled together with $\Phi_{B,0}$

A convenient compact form of the chiral covariant derivative is

$$\mathcal{D}_\mu = \partial_\mu - \mathrm{i}g_3 G_\mu^a T_c^a - \mathrm{i}g_{\mathrm{grav}} \widetilde{G}_\mu^a T_{\mathrm{grav}}^a - \mathrm{i}g_2 W_{L\mu}^i \tau_L^i P_L - \mathrm{i}g_R W_{R\mu}^i \tau_R^i P_R - \mathrm{i}g_1 B_\mu Y - \mathrm{i}g_{\mathrm{dem}} \widetilde{B}_\mu Y_{\mathrm{dem}}, \quad (112)$$

with chiral projectors $P_{L,R} = (1 \mp \gamma^5)/2$. The generators act trivially whenever the field is a singlet under the corresponding factor. In particular, visible color remains vector-like on both quark chiralities, whereas $\mathrm{SU}(2)_L$ and $\mathrm{SU}(2)_R$ act only through the corresponding chiral projectors.

6. The Fermionic Sector and the Exceptional Jordan Algebra

6.1. Fermionic Action from the GTD Lagrangian

The fermionic action comes from the cross sector $\mathcal{L}_{BF}$, not from the purely bosonic sector. In compact form,

$$\mathcal{L}_{BF} = \frac{L_{\mathrm{P}}^4}{2L^4} \mathrm{Tr}\Big(\mathcal{D}_B^\dagger \beta_2 \Psi_F + \beta_1 \Psi_F^\dagger \mathcal{D}_B\Big), \quad (113)$$

with

$$\Psi_F = \Psi_q + \Psi_\ell, \qquad \Psi_q := \frac{\mathrm{i}\alpha}{L} q_F, \qquad \Psi_\ell := \dot{q}_F, \quad (114)$$

its unified fermionic GTD variable. The dotted piece Ψ_ℓ is the split-biquaternionic contribution associated with leptonic modes, while the undotted piece Ψ_q is the octonionic contribution that

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brings in quark modes. Both sectors still decompose into left- and right-handed branches through $q_{F0L}, q_{F0R}, q_{FL}, q_{FR}$.

As noted earlier, this assignment is motivated by the division-algebraic construction of [32]. The dotted sector remains purely split-biquaternionic and is therefore naturally associated with the $Cl(3)$ structure adequate for one generation of chiral leptons on the $(3,3)$ leafwise spacetime scaffold. The undotted sector appears only after the extension to split bioctonions, bringing the $Cl(7)$ structure and the extra internal directions required for quarks and their $\mathrm{SU}(3)_{\mathrm{color}}$ symmetry. Thus leptons are naturally color singlets, whereas quarks require the octonionic enlargement; at the same time, both sectors share the same quaternionic differential scaffold, which explains why the weak/gravitational leafwise operators act universally on both quarks and leptons.

Section 3.4 already gave the effective reduction of this cross sector to a sesquilinear pairing in a Dirac eigenspinor basis. The result can be summarized as follows. If the localized fermion field is expanded as

$$\Psi(x) = \sum_n \hat{a}_n\, \psi_n(x), \qquad \mathcal{D}_A^{\mathrm{phys}} \psi_n = \lambda_n^{\mathrm{phys}} \psi_n, \tag{115}$$

then under the localization map (51) one obtains

$$S_{BF}^{\mathrm{loc}} = \sum_{m,n} \hat{a}_m^\dagger\, \langle \psi_m, \mathcal{D}_A^{\mathrm{phys}} \psi_n \rangle\, \hat{a}_n, \tag{116}$$

which in an eigenbasis reduces to

$$S_{BF}^{\mathrm{loc}} = \sum_n \lambda_n^{\mathrm{phys}}\, \hat{a}_n^\dagger \hat{a}_n = \langle \Psi, \mathcal{D}_A^{\mathrm{phys}} \Psi \rangle = \int \mathrm{d}^4x\, \sqrt{g}\, \bar{\Psi}\, \mathcal{D}_A^{\mathrm{phys}}\, \Psi. \tag{117}$$

This is the precise effective sense in which the GTD action yields the fermionic low-energy theory: the bosonic sector fixes the vector/differential part of $\mathcal{D}_A$, the BF sector yields the fermionic pairing, and the FF sector yields the finite/internal scalar data.

It is useful to separate clearly the two logically distinct steps behind Eqs. (109)–(113). First, the passage from the expanded GTD cross sector Eq. (33) to the compact expression Eq. (109) is an *exact algebraic rewriting*, not an additional dynamical assumption. In fact, once one introduces the normalized bosonic variable $\mathcal{D}_B = \dot{Q}_B$ and the unified fermionic variable $\Psi_F = \Psi_q + \Psi_\ell$ of Eq. (110), the four kinds of cross terms

$$\dot{q}_B^\dagger \dot{q}_F, \qquad q_B^\dagger q_F, \qquad \dot{q}_B^\dagger q_F, \qquad q_B^\dagger \dot{q}_F$$

are repackaged into the single trace

$$L_{BF} = \frac{L_P^4}{2L^4}\, \mathrm{Tr}\big(\mathcal{D}_B^\dagger \beta_2 \Psi_F + \beta_1 \Psi_F^\dagger \mathcal{D}_B\big).$$

Appendix C.1 records the exact reassembly of the full GTD action, and Appendix C.2 summarizes the corresponding operator dictionary.

Second, the passage from Eq. (109) to Eqs. (112)–(113) is *not* an exact algebraic rewriting of the pre-localized GTD variables. Rather, it is the conditional localized reduction described in Section 3.4. One first applies the fermionic localization map $\mathcal{L}_{\mathrm{loc}}^{(F)}$, so that the localized image of the cross sector is

$$S_{BF}^{\mathrm{loc}} := \mathcal{L}_{\mathrm{loc}}^{(F)}[\, L_{BF}\,].$$

Under the hypotheses of Section 3.4, the dotted GTD sector localizes into the color-singlet leptonic subspace and the undotted sector localizes into the color-carrying quark subspace, while the completed operator $\mathcal{D}_A^{\mathrm{phys}}$ is block-diagonal in this split; see Eqs. (52)–(56). Therefore the mixed dotted/undotted

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GTD terms are interpreted as pre-localized bridge terms, but after localization they collapse onto the physical diagonal sesquilinear pairing

$$S_{BF}^{\mathrm{loc}} = \sum_{m,n} \hat{a}_m^\dagger \langle \psi_m, \mathcal{D}_A^{\mathrm{phys}} \psi_n \rangle \hat{a}_n,$$

which in an eigenspinor basis of $\mathcal{D}_A^{\mathrm{phys}}$ becomes Eq. (113). Thus Eq. (113) should be read as the *localized effective form* of the GTD cross sector, not as a direct algebraic rewriting of Eq. (109).

The representation-theoretic content of the localization map may be summarized compactly. For one generation the dotted sector localizes into the color-singlet space $H_\ell^{(1)}$, whereas the undotted sector localizes into the color-carrying space $H_q^{(1)}$ defined in Eq. (56). Because the completed almost-commutative operator is color-diagonal on the observed low-energy truncation, physical matrix elements between these two sectors vanish, Eq. (58). Hence the GTD mixed terms encode pre-localized dotted/undotted bridging, but not observable lepton–quark mixing after localization.

6.2. Octonionic Internal Directions and the Observability of Color

A suggestive structural consequence of the present GTD/$E_8 \times E_8$ bookkeeping is that quarks and leptons do not stand on exactly the same geometric footing. In the broken-phase picture developed above, the split-biquaternionic directions supply the derivative scaffold of the leafwise Dirac operator, whereas the remaining octonionic directions act algebraically as internal, non-derivative fluctuations. The dotted fermionic sector remains purely split-biquaternionic and is already sufficient for the leptonic degrees of freedom, while the undotted sector appears only after the extension to split bioctonions and brings in the additional internal structure needed for quarks and their $SU(3)_{\mathrm{color}}$ symmetry. In this sense, leptons are directly compatible with the quaternionic spacetime scaffold, whereas quarks require an additional octonionic internal sector.

This observation suggests a possible explanation for why color behaves differently from electric charge or mass-related quantum numbers. In the present framework, color is tied to the non-scalar octonionic internal directions rather than to the quaternionic differential structure from which the observed spacetime leaves are assembled. Since the emergence of the low-energy world is assumed to proceed by spontaneous localization to classical spacetime-matter configurations, one may conjecture that isolated excitations carrying unsaturated octonionic-color structure need not descend as independent asymptotic spacetime observables. By contrast, color-singlet composites can be compatible with the emergent spacetime description and hence can appear as observable states.

Read in this way, the non-observation of isolated quarks would have a natural geometric interpretation: a colored quark is not merely an ordinary spacetime excitation carrying one more quantum number, but rather an excitation whose defining internal support still lies in the octonionic sector. A colorless hadron, however, would correspond to a composite configuration for which the net octonionic-color content is neutralized, allowing the state to descend to the spacetime description seen by macroscopic detectors, which themselves belong to the localized classical spacetime sector.

We stress that this is at present only a structural heuristic, not a derivation of QCD confinement. The present paper does not prove that the GTD dynamics forbids finite-energy asymptotic colored states, nor does it derive a flux-tube or area-law mechanism. The more modest point is that the split between quaternionic differential directions and octonionic internal directions makes it plausible that localization should privilege color-singlet states as the observable spacetime sector. Establishing an actual confinement statement in this framework would require a separate dynamical analysis of how localization acts on the octonionic-color sector and whether nonsinglet states fail to survive as asymptotic classical configurations.

6.3. Three generations from the E_6 symmetry

The complexified split bi-octonions generate $\mathrm{Cl}(7) \cong \mathrm{Cl}(6) \oplus \omega\,\mathrm{Cl}(6)$ [32]. From $\mathrm{Cl}(6)$ one obtains one generation of Standard Model chiral quarks and leptons. Three generations arise from the $\mathrm{SU}(3)_F$

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flavour symmetry in the E_6 trinification, with the exceptional Jordan algebra $J_3(\mathbb{O}_\mathbb{C})$ providing the representation space [12]. The use of E_6 as a unifying group for the Standard-Model particle content goes back to the original E_6 grand-unified-theory proposal [55], alongside the earlier SU(5) [57] and SO(10) [58] constructions.

The fermionic Hilbert space decomposes as

$$\mathcal{H}_F = H_{\text{gen}} \otimes \mathbb{C}^3_{\text{gen}}, \qquad \dim H_{\text{gen}} = 32, \qquad \dim \mathcal{H}_F = 96. \tag{118}$$

6.4. The Dirac Equation and the Jordan-Algebra Mass Matrix

Varying the fermionic action (117) with respect to $\bar{\Psi}$ gives the Dirac equation

$$\mathcal{D}_A \Psi = 0. \tag{119}$$

The Dirac operator lives on split bioctonionic space, which is ten-dimensional (per half). Dray and Manogue showed that the Dirac equation in ten-dimensional Minkowski spacetime admits an octonionic E_6 symmetry, with Spin(9, 1) realized in the SL(2, $\mathbb{O}$) sense [39]. The relevant role of E_6 for $J_3(\mathbb{O}_\mathbb{C})$ is as its reduced structure group rather than its automorphism group.

The Jordan matrix then enters the Dirac equation as a mass matrix [12,15]:

- the three eigenvalues of the Jordan matrix correspond to the three fermion generations,
- the eigenvalue spectrum $(q - \delta, q, q + \delta)$ with $\delta^2 = 3/8$ is fixed by the algebra,
- the specific mass ratios follow from the Clebsch–Gordan factors $(2, 1, 1)$ in the $\text{Sym}^3(\mathbf{3})$ representation of flavour SU(3).

This means that the effective internal Dirac operator $\mathcal{D}_F$ is not a free input in the GTD framework: it is determined by the E_6 symmetry of the ten-dimensional octonionic Dirac equation. In the GTD/NCG dictionary, this finite operator is associated primarily with the fermionic/Jordan sector and its bosonized scalar fluctuation, whereas q_B supplies the vector connection. The Yukawa matrices are the Jordan-matrix entries, fixed by the octonionic algebra up to the flavour SU(3) structure [12]. Appendix A implements this statement at the finite-geometry level through the factorized ansatz $Y_f = y_f Y_J$ and computes the resulting traces entering the scalar mass and quartic coupling.

The complete chain is therefore

$$\begin{aligned} \text{GTD action} &\longrightarrow \text{spectral action} \longrightarrow \text{variation of fermionic sector} \\ &\longrightarrow \text{ten-dimensional Dirac equation with } E_6 \text{ symmetry} \\ &\longrightarrow \text{Jordan-algebra eigenvalue problem} \longrightarrow \text{fermion mass ratios.} \end{aligned} \tag{120}$$

Each step is structurally motivated by the algebraic framework, but some identifications still await a fully explicit derivation.

6.5. Three-Generation Fermionic Lagrangian

The resulting fermionic action is best written in chiral form as

$$S_{\text{ferm}} = \int d^4x \sqrt{g} \Big[\bar{\Psi}\, i\gamma^\mu \mathcal{D}_\mu \Psi - \bar{\Psi}_L\, \Phi_{LR}\, Y\, \Psi_R - \bar{\Psi}_R\, \Phi^\dagger_{LR}\, Y^\dagger\, \Psi_L \Big], \tag{121}$$

where $\mathcal{D}_\mu$ is the covariant derivative (112) and Y is the family/Yukawa matrix determined by the Jordan-algebra eigenvalue problem [12,15]. This form makes clear that the differential part of the Dirac operator does not itself mix chiralities: the mixing is carried entirely by the off-diagonal scalar bridge field Φ_{LR}.

In the broken phase one may rewrite (121) in terms of the two Higgs mass eigenstates. Writing the branch fields as Φ_L and Φ_R and diagonalizing the scalar mass matrix as in Eq. (138) below, the

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ordinary visible masses are generated by the Standard-Model-like Higgs eigenstate $H_{\rm SM}$, which is dominantly right-sector in the present interpretation. One therefore writes

$$\mathcal{L}_Y^{\rm SM} = -\bar{Q}_L\, Y_d\, H_{\rm SM}\, d_R - \bar{Q}_L\, Y_u\, \widetilde{H}_{\rm SM}\, u_R \tag{122}$$

$$- \bar{L}_L\, Y_e\, H_{\rm SM}\, e_R - \bar{L}_L\, Y_\nu\, \widetilde{H}_{\rm SM}\, \nu_R + \text{h.c.}, \tag{123}$$

while the second Higgs eigenstate $H_{\rm new}$, dominantly left-sector for small mixing, couples primarily to the complementary right-sector/charge-generating channels. The detailed right-sector Yukawa couplings still depend on the explicit $\mathrm{SU}(2)_R$ representation content and therefore remain part of the open model-building problem.

7. The Higgs Sector

The present GTD/spectral-action construction should not be identified with the earliest minimal spectral Standard Model of Chamseddine–Connes–Marcolli [6]. In that older minimal setting, under the simplest high-scale assumptions, the spectral relations led to a Higgs-mass estimate around 170 GeV. Later work within the spectral-action program showed that extending the scalar sector can modify this conclusion [40], for example through the inclusion of the real singlet already present in the spectral model, and still later through non-minimal spectral extensions such as the Pati–Salam construction [41,56]. The present framework differs structurally from that older minimal model in several ways: the finite geometry used here is only a candidate observed-leaf associative shadow of a deeper exceptional/nonassociative construction; the scalar sector is assembled from branch-resolved bosonic Higgs antecedents together with a composite bifermionic bridge; the low-energy theory includes the additional right-branch sectors $\mathrm{SU}(3)_{\rm grav}$ and $\mathrm{U}(1)_{\rm dem}$; and the matching is not assumed to follow the same high-scale minimal-desert pattern as in the older spectral Standard Model. For these reasons, the old minimal-NCG Higgs-mass estimate does not transfer directly to the present model.

At the same time, the present paper does not yet yield a sharp Higgs-mass prediction. The effective quartic couplings generated from the localized ensemble, the normalization of the Higgs bridge, the detailed right-sector multiplet content, and the final values of the regulator moments all still influence the scalar mass parameters. Accordingly, the present result should be read as fixing the structural location of the Higgs sector within the GTD/spectral-action framework, rather than as a completed numerical Higgs-mass prediction.

7.1. The Scalar Bridge Field and the Broken-Phase Multiplets

In the almost-commutative description the Higgs sector is the scalar fluctuation that combines branch-resolved bosonic dotted zeroth-mode seeds with an off-diagonal bridge on the finite chiral space. The bridge part is represented by Φ_{LR}, which connects the left- and right-chiral finite subspaces and is therefore the unique place where chirality mixing enters the finite Dirac operator. In the GTD language the low-energy two-Higgs sector is assembled from three antecedents: the left bosonic seed $\Phi_{B,L} = \dot{q}_{B0L}$, the right bosonic seed $\Phi_{B,R} = \omega\dot{q}_{B0R}$, and the bosonized e_0-channel of the bifermionic sector.

After symmetry breaking, it is convenient to parameterize the scalar bridge in terms of a left-branch multiplet Φ_L together with a right-branch multiplet Φ_R. In the present $E_8 \times E_8$ interpretation, Φ_R is the Standard-Model-like Higgs precursor associated with the right-sector $\mathrm{U}(1)_{\rm dem}$ branch that generates ordinary particle masses, whereas Φ_L is the new second Higgs associated with the left branch and the emergence of electric charge after left–right/triality breaking [7,12]. This is a broken-phase effective parametrization, not yet a fully derived statement about the irreducible $\mathrm{SU}(2)_R$ representation. The explicit finite geometry still has to determine whether the fundamental bridge field is best regarded as a bidoublet, a pair of doublets, or a larger right-sector multiplet, and how the bosonic seeds $(\Phi_{B,L}, \Phi_{B,R})$ are embedded into that low-energy multiplet structure.

7.2. Broader $E_8 \times E_8$ Interpretation of the Higgs Sector

Ref. [7] motivates a two-Higgs picture at the level of the broken $E_8 \times E_8$ theory: one Higgs sector is associated with the left branch and one with the right branch. This picture has since been sharpened [8]: the bifermionic seed is Hermitian, and its E_6-covariant channels are classified per branch by

$$\mathbf{27} \otimes \mathbf{27} = \mathbf{1} \oplus \mathbf{78} \oplus \mathbf{650}, \tag{124}$$

with each branch's **78** furnishing the gauge currents together with one composite electroweak doublet sitting in the coset of its weak SU(3). The Standard-Model Higgs is the $\mathrm{SU}(3)_R$-coset doublet of the right (pre-gravitational) branch ($\mathbf{78}_R$), whose vacuum expectation value supplies the Dirac mass of the Standard-Model fermions, while the second scalar H_{ch} is the $\mathrm{SU}(3)_L$-coset doublet of the left (visible) branch ($\mathbf{78}_L$); the E_6-singlet **1** is electroweak-inert, and the **650** is a tower of higher composite operators. The calculational claim established in the present paper is the complementary low-energy statement: the physical Higgs antecedents are the branch-resolved bosonic dotted zeroth-mode fields $(\Phi_{B,L}, \Phi_{B,R})$, and the bifermionic sector supplies the visible scalar-bridge channel through its e_0 projection after coarse-graining.

It is nonetheless instructive to compare the residual branchwise E_8 module directly with the Hermitian bifermionic seed, because a tempting reading — now identified and set aside [8] — would try to source the former from the latter. (The temptation is sharpened by a numerical coincidence: each branch carries 24 Weyl fermions, so the branchwise bilinears number $24 \times 24 = 576 = 2 \times 288$, exactly twice the residual count.) We carry out the comparison at one fixed symmetry stage, before any possible slot-family locking or further diagonal identification. At that stage, the first two entries in the branched representation are the independent $\mathrm{SU}(3)_{\mathrm{st}}$ and $\mathrm{SU}(3)_{\mathrm{gen}}$ factors of the parent E_8 module. Using the ordering of Eq. (13) of Ref. [7], with the third and fourth entries denoting the branch-dependent internal factors, the five independent residual irreps from one E_8 are

$$\begin{aligned} U^{(L/R)}_{72,\mathrm{ind}} &= (\mathbf{1},\mathbf{3},\mathbf{3},\mathbf{1})(2) \oplus (\mathbf{3},\bar{\mathbf{3}},\mathbf{3},\mathbf{1})(0) \oplus (\bar{\mathbf{3}},\bar{\mathbf{3}},\mathbf{1},\mathbf{1})(2) \\ &\oplus (\bar{\mathbf{3}},\mathbf{1},\mathbf{3},\mathbf{2})(1) \oplus (\mathbf{3},\mathbf{1},\bar{\mathbf{3}},\mathbf{1})(2), \end{aligned} \tag{125}$$

with the full branchwise residual module given by

$$U^{(L/R)}_{144} = U^{(L/R)}_{72,\mathrm{ind}} \oplus \overline{U^{(L/R)}_{72,\mathrm{ind}}}. \tag{126}$$

The dimensions $9 + 27 + 9 + 18 + 9 = 72$ account for the five independent states, and together with their conjugates one obtains 144 residual labels per E_8, or 288 in total. In particular, 45 of the 72 independent branchwise labels (hence 90 of the 144 including conjugates) lie in fundamental/anti-fundamental $\mathrm{SU}(3)_{\mathrm{gen}}$ sectors, while the remaining 27 (hence 54 including conjugates) are $\mathrm{SU}(3)_{\mathrm{gen}}$ singlets.

This makes the relevant microscopic closure test

$$U^{(L/R)}_{144} \overset{?}{\subseteq} \overline{V^{(L/R)}_F} \otimes V^{(L/R)}_F, \tag{127}$$

where $V^{(L/R)}_F$ is the branchwise fermion module entering the microscopic Hermitian seed

$$\mathcal{B}_F := \beta_1 \Psi_F^\dagger \beta_2 \Psi_F.$$

In the finite-triple shadow used in the present paper one has

$$\mathcal{H}_F = \mathcal{H}_{\mathrm{SM},\nu_R} \otimes \mathbb{C}^3_{\mathrm{fam}},$$

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and correspondingly one writes branchwise

$$V_F^{(L/R)} = W^{(L/R)} \otimes \mathbf{3}_{\rm gen},$$

with $W^{(L/R)}$ the one-generation branchwise module.

Resolution: the residual 288 is an adjoint-lineage label ledger, not composite matter.

Under the present finite-triple ansatz,

$$\overline{V_F^{(L/R)}} \otimes V_F^{(L/R)} = \left(\overline{W^{(L/R)}} \otimes W^{(L/R)}\right) \otimes \left(\overline{\mathbf{3}}_{\rm gen} \otimes \mathbf{3}_{\rm gen}\right). \tag{128}$$

Since

$$\overline{\mathbf{3}}_{\rm gen} \otimes \mathbf{3}_{\rm gen} = \mathbf{1} \oplus \mathbf{8},$$

a Hermitian bifermionic seed $\Psi_F^\dagger M \Psi_F$ carries the generation factor only through $\mathbf{1} \oplus \mathbf{8}$, and therefore contains no fundamental $\mathbf{3}_{\rm gen}$ or $\overline{\mathbf{3}}_{\rm gen}$ channel. One might read this as a *failure* of the closure (127). The subsequent ontology analysis of Ref. [8] shows that the correct reading is the opposite: the seed *should not* host these channels, and the residual 288 should not be sourced from the matter Lagrangian at all. It is an adjoint-lineage representation-label ledger — bookkeeping for the two-branch scaffolding — not a particle spectrum.

Three facts establish this. (i) The bare Hermitian seed is a charge-*difference* object: every channel of $\mathbf{27} \otimes \mathbf{27}$ has $Q(\psi^\dagger \chi) = Q(\chi) - Q(\psi)$, so it contains no charge-sum (Majorana) channel and no fundamental generation triplet — which is exactly the $\mathbf{1} \oplus \mathbf{8}$ obstruction recovered above (Theorem 1 of Ref. [8]). (ii) Once the chiral matter is realized, as it is throughout the program, as $\mathrm{Cl}(6)$ minimal-ideal spinors — which are $\mathrm{SU}(3)_{\rm st}$ singlets, the geometric index being carried by the E_8 scaffolding through $E_8 \supset \mathrm{SU}(3)_{\rm st} \times E_6$ rather than by the E_6-internal matter — no matter bilinear of any kind (Hermitian or Majorana, same- or cross-branch) can carry an $\mathrm{SU}(3)_{\rm st}$ index. The 252 of the 288 residual labels that are $\mathrm{SU}(3)_{\rm st}$-charged are therefore unsourceable from any matter bilinear (Proposition 1 of Ref. [8]), conditional on this spinor ontology. (iii) The only enlargement that could host charge-sum channels at all, a Nambu doubling of the seed, reaches at most the single $\mathrm{SU}(3)_{\rm st}$-singlet residual sector (36 of the 288); reading that sector too as a label, rather than as a composite of an unmotivated doubled reservoir, is a lineage-uniformity choice, flagged honestly in Ref. [8] as methodological rather than a no-go.

The upshot is a clean separation into three lineages: the chiral matter is the $\mathrm{Cl}(6)$ spinor sector; the bifermionic seed does the work of the $\mathbf{78}_{L,R}$ gauge currents and the two electroweak Higgs doublets through Eq. (124) (with the E_6-singlet $\mathbf{1}$ electroweak-inert and the $\mathbf{650}$ a tower of higher composite operators whose binding is a separate dynamical question); and the residual 288 is the linear adjoint-label ledger of the two-branch scaffolding. The matched/residual count $496 = 208 + 288$ is then the dimension of a representation-label ledger, not a count of propagating fields — which directly answers the objection that the $E_8 \times E_8$ scaffolding is "too large," since the residual 288 corresponds to no particles. This realization of the chiral matter as spinors rather than E_8-adjoint components is also what places the chiral sector outside the Distler–Garibaldi no-go theorem, whose hypothesis (fermions as components of an E_8 representation) is not met here; correspondingly, the residual 288, being adjoint-lineage and self-conjugate, is non-chiral and anomaly-mute [8].

With this resolution the broader $E_8 \times E_8$ picture no longer carries an unresolved "288-closure" obligation. The two physical scalars are the $\mathrm{SU}(3)_R$- and $\mathrm{SU}(3)_L$-coset doublets of the per-branch $\mathbf{78}$ channels of $\mathbf{27} \otimes \mathbf{27}$, and the residual 288 is decoupled from the matter Lagrangian. The question that remains is dynamical rather than representation-theoretic: which channels of $\mathbf{27} \otimes \mathbf{27}$ actually bind, and at what scale (the attractive-channel/localization problem of Section 12.3 and Section 11). The present spectral-action analysis should therefore be read as establishing the visible electroweak Higgs bridge inside this now-settled representation-theoretic picture.

7.3. Composite Origin from Bifermionic Terms

We now turn from the broader $E_8 \times E_8$ interpretation to the explicit low-energy e_0-projected bifermionic channel that carries the visible Higgs bridge.

Before writing the visible e_0-projected channel explicitly, it is useful to state the interpretive hierarchy adopted in the present paper. The full bifermionic GTD seed

$$\mathcal{B}_F := \beta_1 \Psi_F^\dagger \beta_2 \Psi_F$$

is taken to be a genuine microscopic prediction of the split bioctonionic GTD Lagrangian. At this microscopic level, $\mathcal{B}_F$ is not itself yet the pair of low-energy Higgs bosons. Rather, it is a composite or bound-state reservoir whose effective broken-phase scalar content is assumed to reorganize only after localization, channel selection, coarse-graining, and left–right / triality breaking.

In the broader $E_8 \times E_8$ program the two effective Higgs sectors are the $\mathbf{78}_{L,R}$ coset doublets of $\mathbf{27} \otimes \mathbf{27}$ (124), while the residual 288 is a decoupled adjoint-lineage label ledger rather than composite scalar content [8]. The present paper works at the complementary low-energy level, showing how the *visible electroweak projection* of the microscopic seed is extracted from the bifermionic sector and assembled with the branch-resolved bosonic Higgs antecedents $(\Phi_{B,L}, \Phi_{B,R})$.

The role of the e_0 projection should therefore be understood as follows. The full bifermionic seed $\mathcal{B}_F$ contains more information than the visible Higgs bridge alone. The projected channel

$$O_H := P_{e_0}\Big(\beta_1 \Psi_F^\dagger \beta_2 \Psi_F\Big)$$

is not assumed to exhaust the whole microscopic bifermionic sector; rather, it isolates the scalar bridge channel relevant for the observed low-energy electroweak description. Thus the logical chain is

$$\mathcal{B}_F \ \rightsquigarrow \ (\mathcal{H}_L \oplus \mathcal{H}_R)^{\text{poss}}_{\text{comp}} \ \rightsquigarrow \ P_{e_0}(\mathcal{B}_F) \ \rightsquigarrow \ O_H \ \rightsquigarrow \ (H_{\text{new}}, H_{\text{SM}})_{\text{eff}}.$$

Here $(\mathcal{H}_L \oplus \mathcal{H}_R)^{\text{poss}}_{\text{comp}}$ is only a schematic placeholder for whatever broader composite scalar content the microscopic bifermionic seed may produce after symmetry breaking. The explicit calculation below uses only the later visible projection $P_{e_0}(\mathcal{B}_F) \to O_H$.

With this understanding, the calculation that follows should be read in a restricted but precise sense: it establishes that the visible e_0-projected color-singlet channels inside the bifermionic seed have the electroweak quantum numbers required of the Higgs bridge, and that after auxiliary-field completion they furnish the composite/off-diagonal scalar data entering the low-energy Dirac operator. The complementary parent-level statement — that the two physical scalars are the $\mathbf{78}_{L,R}$ coset doublets of $\mathbf{27} \otimes \mathbf{27}$, with the residual 288 a decoupled adjoint-lineage label ledger — is established separately in Ref. [8]; what remains genuinely open there is the dynamical question of which $\mathbf{27} \otimes \mathbf{27}$ channels bind.

The bifermionic GTD sector is

$$\mathcal{L}_{FF} = \frac{L_{\text{P}}^6}{2L^6}\, \mathrm{Tr}\Big(\beta_1 \Psi_F^\dagger \beta_2 \Psi_F\Big), \tag{129}$$

with explicit expansion given in Eq. (38). Section 3.5 showed how this sector should be interpreted more carefully. Introducing an auxiliary field Φ_{LR} then yields the Yukawa bridge coupling and the composite/off-diagonal part of the Higgs sector, which must be assembled together with the branch-resolved bosonic dotted zeroth-mode seeds $(\Phi_{B,L}, \Phi_{B,R})$. In this form the GTD-to-Higgs map is

$$q_F^\dagger q_F\big|_{e_0} \longrightarrow \mathcal{O}_H \longrightarrow -G_H\, \mathcal{O}_H^\dagger \mathcal{O}_H \longleftrightarrow \Phi_{LR}^\dagger \Phi_{LR} \qquad \text{after effective auxiliary-field completion.} \tag{130}$$

Thus the Higgs sector is not inferred from the bosonic trace alone: it is fixed by the bifermionic GTD sector and then reappears in the heat-kernel coefficients of the bosonic spectral action once Φ has been assembled into $\mathcal{D}_A$. At the representation-theoretic level, the visible composite channels are the color-singlet doublets $\mathcal{O}_{H_u}$ and $\mathcal{O}_{H_d}$ of Eq. (64), with hypercharges $\mp 1/2$. The NJL mean-field

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estimates (70)–(72) then show that a sufficiently attractive localized quartic channel produces the standard negative mass-squared instability and a positive quartic coupling. This interpretation also avoids double counting: the composite bridge field introduced from the GTD channel is identified with the same finite off-diagonal fluctuation that appears in the non-commutative Dirac operator.

7.4. Effective Higgs Lagrangian

At the level of the broken-phase effective theory we write the two Higgs sectors separately,

$$\mathcal{L}_{H_L} = (D_\mu \Phi_L)^\dagger D^\mu \Phi_L - V_L(\Phi_L), \tag{131}$$

$$\mathcal{L}_{H_R} = (\widetilde{D}_\mu \Phi_R)^\dagger \widetilde{D}^\mu \Phi_R - V_R(\Phi_R), \tag{132}$$

$$\mathcal{L}_{H,\text{mix}} = -m_{LR}^2 \,(\Phi_L^\dagger \Phi_R + \Phi_R^\dagger \Phi_L) - \lambda_{LR}(\Phi_L^\dagger \Phi_L)(\Phi_R^\dagger \Phi_R) - \kappa_{LR}(\Phi_L^\dagger \Phi_R)(\Phi_R^\dagger \Phi_L), \tag{133}$$

so that

$$\mathcal{L}_{\text{Higgs}} = \mathcal{L}_{H_L} + \mathcal{L}_{H_R} + \mathcal{L}_{H,\text{mix}}. \tag{134}$$

Here

$$D_\mu \Phi_L = \partial_\mu \Phi_L - ig_2 W^i_{L\mu} \tau^i_L \Phi_L - ig_1 B_\mu Y^{(\Phi_L)} \Phi_L, \tag{135}$$

$$\widetilde{D}_\mu \Phi_R = \partial_\mu \Phi_R - ig_R W^i_{R\mu} \tau^i_R \Phi_R - ig_{\text{dem}} \widetilde{B}_\mu Y^{(\Phi_R)}_{\text{dem}} \Phi_R, \tag{136}$$

with

$$V_L(\Phi_L) = -\mu_L^2 \,\Phi_L^\dagger \Phi_L + \lambda_L (\Phi_L^\dagger \Phi_L)^2, \qquad V_R(\Phi_R) = -\mu_R^2 \,\Phi_R^\dagger \Phi_R + \lambda_R (\Phi_R^\dagger \Phi_R)^2. \tag{137}$$

The interpretation is branch-resolved: Φ_R is the Standard-Model-like Higgs precursor associated with the right branch and ordinary mass generation, while Φ_L is the new second Higgs associated with the left branch and the charge-generating sector. The visible electroweak Higgs observed at low energies is obtained after mixing. Writing the neutral CP-even modes schematically as a two-component vector, one defines

$$\begin{pmatrix} H_{\text{new}} \\ H_{\text{SM}} \end{pmatrix} = \begin{pmatrix} \cos\theta_H & -\sin\theta_H \\ \sin\theta_H & \cos\theta_H \end{pmatrix} \begin{pmatrix} \Phi_L \\ \Phi_R \end{pmatrix}, \tag{138}$$

so that for small mixing angle θ_H one has $H_{\text{SM}} \simeq \Phi_R$ and $H_{\text{new}} \simeq \Phi_L$. This is the sense in which the Standard-Model Higgs sits in the right sector while a second Higgs survives from the left sector.

The bookkeeping is then as follows. One first assembles the low-energy scalar fluctuation at the GTD/NCG interface as in Eq. (73), namely

$$\Phi_L := \Phi_{B,L} + \eta_L \,\Phi_{LR}, \qquad \Phi_R := \Phi_{B,R} + \eta_R \,\Phi_{LR}, \tag{139}$$

with the same mixing coefficients η_L and η_R as in Eq. (73). In this reading, $\Phi_{B,L}$ and $\Phi_{B,R}$ supply the two bosonic dotted zeroth-mode scalar seeds, Φ_{LR} supplies the off-diagonal chiral/Yukawa bridge, and the bilinear and quartic terms in $\mathcal{L}_{H,\text{mix}}$ measure the interaction between the two branch-resolved Higgs sectors.

8. The Weak Mixing Angle

Following the octonionic analysis of Ref. [9], the rotation from the (W_3, B) basis to the (Z^0, γ) basis is taken to act on spinorial space rather than on conventional gauge space. In that construction one obtains a half-angle relation, quoted here for reference rather than rederived in the present paper:

$$1 = \frac{\sqrt{\cos(\theta_W/2)}}{2} + \sqrt{\sin(\theta_W/2)}, \tag{140}$$

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which yields

$$\sin^2\theta_W \approx 0.2497 \qquad \text{(geometric matching value on octonionic spinorial space).} \tag{141}$$

Writing $r := g_1/g_2$, one has

$$\sin^2\theta_W = \frac{r^2}{1+r^2}, \qquad r_{\text{geom}} \approx 0.577, \tag{142}$$

for the geometric value above. Taking the observed electroweak value to be near $\sin^2\theta_W \simeq 0.231$ gives instead $r_{\text{obs}} \approx 0.548$, so the required correction is only

$$\frac{r_{\text{obs}}}{r_{\text{geom}}} \approx 0.95, \tag{143}$$

i.e. about a five-percent downward shift in the effective g_1/g_2 ratio. In the electroweak-scale matching scenario of the octonionic program, such a correction must come from threshold effects, broken-phase support, localization, and cosmological scaling rather than from a long logarithmic desert running alone.

A standard one-loop threshold estimate illustrates why ordinary matching alone is unlikely to account for the full correction. If heavy fields of masses M_i are integrated out near the matching scale μ_*, then

$$\delta\left(\frac{1}{g_a^2}\right) = -\sum_i \frac{\Delta b_a^{(i)}}{8\pi^2} \ln\frac{M_i}{\mu_*}, \tag{144}$$

and linearizing $\sin^2\theta_W = g_1^2/(g_1^2+g_2^2)$ gives

$$\delta(\sin^2\theta_W) \simeq -\sin^2\theta_W(1-\sin^2\theta_W)\left(g_1^2\,\delta_1 - g_2^2\,\delta_2\right), \qquad \delta_a := \delta(1/g_a^2). \tag{145}$$

For $\Delta b_a^{(i)}$ of order unity and logarithms of order one, these shifts are typically at the sub-percent to percent level. This suggests that the required $\sim 5\%$ correction in g_1/g_2 should come substantially from the broken-phase support/localization mechanism of Section 5.2, with ordinary threshold running providing only part of the effect.

The half-angle factor remains a direct consequence of the spinorial nature of octonionic space and cannot be obtained in a formalism defined on conventional Minkowski spacetime [9]. The present paper does not compute the full threshold correction, but Eqs. (142)–(145) show that the quantitative tension is modest while also clarifying that ordinary one-loop running alone is unlikely to remove it completely.

9. The Candidate Low-Energy Lagrangian

9.1. Collecting All Terms

After spontaneous localization, sectorwise assembly of $\mathcal{D}_A$ from $\mathcal{L}_{BB}$, $\mathcal{L}_{BF}$, and $\mathcal{L}_{FF}$, heat-kernel expansion of the bosonic spectral action, and Wick rotation to Lorentzian signature on each four-dimensional leaf, the candidate low-energy Lagrangian associated with the single-STM-atom GTD action is

$$\mathcal{L}_{\text{universe}} = \mathcal{L}_{\text{EH}} + \mathcal{L}_{\text{SM}} + \mathcal{L}_{\text{new}} + \mathcal{L}_{\text{Higgs}} + \mathcal{L}_{\text{ferm}} + \mathcal{L}_{\text{higher}}. \tag{146}$$

The individual pieces are

$$\mathcal{L}_{\rm EH} = \frac{1}{16\pi G}(R - 2\Lambda_{\rm cosm}), \tag{147}$$

$$\mathcal{L}_{\rm SM} = -\frac{1}{4}G^a_{\mu\nu}G^{a\mu\nu} - \frac{1}{4}W^i_{\mu\nu}W^{i\mu\nu} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu}, \tag{148}$$

$$\mathcal{L}_{\rm new} = -\frac{1}{4}\widetilde{G}^a_{\mu\nu}\widetilde{G}^{a\mu\nu} - \frac{1}{4}\widetilde{B}_{\mu\nu}\widetilde{B}^{\mu\nu}, \tag{149}$$

$$\mathcal{L}_{\rm Higgs} = (D_\mu\Phi_L)^\dagger D^\mu\Phi_L + (\widetilde{D}_\mu\Phi_R)^\dagger\widetilde{D}^\mu\Phi_R - V_{\rm eff}(\Phi_L,\Phi_R), \tag{150}$$

$$\mathcal{L}_{\rm ferm} = \bar{\Psi}\,\mathrm{i}\gamma^\mu\mathcal{D}_\mu\Psi - \bar{\Psi}_L\Phi_{LR}Y\Psi_R - \bar{\Psi}_R\Phi^\dagger_{LR}Y^\dagger\Psi_L + (\text{Majorana/right-sector terms}), \tag{151}$$

$$\mathcal{L}_{\rm higher} = c_1C^2_{\mu\nu\rho\sigma} + c_2R^*R^* + c_3R|\Phi|^2. \tag{152}$$

9.2. Parameter dictionary

The low-energy coefficients are organized by the GTD parameters $(\alpha, L/L_{\rm P})$, the cutoff-function moments (f_0, f_2, f_4), and the microscopic constants $(L_{\rm P}, \tau_{\rm P}, \hbar)$ already present in GTD. In particular, once the microscopic constants $(L_{\rm P}, \tau_{\rm P}, \hbar)$ are specified, Newton's constant is fixed kinematically by

$$G = \frac{L_{\rm P}^5}{\hbar\,\tau_{\rm P}^3}. \tag{153}$$

The spectral relation (89) is therefore interpreted as the matching condition between this microscopic normalization and the coefficient of the four-dimensional Einstein term in the localized low-energy action. In view of the electroweak-scale matching scenario and the broken-phase support mechanism for $\alpha_s/\alpha_{\rm em} = 16$ [9,38], Table 3 distinguishes the bare spectral coefficients from the matched low-energy couplings.

Table 3. Parameter dictionary relating low-energy quantities to the GTD variables.

Physical quantity	**GTD expression**
Dimensionless spectral cutoff	$\Lambda = L/L_{\rm P}$ in normalized variables, with physical cutoff $\Lambda_{\rm phys} = 1/L_{\rm P}$
Physical matching scale	μ_* set by localization / $E_8 \times E_8$ breaking; in the present scenario $\mu_* \sim m_Z$–TeV and is an emergent transition scale rather than a new fundamental gravitational cutoff
Newton constant	Microscopically $G = L_{\rm P}^5/(\hbar\tau_{\rm P}^3)$ with $c = L_{\rm P}/\tau_{\rm P}$; equivalently, the coefficient of R in the localized four-dimensional action satisfies $1/(16\pi G) = 2f_2L^2/(\pi^2L_{\rm P}^2)$
Cosmological constant	proportional to $f_4L^4/L_{\rm P}^4 \cdot 16\pi G$
Common spectral gauge coefficient	$g = \alpha L_{\rm P}/L$
$SU(3)_c$ spectral coefficient	$g_{3,\rm spec} = \alpha L_{\rm P}/L$
$SU(3)_c$ matched low-energy coupling	$\alpha_s(\mu_*) = g_3^2(\mu_*)/(4\pi) \approx 16\,\alpha_{\rm em}(\mu_*)$
$SU(2)_L$ spectral coefficient	$g_{2,\rm spec} = \alpha L_{\rm P}/L$
$U(1)_Y$ spectral coefficient	$g_{1,\rm spec} = \sqrt{3/5}\,\alpha L_{\rm P}/L$
Fine-structure constant	$\alpha_{\rm em} = \alpha^2L_{\rm P}^4/L^4$ in the simplest visible-sector normalization; broken-phase support gives $e = g/4$ and hence $\alpha_s/\alpha_{\rm em} = 16$
Scalar mass parameter	$\mu^2 \sim 2f_2a_{\rm eff}L^2/(\pi^2L_{\rm P}^2)$ after scalar normalization
Scalar quartic	$\lambda_{\rm eff} \sim \pi^2b_{\rm eff}/(2f_0a_{\rm eff}^2)$
Gravi-gluon bare coefficient	$g^{(0)}_{\rm grav} = L_{\rm P}/L$
Weak mixing angle (geometric matching value)	$\sin^2\theta_W \approx 0.2497$
Yukawa couplings	determined by the eigenvalues of $J_3(\mathbb{O}_{\mathbb{C}})$ and their embedding in the finite Dirac operator

10. Reverse-engineering the GTD action from the low-energy spectral data

The forward direction of the paper starts from the GTD single-STM-atom action and derives a candidate low-energy action. It is useful to ask the inverse question as a plausibility check: if one starts from the assembled low-energy spectral action and looks for the minimal trace-dynamical lift on split bioctonionic space, does one recover the fundamental GTD action? The answer is structurally yes. The present section should therefore be read as an inverse consistency argument, not as a replacement for the forward derivation.

10.1. Spectral Variables of the Low-Energy Action

After bosonization of the scalar channel, the low-energy action takes the almost-commutative form

$$S_{\rm LE} = {\rm Tr}[f(\mathcal{D}_A/\Lambda)] + \langle \Psi, \mathcal{D}_A^{\rm phys}\Psi\rangle + S_{\rm res}^{(4F)}, \tag{154}$$

where $S_{\rm res}^{(4F)}$ vanishes if the bifermionic sector is completely bosonized. Let

$$\mathcal{D}_A^{\rm phys}\psi_n = \lambda_n^{\rm phys}\psi_n, \qquad \Psi(x) = \sum_n a_n\,\psi_n(x), \tag{155}$$

with $\{\psi_n\}$ an orthonormal eigenspinor basis. Then

$$S_{\rm LE} = \sum_n f(L\lambda_n^{\rm phys}/\Lambda) + \sum_n \bar{a}_n\,\lambda_n^{\rm phys}\,a_n + S_{\rm res}^{(4F)}. \tag{156}$$

In this form the bosonic spectral variables are the physical Dirac eigenvalues $\lambda_n^{\rm phys}$, whereas the physical fermionic variables are the mode coefficients a_n, in agreement with Landi–Rovelli's discussion of matter coupling [17]. The eigenspinors ψ_n encode the geometric background; the physical fermion field is described by the coefficients multiplying them.

10.2. Trace-Dynamical Lift of the Bosonic Spectral Data

The inverse problem is to replace the c-number spectral data by operator-valued pre-geometric variables. For the bosonic sector, after subtracting the purely constant vacuum term and isolating the bare quadratic precursor of the regulated spectral action, one has schematically

$$S_{\rm bos}^{(2)} \propto \frac{1}{\Lambda^2}\sum_n (L\lambda_n^{\rm phys})^2 = \frac{1}{\Lambda^2}\,{\rm Tr}\Big(\mathcal{D}_A^\dagger\mathcal{D}_A\Big). \tag{157}$$

The natural trace-dynamical lift is therefore a matrix-valued bosonic datum whose localized spectral data coarse-grain to the λ_n. In the GTD bookkeeping this datum splits into a genuine Dirac variable together with two branch-resolved bosonic scalar seeds. On split bioctonionic space the minimal normalized choice is

$$\mathcal{D}_{\rm GTD} = \frac{1}{L}(\mathrm{i}\alpha q_{B,\rm vec} + L\dot{q}_{B,\nabla}) = \mathcal{D}_L + \omega\mathcal{D}_R, \qquad \Phi_{B,L} = \dot{q}_{B0L}, \qquad \Phi_{B,R} = \omega\dot{q}_{B0R}, \tag{158}$$

so that

$$\dot{Q}_B = \mathcal{D}_{\rm GTD} + \Phi_{B,L} + \Phi_{B,R}. \tag{159}$$

The corresponding physical operators are $D_{\rm GTD}^{\rm phys} = \mathcal{D}_{\rm GTD}/L$, $\Phi_{B,L}^{\rm phys} = \Phi_{B,L}/L$, and $\Phi_{B,R}^{\rm phys} = \Phi_{B,R}/L$. The corresponding lifted bosonic action is

$$S_B^{\rm lift} = \frac{1}{2}\int \frac{{\rm d}\tau}{\tau_{\rm Pl}}\,{\rm Tr}\!\left(\frac{L_{\rm P}^2}{L^2}\,\dot{Q}_B^\dagger\dot{Q}_B\right), \tag{160}$$

which matches the bosonic part of the GTD single-STM-atom action. In this inverse lift, the manifold label x of the low-energy fields is no longer fundamental. It is replaced by split bioctonionic STM coordinates encoded in the quadruple $(q_B, \dot{q}_B; q_F, \dot{q}_F)$, but now with an important refinement: the dotted bosonic variables split into a Dirac-type leafwise differential part and two branch-resolved scalar zeroth modes, while the undotted bosonic variables supply only the non-scalar vector/gauge

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directions. In other words, the inverse lift reconstructs not one undifferentiated bosonic variable but the triple $(\mathcal{D}_{\mathrm{GTD}}, \Phi_{B,L}, \Phi_{B,R})$.

10.3. Fermionic mode coefficients and the odd GTD variables

For the physical fermion field, the spectral decomposition makes clear that the quantities to be lifted are the coefficients a_n, not the eigenspinors ψ_n themselves. In the spirit of trace dynamics one therefore promotes

$$a_n \longrightarrow \hat{a}_n, \tag{161}$$

with $\hat{a}_n$ odd matrix variables. These operator-valued coefficients are then grouped into the GTD fermionic coordinates

$$\dot{q}_F = \dot{q}_{F0L} + \omega\dot{q}_{F0R} + \dot{q}_{FL} + \omega\dot{q}_{FR}, \qquad q_F = q_{FL} + \omega q_{FR}, \tag{162}$$

so that the dotted quaternionic sector supplies the split-biquaternionic/leptonic channel, whereas the undotted octonionic sector supplies the quark channel. In this inverse picture the fermionic bilinear in Eq. (156) is the low-energy shadow of the GTD boson–fermion cross sector, while the bosonized part of the odd bilinears supplies the scalar bridge field and any residual four-fermion terms:

$$\sum_n \bar{a}_n\,\lambda_n\,a_n \Longleftarrow S^{\mathrm{lift}}_{BF}, \qquad (\mathcal{D}_F, \Phi) + S^{(4F)}_{\mathrm{res}} \Longleftarrow S^{\mathrm{lift}}_{FF}. \tag{163}$$

Thus the inverse lift treats the bosonic and fermionic sectors differently, exactly as in the Landi–Rovelli expansion: eigenvalues organize the bosonic spectral data, whereas mode coefficients organize the physical fermion field.

10.4. Reconstructing the single-STM-atom action

The minimal bilinear trace action that simultaneously contains the bosonic lift, a linear boson–fermion channel, and a bifermionic channel is obtained by packaging the even and odd variables into

$$\dot{Q}_1^\dagger := \dot{Q}_B^\dagger + \eta\,\beta_1\dot{Q}_F^\dagger, \qquad \dot{Q}_2 := \dot{Q}_B + \eta\,\beta_2\dot{Q}_F, \qquad \eta := \frac{L_{\mathrm{P}}^2}{L^2}, \tag{164}$$

so that the inverse-lift action is

$$\frac{S_{\mathrm{inv}}}{\hbar} = \frac{1}{2}\int \frac{d\tau}{\tau_{\mathrm{Pl}}}\,\mathrm{Tr}\Big(\eta\,\dot{Q}_1^\dagger\dot{Q}_2\Big). \tag{165}$$

Expanding the trace gives

$$\eta\,\dot{Q}_1^\dagger\dot{Q}_2 = \eta\,\dot{Q}_B^\dagger\dot{Q}_B + \eta^2\big(\dot{Q}_B^\dagger\beta_2\dot{Q}_F + \beta_1\dot{Q}_F^\dagger\dot{Q}_B\big) + \eta^3\beta_1\dot{Q}_F^\dagger\beta_2\dot{Q}_F, \tag{166}$$

which reproduces, respectively, the bosonic spectral precursor, the fermionic pairing channel, and the bifermionic/Higgs channel already identified in the forward analysis. Therefore the minimal inverse lift of the assembled low-energy spectral data recovers the same bilinear trace structure as the single-STM-atom GTD action (5), and may be viewed as a natural inverse ansatz for it. This inverse lift is not unique in a purely algebraic sense: many quadratic actions built from generalized Dirac variables would admit analogous lifts. What singles out the GTD ansatz here is the simultaneous requirement of split bioctonionic bookkeeping, the exact $BB/BF/FF$ sector decomposition, and the trace-dynamical normalization inherited from the single-STM action.

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Why the single-STM-atom Lagrangian fixes the classical local Lagrangian.

We recall the relevant discussion in Section 2.2 The role of the single STM atom in the present construction should be understood as fixing the *universal local form* of the classical Lagrangian density. Consider the full universe action as a sum over a very large number of STM atoms,

$$S_{\text{univ}} = \sum_{A=1}^{N_{\text{STM}}} S_{\text{STM}}^{(A)}.$$

In GTD, spontaneous localization is expected to act through the fermionic sector, while the bosonic fields carried by a given STM atom are classicalised along with that fermionic collapse channel. Thus one STM atom should be viewed as a fermion together with the bosonic fields it sources. If one further assumes that, after localization, different fermionic STM atoms occupy different spectral slots, in a manner consistent with Pauli exclusion, then the many-atom sum may be reorganised as a sum over occupied Dirac-eigenvalue sectors.

From this viewpoint, the Dirac spectral decomposition of a *representative* STM atom is sufficient to determine the local functional dependence of the emergent classical action: every localised STM atom contributes the same structural trace-dynamical building block, differing only in the occupied eigenvalue data and in the resulting classical field configuration. The full classical universe action is then obtained by summing or integrating this same local structure over all localised atoms. In this sense, one does not claim that the total many-STM action is literally identical to the action of a single STM atom; rather, one claims that the latter already contains the universal local Lagrangian seed from which the former is assembled.

Equivalently, once localization has selected classical spectral data, the Dirac eigenvalues may be used as invariant labels of the emergent geometry, and the single-STM spectral action captures the basic local contribution whose repetition over the occupied sectors yields the classical low-energy Lagrangian of the universe.

10.5. Status of the Reverse-Engineering Argument

The value of the inverse construction is not that it proves GTD uniquely. Rather, it shows that once the low-energy action is written spectrally, there is a natural minimal lift back to split bioctonionic trace dynamics, and that lift recovers the GTD single-STM-atom action as its simplest bilinear representative. The argument improves plausibility in three ways:

1. it explains why the GTD action is trace-valued and bilinear in the pre-geometric variables,
2. it clarifies why bosonic spectral eigenvalues and fermionic mode coefficients enter differently in the pre-localized theory,
3. it shows that the split bioctonionic bookkeeping is compatible with the sectorwise decomposition of the low-energy action into bosonic, fermionic, and scalar channels.

At the same time, this remains an inverse consistency check. It does not by itself derive spontaneous localization, the explicit finite E_6 triple, the regulator moments (f_0, f_2, f_4), the six- to four-dimensional BF reduction, or the many-STM interaction mechanism. Those ingredients remain necessary for a complete first-principles derivation.

11. Open steps

11.1. What Is Established Here

The following pieces have been completed at the present stage:

1. **Full sectorwise expansion** of the GTD single-STM-atom Lagrangian into bosonic, cross, and bifermionic pieces $\mathcal{L}_{BB}$, $\mathcal{L}_{BF}$, and $\mathcal{L}_{FF}$ (Section 3).
2. **Bosonic spectral identification** of $\mathcal{L}_{BB}$ with the quadratic bosonic functional associated with the vector/differential part of the low-energy Dirac operator, together with the regulator analysis needed to promote it to a genuine spectral action (Sections 3 and Appendix B).

3. **Sectorwise assembly rule** for the completed almost-commutative operator $\mathcal{D}_A$: $\mathcal{L}_{BB}$ fixes the vector/differential part, $\mathcal{L}_{BF}$ fixes the fermionic pairing, and $\mathcal{L}_{FF}$ fixes the finite/internal scalar data (Sections 3, 6, and 7).
4. **Representation-theoretic checks of the effective arrows**: the localization map is now shown to respect the lepton/quark gauge representations, and the composite e_0 scalar channel is shown to contain the color-singlet electroweak doublets with hypercharges $\pm 1/2$ expected of the Higgs bridge, together with NJL mean-field estimates of the critical couplings (Sections 3, 6, and 7).
5. **Heat-kernel expansion** yielding a_0, a_2, and a_4 together with explicit parameter identifications for Newton's constant, the cosmological constant, and gauge couplings, plus structural scalar-sector relations tied to the finite Dirac operator (Section 4).
6. **Candidate low-energy Lagrangian** including Standard Model + gravity + two new forces + an effective two-scalar sector + three fermion generations (Section 9).
7. **Parameter dictionary** relating all physical constants to α, L/L_P, and the cutoff moments (Section 9.2).
8. **Reverse-engineering consistency check** showing that the assembled low-energy spectral action admits a minimal bilinear trace-dynamical lift whose algebraic form matches the single-STM-atom GTD action on split bioctonionic space (Section 10).

11.2. Remaining Computations

11.2.1. Explicit Seeley–DeWitt Coefficients for the E_6 Internal Triple

Appendix A takes a first concrete step on this problem by writing a candidate E_6-compatible finite geometry for the observed one-leaf sector and by computing the corresponding finite traces $\mathrm{tr}_{H_F}(\mathbf{1})$, the gauge traces, and the Yukawa traces a and b in terms of the Jordan eigenvalues. What still remains is the stronger result that this finite geometry should be derived uniquely from the full exceptional/nonassociative GTD construction rather than postulated as its minimal associative shadow.

11.2.2. Renormalization-Group Running with the New Forces

At the level of bare spectral coefficients one has $g_{3,\mathrm{spec}} = g_{2,\mathrm{spec}} = \sqrt{5/3}\, g_{1,\mathrm{spec}}$, but if the GTD matching point lies near $\mu_* \sim m_Z$–TeV the logarithmic running interval is short and threshold effects become at least as important as ordinary renormalization-group flow. The presence of $\mathrm{SU}(3)_{\mathrm{grav}}$, $\mathrm{U}(1)_{\mathrm{dem}}$, the second Higgs sector, and broken-phase support effects modifies the matching conditions. One must therefore determine which states remain light near μ_*, compute the threshold corrections together with the beta functions, and check whether the observed couplings are reproduced.

11.2.3. Detailed Higgs potential from bifermionic terms

The present paper now verifies that the attractive visible color-singlet channels in the e_0 projection carry the Higgs quantum numbers $(\mathbf{1}, \mathbf{2}, \pm 1/2)$ and gives NJL mean-field formulas for the critical couplings and quartics. What still remains is the microscopic derivation of the effective couplings $G_{u,d}$ and of the mixed bridge coupling from the localized ensemble, together with a full treatment of the right-sector multiplets and any residual color-octet channels. One must also isolate the self-adjoint part of the visible e_0-projected channel before the Hubbard–Stratonovich completion, in view of the intrinsic anti-self-adjoint fermionic contribution emphasized in Ref. [29]. Finally, one must reconcile the off-diagonal finite-geometry description of the Higgs with the broken-phase two-Higgs parametrization used here, in which Φ_R is the SM-like mass-generating Higgs and Φ_L is the second Higgs associated with the left branch.

11.2.4. The Wesley–Singh–Isidro Mechanism in Detail

The six-dimensional BF-theory reduction [10] needs to be fully integrated with the spectral-action computation. Specifically: how do the simplicity constraints on Σ_R produce the Einstein–Hilbert action from the $\mathrm{SU}(2)_R$ sector, and how does this connect to the a_2 coefficient in the heat-kernel expansion?

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11.2.5. Determination of f_0, f_2, and f_4

Appendix B takes a first concrete step on the regulator problem by constructing a smooth family of even cutoff functions whose small-u expansion reproduces the GTD bosonic quadratic term and whose moments (f_0, f_2, f_4) are explicit. What remains is the deeper derivation of the cutoff profile from the localization dynamics of the STM ensemble itself. Those values affect the Higgs-mass prediction and the cosmological constant.

11.2.6. Neutrino Masses and the Seesaw Mechanism

The Majorana mass matrix for right-handed neutrinos should emerge from the $\mathrm{SU}(2)_R$ breaking. The seesaw mechanism producing light left-handed neutrino masses should be a natural consequence of the effective two-scalar structure and the large vacuum expectation value of Φ_R.

11.2.7. Verification of Mass-Ratio Predictions

The Jordan-algebra eigenvalue problem gives specific predictions for charged-fermion mass ratios [12,15]. These need to be shown to emerge precisely from the spectral action's fermionic sector, i.e. from the internal Dirac operator $\mathcal{D}_F$ as constrained by the $E_6/J_3(\mathbb{O}_{\mathbb{C}})$ structure.

11.2.8. The 288 Residual Degrees of Freedom: Status

This item, previously posed as an open closure problem, is now resolved at the representation-theoretic level [8]. The observation that the Hermitian seed $\beta_1 \Psi_F^\dagger \beta_2 \Psi_F$ carries the generation factor only through $\mathbf{1} \oplus \mathbf{8}$ — which under the finite-triple ansatz $\mathcal{H}_F = \mathcal{H}_{\mathrm{SM},\nu_R} \otimes \mathbb{C}^3_{\mathrm{fam}}$ forbids the fundamental-$\mathrm{SU}(3)_{\mathrm{gen}}$ residual sectors of Ref. [7] — is not a failure to close but the correct statement that the residual 288 is *not* composite matter. The 288 is read instead as an adjoint-lineage representation-label ledger of the two-branch scaffolding, decoupled from the matter Lagrangian (Section 7.2); the chiral matter lives in the separate $\mathrm{Cl}(6)$ spinor lineage, and the seed's own content is the $\mathbf{78}_{L,R}$ gauge currents and the two electroweak Higgs doublets of $\mathbf{27} \otimes \mathbf{27}$. The residual computation that genuinely remains is therefore dynamical, not representation-theoretic: a bound-state/gap-equation treatment of which $\mathbf{27} \otimes \mathbf{27}$ channels condense (the attractive-channel hypothesis), which would fix the masses and mixing of the two scalars.

11.3. Assessment

The architecture of the program is coherent, and several nontrivial structural links are already in place: the GTD bosonic sector is written as a genuine normalized Dirac variable together with a pair of branch-resolved bosonic dotted zeroth-mode scalar seeds and their physical L^{-1} counterparts, the BF reduction supplies a plausible route to emergent four-dimensional geometry, the Jordan-algebra sector provides a concrete candidate source for flavour structure, the structural classes of low-energy terms are obtained at the level of formal heat-kernel bookkeeping, and the reverse-engineering analysis shows that the assembled low-energy spectral action lifts naturally back to the bilinear trace structure of the single-STM-atom GTD action as a minimal inverse ansatz. Several load-bearing arrows are made more explicit: under stated hypotheses, the cross sector reduces to a sesquilinear fermionic pairing in a Dirac eigenspinor basis, the localization map is checked for compatibility with the lepton/quark gauge representations, the bifermionic sector admits an effective auxiliary-field completion yielding the composite/off-diagonal scalar bridge together with Higgs-channel quantum-number checks and mean-field critical-coupling formulas, and the BF-to-Dirac chain is made explicit at the level of the principal symbol of the leafwise Dirac operator. But the derivation is not yet complete in the strict spectral-action sense.

The most load-bearing unresolved steps are now narrower. The dynamical localization mechanism — whose deterministic reduction skeleton is now established at the emergent Hilbert level [30], though its Born-rule completion remains conditional on the Adler–Millard fluctuation sector — and the explicit many-STM relational or entanglement structure remain open. In the scalar sector, one still has to derive the effective quartic coupling G_H and the mixed bridge coupling quantitatively from the localized

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ensemble rather than postulate their infrared form. In the fermionic sector, the localization map of odd GTD matrices to Dirac-mode coefficients still needs a microscopic derivation from the GTD dynamics. In the BF sector, the full Wesley–Singh–Isidro reduction still has to be completed beyond the principal-symbol chain worked out here. These are not cosmetic omissions: they are exactly the points that must eventually convert the present architecture into a full derivation.

The appendices have already supplied a candidate E_6-compatible finite triple with explicit trace invariants and a workable regulator family with explicit moments. Accordingly, the decisive tasks are now not to write down those objects in abstract form, but to derive them from GTD and to close the many-STM step:

1. derive the candidate E_6-compatible finite spectral triple, together with its finite traces, uniquely from the full exceptional/nonassociative GTD construction rather than postulate its minimal associative shadow,
2. derive the spectral cutoff profile and the moments (f_0, f_2, f_4) dynamically from GTD localization/coarse-graining rather than select a phenomenologically convenient regulator family,
3. specify the many-STM interaction, correlation, or entanglement mechanism needed to justify the passage from a single localized STM atom to the universe action.

Alongside these, the BF reduction beyond the principal symbol, the microscopic localization map, and the scalar-sector couplings still require explicit completion. Until then, the strongest accurate claim is that GTD furnishes a promising partially constructed framework for deriving the low-energy universe, not that the derivation is finished.

12. Phenomenological Implications

12.1. Electroweak-Scale Matching with Fundamental G

In the octonionic program the $E_8 \times E_8$ breaking, the left–right breaking, and the electroweak breaking are argued to coincide, with the freeze-out of the quantum-to-classical transition expected between 100 GeV and a TeV rather than at a conventional Planckian unification scale [9,37]. The present spectral-action derivation should therefore be read as an electroweak-scale matching calculation. However, this does *not* mean that the gravitational constant is reset to the electroweak scale. In the GTD viewpoint the independent microscopic constants are L_P, τ_P, and $\hbar$; once $c = L_P/\tau_P$, Newton's constant is already fixed as

$$G = \frac{L_P^5}{\hbar\, \tau_P^3}. \tag{167}$$

The scale $\mu_* \sim m_Z$–TeV is instead interpreted as the emergent many-STM localization and symmetry-breaking scale at which the low-energy action is matched to observation.

This reading is consistent with the cosmological argument of Ref. [37], where the observed particle number $N_U \sim 10^{80}$ is used to estimate the end of inflation and a primordial composite-Higgs scale of order 10^{23} GeV is scaled down to the TeV regime. The present paper does not rederive that argument, but it is compatible with it: the spectral-action analysis supplies the form of the localized low-energy Lagrangian, while the independent cosmological argument explains how the broken-phase matching scale can emerge far below the microscopic Planck data.

With this interpretation, the algebraic heat-kernel formulas of Sections 4 and 9 remain valid, but their phenomenological reading changes in three ways. First, there is no long logarithmic desert between the GTD matching point and the observed weak scale. Consequently, deviations of the geometric value (141) from the measured electroweak quantity, and deviations of the bare spectral relation (96) from the observed gauge couplings, must be attributed substantially to threshold effects, broken-phase support, localization, and cosmological scaling rather than to many decades of ordinary renormalization-group evolution. Second, the recent gauge-sector analysis of the octonionic framework isolates a broken-phase mechanism under which a common visible Yang–Mills coefficient yields

$$\frac{\alpha_s}{\alpha_{\rm em}} = 16, \qquad e = \frac{g}{4}, \tag{168}$$

providing a concrete route from the common spectral coefficient g to the larger observed visible color coupling [38]. Third, Eq. (89) should therefore be viewed as a consistency relation between the microscopic GTD normalization of G and the coefficient of the four-dimensional Einstein term after localization and dimensional reduction, not as an independent prediction of the spectral sector. The open issue is therefore not to generate G from electroweak physics, but to show in detail how the one-STM normalization survives the many-STM transition and yields the observed four-dimensional coefficient without spoiling the other matched couplings.

12.2. CKM, PMNS, and Neutrino Masses After the Present Paper

The main phenomenological gain of the present paper is that the location of the flavour data in the effective low-energy action is now explicit. The charged-fermion and neutrino mixings are determined by the canonically normalized Yukawa blocks of the finite Dirac operator:

$$V_{\rm CKM} = U_{u,L}^\dagger U_{d,L}, \qquad U_{\rm PMNS} = U_{e,L}^\dagger U_{\nu,L}, \tag{169}$$

where the unitary matrices diagonalize the canonically normalized Yukawas,

$$U_{f,L}^\dagger Y_f^{\rm can} U_{f,R} = {\rm diag}(y_{f1}, y_{f2}, y_{f3}), \qquad Y_f^{\rm can} = Z_{f,L}^{-1/2} Y_f \, Z_{f,R}^{-1/2}. \tag{170}$$

The Jordan-algebra program already supplies candidate three-family eigenvalue data, charged-fermion mass ratios, and leading CKM root–sum rules for the charged sectors [12], and the companion flavor-transport analysis [13] fixes the CP structure of these mixings: a single correlated CKM phase in the quark sector and Dirac CP conservation in the minimal lepton sector (Section 12.3). What the present paper adds is the missing effective-Lagrangian context: it identifies the kinetic terms, gauge assignments, Higgs bridge field, and the finite-operator slot in which the Jordan matrices must be inserted. This should make any future derivation of the CKM matrix sharper, because the remaining ambiguity is no longer "what is the low-energy Lagrangian?" but rather "what exact flavour basis, scalar normalization, and left-handed intertwiner follow from the full finite triple?"

The neutrino sector is similar but more delicate. Once the right-sector breaking pattern is fixed, one expects either a Majorana block M_R in the finite Dirac operator or an effective Weinberg operator, leading to

$$\mathcal{M}_\nu = \begin{pmatrix} 0 & m_D \\ m_D^T & M_R \end{pmatrix}, \qquad m_\nu^{\rm light} \simeq -m_D^T M_R^{-1} m_D. \tag{171}$$

The present paper therefore improves the prospects for deriving PMNS angles and neutrino masses, but it does not yet fix them: one still needs the explicit right-sector representation content, the neutrino Majorana mechanism, and the low-energy normalization of the Higgs bridge.

It is not natural to derive $V_{\rm CKM}$ or $U_{\rm PMNS}$ directly as functions only of the regulator moments (f_0, f_2, f_4). Those moments primarily control the bosonic normalization of the spectral action: the Einstein term, gauge kinetic terms, scalar mass term, and scalar quartic. They influence flavour only indirectly through canonical normalization, vacuum alignment, and renormalization-group or threshold evolution. The leading flavour structure should instead come from the Jordan/Yukawa sector and, for neutrinos, from the Majorana block. In that sense the regulator moments are important, but they are not the primary flavour data.

A realistic next-step program is therefore:

1. write the explicit finite Dirac operator with separate up, down, charged-lepton, and neutrino Yukawa blocks consistent with the $E_6/J_3(\mathbb{O}_\mathbb{C})$ data;

2. derive the Higgs-bridge normalization and the right-sector breaking pattern;
3. evaluate the CKM and PMNS matrices at the electroweak matching scale μ_*;
4. only then include the indirect dependence on (f_0, f_2, f_4) through scalar normalization and threshold/RG effects.

Thus the present paper does improve the plausibility of an eventual first-principles derivation of CKM, PMNS, and neutrino masses: it does so by fixing the effective-Lagrangian framework in which those quantities must now be computed.

12.3. Predictive Signatures

The framework outlined above is conditional in many respects, but the structural ingredients identified here do entail several testable signatures. We collect them here, distinguishing what is sharp from what is qualitative.

Two-Higgs structure (H_L, H_R).

The broken-phase parametrization of Section 7 contains two scalar multiplets: Φ_R, the Standard-Model-like Higgs precursor associated with the right branch and ordinary particle masses, and Φ_L, the second Higgs associated with the left branch and the emergence of electric charge after left–right/triality breaking. At the $E_8 \times E_8$ level these are the $\mathrm{SU}(3)_R$- and $\mathrm{SU}(3)_L$-coset doublets of the $\mathbf{78}_{L,R}$ channels of $\mathbf{27} \otimes \mathbf{27}$ [8]; both are composite. The specific masses and mixings of these two scalars are not fixed by the present construction; they depend on the right-sector breaking scale, the bridge-curvature normalisation, and the localization dynamics that are not derived here. We add one honest caveat on the compositeness signatures. If, as argued in Ref. [8], the compositeness (form-factor) scale of these condensates is of order M_{Pl} while the electroweak scale v is set separately at graviweak breaking, then all compositeness deviations of the observed 125 GeV scalar are suppressed by $\mathcal{O}(v^2/M_{\mathrm{Pl}}^2) \sim 10^{-34}$: signature (i), modified $h \to b\bar{b}$ and $h \to \tau\bar{\tau}$ couplings, is then unobservably small, and the 125 GeV scalar behaves as an effectively fundamental, Standard-Model-like Higgs — consistent with current data but not a discriminator. The genuinely distinctive content is therefore (ii) the second scalar H_{ch} (whose mass and couplings are not computed here) and the right-handed/sterile neutrinos, rather than precision deviations in the 125 GeV state. Contributions to oblique S and T parameters from the second-Higgs sector remain possible but likewise depend on its undetermined mass scale; quantitative collider predictions for Φ_L require the right-sector phenomenology to be developed further.

Gauge-coupling structure: $\alpha_s/\alpha_{em} = 16$.

The framework's broken-phase support mechanism gives the structural ratio $\alpha_s/\alpha_{em} = 16$ together with $e = g/4$, following the analysis of [38]. This is a consistency check on the framework's gauge-sector reduction at the matching scale rather than a separately tunable observable. The detailed running of these couplings, including threshold effects from the new forces $\mathrm{SU}(3)_{\mathrm{grav}}$ and $\mathrm{U}(1)_{\mathrm{dem}}$, is required for a complete comparison to data.

Fermion mass ratios and CP-violating phases.

The exceptional Jordan algebra eigenvalue spectrum [12] produces specific predictions for charged-fermion mass ratios. The v6 revision of [12] (under review at *Annalen der Physik*), together with the companion flavor-transport analysis [13], sharpens the structure of the CP-violating phases. The result is *asymmetric* between the two sectors, and it *supersedes* the earlier expectation of a maximal leptonic phase.

For quarks, the Cabibbo $(1,2)$ block is generated by a single ladder rung whose orientation angle χ obeys the exact law $\phi_{12} = -2\chi$. The geometric rotor sits at the quadrature-balanced point $\phi_{12} = \pi/2$, and the same phenomenological tilt $\varepsilon \simeq -26.1^\circ$ that reproduces $|V_{us}| \simeq 0.225$ displaces the physical phase to

$$|\delta_{CP}^{\mathrm{quark}}| = \tfrac{\pi}{2} + \varepsilon \simeq 63.9^\circ. \tag{172}$$

This is a *one-parameter structural correlation*, not a parameter-free prediction: a single complex amplitude in the $(1,2)$ block fixes both $|V_{us}|$ and the CKM phase. It agrees with the PDG-2024 global-fit value $\delta_{CP}^{\text{quark}} = 65.7^\circ \pm 1.5^\circ$ at the $\sim 1.2\sigma$ level.

For leptons, by contrast, the minimal real-symmetric texture yields a vanishing rephasing-invariant Jarlskog,

$$J_\ell = 0, \qquad \delta_{CP}^{\text{lepton}} \in \{0, \pi\} \quad \text{(leptonic Dirac CP conservation).} \tag{173}$$

The earlier "maximal" reading $|\delta_{CP}^{\text{lepton}}| = \pi/2$ is an artifact: a row-global factor of i in U_{PMNS} is a one-sided diagonal rephasing, removable by charged-lepton field redefinitions, and contributes nothing to the Jarlskog. A reality theorem establishes that every lepton flavor-transport amplitude is exactly real under the entire class of transports the framework employs (G_2 automorphisms and rung rotors that do not mix the identity line with the lepton flavor plane), the sole loophole being a Higgs-bridge operator that mixes the identity direction into that plane [13]. Leptonic Dirac CP conservation is therefore a sharp *conditional* prediction; possible Majorana phases are a separate question not fixed by this Dirac/Jarlskog statement.

This asymmetric pattern — physical CP violation confined to the quark sector, leptonic Dirac CP conserved — is directly falsifiable. A long-baseline determination of $\delta_{CP}^{\text{lepton}}$ away from $\{0, \pi\}$ by DUNE or Hyper-Kamiokande would exclude the minimal lepton texture. We state the present tension explicitly rather than soften it: current global fits in fact lean against the prediction, favouring near-maximal leptonic CP violation ($\delta_{CP} \simeq 270^\circ$) over CP conservation at more than 3σ within the inverted ordering [13].

Neutrino mass ordering, masses, and the seesaw scale.

At leading structural order the neutrino Jordan spectrum is $(-\delta_\nu, 0, +\delta_\nu)$ with $\delta_\nu^2 = 3/4$ [12] — one vanishing eigenvalue and a pair of equal magnitude and opposite sign, i.e. absolute amplitudes $\{0, \delta_\nu, \delta_\nu\}$. Under a stated minimality assumption (the subleading lifts that generate the solar splitting are perturbative, so they split the quasidegenerate pair without reordering the leading symmetric spectrum), this maps to the *inverted* ordering with $m_1 \simeq m_2 \simeq 0.050$ eV, $m_3 \simeq 0$, and opposite Majorana parities. The general spectral fit accommodates either ordering, so it is specifically the minimal, perturbative-lift reading that commits to inversion — which is precisely what makes it falsifiable [13]. The opposite parities drive the effective neutrinoless-double-beta-decay mass to the lower edge of the inverted band, $m_{\beta\beta} \simeq 19$ meV, with summed mass $\Sigma m_\nu \simeq 0.10$ eV. The absolute scale is consistent with a seesaw $M_R \sim 10^{14}$ GeV tied to the right-sector breaking.

We again quantify the present experimental lean rather than soften it. Global oscillation fits mildly prefer normal ordering while allowing the inverted one, and within the inverted ordering they favour near-maximal CP violation over conservation; the predicted combination (inverted ordering with conserved leptonic CP) therefore stands against current oscillation data at roughly the 3σ level [13]. In cosmology, the tightest minimal-ΛCDM combinations (DESI DR2 BAO with CMB, $\Sigma m_\nu < 0.064$ eV at 95%) already lie below the predicted sum, although the bound relaxes under extended dark-energy and alternative-likelihood assumptions, so the cosmological verdict is model-dependent today. The package is thus triply falsifiable in the near term: by JUNO (mass ordering, independently of δ_{CP} and θ_{23}), by DUNE and Hyper-Kamiokande (leptonic CP conservation), and by next-generation $0\nu\beta\beta$ searches such as LEGEND-1000 and nEXO ($m_{\beta\beta}$).

Dark sector signatures.

The framework's unbroken $\mathrm{U}(1)_{\text{dem}}$ (dark electromagnetism, sourced by the square root of mass) and the broken $\mathrm{SU}(2)_R$ associated with gravitation generate distinguishable structural signatures from a single-Higgs Standard Model: anomalous long-range forces, modified gravitational coupling at scales tied to the right-sector breaking, and possible signatures in early-universe cosmology. The detailed phenomenology of these sectors is the subject of separate work.

In summary, the framework makes several falsifiable predictions: an asymmetric CP pattern (a quark CKM phase correlated with $|V_{us}|$, and a CP-conserving minimal lepton sector), an inverted neutrino mass ordering at leading structural order, a structural gauge-coupling ratio $\alpha_s/\alpha_{em} = 16$, and a two-Higgs scalar sector with distinguishable signatures from the Standard Model. Detailed numerical predictions for collider signatures, however, require inputs (the right-sector breaking scale, the regulator moments, threshold matching) that the present manuscript does not fix.

13. Conclusions and Outlook

We have presented a conditional structural framework that connects Generalized Trace Dynamics to the low-energy action of the observed universe via the spectral action principle. The principal structural result is the sectorwise decomposition of the GTD Lagrangian into a bosonic sector (which supplies the spectral-action quadratic functional), boson–fermion cross terms (which supply the fermionic pairing under a localization hypothesis), and bifermionic terms (which supply the visible scalar/Higgs bridge channel). Combined with the Wesley–Singh–Isidro six- to four-dimensional BF reduction and the candidate E_6-compatible finite spectral triple, this decomposition allows the standard Seeley–DeWitt heat-kernel expansion to produce the structural classes of low-energy terms: Einstein–Hilbert gravity, Yang–Mills kinetic terms, and scalar kinetic and potential terms.

The status of the construction has been made explicit through Table 1 in Section 1.3 and the assessment in Section 11. The framework is conditional in several respects — on the localization map, on the auxiliary-field completion of the visible quartic channel, on the broken-phase support factors, and on the use of an associative shadow for the observed-leaf finite geometry. None of these is presented as derived in the strict sense. The role of Appendix D, which records a programmatic pre-breaking nonassociative scaffold, is also made explicit: it is not load-bearing for the main-text construction.

The predictive content surveyed in Section 12.3 includes falsifiable structural correlations, the most sharply testable being leptonic Dirac CP conservation ($\delta_{CP}^{\text{lepton}} \in \{0, \pi\}$, conditional on the minimal flavor-transport texture), the correlated quark CKM phase $|\delta_{CP}^{\text{quark}}| = \pi/2 + \varepsilon \simeq 63.9°$, the gauge-coupling ratio $\alpha_s/\alpha_{em} = 16$, the preference for inverted neutrino mass ordering at leading structural order, and the existence of a second Higgs scalar tied to the right-sector breaking. Several of these will be probed by DUNE, Hyper-K, and JUNO as their datasets mature; we note that current global fits already lean against the inverted-ordering/CP-conserving lepton package at roughly the 3σ level.

The outlook for converting the present framework into a complete first-principles derivation has been outlined in Section 11. The key open steps are: (i) a microscopic derivation of the localization map; (ii) a microscopic derivation of the attractive quartic channel; (iii) the construction of a fully exceptional/nonassociative pre-breaking spectral triple of which the present associative shadow is a localization; and (iv) the threshold-matching computation needed to convert the conditional consistency checks into sharp low-energy predictions. Each of these is a self-contained research problem; the present paper is intended as an organizing document on top of which these problems can now be approached.

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Appendix A. A Candidate E_6-Compatible Finite Spectral Triple and the Finite Traces

Appendix A.1. Finite Data

The purpose of this appendix is to make explicit the finite data required by the heat-kernel coefficients used in Section 4. We do not claim here that the full exceptional/nonassociative internal geometry has been uniquely derived from GTD. Rather, we write a minimal finite geometry which is compatible with the low-energy content assumed in the main text and which inserts the $E_6/J_3(\mathbb{O}_\mathbb{C})$ input through the family/Yukawa sector. In the GTD/NCG dictionary, the bosonic sector q_B supplies the vector connection, while the finite matrix D_F is associated with the fermionic/Jordan sector and its bosonized scalar fluctuation.

At the branchwise trinification scale, the associative shadow of the internal symmetry is not most naturally viewed as a single symmetric $SU(3)_c \times SU(3)_L \times SU(3)_R$ trinification, but rather as the two-branch structure motivated in the $E_8 \times E_8$ framework:

$$E_{6L} \longrightarrow SU(3)_c \times SU(3)_{F,L} \times SU(3)_L, \qquad E_{6R} \longrightarrow SU(3)_{c'} \times SU(3)_{F,R} \times SU(3)_R.$$

Here $SU(3)_{F,L}$ and $SU(3)_{F,R}$ are flavour/family symmetries and are treated as global rather than gauged, while $SU(3)_{c'}$ is the would-be right-sector color factor. In the minimal low-energy limit adopted here, the observed QCD group is the unique gauged $SU(3)_c$, the flavour $SU(3)$'s are encoded through the family factor and Jordan data, and any additional right-sector color factor is assumed to be decoupled, Higgsed away, or otherwise absent from the explicit observed-leaf finite algebra.

Accordingly, the finite almost-commutative data used in the present appendix are those of the observed leaf,

$$T_F^{(E_6)} = (\mathcal{A}_F, \mathcal{H}_F, D_F; J_F, \gamma_F), \qquad \mathcal{A}_F = \big(\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})\big) \otimes \mathbf{1}_{\text{fam}},$$

with

$$\mathcal{H}_F = \mathcal{H}_{\text{SM},\nu_R} \otimes \mathbb{C}^3_{\text{fam}}, \qquad \dim \mathcal{H}_{\text{SM},\nu_R} = 32, \qquad \dim \mathcal{H}_F = 96.$$

Thus the global flavour information of the branchwise trinification is not represented by additional gauge generators in $\mathcal{A}_F$, but by the family factor $\mathbb{C}^3_{\text{fam}}$ and by the Jordan operator Y_J introduced below.

We decompose

$$\mathcal{H}_{\text{SM},\nu_R} = H_L \oplus H_R \oplus H_L^c \oplus H_R^c,$$

with one-generation particle content

$$Q_L \sim (3, 2, +1/6), \qquad L_L \sim (1, 2, -1/2),$$

$$u_R \sim (3, 1, +2/3), \qquad d_R \sim (3, 1, -1/3),$$

$$\nu_R \sim (1, 1, 0), \qquad e_R \sim (1, 1, -1).$$

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Appendix A.2. Branchwise Trinification and the Low-Energy Truncation

The fundamental representation of E_6 branches in the usual trinification way as

$$27 = (3,3,1) \oplus (\bar{3},1,\bar{3}) \oplus (1,\bar{3},3),$$

but in the present GTD/$E_8 \times E_8$ setting this branching is to be read branchwise:

$$27_L \quad \text{under} \quad SU(3)_c \times SU(3)_{F,L} \times SU(3)_L,$$

$$27_R \quad \text{under} \quad SU(3)_{c'} \times SU(3)_{F,R} \times SU(3)_R.$$

The visible left-handed electroweak sector is obtained from the breaking

$$SU(3)_L \longrightarrow SU(2)_L \times U(1)_Y,$$

whereas the complementary right branch is broken as

$$SU(3)_R \longrightarrow SU(2)_R \times U(1)_{Y_{\text{dem}}} \longrightarrow U(1)_{\text{dem}}.$$

The two flavour groups $SU(3)_{F,L} \times SU(3)_{F,R}$ remain global and are not part of the gauged finite algebra. Their rôle is instead to organize the three-family structure through the family factor $\mathbb{C}^3_{\text{fam}}$ and the Jordan operator Y_J.

In this sense Appendix A is a low-energy truncation of the branchwise trinification, not a full finite geometry for the entire unbroken $E_8 \times E_8$ stage. The observed multiplets $Q_L, L_L, u_R, d_R, e_R, \nu_R$ embed naturally in the broken left/right branch data, while any additional vector-like or exotic states are assumed to be lifted by symmetry breaking, localization, or decoupling. Consequently, the finite triple written below is constrained by the branchwise E_6 structure but not uniquely fixed by it.

This branchwise refinement affects mainly the interpretation of the finite data, rather than the explicit observed-leaf traces. Because the flavour $SU(3)$'s are global, the finite gauge traces in Section A.4 are unchanged. If one wishes to keep both leaves explicitly, one should duplicate the same formulas with $L \leftrightarrow R$ and with the corresponding replacement of the left electroweak labels by the right-sector labels $\big(SU(2)_R, U(1)_{Y_{\text{dem}}}\big)$.

Appendix A.3. Finite Dirac Operator

A minimal ansatz compatible with the main text is

$$D_F = \begin{pmatrix} 0 & M^\dagger & 0 & 0 \\ M & 0 & 0 & M_R^\dagger \\ 0 & 0 & 0 & \overline{M}^\dagger \\ 0 & M_R & \overline{M} & 0 \end{pmatrix}, \tag{A1}$$

acting on $H_L \oplus H_R \oplus H_L^c \oplus H_R^c$, with

$$M = \text{diag}\big(Y_\nu,\, Y_e,\, Y_u \otimes \mathbf{1}_3,\, Y_d \otimes \mathbf{1}_3\big), \qquad Y_f = y_f\, Y_J. \tag{A2}$$

Here $\mathbf{1}_3$ in the quark blocks acts on color. If a Majorana sector is present we write

$$M_R = y_R\, Y_J. \tag{A3}$$

This is the simplest factorized ansatz realising the claim that the three-family structure is controlled by the exceptional Jordan algebra.

The corresponding fluctuated finite operator is

$$D_{F,\Phi} = D_F + \Phi_{LR} + J_F \Phi_{LR} J_F^{-1}, \tag{A4}$$

with $\Phi_{LR} : H_R \to H_L$ off-diagonal. In the broken-phase effective description one may parameterize Φ_{LR} by a left-branch scalar Φ_L together with a right-branch scalar Φ_R, with the physical Standard-Model-like Higgs arising predominantly from the right branch after mixing; the exact $\mathrm{SU}(2)_R$ representation remains an open choice to be fixed by the full finite geometry.

Appendix A.4. Finite Traces Entering the Heat-Kernel Coefficients

The identity trace is immediate:

$$\mathrm{tr}_{H_F}(\mathbf{1}) = 96. \tag{A5}$$

If both four-dimensional leaves are retained explicitly, all finite traces below double.

For the observed gauge sector, using the normalization

$$\mathrm{tr}_{\mathbf{N}}(T^a T^b) = \frac{1}{2}\,\delta^{ab} \tag{A6}$$

in the fundamental representation, one finds for each fixed generator

$$\mathrm{tr}_{H_F}(T_c^a T_c^b) = 12\,\delta^{ab}, \tag{A7}$$

$$\mathrm{tr}_{H_F}(t_L^i t_L^j) = 12\,\delta^{ij}, \tag{A8}$$

$$\mathrm{tr}_{H_F}(Y^2) = 20. \tag{A9}$$

Indeed, per generation the color generator acts on eight copies of the fundamental/anti-fundamental index (two from Q_L, one from each of u_R and d_R, and the corresponding conjugates), giving $8 \times \frac{1}{2} = 4$; multiplying by three families gives 12. The same counting holds for $\mathrm{SU}(2)_L$. For hypercharge,

$$2\left[6\left(\frac{1}{6}\right)^2 + 3\left(\frac{2}{3}\right)^2 + 3\left(-\frac{1}{3}\right)^2 + 2\left(-\frac{1}{2}\right)^2 + (-1)^2\right] \times 3 = 20. \tag{A10}$$

Hence

$$\frac{3}{5}\,\mathrm{tr}_{H_F}(Y^2) = 12, \tag{A11}$$

which is the standard GUT normalization behind

$$g_3^2 = g_2^2 = \frac{5}{3}\,g_1^2. \tag{A12}$$

Now define the Jordan invariants

$$s_2(J) = \sum_{i=1}^{3} |\lambda_i|^2, \qquad s_4(J) = \sum_{i=1}^{3} |\lambda_i|^4. \tag{A13}$$

Then the Yukawa traces entering the a_2 and a_4 coefficients are

$$a := \mathrm{tr}(M^\dagger M) = s_2(J)\Big(|y_\nu|^2 + |y_e|^2 + 3|y_u|^2 + 3|y_d|^2\Big), \tag{A14}$$

$$b := \mathrm{tr}\big[(M^\dagger M)^2\big] = s_4(J)\Big(|y_\nu|^4 + |y_e|^4 + 3|y_u|^4 + 3|y_d|^4\Big). \tag{A15}$$

If the Majorana block (A3) is retained, the corresponding invariants are

$$a_R = \mathrm{tr}(M_R^\dagger M_R) = |y_R|^2\, s_2(J), \qquad b_R = \mathrm{tr}\big[(M_R^\dagger M_R)^2\big] = |y_R|^4\, s_4(J). \tag{A16}$$

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Equations (A5)–(A16) are the explicit finite traces needed for the heat-kernel coefficients used in Section 4. What remains open is the stronger statement that the above finite geometry follows uniquely from the full exceptional/nonassociative GTD construction, rather than being the minimal associative shadow consistent with it.

Appendix B. Regularizing the GTD Bosonic Quadratic Functional into a Genuine Spectral Action

The identification in Section 3 is exact at the algebraic level, but a bare trace of $\mathcal{D}^\dagger\mathcal{D}$ is not the Chamseddine–Connes spectral action on an infinite-dimensional Hilbert space. Here $\mathcal{D}$ denotes the *normalized dimensionless* GTD Dirac variable; the corresponding physical operator is $\mathcal{D}/L$. This appendix gives an explicit smooth regularization which preserves the quadratic GTD term while producing well-defined cutoff moments (f_0, f_2, f_4).

Appendix B.1. From the Bare Quadratic Trace to a Smooth Cutoff

Write

$$u = \frac{|\mathcal{D}|}{\Lambda}, \qquad |\mathcal{D}| := (\mathcal{D}^\dagger\mathcal{D})^{1/2}, \qquad \Lambda = \frac{L}{L_P}. \tag{A17}$$

The bare GTD bosonic functional is

$$S_{\text{quad}}(\mathcal{D};\Lambda) = \frac{1}{2\Lambda^2}\,\mathrm{Tr}(\mathcal{D}^\dagger\mathcal{D}) = \frac{1}{2}\,\mathrm{Tr}(u^2), \tag{A18}$$

At this stage one should also be explicit about predictive content. The three low-energy quantities usually read from the bosonic spectral action – a gauge normalization, the coefficient of the Einstein term, and the cosmological term – may be used to fix the three moments (f_0, f_2, f_4) or, equivalently, the regulator parameters in the family constructed below. Therefore the bosonic regulator sector by itself is a matching framework rather than a source of three independent predictions. Its nontrivial content lies in the fact that the same moments must then simultaneously feed every bosonic coefficient, so the predictive relations reside in the interdependence of gauge, gravitational, and scalar sectors rather than in the separate values of f_0, f_2, and f_4 themselves. which diverges on the full Hilbert space. We therefore replace it by the genuine spectral action

$$S_f(\mathcal{D};\Lambda) = \mathrm{Tr}[f(u)], \tag{A19}$$

with $f : \mathbb{R} \to \mathbb{R}$ even, positive, smooth, and rapidly decaying.

A pure quadratic regulator $f(u) = u^2\chi(u^2)$ is not sufficient, because $f(0) = 0$ and hence $f_0 = 0$; the a_4 coefficient would then fail to generate the canonically normalized Yang–Mills terms and Higgs quartic coupling. One therefore needs a nonzero constant piece.

Appendix B.2. A Simple Regulator Family

A convenient choice is

$$f_{c,\beta}(u) = (c + \beta u^2)e^{-u^2}, \qquad c > 0, \tag{A20}$$

which is even and rapidly decaying. The corresponding spectral action is

$$S_{c,\beta}(\mathcal{D};\Lambda) = \mathrm{Tr}\left[f_{c,\beta}\left(\frac{|\mathcal{D}|}{\Lambda}\right)\right] = c\,\mathrm{Tr}\left(e^{-\mathcal{D}^\dagger\mathcal{D}/\Lambda^2}\right) + \frac{\beta}{\Lambda^2}\,\mathrm{Tr}\left(\mathcal{D}^\dagger\mathcal{D}\,e^{-\mathcal{D}^\dagger\mathcal{D}/\Lambda^2}\right). \tag{A21}$$

Its heat-kernel asymptotic expansion is

$$\mathrm{Tr}\left[f_{c,\beta}(\mathcal{D}_A/\Lambda)\right] \approx 2f_4\Lambda^4 a_0 + 2f_2\Lambda^2 a_2 + f_0 a_4 + \mathcal{O}(\Lambda^{-2}), \tag{A22}$$

with moments

$$f_0 = f_{c,\beta}(0) = c, \tag{A23}$$

$$f_2 = \int_0^\infty f_{c,\beta}(u)\, u\, \mathrm{d}u = \frac{c+\beta}{2}, \tag{A24}$$

$$f_4 = \int_0^\infty f_{c,\beta}(u)\, u^3\, \mathrm{d}u = \frac{c}{2} + \beta. \tag{A25}$$

The small-u expansion is

$$f_{c,\beta}(u) = c + (\beta - c)u^2 + \left(\frac{c}{2} - \beta\right)u^4 + \mathcal{O}(u^6). \tag{A26}$$

Therefore, if one wants the regulated spectral action to reproduce the quadratic GTD coefficient $\frac{1}{2}\,\mathrm{Tr}(u^2)$ at low order, one imposes

$$\beta - c = \frac{1}{2}. \tag{A27}$$

Under this matching condition the moments become

$$f_0 = c, \qquad f_2 = c + \frac{1}{4}, \qquad f_4 = \frac{3}{2}c + \frac{1}{2}. \tag{A28}$$

Appendix B.3. Relation to the parameters in the main text

The main-text normalization conditions can now be read as equations for the regulator moments. From the gauge-sector normalization,

$$g^2 = \frac{\pi^2}{f_0} = \frac{\pi^2}{c}, \tag{A29}$$

so that

$$c = \frac{\pi^2}{g^2}. \tag{A30}$$

From Newton's constant,

$$\frac{1}{16\pi G} = \frac{2 f_2 L^2}{\pi^2 L_{\mathrm{P}}^2}, \tag{A31}$$

we obtain

$$f_2 = \frac{\pi^2 L_{\mathrm{P}}^2}{32\pi G\, L^2}, \qquad \beta = 2f_2 - c. \tag{A32}$$

Here again G is best regarded as a microscopic GTD constant fixed by $(L_{\mathrm{P}}, \tau_{\mathrm{P}}, \hbar)$, while Eq. (A32) determines the regulator moment required for the localized low-energy spectral matching. Finally,

$$f_4 = \frac{c}{2} + \beta \tag{A33}$$

controls the cosmological-constant term. The important structural point is that the GTD quadratic functional has now been promoted to a genuine spectral action with explicit, finite moments.

What still remains open is the deeper dynamical question: can the cutoff profile f be derived from the localization dynamics of the STM ensemble, rather than chosen phenomenologically? The present appendix does not solve that problem, but it removes the purely formal obstacle that the bare quadratic trace has no well-defined spectral moments.

Appendix C. Operator Dictionary and Exact Reassembly of the Single-STM Action

This appendix records two technical facts used repeatedly in the main text: first, the exact algebraic reassembly of the fundamental single-STM GTD action into the three sectors $\mathcal{L}_{BB} + \mathcal{L}_{BF} + \mathcal{L}_{FF}$; second, a compact dictionary for the various Dirac-type operators that appear in the paper.

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Appendix C.1. Exact reassembly of the GTD action

Starting from the shifted variables

$$q_1^\dagger = q_B^\dagger + \eta\, \beta_1 q_F^\dagger, \qquad q_2 = q_B + \eta\, \beta_2 q_F, \qquad \eta = \frac{L_P^2}{L^2}, \tag{A34}$$

one has

$$\dot{q}_1^\dagger \dot{q}_2 = \dot{q}_B^\dagger \dot{q}_B + \eta\left(\dot{q}_B^\dagger \beta_2 \dot{q}_F + \beta_1 \dot{q}_F^\dagger \dot{q}_B\right) + \eta^2 \beta_1 \dot{q}_F^\dagger \beta_2 \dot{q}_F, \tag{A35}$$

$$q_1^\dagger q_2 = q_B^\dagger q_B + \eta\left(q_B^\dagger \beta_2 q_F + \beta_1 q_F^\dagger q_B\right) + \eta^2 \beta_1 q_F^\dagger \beta_2 q_F, \tag{A36}$$

$$\begin{aligned}\dot{q}_1^\dagger q_2 - q_1^\dagger \dot{q}_2 = \dot{q}_B^\dagger q_B - q_B^\dagger \dot{q}_B + \eta\left(\dot{q}_B^\dagger \beta_2 q_F + \beta_1 \dot{q}_F^\dagger q_B - q_B^\dagger \beta_2 \dot{q}_F - \beta_1 q_F^\dagger \dot{q}_B\right) \\ + \eta^2\left(\beta_1 \dot{q}_F^\dagger \beta_2 q_F - \beta_1 q_F^\dagger \beta_2 \dot{q}_F\right).\end{aligned} \tag{A37}$$

Substituting Eqs. (A35)–(A37) into the fundamental GTD Lagrangian (28), multiplying by the overall (dimensionless) factor $L_P^2/(2L^2) = \eta/2$, and collecting the terms of order η^0, η^1, and η^2 yields exactly the three sectors (33)–(35), or equivalently the expanded forms (36), (37), and (38). This is the exact algebraic justification for the reassembly used throughout the paper.

Appendix C.2. Dictionary of Dirac-type operators and scalar data

Symbol	Dimension	Origin	Role in the derivation
$\mathcal{D}_B = \dot{Q}_B$	dimensionless	full normalized bosonic GTD variable	splits as $\mathcal{D}_{\rm GTD} + \Phi_{B,L} + \Phi_{B,R}$ inside the single-STM action
$\mathcal{D}_{\rm GTD}$	dimensionless	normalized GTD Dirac variable	packages the quaternionic leaf operator together with the non-scalar octonionic vector/gauge fluctuation
$D_{\rm GTD}^{\rm phys} = \mathcal{D}_{\rm GTD}/L$	L^{-1}	physical GTD Dirac operator	physical operator compared with the non-commutative-geometry Dirac operator
$\Phi_{B,L}^{\rm phys} = \Phi_{B,L}/L$	L^{-1}	left bosonic dotted zeroth-mode seed	second-Higgs antecedent extracted from $\dot{q}_{B0L}$
$\Phi_{B,R}^{\rm phys} = \Phi_{B,R}/L$	L^{-1}	right bosonic dotted zeroth-mode seed	SM-like Higgs antecedent extracted from $\omega \dot{q}_{B0R}$
$D_\Sigma^{\rm phys}$	L^{-1}	BF-reduced leafwise Dirac operator	curved four-dimensional differential operator on the emergent leaf
$A_\Sigma^{\rm phys}$	L^{-1}	localized non-scalar octonionic bosonic sector	vector/gauge fluctuation on the four-dimensional leaf
$D_{\rm vec}^{\rm phys}$	L^{-1}	$D_\Sigma^{\rm phys} \otimes \mathbf{1} + A_\Sigma^{\rm phys}$	bosonic differential-plus-vector operator entering the quadratic spectral precursor
Ψ_F	dimensionless odd variable	normalized fermionic GTD Dirac variable $\dot{Q}_F$	unified pre-localized fermionic variable
$\mathcal{D}_F^{\rm phys}$	L^{-1}	finite/internal Dirac operator	carries Yukawa and Majorana data in the almost-commutative geometry; not altered by the bosonic zeroth-mode split

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Symbol	Dimension	Origin	Role in the derivation
Φ_{LR}^{phys}	L^{-1}	auxiliary-field completion of the localized FF channel	composite/off-diagonal scalar bridge entering the finite fluctuation
$\mathcal{D}_A^{\text{phys}}$	L^{-1}	$D_\Sigma^{\text{phys}} \otimes \mathbf{1} + \gamma^5 \otimes \mathcal{D}_F^{\text{phys}} + A_\Sigma^{\text{phys}} + \gamma^5 \otimes \Phi^{\text{phys}}$	completed almost-commutative operator controlling the low-energy action
$\mathcal{D}_A = L\mathcal{D}_A^{\text{phys}}$	dimensionless	normalized counterpart of $\mathcal{D}_A^{\text{phys}}$	operator appearing in the normalized bosonic spectral action $\text{Tr}[f(\mathcal{D}_A/\Lambda)]$

The crucial bookkeeping rule is therefore the following. The bosonic GTD sector first splits into a genuine Dirac/vector part $(D_\Sigma^{\text{phys}} + A_\Sigma^{\text{phys}})$ and two separate bosonic scalar seeds $(\Phi_{B,L}^{\text{phys}}, \Phi_{B,R}^{\text{phys}})$. The cross sector fixes the fermionic pairing with the completed operator. The bifermionic sector fixes the composite/off-diagonal scalar bridge and hence the chirality-mixing part of the finite fluctuation. Only after these pieces are assembled into the branch-resolved two-Higgs fluctuation does one obtain the full almost-commutative operator $\mathcal{D}_A^{\text{phys}}$ used in the spectral action.

Appendix D. A Possible Pre-Breaking Nonassociative, Jordan, and Phase-Space-like Scaffold

Scope and status of this appendix. The body of this paper uses throughout the associative shadow of the internal geometry, controlled by the split-biquaternionic leafwise operator D_6. The main-text construction — including the Higgs sector, the heat-kernel expansion, and the spectral assembly of $\mathcal{D}_A$ — does not depend on the pre-breaking nonassociative structure recorded in this appendix. This appendix is included as a programmatic look-ahead: it records a mathematically motivated candidate pre-breaking scaffold that is compatible with the structural lessons of the main text and that may, in future work, be shown to be the natural ancestor of the broken-phase associative geometry used here. It is not part of the derivation carried out in the body of the paper, and no claim is made here that a full pre-breaking geometry has already been constructed. The role of this appendix is interpretive and programmatic.

The main text is written after symmetry breaking and localization, where the differential sector is controlled by the split-biquaternionic leafwise operator D_6, the internal sector is described by its almost-commutative associative shadow, and the low-energy scalar sector is assembled from branch-resolved bosonic antecedents together with a composite bifermionic bridge. It is natural to ask what geometric structure may exist *before* symmetry breaking, when the full split bioctonionic variables and the branchwise $E_{6L} \times E_{6R}$ matter-gauge sector are still unbroken. The purpose of this appendix is to record a mathematically motivated *candidate* pre-breaking scaffold compatible with the structural lessons of the main text.

The key distinction is the following. The controlled differential operator in the present paper is the associative split-biquaternionic operator D_6 and its leafwise descendants. The larger split-bioctonionic, Jordan, and phase-space-like structures discussed below are proposed as possible *ancestors* of that broken-phase geometry. The appendix suggests how the broken-phase spectral-action framework might sit inside a fuller pre-breaking mathematical structure, without claiming that the latter has already been derived.

This viewpoint is strengthened by the automorphism-invariance discussion in Appendix E of Ref. [10]. There, the deepest symmetry of the pre-geometric octonionic theory is formulated not as ordinary diffeomorphism invariance and not as ordinary Yang–Mills gauge invariance, but as covariance under admissible automorphisms of the underlying noncommutative, nonassociative

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coordinate algebra. In that language, the pre-breaking scaffold is naturally described by algebraic data first, and only later by the emergent spacetime/gauge split familiar from the broken-phase theory.

Appendix D.1. The Split-Bioctonionic Scaffold Before Symmetry Breaking

Before symmetry breaking, the natural algebraic scaffold of one STM atom is the split bioctonion

$$\mathbb{O}_s \oplus \omega \mathbb{O}_s, \qquad \omega^2 = +1,$$

with sixteen real directions in total. It is useful to distinguish the fourteen imaginary directions from the two real directions. The imaginary directions are the natural candidates for differential/Clifford-type data, whereas the real directions are the natural scalar antecedents [42]. Accordingly, a generic split-bioctonionic symbol may be written schematically as

$$X_{\text{pre}} = x_{0L} + \sum_{a=1}^{7} x_L^a e_a + \omega\left(x_{0R} + \sum_{a=1}^{7} x_R^a e_a \right).$$

Here the fourteen imaginary directions $(x_L^a, \omega x_R^a)$ form the natural candidate pre-breaking differential scaffold, whereas the two real directions $(x_{0L}, \omega x_{0R})$ are the pre-breaking scalar antecedents.

This distinction is consistent with the broken-phase analysis of the main text. There, the dotted branchwise bosonic zeroth modes survive as Higgs antecedents, while the differential sector is ultimately organized by the split-biquaternionic leafwise operator. The present appendix simply pushes that split one level higher, to the pre-breaking bioctonionic stage.

Appendix D.2. A Candidate 7D and $(7,7)$ Dirac-Type Structure

Let

$$V_7 := \operatorname{Im}\mathbb{O} \simeq \mathbb{R}^7, \qquad V_{7,7} := V_7 \oplus \omega V_7.$$

The division-algebra and Clifford-algebra analysis used elsewhere in the GTD program suggests that the imaginary octonionic directions are naturally associated with a $Cl(7)$-type structure, whereas the split doubling suggests a corresponding $(7,7)$-signature refinement. This motivates the introduction of a *candidate* pre-breaking differential operator of the schematic form

$$D_7 = \sum_{a=1}^{7} \Gamma^a \nabla_a, \qquad D_{7,7} = \sum_{a=1}^{7} \Gamma^a \nabla_a + \sum_{a=1}^{7} \widetilde{\Gamma}^a \widetilde{\nabla}_a, \tag{A38}$$

with Clifford relations chosen so that, at the level of principal symbols, $D_{7,7}^2$ yields the split Laplacian of signature $(7,7)$.

This should be read carefully. The present paper does *not* use $D_{7,7}$ as its working Dirac operator, and does *not* establish a clean Dirac-square identity on generic octonion-valued fields. Nonassociativity obstructs such a direct spinorial use of all octonionic directions at once. What can be stated more safely is that a $(7,7)$-type principal-symbol scaffold is plausible before symmetry breaking, whereas the clean first-order Dirac operator actually under control in the main text is the associative split-biquaternionic descendant D_6. In that sense, the reduction chain

$$D_{7,7} \rightsquigarrow D_6 \rightsquigarrow D_{4,L} \oplus D_{4,R}$$

should be read as a structural program, not as a theorem proved in the present article.

Appendix D.3. The Pre-Breaking Internal Sector as a Jordan/Nonassociative Background

Before symmetry breaking, the internal sector is more naturally described by a branchwise Jordan/nonassociative structure than by the associative finite algebra $\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$ used in Appendix A. That associative finite algebra is the observed-leaf shadow of the internal geometry after

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symmetry breaking, localization, and low-energy truncation. Before breaking, the relevant exceptional matter-gauge symmetry is instead

$$E_{6L} \times E_{6R},$$

motivated in the octonionic $E_8 \times E_8$ program by the two branchwise exceptional matter-gauge sectors living on the split bioctonionic scaffold.

For this stage, the more natural mathematical language is that of nonassociative or Jordan spectral geometry. In particular, Boyle and Farnsworth extend the spectral action principle to nonassociative geometries; Boyle and Farnsworth also show that Jordan geometry provides a natural language for Standard-Model and Pati–Salam structure; Besnard and Farnsworth define particle models based on Jordan backgrounds; and Farnsworth's later work shows that finite-dimensional exceptional spectral geometries can carry gauge-covariant Dirac operators and charged scalar sectors [22,23,26,43].

A schematic pre-breaking algebraic background compatible with the present program is therefore of the form

$$\mathfrak{B}_{\text{pre}} = (\mathcal{A}_{\text{pre}}, \mathcal{H}_{\text{pre}}, D_{\text{pre}}; J_{\text{pre}}, \Gamma_{\text{pre}}),$$

where $\mathcal{A}_{\text{pre}}$ is not yet an ordinary associative algebra but a branchwise Jordan/nonassociative coordinate structure carrying the $E_{6L} \times E_{6R}$ data, and where D_{pre} is a pre-breaking Dirac-type operator whose broken-phase shadow is the pair (D_6, D_F) used in the main text. In this interpretation, the finite/internal operator D_F of Appendix A is the observed-leaf associative shadow of a deeper Jordan-exceptional operator, while the family/Yukawa ansatz $Y_f = y_f Y_J$ is the low-energy remnant of a deeper pre-breaking Jordan structure rather than the final form of the unbroken internal geometry.

The same point may be phrased as a covariance principle. Following Appendix E of Ref. [10], one may regard the pre-geometric background as a tuple

$$(A_{\text{pre}}, E, D, J, \Gamma, \nabla, S),$$

where A_{pre} is the noncommutative, possibly nonassociative coordinate algebra, E is the matter module, D is a Dirac-type operator, J and Γ are the real structure and grading when defined, ∇ is the generalized connection, and S is the underlying action functional. The natural symmetry principle is then invariance under admissible automorphisms $\phi \in \mathrm{Aut}_{\text{adm}}(A_{\text{pre}})$,

$$S[\Psi, \nabla, D; A_{\text{pre}}] = S[\Psi^{\phi}, \nabla^{\phi}, D^{\phi}; A_{\text{pre}}],$$

rather than separate postulated invariances under spacetime coordinate transformations and internal gauge transformations. In this sense, the Jordan/nonassociative background proposed here should be viewed as an algebraic precursor from which the later spacetime/gauge split emerges, not as an ordinary manifold geometry with extra internal factors already attached.

Appendix D.4. Dotted/Undotted Polarization and the Limited Role of Metaplectic Geometry

The dotted/undotted split in GTD suggests a phase-space-like polarization of the pre-breaking variables: the dotted sectors behave like outer/differential data, whereas the undotted sectors behave like complementary algebraic/internal data. However, the present paper does not construct an explicit symplectic form on this pre-breaking space, and therefore does not prove that the dotted/undotted split is a canonical phase space in the strict symplectic sense.

If one wishes to make this phase-space interpretation mathematically sharp, metaplectic or Mp^c geometry is the natural language. Given a symplectic manifold $(\mathcal{P}, \Omega)$, a metaplectic or Mp^c structure is the symplectic analogue of a spin structure and allows the definition of symplectic spinors and symplectic Dirac operators [44,45]. In the present context, however, this should be viewed as a *possible future completion*, not as part of the derivation carried out in the main text. The broken-phase spectral-action analysis does not require a metaplectic structure, and nothing in the present paper depends quantitatively on one.

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Thus metaplectic geometry, if relevant here at all, should be regarded as optional and pre-breaking. It would become mathematically relevant only after an explicit symplectic form Ω_{pre} on the pre-breaking GTD variables has been constructed. Until then, the phase-space reading of dotted versus undotted variables remains a structural polarization heuristic, not a completed symplectic geometry.

Nothing in Appendix E of Ref. [10] requires such a symplectic or metaplectic completion; the automorphism principle discussed there concerns a deeper algebraic covariance, whereas the metaplectic completion proposed here would be an additional phase-space refinement.

Appendix D.5. The Bifermionic Seed as a Microscopic Higgs Reservoir

The main text analyzes the Higgs sector only after branch projection, e_0 projection, and low-energy truncation. At the microscopic pre-breaking level, the primitive object is instead the full bifermionic seed

$$\mathcal{B}_F := \beta_1 \Psi_F^\dagger \beta_2 \Psi_F,$$

or, in raw GTD language, the corresponding dotted-fermion trace bilinear. This microscopic seed should not be identified directly with the pair of low-energy Higgs bosons. Rather, it is best viewed as a *microscopic composite reservoir* from which the effective scalar sector emerges after localization, channel selection, coarse-graining, and broken-phase mixing.

This interpretation is stronger than saying only that the visible e_0-projected channel yields a Higgs bridge. It is not, however, a claim that the residual non-SM $E_8 \times E_8$ degrees of freedom are themselves bifermionic composites: the Hermitian seed carries the generation factor only through $\mathbf{1} \oplus \mathbf{8}$, and the subsequent ontology analysis [8] establishes that the residual 288 is an adjoint-lineage label ledger decoupled from the matter Lagrangian rather than composite matter to be "closed" into the seed (Section 7.2). What the present paper establishes is the visible low-energy electroweak projection of the bifermionic seed; the two composite Higgs doublets sit in the $\mathbf{78}_{L,R}$ channels of $\mathbf{27} \otimes \mathbf{27}$.

A useful algebraic refinement is to distinguish the quadratic and cubic exceptional invariants. If $J \in J_3(\mathbb{O}_{\mathbb{C}})$, then the quadratic Jordan form

$$B(J_1, J_2) := \mathrm{Tr}_J(J_1 \circ J_2)$$

is the natural bilinear exceptional invariant, whereas the cubic norm $N(J)$ is the familiar E_6-invariant determinant-like form. In the present framework the microscopic composite Higgs seed is more naturally tied to the *quadratic* Jordan form than to the cubic norm, because the bifermionic GTD seed is bilinear rather than trilinear. The cubic norm is better interpreted as an ancestor of the post-breaking Yukawa structure, not as the Higgs field itself.

Accordingly, the schematic chain

$$\mathcal{B}_F \rightsquigarrow (\mathcal{H}_L \oplus \mathcal{H}_R)^{\text{poss}}_{\text{comp}} \rightsquigarrow P_{e_0}(\mathcal{B}_F) \rightsquigarrow O_H \rightsquigarrow \Phi_{LR}$$

should be read as follows. The first step is only a broader interpretive possibility of the $E_8 \times E_8$ program: the full bifermionic seed may act as a microscopic reservoir for some larger composite scalar sector after symmetry breaking. The second step isolates the visible scalar projection. Only the later steps enter the quantitative low-energy analysis of the present paper. Thus the full two-Higgs composite closure remains open, whereas the visible e_0-projected color-singlet electroweak channels are the genuinely checked part of the present derivation.

Appendix D.6. How the pre-breaking scaffold may reduce to the broken-phase geometry

The main text works with a more concrete and economical geometric structure. Appendix E of Ref. [10] suggests the schematic emergence pattern

$$A_{\text{pre}} \rightsquigarrow C^\infty(M_4) \otimes A_{\text{int}}, \qquad \mathrm{Aut}_{\text{adm}}(A_{\text{pre}}) \rightsquigarrow \mathrm{Diff}(M_4) \ltimes G_{\text{int}}.$$

In this picture, ordinary spacetime covariance and ordinary internal gauge symmetry are not primitive independent inputs. Rather, they appear as two classical descendants of one deeper automorphism principle after symmetry breaking and approximate commutativity set in. Moreover, on the gravitational branch the right-handed gauge variables are expected to admit a constrained BF/Plebanski description whose on-shell gauge-covariant transformations reproduce the diffeomorphism redundancy of the emergent spacetime geometry [10]. This gives a precise sense in which the broken-phase spin/Riemannian geometry used in the main text may descend from a deeper pre-geometric octonionic covariance:

1. a split-biquaternionic differential operator D_6 on a $(3,3)$ base;
2. two leafwise four-dimensional descendants $D_{4,L}$ and $D_{4,R}$;
3. a branchwise E_6-motivated internal/family sector whose observed-leaf associative shadow is encoded by Appendix A;
4. and branchwise bosonic scalar antecedents together with a composite bifermionic bridge.

A pre-breaking nonassociative/Jordan scaffold is useful only if it can reduce consistently to this later structure. The most plausible reduction chain is:

1. symmetry breaking selects a preferred quaternionic subalgebra inside each octonionic branch, so that differential directions aligned with this subalgebra survive as leafwise Clifford directions while complementary octonionic directions are reinterpreted as internal vector/gauge data;
2. the BF mechanism then reduces the effective differential structure to the six-dimensional $(3,3)$ operator D_6 and thence to the two overlapping four-dimensional leaves [10];
3. the Jordan/nonassociative internal background reduces to the observed-leaf associative finite geometry of Appendix A, with family structure retained through Y_J;
4. localization of the many-STM ensemble then yields the classical bundle/geometry whose effective action is described by the spectral-action analysis of the main text [16].

At present, this chain is a structural program rather than a finished theorem. The present appendix should therefore be read as a mathematically motivated proposal for what a fuller pre-breaking completion of the GTD-to-spectral-action program might look like, not as an additional derivation on top of the main text.

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