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Exceptional Differential Polynomial Systems Formed by Simple Pseudo-Wronskians of Jacobi Polynomials and Their Infinite and Finite X-Orthogonal Reductions

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Abstract: The paper advances the new technique for constructing the exceptional differential polynomial systems (X-DPSs) and their infinite and finite orthogonal subsets. First, using Wronskians of Jacobi polynomials (JPWs) with a common pair of the indexes, we generate the Darboux-Crum nets of the rational canonical Sturm-Liouville equations (RCSLEs). It is shown that each RCSLE in question has four infinite sequences of quasi-rational solutions (q-RSs) such that their polynomial components from each sequence form a X-Jacobi DPS composed of simple pseudo-Wronskian polynomials (*p*-WPs). For each p-th order rational Darboux Crum transform of the Jacobi-reference (JRef) CSLE, used as the starting point, we formulate two rational Sturm-Liouville problems (RSLPs) by imposing the Dirichlet boundary conditions on the solutions of the so-called 'prime' SLE (*p*-SLE) at the ends of the intervals (-1,+1) or (+1,∞). Finally, we demonstrate that the polynomial components of the q-RSs representing the eigenfunctions of these two problems have the form of simple *p*-WPs composed of p Romanovski-Jacobi (R-Jacobi) polynomials with the same pair of the indexes and a single classical Jacobi polynomial or accordingly p classical Jacobi polynomials with the same pair of positive indexes and a single R-Jacobi polynomial. The common fundamentally important feature of all the simple *p*-WPs involved is that they do not vanish at the finite singular endpoints –the main reason of why they were selected for the current analysis in the first place. The discussion is accompanied by a sketch of the one-dimensional quantum-mechanical problems exactly solvable by the aforementioned infinite and finite EOP sequences.

Keywords: rational Sturm-Liouville equation; pseudo-Wronskian polynomial; Darboux-Crum transformation; exceptional differential polynomial system; exceptional orthogonal polynomial system; exceptional orthogonal polynomials; Romanovski-Jacobi polynomials; Dirichlet problem

MSC: 34B24

1. Introduction

In the recently published paper [1] the author presented the systematic analysis of the Xm-Jacobi 'differential polynomial systems' (DPSs), with the term 'DPS' used in exactly the same sense as it was done by Everitt et al [2,3] for the conventional polynomial sequences satisfying the (generally complex) second-order differential equations with polynomial coefficients (PDEs). The polynomial sequences are referred to as 'exceptional' [4,5], since each sequence either does not start from a constant or lacks the first-degree polynomial, the discovered polynomials are not covered by Bochner's classical proof [6]. As initially stressed by Kwon and Littlewood [7], Bochner himself did not mention the orthogonality of the polynomial systems that he found' and made no attempt to expand his argumentation to the real field – the problem initially analyzed in the 'ill-fated' [8] paper by Routh [9].

It has been proven by Kwon and Littlejohn [7] more recently that all the real field reductions of the complex DPSs constitute quasi-definite orthogonal polynomial sequences [10] and for this reason

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the cited authors refer to the latter as 'OPSs'. However this is not true for the X-DPSs and we thus preserve the term 'X-OPS' solely for the sequences formed by positively definite orthogonal polynomials.

By further advancing the formalism put forward in [1], we then apply the sequential rational Rudjak-Zakhariev [11] transformations (RRZTs) to construct the new RCSLEs with quasi-rational solutions (q-RSs). For convenience we sketch the main features of the generic Rudjak-Zakhariev transformations (RZTs) in Appendix A. We term a RZT rational if it uses a quasi-rational transformation function (q-RTF).

We then take advantage of Schulze-Halberg's formalism for the so-called 'foreign auxiliary equations' [12] to generalize the notion of the Darboux-Crum [13,14] transformations (DCTs) to the CSLEs. It is proven in Appendix B that sequential RZTs give rise to the DCT defined in the aforementioned way. Again we refer to a DCT of the RCSLE as 'rational' (RDCT) if it uses quasi-rational seed functions. Below these seed functions are represented by the four infinite sequences of the quasi-rational solutions (q-RSs) of the Jacobi-reference (JRef) CSLE which is defined via (1)–(3) in Section 2.

In this paper we focus solely on the RDCTs using quasi-rational seed functions with polynomial components formed by Jacobi polynomials with a common pair of the indexes λ_-,λ_+ [15]. The resultant net of the RCSLEs is introduced in Section 4. However, before switching the discussion to these RCSLEs, we found it useful to draw the reader's attention to one of most important element of our technique for generating infinite and finite sequences of exceptional orthogonal polynomials (EOPs), which has been already utilized in [1] for constructing rational Darboux transforms ($\mathrm{RD}\mathfrak{I}$s) of the Romanovski-Jacobi (R-Jacobi) polynomials [16–18].

Namely, to formulate the Sturm-Liouville problem (SLP), we introduce the so-called [19] 'prime' SLEs (*p*-SLEs) chosen in such a way that the two characteristic exponents (ChExps) for the poles at the endpoints differ only by sign. As a result, the energy spectrum of the given Sturm-Liouville problem can be obtained by solving the given *p*-SLE under the Dirichlet boundary conditions (DBCs). This in turn allows one to take advantage of the rigorous theorems proven in [20] for eigenfunctions of the generic SLE solved under the DBCs.

In Section 3 we introduce the prime forms of the JRef CSLE on the orthogonalization intervals (-1,+1) and (1,∞) and then make use of the DBCs to select quasi-rational principal Frobenius solutions (q-RPFSs) near the singular endpoints in question. Our main interests lies q-RPFSs lying below the lowest eigenvalue of the JRef CSLEs. We have already used these solutions as the q-RTFs in [1] to construct infinite and finite subsets of X_m-Jacobi DPSs. In this paper we extend this approach to the RDCTs, using the sequential RRZTs to generate sequences of q-RPFSs (see Appendix B for details).

In Section 5 we again take advantage of Schulze-Halberg's [12] technique to show that the RCSLEs constructed in Section 4 has four infinite sequences of the q-RSs with the polynomial components. While one of these sequences is formed by Jacobi polynomial Wronskians (JPWs), the polynomials components of three others are represented by the so-called [1] 'simple' pseudo-Wronskian polynomials (p-WPs }. Namely, we refer to a pseudo-Wronskian of Jacobi polynomials [21] as 'simple' if only a single polynomial in the given set of seed Jacobi polynomials has at least one Jacobi index with a different sign (compared with the sign of the common index of other Jacobi polynomials in the given set). As proven in Section 6, the simple p-WPs obey the Fuchsian differential equations with polynomial coefficients (FPDE), forming a X-Jacobi DPS. We have here a very specific example of Durán's fundamental theory of the X-Jacobi OPSs [22].

Section 7 constitutes the culminating point of our analysis. Namely, we prove that the rational Darboux-Crum transform ($\mathrm{RDC}\mathfrak{I}$) of eigenfunctions of the Jacobi *p*-SLEs on either finite or infinite interval in question represent the quasi-rational eigenfunctions of the corresponding Dirichlet problem formulated for the given $\mathrm{RDC}\mathfrak{I}$ of the JRef CSLE and therefore the polynomial components of these q-RSs form an orthogonal polynomial set satisfying (by the way it is constructed) to the FPDE .

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While the RDC𝔍s of the classical Jacobi polynomials have been thoroughly covered in [22], the admissibility of the JPWs on the infinite interval (1,∞) represent the important new result. In subsection 7.1 we thus verify our conclusions based on the more general theorems proven in [22] and then formulate the novel approach to the theory of the RDC𝔍s of the R-Jacobi polynomials.

Theorems proven in Section 7 are heavily based on the two cornerstones of the developed formalism thoroughly justified in Appendices C and D. Namely, Appendix C presents the proof that the RRZ𝔍 of the q-RPFS itself is the q-RPFs of the transformed RCSLE. In Appendix D we use this very broad result to prove that the latter RCSLE is exactly solvable and therefore the mentioned q-RPF lies below the lowest eigenvalue. This result in turn lays down the foundation for our crucial proof that the JPWs of our choice do not have zeros inside the given orthogonalization interval.

We illustrate this assertion in Appendix E, using the second-degree JPW as an example.

2. Four Distinguished Infinite Sequences of q-RSs

Let us start our analysis with the Jacobi-reference (JRef) CSLE

$$\left\{\frac{d^2}{d\eta^2}+I^o[\eta;\vec{\lambda}_o]+sgn(1-\eta^2)\,\varepsilon\,\rho[\eta]\right\}\Phi[\eta;\vec{\lambda}_o;\varepsilon]=0 \qquad (1)$$

with the single pole density function

$$\rho[\eta]:=\frac{1}{|\eta^2-1|} \qquad (2)$$

and the reference polynomial fraction (RefPF) parameterized as follows:

$$I^o[\eta;\vec{\lambda}_o]\equiv\sum_{\aleph=\pm}\frac{1-\lambda^2_{o;\aleph}}{4(1-\aleph\eta)^2}+\frac{1-\lambda^2_{o;+}-\lambda^2_{o;-}}{4(1-\eta^2)} \qquad (3)$$

$$=\frac{1}{2(1-\eta^2)}\sum_{\aleph=\pm}\frac{1-\lambda^2_{o;\aleph}}{1-\aleph\eta}-\frac{1}{4(1-\eta^2)}, \qquad (4)$$

where $\lambda_{o;\pm}$ are the ExpDiffs for the poles at ±1 and the energy reference point is chosen by the requirement that the ExpDiff for the singular point at infinity vanishes at zero energy, i. e.,

$$\lim_{|\eta|\to\infty}\left(\eta^2 I^o[\eta;\vec{\lambda}_o]\right)=\tfrac{1}{4}. \qquad (5)$$

The energy sign is chosen in such a way:

$$sgn(\varepsilon)=sgn(1-\eta^2) \qquad (6)$$

that the sought-for eigenvalues are positive (negative) when the Sturm-Liouville problem in question is formulated on the finite interval $-1<\eta<1$ (or respectively on the positive infinite interval $1<\eta<\infty$). An analysis of solutions of the CSLE (1) on the negative infinite interval $-\infty<\eta<-1$ can be skipped without loss of generality due to the symmetry of the RefPF (2) under reflection of its argument, accompanied by the interchange of the exponents differences (ExpDiffs) $\lambda_{o;\pm}$ for the CSLE poles at ±1.

Let us now consider the gauge transformation

$$\Phi[\eta;\vec{\lambda}_o;\varepsilon]=\phi_0[\eta;\vec{\lambda}]F[\eta;\vec{\lambda};\varepsilon], \qquad (7)$$

where

$$\phi_0[\eta;\vec{\lambda}]:=(1+\eta)^{\frac{1}{2}(\lambda_-+1)}|1-\eta|^{\frac{1}{2}(\lambda_++1)}\quad(-1<\eta\neq1). \qquad (8)$$

Keeping in mind that

$$ld\,\phi_0[\eta;\vec{\lambda}] := \dot{\phi}_0[\eta;\vec{\lambda}]/\phi_0[\eta;\vec{\lambda}] = \frac{\lambda_- + 1}{2(\eta+1)} + \frac{\lambda_+ + 1}{2(\eta-1)}, \qquad (9)$$

coupled with (3), one finds

$$\ddot{\phi}_0[\eta;\vec{\lambda}]/\phi_0[\eta;\vec{\lambda}] = ld^2\phi_0[\eta;\vec{\lambda}] + l\dot{d}\,\phi_0[\eta;\vec{\lambda}] \qquad (10)$$

$$= -\mathrm{I}^o[\eta;\vec{\lambda}_o] - \frac{(\lambda_- + \lambda_+ + 1)^2}{4(1-\eta^2)}, \qquad (11)$$

with dot standing for the derivative with respect to η. i.e., the quasi-rational function (8) is the solution of the JRef CSLE at |ε| equal to

$$\varepsilon_0(\vec{\lambda}) = \tfrac{1}{4}(\lambda_- + \lambda_+ + 1)^2. \qquad (12)$$

It then directly follows from the identity

$$\ddot{\Phi}[\eta;\vec{\lambda};\varepsilon]/\phi_0[\eta;\vec{\lambda}] \equiv -\left\{\mathrm{I}^o[\eta;\vec{\lambda}_o] + \frac{\varepsilon_0(\vec{\lambda})}{1-\eta^2}\right\}\mathrm{F}[\eta;\vec{\lambda};\varepsilon] + \ddot{\mathrm{F}}[\eta;\vec{\lambda};\varepsilon]$$

$$+2ld\phi_0[\eta;\vec{\lambda}] \times \dot{\mathrm{F}}[\eta;\vec{\lambda};\varepsilon] \qquad (13)$$

that the function (7) satisfies the FPDE

$$\left[(\eta^2-1)\frac{d^2}{d\eta^2} + 2P_1^{(\lambda_+,\lambda_-)}(\eta)\frac{d}{d\eta} + \tfrac{1}{4}(\lambda_- + \lambda_+ + 1)^2 - |\varepsilon|\right] \times \mathrm{F}[\eta;\vec{\lambda};\varepsilon] = 0 \quad (14)$$

with the polynomial coefficients. It is essential that the resultant FPDE is well-defined for any real values of the variable η, including the border points $|\lambda_-|=1$ or $|\lambda_+|=1$ between the LP and LC regions (which require a special attention and were sidelined for this reason in our current discussion). The FPDE (14) turns into the conventional Jacobi equation

$$(\eta^2-1)\ddot{P}_m^{(\lambda_+,\lambda_-)}(\eta) + 2P_1^{(\lambda_+,\lambda_-)}(\eta)\dot{P}_m^{(\lambda_+,\lambda_-)}(\eta) + \qquad (15)$$

$$m(\lambda_+ + \lambda_- + m + 1)P_m^{(\lambda_+,\lambda_-)}(\eta) = 0$$

at the energies

$$|\varepsilon| = \varepsilon_m(\vec{\lambda}) = \tfrac{1}{4}(\lambda_+ + \lambda_- + 2m + 1)^2. \qquad (16)$$

In following [2,3], we say that the polynomials in question form the Jacobi DPS.

Note that, in addition with the renowned polynomial solutions, the FPDE (14) has 3 other infinite sequences of the q-RSs listed in Table 1 in [22] (or Table 2 in [23]). It is worth pointing out to the difference in our terminology, compared with that in {22,23]. Namely, we restrict the term 'eigenfunction' only to a solution of a Sturm-Liouville problem (SLP), i.e., in our terms only the classical Jacobi polynomials constitute the eigenfunctions of the Sturm-Liouville differential expression (15), assuming that the corresponding polynomial SLP (PSLP] is formulated on the interval (-1,+1), using the boundary conditions (10) in [24].

By choosing

$$\lambda_-,\ \lambda_+,\ \lambda_- + \lambda_+ + m \neq -k \qquad \text{for any positive integer } k \le m \qquad (17)$$

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(see §4.22(3) in [25]), we assure that the Jacobi polynomial in question has exactly m simple zeros $\eta_l(\vec{\lambda};m)$, i.e., using its monic form,

$$\hat{P}_m^{(\lambda_+,\lambda_-)}(\eta)=\Pi_m[\eta;\bar{\eta}(\vec{\lambda};m)], \tag{18}$$

where by definition

$$\Pi_m[\eta;\bar{\eta}]:=\prod_{l=1}^{m}[\eta-\eta_l]. \tag{19}$$

It is crucial that the Jacobi indexes do not depend on the polynomial degree, in contrast with the general case [26,27]. This remarkable feature of the CSLE under consideration is the direct consequence of the fact that the density function (2) has only simple poles in the finite plane [15] and as a result the ExpDiffs for the CSLE poles at ±1 become energy-independent [28].

We conclude that the JRef CSLE with the density function (2) has four infinite sequences of the q-RSs

$$\phi_m[\eta;\vec{\lambda}]=|1+\eta|^{\frac{1}{2}(\lambda_-+1)}|1-\eta|^{\frac{1}{2}(\lambda_++1)}\,P_m^{(\lambda_+,\lambda_-)}(\eta) \tag{20}$$

$$(|\lambda_\pm|=\lambda_{0;\pm})$$

at the energies (16), with the vector parameter $\vec{\lambda}$ restricted to the one of the four quadrants for each sequence.

Each infinite sequence starts from the q-RS (8) with $\vec{\lambda}$ restricted to the corresponding quadrant. Substituting (9) into the identity

$$\ddot{\phi}_0[\eta;\vec{\lambda}]/\phi_0[\eta;\vec{\lambda}]\equiv ld^2\phi_0[\eta;\vec{\lambda}]+l\dot{d}\,\phi_0[\eta;\vec{\lambda}], \tag{21}$$

where the symbolic expression *ld* f[η] denotes the logarithmic derivative of the function f[η], we find that the function (8) is the solution of the Riccati equation

$$l\dot{d}\;\phi_0[\eta;\vec{\lambda}]-ld^2\phi_0[\eta;\vec{\lambda}]+I^0[\eta;\vec{\lambda}]+\frac{\varepsilon_0(\vec{\lambda})}{1-\eta^2}=0 \tag{22}$$

3. Use of ‘Prime’ Forms of J-Ref CSLE to Select q-RPFSs

The gauge transformation

$$\Psi_{\wp}[\eta;\vec{\lambda};\varepsilon]=\wp^{-\frac{1}{2}}[\eta;\vec{\lambda}]\Phi[\eta;\vec{\lambda};\varepsilon], \tag{23}$$

with an arbitrarily chosen positive function $\wp[\eta]$, converts the JRef CSLE (1) into the SLE of the generic form:

$$\{\frac{d}{d\eta}\wp[\eta;\vec{\lambda}]\frac{d}{d\eta}-q_{\wp}[\eta;\vec{\lambda}]+sgn(1-\eta^2)\varepsilon\, w_{\wp}[\eta;\vec{\lambda}]\}\Psi[\eta;\vec{\lambda};\varepsilon]=0, \tag{24}$$

with the weight

$$w_{\wp}[\eta;\vec{\lambda}]:=\rho[\eta]/\wp[\eta;\vec{\lambda}]. \tag{25}$$

The PF representing the zero-energy free term is given by the following generic formula [19]:

$$q_{\wp}[\eta;\vec{\lambda}]=\wp[\eta;\vec{\lambda}]I^0[\eta;\vec{\lambda}]+\mathcal{S}\{\wp[\eta;\vec{\lambda}]\} \tag{26}$$

with

$$\mathfrak{S}\{f[\eta]\} := \tfrac{1}{4}\dot{f}^{2}[\eta]/f[\eta] - \tfrac{1}{2}\ddot{f}[\eta] \tag{27}$$

and the sign of the sought-for spectral parameter ε is dictated by the constraint (6).

Let us choose the leading coefficient function in such a way:

$$p[\eta] = \wp[\eta] := \begin{cases} 1-\eta^2 & \text{for } -1<\eta<+1, \\ \eta-1 & \text{for } \eta>1 \end{cases} \tag{28}$$

that the SLP of our interest can be formulated as the Dirichlet problem:

$$\lim_{\eta\to\eta_{\mp}} \psi_j[\eta;\vec{\lambda}] = 0 \tag{29}$$

at the ends of the given interval of orthogonalization $\eta_{\mp} = \mp 1$ or $\eta_{-} = 1$, $\eta_{+} = \infty$. It has been proven in [20] that the eigenfunctions of this Dirichlet problem must be square-integrable:

$$\int_{\eta_-}^{\eta_+} d\eta\, \psi_j^2[\eta;\vec{\lambda}]w[\eta] < \infty \tag{30}$$

and mutually orthogonal:

$$\int_{\eta_-}^{\eta_+} d\eta\, \psi_j[\eta;\vec{\lambda}_o]\psi_{j'}[\eta;\vec{\lambda}_o]w[\eta] = 0 \quad (j \neq j'), \tag{31}$$

with the weight

$$w[\eta] := w_{\wp}[\eta] = \begin{cases} 1 & \text{for } -1<\eta<+1, \\ (1+\eta)^{-1} < \tfrac{1}{2} & \text{for } \eta>1. \end{cases} \tag{32}$$

Due to the very special choice of the leading coefficient function (28) for the *p*- SLE (24), the two ChExps for each singular endpoint differ by sign, while having exactly the same absolute value, which assures [19] that each DBC unambiguously selects PFS near the given end. In other words, the DBCs (28) unequivocally determine the PFSs near the both singular ends of the given interval of orthogonalization.

Substituting (28) into (27) gives [29]:

$$\mathfrak{S}\{\wp[\eta]\} = \begin{cases} (1-\eta^2)^{-1} & \text{for } -1<\eta<+1, \\ \tfrac{1}{4}(\eta-1)^{-1} & \text{for } \eta>1, \end{cases} \tag{33}$$

which shows the free-energy term of the *p*-SLE with the leading coefficient function (28) has simple poles in the finite plane.

As illuminated in Section 7, the concept of the *p*-SLEs allows one to select the sequences of the nodeless PFSs, which assures that the corresponding 'X-Bochner operators' (in terms of [30]) are regular inside the given orthogonalization interval. This is one of the most important achievements of this paper.

3.1. Dirichlet Problem on Interval (-1,+1)

The crucial advantage of representing the conventional Jacobi equation in the *p*-SLE form is that the q-RS

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$$\psi_j[[\eta;\vec{\lambda}] = (1+\eta)^{\frac{1}{2}\lambda_-}(1-\eta)^{\frac{1}{2}\lambda_+} P_j^{(\lambda_+,\lambda_-)}(\eta) \tag{34}$$

represents the PFS near the poles at ∓ 1 iff the corresponding Jacobi index $\lambda_\mp$ is positive. In particular, the q-RSs

$$\psi_j[[\eta;\vec{\lambda}_o] = (1+\eta)^{\frac{1}{2}\lambda_{o;-}}(1-\eta)^{\frac{1}{2}\lambda_{o;+}} P_j^{(\lambda_{o;+},\lambda_{o;-})}(\eta) \quad (-1<\eta<1) \tag{35}$$

formed by the classical Jacobi polynomials with positive indexes necessarily satisfy the DBCs at ∓ 1 and as a result constitute the eigenfunctions of the given Dirichlet problem. The orthogonality relations (30) thus turn into the conventional orthogonality relations for the classical Jacobi polynomials

$$\int_{-1}^{+1} d\eta\, P_j^{(\lambda_{o;+},\lambda_{o;-})}(\eta) P_{j'}^{(\lambda_{o;+},\lambda_{o;-})}(\eta)\mathcal{W}_m[\eta;\vec{\lambda}_o] = 0 \quad (j \neq j') \tag{36}$$

with the weight function

$$\mathcal{W}_m[\eta;\vec{\lambda}] := (1+\eta)^{\lambda_-}(1-\eta)^{\lambda_+} \quad \text{for } -1<\eta<1. \tag{37}$$

Since the j^{th}+1-solution has exactly j zeros between -1 and +1 and the positive eigenvalues converges to 0 as the polynomial degree tends to infinity, the Dirichlet problem in question may not have any other eigenfunctions.

One still needs to prove that the q-RSs (34) form the complete set of the eigenfunctions of the given Dirichlet problem. This can be performed, for example, by converting the JRef CSLE (1) to the hypergeometric equation on the interval (0,+1) and then follow the arguments presented by the author [26] for the exactly solvable JRef CSLE with the properly chosen density function.

On other hand, choosing

$$\mathcal{p}[\eta;\vec{\lambda}] := (1-\eta^2)\mathrm{W}[\eta;\vec{\lambda}] = (1+\eta)^{\lambda_-+1}(1-\eta)^{\lambda_++1}, \tag{38}$$

we come to the Sturm-Liouville form [31] of the Jacobi equation:

$$\left\{\frac{d}{d\eta}\mathcal{p}[\eta;\vec{\lambda}]\frac{d}{d\eta} - q_{\mathcal{p}}[\eta;\vec{\lambda}] + \varepsilon_j(\vec{\lambda}_o)\mathrm{W}[\eta;\vec{\lambda}]\right\} P_j^{(\lambda_+,\lambda_-)}(\eta) = 0. \tag{39}$$

It is crucial that the leading coefficient function (38) for $\lambda_\mp > -1$ vanishes at the ends of the interval [–1,+1], which assures that the 'generalized' [32] Wronskian ($\mathcal{g}$-W) of two classical Jacobi polynomials

$$\mathcal{W}_{\mathcal{p}}\{P_j^{(\lambda_+,\lambda_{o;-})}(\eta), P_{j'}^{(\lambda_+,\lambda_-)}(\eta)\} := \mathcal{p}[\eta;\vec{\lambda}]\mathrm{W}\{P_j^{(\lambda_+,\lambda_{o;-})}(\eta), P_{j'}^{(\lambda_+,\lambda_-)}(\eta)\} \tag{40}$$

$$\text{for } 0 \le j' < j \le j_{max}$$

vanishes at ±1.

Our next step is first to consider all the q-RSs of the given *p*-SLE, which vanish at one of the endpoints of the infinite interval (-1, +1) and then select the subsets of the collected PFSs below the lowest eigenvalue.

Since our approach allows one to formulate the spectral problem only for positive values of the Jacobi indexes, this limitation restricts one's ability to construct the X-Jacobi OPSs formed by the RDC𝔖s of the classical Jacobi polynomials with negative indexes, as it has already become clear from our analysis [1] of the Xm-Jacobi OPSs. However, as demonstrated in [1], the certain advantage of our approach is that it allows one to treat in parallel the RSLPs for both intervals (-1,+1) and (1,∞) and moreover to prove that the RDC𝔖s of the PFSs of the JRef CSLE are themselves the PFSs of the resultant RCSLEs. We refer the reader to Appendix C, where we present the proof of the latter assertion representing one of the pillars of our formalism.

In following our olden study [33] on the Darboux transforms (D𝔖s) of radial potentials, we use the letters **a** and **b** to specify the PFS near the singular endpoints ∓ 1 (cases I and II in Quesne's [34] commonly used classification scheme of q-RSs according to their behavior near the endpoints). We use the letters **c** and **d** [33] to identify the $n_{ç}$ eigenfunctions and respectively all the q-RSs (34) not vanishing at both ends (case III in Quesne's classification scheme). For the given SLP there is the one-to-one correlation between the labels **t**=**a**, **b**, **c**, **d** and the sign $\sigma_{\mp}$ of the Jacobi indexes $\lambda_{\mp}$, as specified in Table 1.

Table 1. Correlation between labels **t** and signs of Jacobi indexes.

t	**a**	**b**	**c**	**d**
σ- σ+	+ -	- +	+ +	+ -

Re-writing the dispersion formula (16) for $\vec{\sigma} = \pm\mp$ as

$$\varepsilon_{\pm\mp,m}(\vec{\lambda}_o) = (\mp\lambda_{o;-} \pm \lambda_{o;+} - 2m - 1)^2 \qquad (41)$$

we find that the PFSs of either type **a** or type **b** lie below the lowest eigenvalue

$$\varepsilon_{\mathbf{c},0}(\vec{\lambda}_o) = (\lambda_{o;-} + \lambda_{o;+} + 1)^2, \qquad (42)$$

iff

$$m < -\lambda_+ = \lambda_{o;+} \quad (\lambda_{\mp} = \pm\lambda_{o;\mp}) \qquad (43)$$

or

$$m < -\lambda_- = \lambda_{o;-} \quad (\lambda_{\mp} = \mp\lambda_{o;\mp}) \qquad (44)$$

accordingly.

Using the Klein formulas [25], we have proved [28] that the m-degree Jacobi polynomial with the indexes λ_- and λ_+ does not have zeros between –1 and +1 iff

$$m \le \tfrac{1}{2}(|\lambda_-| - \lambda_- + |\lambda_+| - \lambda_+) \qquad (45)$$

and

$$(-)^m \langle\lambda_- + 1\rangle_m \langle\lambda_+ + 1\rangle_m =$$

$$\langle -\lambda_- - m\rangle_m \langle\lambda_+ + 1\rangle_m = \langle\lambda_- + 1\rangle_m \langle -\lambda_+ - m\rangle_m > 0\,, \qquad (46)$$

where $\langle\nu\rangle_m$ is the rising factorial [35]. One can easily verify that the latter conditions do hold under the both constraints (43) and (44).

3.2. Dirichlet Problem on Interval (1,∞)

Examination of the q-RSs

$$\psi_j[[\eta; -\lambda_{o;-}, \lambda_{o;+}] = (1+\eta)^{1/2 - 1/2\lambda_{o;-}} (\eta - 1)^{1/2\lambda_{o;+}} P_j^{(\lambda_{o;+}, -\lambda_{o;-})}(\eta) \quad (47)$$

reveals that they satisfy the DBCs at the both ends of the interval (1,∞) for

$$0 \le j < \tfrac{1}{2}(\lambda_{o;-}, -\lambda_{o;+} - 1) \quad (48)$$

and therefore represent the eigenfunctions of the RSLP in question, which brings us to the orthogonality relations

$$\int_1^\infty d\eta\, P_j^{(\lambda_{o;+},-\lambda_{o;-})}(\eta) P_{j'}^{(\lambda_{o;+},-\lambda_{o;-})}(\eta) \mathcal{W}_m[\eta;\vec{\lambda}_o] = 0 \quad (j \ne j') \quad (49)$$

with the weight function

$$\mathcal{W}_m[\eta;\vec{\lambda}_o] := (1+\eta)^{-\lambda_{o;-}} (\eta-1)^{\lambda_{o;+}} \quad \text{for} \quad \eta > 1. \quad (50)$$

One can easily verify that (49) is nothing but another form of the conventional orthogonality relations for the R-Jacobi polynomials

$$\int_0^\infty d\underset{\to}{z}\; J_j^{(\alpha,\beta)}(\underset{\to}{z}) J_{j'}^{(\alpha,\beta)}(\underset{\to}{z})\, \underset{\to}{\omega}_{\alpha,\beta}[\underset{\to}{z}] = 0 \quad (j \ne j') \quad (51)$$

with the weight function

$$\underset{\to}{\omega}_{\alpha,\beta}[\underset{\to}{z}] := \underset{\to}{z}^{\alpha} (\underset{\to}{z}+1)^{-|\beta|} \quad \text{for} \quad \eta \in [1,\infty) \quad (52)$$

under constraint $\alpha > 0$, $\beta < 0$, where we adopted Askey's [36] definition of the R-Jacobi polynomials which, as proven by Chen and Srivastava [37], is equivalent to the elementary formula

$$J_n^{(\alpha,\beta)}(\underset{\to}{z}) := P_n^{(\alpha,\beta)}(2\underset{\to}{z}+1) \text{ for } \alpha > -1,\ \beta + 2n < 0\,, \quad (53)$$

with

$$\underset{\to}{z} := \tfrac{1}{2}(\eta - 1)\,. \quad (54)$$

Note that we [38] (see also [39]) changed the symbol R for J to avoid the confusion with R-Routh (Romanovski/pseudo-Jacobi [17,18]) polynomials denoted in the recent publications [40–43] by the same letter 'R'.

Our next step is to determine all the q-RSs vanishing at one of the endpoints of the infinite interval [+1, +∞) and then select the subsets of the collected PFSs which lie below the lowest eigenvalue. To explicitly reveal the behavior of the Jacobi-seed (JS) q-RSs (54) near the singular endpoints in question, we label them as indicated in Table 2 below, with σ_∞ specifying either the decay (+) or growth (-) of the given JS at infinity.

To indicate that the classification of the JS solutions is done on the infinite interval (1, ∞), we underline the symbol $\dagger$ by tilde. We then mark the given symbol by prime if the polynomial components of the given sequence of the q-RSs do not include a constant. (Note that the 'secondary' sequences of such a type do not exist for the potentials with infinitely many discrete energy levels which were the focal point of Quesne's analysis [34].)

By definition

$$\underset{\sim}{\varepsilon}_{\underset{\sim}{\dagger}\bar{\sigma},m},{}_m(\vec{\lambda}_o) \equiv -\varepsilon_{\bar{\sigma},m}(\vec{\lambda}_o), \quad (55)$$

Note that the PFSs of the series $\underset{\sim}{b}'$ may exist only if the SLE does not have the discrete energy spectrum. We thus need to consider the three sequences of the quasi-rational PFSs: two *primary* (starting from m=0) sequences $\underset{\sim}{a}$ and $\underset{\sim}{b}$ as well as the infinite secondary sequence $\underset{\sim}{a}'$ starting from $m = n_{\underset{\sim}{c}}$.

The primary sequence $\underset{\sim}{a}$ is formed by classical Jacobi polynomials and consequently may not have zeros between 1 and ∞. As expected, all the PFSs of this type lie at the energies

Table 2. Classification of JS solutions on the infinite interval (1, ∞) based on their asymptotic behavior near the endpoints.

$\mathbf{t}_{\tilde{\sigma},m}$	σ_- σ_+ σ_∞	m
$\underset{\sim}{\mathbf{a}}$	+ + −	$0 \le m < \infty$
$\underset{\sim}{\mathbf{a}}'$	− + −	$m \ge n_{\underset{\sim}{\mathbf{c}}} = j_{max} + 1$
$\underset{\sim}{\mathbf{b}}$	− − +	$0 \le m < \frac{1}{2}(\lambda_{o;-} + \lambda_{o;+} - 1)$
$\underset{\sim}{\mathbf{b}}'$	− + +	$0 \le m < \frac{1}{2}(\lambda_{o;+} - \lambda_{o;-} - 1)$
$\underset{\sim}{\mathbf{c}}$	− + +	$0 \le m \le j_{max}$
$\underset{\sim}{\mathbf{d}}$	+ − −	$\frac{1}{2}(\lambda_{o;+} - \lambda_{o;-} - 1) \le m < \infty$
$\underset{\sim}{\mathbf{d}}'$	− − −	$m > \frac{1}{2}(\lambda_{o;-} + \lambda_{o;+} - 1)$

$$\varepsilon_{\underset{\sim}{\mathbf{a}},m}(\vec{\lambda}_o) \equiv -\varepsilon_{++,m}(\vec{\lambda}_o) \qquad (56)$$

below the lowest eigenvalue

$$\varepsilon_{\underset{\sim}{\mathbf{c}},0}(\vec{\lambda}_o) \equiv -\varepsilon_{+-,0}(\vec{\lambda}_o). \qquad (57)$$

The PFSs from the primary sequence $\underset{\sim}{\mathbf{b}}$ at the energies

$$\varepsilon_{\underset{\sim}{\mathbf{b}},m}(\vec{\lambda}_o) \equiv -\varepsilon_{--,m}(\vec{\lambda}_o) \qquad (58)$$

for

$$0 \le m < \frac{1}{2}(\lambda_{o;-} + \lambda_{o;+} - 1) \qquad (59)$$

do not have real zeros larger than 1 iff

$$\varepsilon_{\underset{\sim}{\mathbf{b}},\mathbf{m}}(\vec{\lambda}_o) - \varepsilon_{\underset{\sim}{\mathbf{c}},0}(\vec{\lambda}_o) = -4(\lambda_{o;-} - \mathbf{m} - 1)(\lambda_{o;+} - \mathbf{m}) < 0, \qquad (60)$$

i.e., iff

$$0 \le \mathbf{m} < \lambda_{o;+} < \lambda_{o;-} - 1. \qquad (61)$$

Similarly the PFSs from the secondary sequence $\underset{\sim}{\mathbf{a}}'$ at the energies

$$\varepsilon_{\underset{\sim}{\mathbf{a}}',m}(\vec{\lambda}_o) = -\varepsilon_{+-,m}(\vec{\lambda}_o) \quad \text{for} \quad m \ge n_{\underset{\sim}{\mathbf{c}}} \qquad (62)$$

do not have real zeros larger than 1 iff

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$$\varepsilon_{\mathbf{a}',m}(\vec{\lambda}_o) - \varepsilon_{\mathbf{c},0}(\vec{\lambda}_o) = -4m(\lambda_{o;+} - \lambda_{o;-} + m + 1) < 0 \tag{63}$$

or, in other words, iff

$$m > \lambda_{o;-} - \lambda_{o;+} - 1. \tag{64}$$

4. RDCTs of JRef SLE Using Seed Jacobi Polynomials with Common Pair of Indexes

We call the DCT rational if it uses quasi-rational seed functions. In this Section we focus solely on the RDCTs using the seed functions (20) with the common Jacobi indexes $\vec{\lambda}$. Let us consider the RDCT using an arbitrary set of p seed functions,

$$\bar{M}_p := m_1, ..., m_p .$$

Denoting the Jacobi polynomial Wronskian (JPW) as

$$W^{(\lambda_+,\lambda_-)}_{\mathfrak{N}(\bar{M}_p)}[\eta\,|\,\bar{M}_p] := W\{P^{(\lambda_+,\lambda_-)}_{m_{k=1,...,p}}(\eta)\} \tag{65}$$

and substituting the Wronskian

$$W\{\phi_{m_{k=1,...,p}}[\eta;\vec{\lambda}]\} = \phi_0^p[\eta;\vec{\lambda}]W^{(\lambda_+,\lambda_-)}_{\mathfrak{N}(\bar{M}_p)}[\eta\,|\,\bar{M}_p] \tag{66}$$

into (A18), we come to the RCSLE

$$\left\{\frac{d^2}{d\eta^2} + I^o[\eta;\vec{\lambda}|\bar{M}_p] + sgn(1-\eta^2)\varepsilon\,\rho[\eta]\right\}\Phi[\eta;\vec{\lambda};\varepsilon|\bar{M}_p] = 0 \tag{67}$$

with the RefPF [12]

$$I^o[\eta;\vec{\lambda}|\bar{M}_p] = I^o[\eta;\vec{\lambda}_o] + 2p\sqrt{\rho[\eta]}\,\frac{d}{d\eta}\frac{ld\,\phi_0[\eta;\vec{\lambda}]}{\sqrt{\rho[\eta]}} - p(p-2)\mathfrak{J}\{\rho[\eta]\}$$

$$+2\sqrt{\rho[\eta]}\,\frac{d}{d\eta}\frac{ld\,W^{(\lambda_+,\lambda_-)}_{\mathfrak{N}(\bar{M}_p)}[\eta\,|\,\bar{M}_p]}{\sqrt{\rho[\eta]}}. \tag{68}$$

Let us now show that the first three summands can be then re-arranged as

$$I^o[\eta;\vec{\lambda}+p\vec{1}] = I^o[\eta;\vec{\lambda}_o] + 2p\sqrt{\rho[\eta]}\,\frac{d}{d\eta}\frac{ld\,\phi_0[\eta;\vec{\lambda}]}{\sqrt{\rho[\eta]}} \tag{69}$$

$$+2\sqrt{\rho[\eta]}\,\frac{d}{d\eta}\frac{ld\,W^{(\lambda_+,\lambda_-)}_{\mathfrak{N}(\bar{M}_p)}[\eta\,|\,\bar{M}_p]}{\sqrt{\rho[\eta]}} - p(p-2)\mathfrak{J}\{|1-\eta^2|\}$$

and then prove that

$$I^o[\eta;\vec{\lambda}|\bar{M}_p] = I^o[\eta;\vec{\lambda}+p\vec{1}] + 2\,\dot{ld}\,W^{(\lambda_+,\lambda_-)}_{\mathfrak{N}(\bar{M}_p)}[\eta\,|\,\bar{M}_p]$$

$$-ld\,\rho[\eta]\,ld\,W^{(\lambda_+,\lambda_-)}_{\mathfrak{N}(\bar{M}_p)}[\eta\,|\,\bar{M}_p] \qquad (70)$$

which represents one of the most important results of this section. To prove (70), we first re-write the second summand in (69) as

$$2p\sqrt{\rho[\eta]}\,\frac{d}{d\eta}\,\frac{ld\,\phi_0[\eta;\vec{\lambda}]}{\sqrt{\rho[\eta]}} = 2p\,l\dot{d}\,\phi_0[\eta;\vec{\lambda}] - p\,ld\,\rho[\eta]\,ld\,\phi_0[\eta;\vec{\lambda}]\,. \qquad (71)$$

Taking into account (9), coupled with

$$ld\,\rho[\eta] = -\frac{1}{\eta+1} - \frac{1}{\eta-1}, \qquad (72)$$

gives

$$2p\sqrt{\rho[\eta]}\,\frac{d}{d\eta}\,\frac{ld\,\phi_0[\eta;\vec{\lambda}]}{\sqrt{\rho[\eta]}} = 2p\,l\dot{d}\,\phi_0[\eta;\vec{\lambda}] - p\,ld\,\rho[\eta]\,ld\,\phi_0[\eta;\vec{\lambda}] \qquad (73)$$

$$= -\frac{p(\lambda_+ + 1)}{2(\eta-1)^2} - \frac{p(\lambda_- + 1)}{2(\eta+1)^2} + \frac{p(\lambda_- + \lambda_+ + 2)}{2(\eta^2-1)}\,. \qquad (74)$$

Combining (74) with the definition (3) of the RefPF of the JRef CSLE (1), and also taking into account that [29]

$$\mathfrak{I}\{|1-\eta^2|\} = \frac{1}{4(\eta-1)^2} + \frac{1}{4(\eta+1)^2} - \frac{1}{2(\eta^2-1)}, \qquad (75)$$

one can directly confirm that the three distinguished singularities appearing in the right-hand side of (74) can be grouped as follows

$$2p\sqrt{\rho[\eta]}\,\frac{d}{d\eta}\,\frac{ld\,\phi_0[\eta;\vec{\lambda}]}{\sqrt{\rho[\eta]}} = I^o[\eta;\vec{\lambda} + p\vec{1}] - I^o[\eta;\vec{\lambda}_o] + \qquad (76)$$

$$+p(p-2)\mathfrak{I}\{|1-\eta^2|\}.$$

Before proceeding with the further analysis of the RefPF (66), let us first illuminate some remarkable features of the JPW (65). First, let us prove that the Wronskian of the Jacobi polynomials with the common positive integer does not vanish at the corresponding pole of the JRef CSLE (1), which simplifies the computation of the ExpDiffs for the pole of the RefPF (66) at this point.

Theorem 1: *The JPW (65) is finite at the singular point* ∓ 1 *if* $\lambda_\mp > 0$.

Proof. To verify this assertion, we take advantage of our generic observation that any DCT can be decomposed into the sequence of RZTs with the TFs given by the recurrence formulas (A19) in Appendix B. Making use of (66), one can easily verify that these TFs for the RDCTs under consideration have the following quasi-rational form

$$\phi_{m_{p+1}}[\eta;\vec{\lambda}|\bar{M}_p] = \frac{\phi_0[\eta;\vec{\lambda}]\,W^{(\lambda_+,\lambda_-)}_{\mathfrak{N}(\bar{M}_{p+1})}[\eta\,|\,\bar{M}_{p+1}]}{\rho^{\frac{1}{2}p}[\eta]\,W^{(\lambda_+,\lambda_-)}_{\mathfrak{N}(\bar{M}_p)}[\eta\,|\,\bar{M}_p]}, \qquad (77)$$

i.e., taking into account (2) and (8),

$$\phi_{m_{p+1}}[\eta;\vec{\lambda}|\bar{M}_p] = \frac{\phi_0[\eta;\vec{\lambda}+p\vec{1}]W^{(\lambda_+,\lambda_-)}_{\mathfrak{N}(\bar{M}_{p+1})}[\eta\,|\,\bar{M}_{p+1}]}{W^{(\lambda_+,\lambda_-)}_{\mathfrak{N}(\bar{M}_p)}[\eta\,|\,\bar{M}_p]}. \tag{78}$$

Let us assume that the JPW in the denominator of the PF on the right remains finite at ∓ 1. Then examination of the RefPF (70), coupled with the definition of the JRef RefPF (3), reveals that the power exponent of η±1 in the numerator of the PF (78) coincides with the positive ChExp for the pole of the RCSLE (67) at ∓ 1. Since the second ChExp, according to (3), is negative, the TF (78) necessarily represents the PFS near the pole in question if $\lambda_\mp > 0$, and therefore the numerator of the PF may not have the zero at ∓ 1. This completes the proof of Theorem 1 by mathematical induction, since the theorem necessarily holds for p=1 due to the constraint (17) imposed on the seed Jacobi polynomial. □

In Appendix E we explicitly confirm the theorem for the simplest second-degree JPW Wronskian formed by the Jacobi polynomials of degrees 1 and 2

Corollary 1: *The Wronskian of the classical Jacobi polynomials with positive indexes may not have zeros at* ∓ 1.

As illuminated in subsection 7.1, this corollary plays the crucial role in the theory of the $RDC\mathfrak{S}$s of the R-Jacobi polynomials using the quasi-rational seed functions formed by the classical Jacobi polynomials with positive indexes.

We were unable to prove Theorem 1 for the case when the 2D vector $\vec{\lambda}$ lies in the third quarter so we make the following assumption specifically for this case:

Corollary 2: *The JPW does not generally have zeros at* ∓ 1 *, regardless of the sign of* $\lambda_\mp$.

Proof. After representing the JPW as a polynomial in either $\eta - 1$ or $\eta + 1$ (instead of η), let us take advantage of the fact that the common leading coefficient is the polynomial in both λ_- and λ_+. According to Theorem 1, this coefficient remains finite at positive values of $\lambda_\mp$ and therefore does not vanish identically for any values of $\lambda_\mp$, which completes the proof. □

In [1] we implicitly used this assumption to construct the finitely many sequences of the $RD\mathfrak{S}$s of the R-Jacobi polynomials using the TFs of type $\underset{\sim}{\mathbf{b}}$ ($\lambda_\mp < 0$, without going into more details.

As explained below, we also have to disregard some specially designed exceptions [44], when the quasi-rational function (77) becomes regular at the two poles of the RCSLE (67), whish leads us to the following assertion:

Preposition 1: *As a rule, the JPWs in the numerator and denominator of the fraction (77) do not have common zeros.*

Theorem 2: *The JPW in the numerator of the fraction (77) has only simple zeros as far as the Preposition 1 holds.*

Proof. Based on our prepositions, any zero of the JPW in the numerator of the fraction (77) is a regular point of the RCSLE (67) and therefore the polynomial in question may not have zeros of order higher than 1. (Otherwise the solution (77) of the RCSLE (67) and its first derivative would vanish at the same point which is possible only for the trivial solution identically equal to zero). □

Let

$$\bar{\eta}(\vec{\lambda}\,|\,\bar{M}_p) := \eta_{l=1,\ldots,\mathfrak{N}(\bar{M}_p)}(\vec{\lambda}\,|\,\bar{M}_p) \qquad (79)$$

be the $\mathfrak{N}(\bar{M}_p)$ zeros of the JPW (65), i.e.,

$$\hat{W}^{(\lambda_+,\lambda_-)}_{\mathfrak{N}(\bar{M}_p)}[\eta\,|\,\bar{M}_p] = \Pi_{\mathfrak{N}(\bar{M}_p)}[\eta;\bar{\eta}^{(\mathfrak{N})}(\vec{\lambda}|\bar{M}_p)] :=$$

$$\prod_{\ell=1}^{\mathfrak{N}(\bar{M}_p)} [\eta - \eta_\ell(\vec{\lambda}|\bar{M}_p)] \qquad (80)$$

Re-writing (72) as

$$ld\,\rho[\eta] = -\frac{2\eta}{\eta^2 - 1} \qquad (81)$$

and taking into account that

$$Q[\eta;\bar{\eta}] := \dot{ld}\,\Pi_m[\eta;\bar{\eta}] = -\sum_{l=1}^{m} \frac{1}{[\eta - \eta_l]^2}, \qquad (82)$$

we can decompose the RefPF (70) as follows

$$I^o[\eta;\vec{\lambda}|\bar{M}_p] = I^o[\eta;\vec{\lambda} + p\vec{1}] - \sum_{\ell=1}^{\mathfrak{N}(\bar{M}_p)} \frac{2}{[\eta - \eta_l(\vec{\lambda}|\bar{M}_p)]^2} \qquad (83)$$

$$+ \frac{2\eta}{\eta^2 - 1} \sum_{\ell=1}^{\mathfrak{N}(\bar{M}_p)} \frac{1}{\eta - \eta_l(\vec{\lambda}|\bar{M}_p)},$$

in agreement with (87) in [1] for p=1.

The indicial equation for all the extraneous poles of the RCSLE (67) has exactly the same form:

$$\rho(\rho - 1) - 2 = 0. \qquad (84)$$

The equation has two roots -1 and 2, which implies that the JPW in the numerator of the fraction (78) can formally have a zero of the third order [44]. However, as it becomes obvious from the analysis presented in [44], this is a relatively exotic case, when the solution becomes regular at two singular points, which will be simply disregarded here.

5. Four Infinite Sequences of q-RSs with Polynomial Components Represented by Simple p-WPs

In addition to (77), the RCSLE (67) has three infinite sequences of the q-RSs:

$$\phi_{\phi,j}[\eta;\vec{\lambda}|\bar{M}_p] = \frac{W\{\phi_{m_{k=1,\ldots,p}}[\eta;\vec{\lambda}], \phi_j[\eta;\vec{\lambda}']\}}{\rho^{-\frac{1}{2}p}[\eta]W\{\phi_{m_{k=1,\ldots,p}}[\eta;\vec{\lambda}]\}} \quad (\vec{\lambda}' \neq \vec{\lambda}), \qquad (85)$$

where the indexes $\vec{\lambda}'$ may differ only by sign from the common indexes $\vec{\lambda}$ of the seed Jacobi polynomials:

$$\lambda'_\mp = \phi_\mp \vec{\lambda}_\mp \qquad (86)$$

and

$$|\lambda'_{\mp}|=|\lambda_{\mp}|=\lambda_{o;\mp}\,. \quad (87)$$

Here we come to the most important result of this section: introducing the notion of the *simple* *p-WPs* which, by analogy with the JPWs, remain finite at ∓ 1.

Theorem 3: *The quasi-rational numerators of the fractions (85) can be expressed in terms of simple p-WPs defined via (92) below.*

Proof. In following [22], let us first introduce the eigenfunctions of the Jacobi operator:

$$f[\eta;\vec{\lambda}\,|\,\vec{\sigma},j]:=f[\eta;\vec{\lambda}\,|\,\vec{\sigma},0]\,P_j^{(\sigma_+\lambda_+,\sigma_-\lambda_-)}(\eta)\,, \quad (88)$$

where

$$f[\eta;\vec{\lambda}\,|\,\vec{\sigma},0]:=\phi_0[\eta;\vec{\sigma}\times\vec{\lambda}\,]/\,\phi_0[\eta;\vec{\lambda}\,]\,, \quad (89)$$

$$=\prod_{\aleph=\pm}|1-\aleph\eta|^{\frac{1}{2}(\sigma_{\aleph}1-1)\lambda_{\aleph}}\,, \quad (90)$$

(see Table 1 in [45] for details). We can then re-write the Wronskian in the numerator of the PF (85) as

$$W\{\phi_{m_{k=1,\ldots,p}}[\eta;\vec{\lambda}\,],\phi_j[\eta;\vec{\lambda}'\,]\} \quad (91)$$

$$=\phi_0^{p+1}[\eta;\vec{\lambda}\,]\times W\{P_{m_{k=1,\ldots,p}}^{(\lambda_+,\lambda_-)}(\eta),\,f[\eta;\vec{\lambda}\,|\,\vec{\sigma},j]\}\,.$$

Though this paper is devoted solely to the EOP sequences associated with the particular case $\vec{\sigma}=-+$ (as advocated in [45]), it seems enlightening to discuss in parallel all the three sequences of the simple p-WPs. First, it is worth noting that the net of X-Jacobi OPSs of our choice starts from the Xm-Jacobi OPSs of series J1, but not with the traditional Xm-Jacobi OPSs [24,46], referred to in our works [1,38] as being of series J2.

This is true [45] that the Xm-Jacobi OPSs in case $\vec{\sigma}=--$ can be obtained by eliminating certain pairs of juxtaposed eigenfunctions [47,48]; however, the analysis of the RDC𝔍s of the JRef CSLE (1) obtained by sequential RRZTs is much easier as illustrated by the proofs presented in Appendices C and D. While the admissibility of the partitions composed of even-length segments for X-Jacobi OPSs has been proven by Durán in his renowned treatise [21], an extension of this assertion to the RDC𝔍s of the R-jacobi polynomials constitutes a much more challenging problem (cf. the bulky arguments presented by us in [49] for the RDC𝔍s of the R-Routh polynomials).

Making use of Jacobi polynomial relations (92) in [21], we can tn represent the derivatives of functions (88) in in the explicitly quasi-rational form:

$$\frac{d^l}{d\eta^l}f[\eta;\vec{\lambda}\,|\,\vec{\sigma},j]=d_{\vec{\sigma},j}^{(l)}(\vec{\lambda}')\,f[\eta;\vec{\lambda}\,|\,\vec{\sigma},0]\Theta_{\vec{\sigma}}^{-l}[\eta]\,P_{j-[\vec{\sigma}]l/2}^{(\lambda'_++\sigma_+l,\lambda'_-+\sigma_-l)}(\eta), \quad (92)$$

where

$$\Theta_{\vec{\sigma}}[\eta]:=\prod_{\aleph=\pm}(1-\aleph\eta)^{\frac{1}{2}(1-\sigma_{\aleph}1)}\,, \quad (93)$$

$$[\vec{\sigma}]:=\sigma_+1+\sigma_-1\,, \quad (94)$$

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and the indexes $\vec{\lambda}'$ appearing on the right are related to the indexes $\vec{\lambda}$ of the seed Jacobi polynomials via (86). The constant coefficient factors in (92) are determined by the elementary formulas [21]:

$$d^{(l)}_{\vec{\sigma},j}(\vec{\lambda}') = \begin{cases} 2^{-l}\langle \lambda'_+ + \lambda'_- + j + 1\rangle_l & \text{if } \vec{\sigma} = ++, \\ (-2)^l \langle j+1\rangle_l & \text{if } \vec{\sigma} = --, \\ (-1)^l \langle j+\lambda'_+\rangle_l & \text{if } \vec{\sigma} = -+, \\ \langle j+\lambda'_-\rangle_l & \text{if } \vec{\sigma} = +-. \end{cases} \quad (95)$$

The listed formulas can be directly verified by expressing the hypergeometric functions in terms of Jacobi polynomials in 2.1(20), 2.1(27), 2.1(24), and 2.1(22) in [49], with $a = -\text{m}$. While all four Jacobi polynomial relations (92) were obtained in [21] based on the translational shape-invariance of the trigonometric Pöschl-Teller (t-PT) potential, we prefer to refer the reader to the more general relations 2.1(7), 2.1(9), 2.1(8), and 2.1(22) for hypergeometric functions in [49], as the starting point for validating (92). The cited relations are valid within a broader range of the parameters, beyond the limits of the Liouville transformation implicitly used in [21].

Substituting the derivatives (92) into the Wronskian in the right-hand side of (86), we can represent the quasi-rational form

$$W\{P^{(\lambda_+,\lambda_-)}_{m_{k=1,\dots,p}}(\eta), f[\eta;\vec{\lambda}\,|\,\vec{\sigma}, j]\} = f[\eta;\vec{\lambda}'\,|\,\vec{\sigma},0]\Theta^{-p}_{\vec{\sigma}}[\eta]\mathcal{P}_{\mathfrak{N}(\bar{M}_p;\vec{\sigma},j)}[\eta;\vec{\lambda}\,|\,\bar{M}_p;\vec{\sigma}, j], \quad (96)$$

with the polynomial component represented by the simple p-WP $(\vec{\sigma} \neq ++)$:

$$\mathcal{P}_{\mathfrak{N}(\bar{M}_p;\vec{\sigma},j)}[\eta;\vec{\lambda}\,|\,\bar{M}_p;\vec{\sigma}, j] := \quad (97)$$

$$\begin{vmatrix} P^{(\lambda_+,\lambda_-)}_{m_1}(\eta) & P^{(\lambda_+,\lambda_-)}_{m_2}(\eta) & \dots & \Theta^{p}_{\vec{\sigma}}[\eta]P^{(\lambda'_+,\lambda'_-)}_{j}(\eta) \\ \frac{d}{d\eta}P^{(\lambda_+,\lambda_-)}_{m_1}(\eta) & \frac{d}{d\eta}P^{(\lambda_+,\lambda_-)}_{m_2}(\eta) & \dots & P^{(p,1)}_{n_{j;[\vec{\sigma}]}(p-1)}[\eta;\vec{\lambda}'|\vec{\sigma}] \\ & \dots & & \\ \frac{d^p}{d^p\eta}P^{(\lambda_+,\lambda_-)}_{m_1}(\eta) & \frac{d^p}{d^p\eta}P^{(\lambda_+,\lambda_-)}_{m_2}(\eta) & \dots & P^{(p,p)}_{n_{j;[\vec{\sigma}]}(0)}[\eta;\vec{\lambda}'|\vec{\sigma}] \end{vmatrix},$$

where

$$P^{(p,l)}_{n_{j;[\vec{\sigma}]}(p-l)}[\eta;\vec{\lambda}'\,|\,\vec{\sigma}] := d^{(l)}_{\vec{\sigma},j}(\vec{\lambda}')\,\Theta^{p-l}_{\vec{\sigma}}[\eta]\; P^{(\lambda'_+ + \sigma_+ l,\,\lambda'_- + \sigma_- l)}_{j-[\vec{\sigma}]\,l/2}(\eta) \quad (98)$$

are the polynomials of the degree

$$n_{j;[\vec{\sigma}]}(k) = j - 1 + (k+1)(1 - \tfrac{1}{2}[\vec{\sigma}]), \quad (99)$$

with the integer $[\vec{\sigma}]$ defined via (94), which completes the proof. □

Here and below we use the symbol $\mathcal{P}$ for the polynomials forming a X-Jacobi DPS , with $\mathfrak{N}$ standing for their degrees, while the nonnegative integer j counts the polynomials within the given X-DPS which is constructed using p seed Jacobi polynomials of degrees $m_1,\dots,m_p$ with the common

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pair of the indexes $\vec{\lambda}$. The polynomial (97) thus represents the RDC$\mathfrak{J}$ of the Jacobi polynomial of the degree j with the indexes $\vec{\lambda}'$ defined via (86) and (87).

Keeping in mind that

$$n_{j;[\vec{\sigma}]}(0) = j - \tfrac{1}{2}[\vec{\sigma}] \quad (100)$$

and using the cofactor expansion of the determinant (99) in terms of the ($p-l$, p+1) minors (l =0,…,p), we find that the first term in the sum has the degree

$$\mathfrak{N}(\bar{M}_p;\vec{\sigma}, j) = \mathfrak{N}(\bar{M}_p) + j - \tfrac{1}{2}[\vec{\sigma}], \quad (101)$$

where [51]

$$\mathfrak{N}(\bar{M}_p) = \sum_{k=1}^{p} m_k - \tfrac{1}{2}p(p-1). \quad (102)$$

One can directly verify that

$$\mathfrak{N}(\bar{M}_p;++,m_{p+1}) = \mathfrak{N}(\bar{M}_p) + m_{p+1} - 1 = \mathfrak{N}(\bar{M}_{p+1}), \quad (103)$$

as expected. Taking into account that $[\mp\pm] = 0$, we also find

$$n_{j;\mp\pm}(k) = j + k \quad (104)$$

and

$$\mathfrak{N}(\bar{M}_p;\mp\pm, j) = \mathfrak{N}(\bar{M}_p) + j\,. \quad (105)$$

Preposition 2: *In general* the *degree of the simple* p*-WP (97) is equal to the positive integer specified by (101).*

Proof. One can easily verify that the degree of the ($p+1-l$)-th column element and degree of the corresponding cofactor polynomial minor increases and respectively decreases by 1 as l grows, confirming that all the polynomial summands have the common degree (101). Based on Bonneux's [45] formula (2.10), we assert that the simple p-WP (97) has the degree (101) iff

$$\lambda_+ + \lambda_- + m_k \notin \{-1,-2,\ldots,-m_k\},\ \lambda'_+ + \lambda'_- + j \notin \{-1,-2,\ldots,-j\}\,. \quad (106)$$

The simplest way to avoid the degree reduction is to assume that the Jacobi indexes $\vec{\lambda}$ are non-integers. □

In [45] the positive integer (102) is termed 'length' $|\lambda|$ of the partition $\lambda := \lambda_1, \lambda_2, \ldots \lambda_p$ with $\lambda_k := m_k + p - k$ (k=1,…,p). Setting $\vec{\sigma} = -+$, $\mu = j = |\mu|$, $r_1 = p$, $r_2 = 1$ confirms that the derived formula for the degree of the simple p-WP of the given type agrees with the more general formula $|\lambda| + |\mu|$ for the two partitions λ and μ [45].

To relate our analysis to Durán's theory [21], let us elaborate the case $\vec{\sigma} = -+$ in more details, Keeping in mind that

$$\Theta_{-+}[\eta] = 1 + \eta \quad (107)$$

and choosing $\lambda_+ = \alpha$, $\lambda_- = -\beta$, so the function (89) takes form:

$$f[\eta; -\beta, \alpha\,|\,-+, 0] = (1+\eta)^{-\beta}, \quad (108)$$

we can re-write the Wronskian in the denominator of the fraction (85) as

$$W\{\phi_{m_{k=1,2,\ldots,p}}[\eta; -\beta, \alpha]\} = \phi_0^p[\eta; -\beta, \alpha]\,\Omega^{(\alpha,\beta)}_{\emptyset,\mu}, \quad (109)$$

where the polynomial

$$\Omega^{(\alpha,\beta)}_{\emptyset,\mu} = (1+\eta)^{p\beta}\, W\{f[\eta;-\beta,\alpha\,|\,-+, m_1],\ldots, f[\eta;-\beta,\alpha\,|\,-+, m_p]\} \quad (110)$$

is defined via (2.7) in [45] with $\mu := m_1 + p - 1, m_2 + p - 2, \ldots m_p$ and we also took into account that

$$\phi_0[\eta;\beta,\alpha](1+\eta)^{-\beta} = \phi_0[\eta;-\beta,\alpha]. \quad (111)$$

Comparing (109) with (66), we conclude that

$$\Omega^{(\alpha,\beta)}_{\emptyset,\mu} = W^{(\alpha,-\beta)}_{\mathfrak{N}(\bar{M}_p)}[\eta\,|\,\bar{M}_p] \quad (112)$$

and therefore the requirement for the JPW in question not to have zeros in the closed interval [-1,+1] is the particular case of Lemma 5.1 and Theorem 6.3 in [22].

Similarly, we re-write the Wronskian in the numerator of the fraction (85) as

$$W\{\phi_{m_{k=1,\ldots,p}}[\eta;-\beta,\alpha],\phi_j[\eta;\beta,\alpha]\} = \phi_0^{p+1}[\eta;\beta,\alpha] \quad (113)$$

$$\times W\{f[\eta;\alpha,-\beta\,|\,-+, m_1],\ldots, f[\eta;\alpha,-\beta\,|\,-+, m_p], P_j^{(\alpha,\beta)}(\eta)\},$$

while setting $\vec{\sigma} = -+$ and $\lambda_+ = \alpha$, $\lambda_- = -\beta$, in (91) and (96) gives

$$W\{\phi_{m_{k=1,\ldots,p}}[\eta;-\beta,\alpha],\phi_j[\eta;\beta,\alpha]\} = \phi_0^{p+1}[\eta;-\beta,\alpha]$$

$$\times(1+\eta)^{-\beta-p}\, \mathcal{P}_{\mathfrak{N}(\bar{M}_p)+j}[\eta;-\beta,\alpha\,|\,\bar{M}_p;-+, j]. \quad (114)$$

Re-expressing (113) in terms of X-Jacobi polynomials (2.31) in [45]:

$$P^{(\alpha,\beta)}_{\emptyset,\mu,p+j} := (1+\eta)^{(\beta+1)p}$$
$$\times W\{f[\eta;\alpha,-\beta\,|\,-+, m_1],\ldots, f[\eta;\alpha,-\beta\,|\,-+, m_p], P_j^{(\alpha,\beta)}(\eta)\}, \quad (115)$$

making use of (111), and comparing the resultant expression

$$W\{\phi_{m_{k=1,\ldots,p}}[\eta;-\beta,\alpha],\phi_j[\eta;\beta,\alpha]\} = \phi_0^{p+1}[\eta;-\beta,\alpha](1+\eta)^{-\beta-p} P^{(\alpha,\beta)}_{\emptyset,\mu,p+j} \quad (116)$$

with (114) shows that

$$P^{(\alpha,\beta)}_{\emptyset,\mu,p+j} \equiv \mathcal{P}_{\mathfrak{N}(\bar{M}_p)+j}[\eta;-\beta,\alpha\,|\,\bar{M}_p;-+, j]. \quad (117)$$

Coming back to the q-RSs (85) for the simple p-WPs of the general type $\vec{\sigma} \neq ++$, note that the numerator of the fraction has the form:

$$W\{\phi_{m_{k=1,\ldots,p}}[\eta;\vec{\lambda}],\phi_j[\eta;\vec{\lambda}']\} = \phi_0^p[\eta;\vec{\lambda}]\Theta_{\vec{\sigma}}^{-p}[\eta]\phi_0[\eta;\vec{\lambda}'] \quad (118)$$

$$\times\mathcal{P}_{\mathfrak{N}(\bar{M}_p;\vec{\sigma},0)+j}[\eta;\vec{\lambda}\,|\,\bar{M}_p;\vec{\sigma}, j]$$

Making use of the identity

$$\Theta_{\vec{\sigma}}^{-p}[\eta]\phi_0[\eta;\vec{\lambda}'] = \phi_0[\eta;\vec{\lambda}' + p(\vec{\sigma}\times\vec{1} - \vec{1})], \quad (119)$$

we can then re-write (118) as

$$W\{\phi_{m_{k=1,\ldots,p}}[\eta;\vec{\lambda}],\phi_j[\eta;\vec{\lambda}']\} = \phi_0^p[\eta;\vec{\lambda}]\phi_0[\eta;\vec{\lambda}' + p(\vec{\sigma}\times\vec{1} - \vec{1})]$$

$$\times\mathcal{P}_{\mathfrak{N}(\bar{M}_p;\vec{\sigma},0)+j}[\eta;\vec{\lambda}\,|\,\bar{M}_p;\vec{\sigma}, j], \quad (120)$$

so the fraction takes the sought-for form:

$$\phi_{\vec{\sigma},j}[\eta;\vec{\lambda}|\bar{M}_p] = \phi_0[\eta;\vec{\sigma}\times(\vec{\lambda}+p\vec{1})]\frac{\mathcal{P}_{\mathfrak{N}(\bar{M}_p;\vec{\sigma},0)+j}[\eta;\vec{\lambda}\,|\,\bar{M}_p;\vec{\sigma},j]}{W^{(\lambda_+,\lambda_-)}_{\mathfrak{N}(\bar{M}_p)}[\eta\,|\,\bar{M}_p]}. \quad (121)$$

Setting $p=1$, $\bar{M}_1 = m$ brings us to (125) in [1], as expected.

Let us now point to the very unique feature of the simple p-WP in the numerator of the PF in the right-hand side of (121):

Theorem 4: The simple p-*WP* **(97) does not have zeros at** ∓ 1**, a***ssuming that both Jacobi indexes are non-integers.*

Proof: First, let us remind the reader that the ExpDiff for the pole of the CSLE (63) at ∓ 1 and the corresponding ChExps are equal to $|\lambda_\pm + p|$ and respectively $\tfrac{1}{2} + |\lambda_\pm + p|$, $\tfrac{1}{2} - |\lambda_\pm + p|$. On other hand, examination of the power function in front of the PF reveals that the power exponent of $\eta \pm 1$ coincides with one of these ChExps. For the p-WP in question to vanish at ∓ 1 the corresponding ExpDiff must be a positive integer, contradicts to the assumption that both Jacobi indexes are non-integers. □

As illuminated in the next Section, Theorem 4 assures that the p-WPs in question satisfy a FPDE and therefore form a X-DPS in our terminology.

6. X-Jacobi DPSs Composed of Simple p-WPs

Our next step is to prove that both JPW (65) and all three p-WPs in the numerator of the fraction in the right-hand side of (121) for $\vec{\sigma} \neq ++$ constitute polynomial solutions of the second-order FPDEs. To construct the latter FPDEs, we make the four alternative gauge transformations

$$\Phi[\eta;\vec{\lambda}_o;\varepsilon|\bar{M}_p] = \phi_0[\eta;\vec{\lambda}_o|\bar{M}_p;\vec{\sigma}]\times F[\eta;\vec{\lambda}_o;\varepsilon|\bar{M}_p;\vec{\sigma}] \quad (122)$$

with the gauge functions

$$\phi_0[\eta;\vec{\lambda}|\bar{M}_p;\vec{\sigma}] = \frac{\phi_0[\eta;\vec{\sigma}\times(\vec{\lambda}+p\vec{1})]}{W^{(\lambda_+,\lambda_-)}_{\mathfrak{N}(\bar{M}_p)}[\eta\,|\,\bar{M}_p]} \quad (122)$$

satisfying the RCSLE

$$\ddot{\phi}_0[\eta;\vec{\lambda}|\bar{M}_p;\vec{\sigma}] + \left\{ I^0_0[\eta;\vec{\lambda}|\bar{M}_p] - \frac{\varepsilon_0(\vec{\lambda}'+p\vec{\sigma}\times\vec{1})}{\eta^2-1} \right\} \phi_0[\eta;\vec{\lambda}|\bar{M}_p;\vec{\sigma}] = 0. \quad (123)$$

However, before proceeding with this step, let us first rewrite the RefPF (70) in a more convenient form

$$I^0_0[\eta;\vec{\lambda}|\bar{M}_p] := I^0[\eta;\vec{\lambda}+p\vec{1}] + 2\hat{Q}[\eta;\bar{\eta}^{(\mathfrak{N})}(\vec{\lambda}|\bar{M}_p)] \quad (124)$$

$$+2 ld\phi_0[\eta;\vec{\sigma}\times(\vec{\lambda}+p\vec{1})] ld\, W^{(\lambda_+,\lambda_-)}_{\mathfrak{N}(\bar{M}_p)}[\eta\,|\,\bar{M}_p],$$

where [28]

$$\hat{Q}[\eta;\bar{\eta}] := -\tfrac{1}{2}\Pi_m[\eta;\bar{\eta}]\frac{d^2}{d\eta^2}\Pi_m^{-1}[\eta;\bar{\eta}] \quad (125)$$

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$$= \tfrac{1}{2}\frac{\ddot{\Pi}_m[\eta;\bar{\eta}]}{\Pi_m[\eta;\bar{\eta}]} - \frac{\dot{\Pi}^2_m[\eta;\bar{\eta}]}{\Pi^2_m[\eta;\bar{\eta}]}. \tag{126}$$

The PF (123) is related to the Quesne PF [52–54]

$$Q[\eta;\bar{\eta}] := \frac{\ddot{\Pi}_m[\eta;\bar{\eta}]}{\Pi_m[\eta;\bar{\eta})]} - \frac{\dot{\Pi}^2_m[\eta;\bar{\eta}]}{\Pi^2_m[\eta;\bar{\eta}]} \tag{127}$$

in the elementary fashion:

$$\hat{Q}[\eta;\bar{\eta}] = Q[\eta;\bar{\eta}] + \frac{\ddot{\Pi}_m[\eta;\bar{\eta}]}{2\Pi_m[\eta;\bar{\eta}]} \tag{128}$$

$$= -\sum_{l=1}^{m} \frac{1}{[\eta-\eta_l]^2} + \frac{\ddot{\Pi}_m[\eta;\bar{\eta}]}{2\Pi_m[\eta;\bar{\eta}]}. \tag{129}$$

In our earlier works [28,55] we adopted the Quesne PF in the form (126) (see, i.g., (39) in [54], with $g^{(\alpha)}_{\mu}$ standing for $\Pi_m[\eta;\bar{\eta}]$ here), overlooking its alternative form (82) without any mixed simple poles at $\eta_{l=1,\dots,m}$.

We are now ready to prove the assertion made by us at the beginning of the section:

Theorem 5: *The polynomials (97) satisfy the second-order FPDEs and therefore form four distinguished X-Jacobi DPSs.*

Proof: Substituting (121) into the RCSLE (67) and taking advantage of (124), coupled with (123) and (128), we come to the second-order PDE:

$$\{\mathbf{D}_\eta(\vec{\lambda}|\bar{M}_p;\vec{\sigma}) + C_{\mathfrak{N}(\bar{M}_p)}[\eta;\vec{\lambda};\varepsilon\,|\,\bar{M}_p;\vec{\sigma}]\}F[\eta;\vec{\lambda}_o;\varepsilon|\bar{M}_p;\vec{\sigma}] = 0, \tag{130}$$

where $\mathbf{D}_\eta(\vec{\lambda}|\bar{M}_p;\vec{\sigma})$ is an abbreviated notation for the second-order differential operator in η:

$$\mathbf{D}_\eta(\vec{\lambda}|\bar{M}_p;\vec{\sigma}) = (\eta^2-1)\Pi_{\mathfrak{N}(\bar{M}_p)}[\eta;\bar{\eta}^{(\mathfrak{N})}(\vec{\lambda}\,|\,\bar{M}_p)]\frac{d^2}{d\eta^2} \tag{131}$$

$$+2B_{\mathfrak{N}(\bar{M}_p)+1}[\eta;\vec{\lambda}\,|\,\bar{M}_p;\vec{\sigma}]\frac{d}{d\eta}$$

with the polynomial coefficient function of the first derivative

$$B_{\mathfrak{N}(\bar{M}_p)+1}[\eta;\vec{\lambda}\,|\,\bar{M}_p;\vec{\sigma}] := (\eta^2-1)\Pi_{\mathfrak{N}(\bar{M}_p)}[\eta;\bar{\eta}^{(\mathfrak{N})}(\vec{\lambda}\,|\,\bar{M}_p)]$$

$$\times\left(\sum_{\aleph=\pm}\frac{\sigma_\aleph(\lambda_\aleph+1)+1}{2(\eta-\aleph 1)} - \sum_{l=1}^{\mathfrak{N}(\bar{M}_p)}\frac{1}{\eta-\eta_l(\vec{\lambda};\bar{M}_p)}\right). \tag{132}$$

The ε-dependent polynomial of degree m representing the free term of the PDE (130) is linear in the energy:

$$C_{\mathfrak{N}(\bar{M}_p)}[\eta;\vec{\lambda};\varepsilon|\bar{M}_p;\vec{\sigma}] = C_{\mathfrak{N}(\bar{M}_p)}[\eta;\vec{\lambda}|\bar{M}_p;\vec{\sigma}] - \varepsilon\Pi_{\mathfrak{N}(\bar{M}_p)}[\eta;\bar{\eta}^{(\mathfrak{N})}(\vec{\lambda}\,|\,\bar{M}_p)], \tag{133}$$

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with the energy-independent part represented by the following polynomial of degree $\mathfrak{N}(\bar{M}_p)$:

$$C_{\mathfrak{N}(\bar{M}_p)}[\eta;\vec{\lambda}|\bar{M}_p;\vec{\sigma}]+\varepsilon_0(\vec{\lambda}'+p\vec{\sigma}\times\vec{1})\bar{\eta}^{(\mathfrak{N})}(\vec{\lambda}|\bar{M}_p)$$
$$=(\eta^2-1)W^{(\lambda_+,\lambda_-)}_{\mathfrak{N}(\bar{M}_p)}[\eta\,|\,\bar{M}_p]\left\{I^o[\eta;\vec{\lambda}|\bar{M}_p]-I^o_0[\eta;\vec{\lambda}|\bar{M}_p]\right\} \quad (134)$$

$$=2\left[\eta-P_1^{(\lambda'_+ +\sigma_+ 1,\lambda'_- +\sigma_- 1)}(\eta)\right]\dot{W}^{(\lambda_+,\lambda_-)}_{\mathfrak{N}(\bar{M}_p)}[\eta\,|\,\bar{M}_p]$$

$$-(\eta^2-1)\ddot{W}^{(\lambda_+,\lambda_-)}_{\mathfrak{N}(\bar{M}_p)}[\eta\,|\,\bar{M}_p]. \quad (135)$$

The crucial point is that the polynomial coefficient of the second derivative in (131) has exactly $\mathfrak{N}(\bar{M}_p)+2$ simple zeros while the degrees of the polynomials (132) and (135) are equal to $\mathfrak{N}(\bar{M}_p)+1$ and $\mathfrak{N}(\bar{M}_p)$ [25], which completes the proof. □

Making use of the Jacobi equation (15) for p=1 ($\bar{M}_1=m$), one can verify that (135) turns into (151) in [1], as expected.

7. Prime Forms of RDCℑs of the J-Ref CSLE Solved Under DBCs on Intervals (-1,+1) and (1,∞)

Starting from this point, we discuss only the admissible sets $\bar{\mathbf{M}}_p=\mathbf{m}_1,\ldots,\mathbf{m}_p$ of JS solutions assuring that the corresponding JPWs do not have nodes within the given orthogonalization interval for the specified ranges of the parameters λ_-,λ_+.

Using the gauge transformations

$$\Psi[\eta;\vec{\lambda};\varepsilon\,|\,\bar{\mathbf{M}}_p]=(1-\eta^2)^{-1/2}\,\Phi[\eta;\vec{\lambda};\varepsilon\,|\,\bar{\mathbf{M}}_p] \quad (136)$$

and

$$\Psi[\eta;\vec{\lambda};\underset{\sim}{\varepsilon}\,|\,\bar{\mathbf{M}}_p]=(\eta-1)^{-1/2}\,\Phi[\eta;\vec{\lambda};\underset{\sim}{\varepsilon}\,|\,\bar{\mathbf{M}}_p], \quad (137)$$

we then convert the RCSLE (67) to its prime forms on the intervals $(-1,+1)$ and $(+1,\infty)$:

$$\left\{\frac{d}{d\eta}(1-\eta^2)\frac{d}{d\eta}-\not{q}[\eta;\vec{\lambda}\,|\,\bar{\mathbf{M}}_p]+\varepsilon\right\}\Psi[\eta;\vec{\lambda};\varepsilon\,|\,\bar{\mathbf{M}}_p]=0 \quad (138)$$

for $\eta\in(-1,+1)$

and

$$\left\{\frac{d}{d\eta}(\eta-1)\frac{d}{d\eta}-\not{q}[\eta;\vec{\lambda}\,|\,\bar{\mathbf{M}}_p]+\frac{\underset{\sim}{\varepsilon}}{\eta+1}\right\}\Psi[\eta;\vec{\lambda};\underset{\sim}{\varepsilon}\,|\,\bar{\mathbf{M}}_p]=0 \quad (139)$$

for $\eta\in(1,\infty)$,

with the leading coefficient function and weight function defined via to (28) and (32) respectively. It is worth reminding the reader the main reasoning behind these particular transformations [19], namely, the leading coefficient functions in both cases were chosen in such a way that the ChExps at each of the singular endpoints have opposite signs and as a result the DBC imposed at the given end unambiguously determines the regular solution (or the PFS as we term it here).

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In this paper we only discuss the seed solutions represented by the PFSs near the same endpoint under condition that they lie below the lowest eigenvalue. Since the RDCTs using the seed functions of types + – and – + are specified by same series of the Maya diagrams [21], any RCSLE using an arbitrary combination of these seed functions can be alternatively obtained by considering only infinitely many combinations $\bar{M}_p := \{m_1, m_2, ..., m_p\}$ of the PFSs of the same type + – or – + [21,56,57]. In particular, the Jacobi polynomial of order m with the indexes $\vec{\lambda}$ can be represented as the Wronskian of Jacobi polynomials of the sequential degrees $\tilde{m}$=1,...,m with the indexes $-\vec{\lambda}$. The simplest case m=2 is discussed in Appendix E.

Note that Gȯmez-Ullate et al. [24,45] derived the general expression for the Xm-Jacobi OPS, taking advantage of the Klein formulas [25] to select all the Jacobi polynomials without zeros between -1 and +1 under the constraint

$$\lambda_+ = -\alpha - 1 < 0, \ \lambda_- = \beta - 1 > -2 \,, \tag{140}$$

whereas our approach allows us to identify only the bulk part of those polynomials with one of the first Jacobi indexes restricted solely to positive values. For the multi-indexed X-Jacobi OPSs the admissibility of the given set of seed Jacobi polynomials for the finite orthogonalization interval was thoroughly analyzed by Durán [22].

In this Section we consider the admissibility problem in parallel for both the finite and infinite orthogonalization intervals. While our study of the multi-indexed X-Jacobi OPSs constitutes the particular case of Durán's analysis, the results for the RDC$\mathfrak{I}$s of the R-Jacobi polynomials constitute the state-of-the-art development. In particular, we prove that the Wronskians of the R-Jacobi polynomials with the common first and second Jacobi indexes respectively positive and negative do not have zeros in the interval [-1,+1] while the Wronskians of the classical Jacobi polynomials with common positive indexes may only have real zeros smaller than 1.

To determine the admissible sets $\bar{\mathbf{M}}_p$ of seed Jacobi polynomials, we introduce the sequence of the q-RTFs

$$\psi_{++,\mathbf{m}_{p+1}}[\eta;\vec{\lambda}|\bar{\mathbf{M}}_p] = \psi_0[\eta;\lambda_- + p, \lambda_+ + p] \frac{\mathcal{P}_{\mathfrak{N}(\bar{\mathbf{M}}_{p+1})}[\eta;\vec{\lambda}\,|\,\bar{\mathbf{M}}_p;++\,,\mathbf{m}_{p+1}]}{W^{(\lambda_+,\lambda_-)}_{\mathfrak{N}(\bar{\mathbf{M}}_p)}[\eta\,|\,\bar{\mathbf{M}}_p]} \tag{140}$$

by setting $\vec{\sigma} = +,+$ in (85) and also taking into account that the solutions of the *p*-SLEs (138) and (139) are related to the solutions of the RCSLE (67) via the gauge transformations (136) and (137), i.e.,

$$\psi_0[\eta;\beta,\alpha] = \begin{cases} (1+\eta)^{\frac{1}{2}\beta}\,(1-\eta)^{\frac{1}{2}\alpha} & \text{for } |\,\eta\,| < 1, \\ (1+\eta)^{\frac{1}{2}(\beta+1)}\,(\eta-1)^{\frac{1}{2}\alpha} & \text{for } \eta > 1. \end{cases} \tag{141}$$

While the sequences of the q-RSs (140) on the infinite interval are unlimited, we have to truncate the chain of the sequential RRZTs of the *p*-SLE (135) when the ExpDiff for the pole at -1 reaches its minimum value

$$0 < {}^*\lambda_{0;-}^{(p_{\max})} = \lambda_{0;-} - p_{\max} < 1 \tag{142}$$

with

$$p_{\max} = \lfloor \lambda_{0;-} \rfloor. \tag{143}$$

Below we always assume that p in (140) for $|\,\eta\,| < 1$ does not exceed (143), without explicitly mentioning this restriction.

Below we consider only the Wronskian net of the Jacobi polynomials with the indexes

$$\lambda_{\mp} = \mp\lambda_{0;\mp} \ \text{ for } |\eta|<1 \text{ and } \vec{\lambda} = \vec{\lambda}_0 \ \text{ for } \eta > 1, \tag{144}$$

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while

$$\lambda'_{\mp} = \mp\lambda_{\mp} \quad (\vec{\sigma} = -+) \text{ in both cases.} \tag{145}$$

We refer to the X-Jacobi DPS constructed using p seed Jacobi polynomials of the degrees $m_1, m_2, ..., m_p$ as being of series J1(p). The selection (144), (145) for the p-WEOP sequences under consideration is consistent with (2.9) in [45], with $\alpha = \lambda_{o;+}$, $\beta = \lambda_{o;-}$.

Our next step is to prove that the q-RSs (140) constitute the PFSs near the singular endpoint +1 whether or not the polynomial denominator of the PF on the right has zeros inside the corresponding orthogonalization interval. Though the theorem stated below represents the very specific case of the general proposition proven in Appendix C, we feel useful to present an independent proof for the case of our current interest, which may be more appealing to the reader due to its simplicity.

Theorem 6. *The q-RSs (140) satisfy the DBC at the endpoint +1.*

Proof: Based on Theorem 1, we first confirm that the JPW in the denominator of the PF in the right-hand side of (140) remains finite at η = +1, keeping in mind that $\lambda_+ = \lambda_{o;+} > 0$ in both cases. Examination of the q-RS (140) with $\alpha = \lambda_{o;+} + p$ then shows that it vanishes at η = +1, which completes the proof. □

It will be proven later that the q-RSs

$$\psi_{-+,j}[\eta;\vec{\lambda}|\bar{\mathbf{M}}_p] = \psi_0[\eta;-\lambda_- - p, \lambda_+ + p]\frac{\mathcal{P}_{\mathfrak{N}(\bar{\mathbf{M}}_p,-+,0)+j}[\eta;\vec{\lambda}\,|\,\bar{\mathbf{M}}_p;-+\,,j]}{W^{(\lambda_+,\lambda_-)}_{\mathfrak{N}(\bar{\mathbf{M}}_p)}[\eta\,|\,\bar{\mathbf{M}}_p]} \tag{146}$$

for $0 \le j \le j_{max}$

represent the eigenfunctions of the *p*-SLEs (138) or (139) if $\bar{\mathbf{M}}_p$ is an admissible set, but before making the latter assumption, let us first show that

Theorem 7. *The q-RSs (146), with $\vec{\lambda}$ given by (144), satisfy the DBCs at the ends of the corresponding orthogonalization interval,l whether or not the JPW at the denominator of the PF in the right-hand side of (146) have zeros inside this interval.*

Proof: Setting $\alpha = \lambda_{o;+} + p$, $\beta = \pm\lambda_{o;-} - p$ in (146) confirms that the q-RSs in question vanishes at both ends of the interval [-1,+1] as well as at the lower end of the interval $(1,\infty)$.

Furthermore, since the functions $\phi_0[\eta;-\lambda_{o,-},\lambda_{o;+}]$ and $\phi_0[\eta;-\lambda_{o,-} - p, \lambda_{o;+} + p]$ have exactly the same asymptotics at infinity and the eigenfunction (47) of the *p*-SLE (24) on the interval $(+1,\infty)$ vanishes at the upper end by definition, we conclude that the q-RSs (146) also obey the DBC at infinity. □

Starting from this point, we assume that all the RCSLEs generated by the RDCTs with the seed solutions $\mathbf{m}_1, \bar{\mathbf{M}}_2, ..., \bar{\mathbf{M}}_p$ have exactly the same energy spectrum and also that none of their poles lies inside the corresponding orthogonalization interval. We then need to prove that this also true for the set $\bar{\mathbf{M}}_p, \mathbf{m}_{p+1}$.

Corollary 3: *If $\bar{\mathbf{M}}_p$ is an admissible set then the q-RSs (146) on the interval (-1,+1) or $(1,\infty)$ represent the eigenfunctions of the p-SLE (138) or respectively (139).*

Proof: The assertion directly follows from Theorem 6, since the RCSLE in question does not have singularities inside the orthogonalization interval. □

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As pointed out in subsection 3.1, the q-RS solutions of the JRef CSLE (1) with the polynomial components composed of the R-Jacobi polynomials (type **b**) represent PFSs of the *p*-SLE (24) near the pole at the upper end of the interval (-1,+1) and lie below the lowest eigenvalue (42) of the corresponding SLP. We thus conclude that the PFSs (140) of the *p*-SLE (138) lie below the eigenvalue $\varepsilon_0(\vec{\lambda}')$.

Similarly, the q-RS solutions of the JRef CSLE (1) with the polynomial components composed of the classical Jacobi polynomials (type $\underset{\sim}{\mathbf{a}}$) represent PFSs of the *p*-SLE (24) near the pole at the lower end of the interval (1,∞) and lie below the lowest eigenvalue (56). We thus conclude that the PFSs (140) of the *p*-SLE (136) lie below the eigenvalue $\underset{\sim}{\varepsilon}_0(\vec{\lambda}') = -\varepsilon_0(\vec{\lambda}')$.

To confirm that the q-RPFs (140) do not have zeros inside the orthogonalization interval (-1,+1) or (1,∞), we first need to prove that there is no eigenfunctions below the energies $\varepsilon_0(\vec{\lambda}')$ or accordingly $\underset{\sim}{\varepsilon}_0(\vec{\lambda}')$, i.e., that the latter are indeed the lowest eigenvalues of the *p*-SLE (138) or respectively (139). To verify the latter assertion, we take advantage of the powerful theorem proven in Appendix D, which assures that the q-RPF (140), starting the sequence (j=0), does not have zeros inside the given the orthogonalization interval.

The final preposition to prove that the PFSs lying below the eigenvalue in question do not have zeros inside this interval either.

Theorem 8. *A PFS near one of the endpoints ±1 may not have zeros inside the given interval of orthogonalization if it lies below the lowest eigenvalue of the given Sturm-Liouville problem.*

Proof: For the Sturm-Liouville problem on the orthogonalization interval (-1,+1) the formulated assertion directly follows from the Sturm comparison theorem (see, i.g., Theorem 3.1 in Section XI of Hartman's monograph [58]), keeping in mind that the logarithmic derivatives for all the PFSs (including the eigenfunction in question) have the same asymptotics near the pole in question:

$$\lim_{\eta\to-1}\left[(1-\eta^2)\,ld\Psi[\eta;\boldsymbol{\lambda}_{o;-},\boldsymbol{\lambda}_{+};\varepsilon\,|\,-+\vdots\bar{\boldsymbol{M}}_{p}]\right] = -{}^{*}\lambda_{o;-}^{(p)}, \tag{147}$$

and as a result the condition (3.4) in [58] turns into the identity. To apply the Sturm Theorem to the PFSs near the upper end +1, one simply needs to replace η for the reflected argument -η.

It is a more challenging problem to satisfy Sturm's constraint for the logarithmic derivatives in the limit η→∞ and we refer the reader to the proof of this assertion given in Appendix B in [1] for the PFSs of the *p*-SLE (24) solved under the DBCs at the ends of the interval (+1,∞). The arguments presented in support of this proof are equally applied to the *p*-SLE (139) without any modification. □

Corollary 4: *The JPW (65) does not have zeros inside the corresponding orthogonalization interval assuming that the Jacobi indexes restricted by the conditions (144) and (145) and the order of the given RDCT on the interval (-1,+1) does not exceed the upper bound (143).*

It has been proven in [20] that the eigenfunctions of the generic SLE solved under the DBCs must be mutually orthogonal with the SLE weight function on the interval in question. Therefore

$$\int_{\eta_-}^{\eta_+} d\eta\, \psi_{\mathbf{c},j}[\eta;\vec{\lambda}\,|\,\bar{\boldsymbol{M}}_p]\psi_{\mathbf{c},j'}[\eta;\vec{\lambda}\,|\,\bar{\boldsymbol{M}}_p]w[\eta;\vec{\boldsymbol{\lambda}}_o] = 0 \tag{148}$$

$$\text{for } 0 \le j' < j \le j_{max},\; p \le p_{max}.$$

Consequently, the polynomial components of the quasi-rational eigenfunctions (146) must be mutually orthogonal with the weight function

$$W[\eta;\vec{\boldsymbol{\lambda}}\,|\,\bar{\boldsymbol{M}}_p;-+] := \frac{\psi_0^2[\eta;-\boldsymbol{\lambda}_- - p,\boldsymbol{\lambda}_+ + p]}{W^{(\lambda_+,\lambda_-)}_{\mathfrak{N}(\bar{M}_p)}[\eta\,|\,\bar{\boldsymbol{M}}_p]}\, w[\eta;\vec{\boldsymbol{\lambda}}_o] \quad (p \le p_{max}) \tag{149}$$

$$\text{for } |\eta| < 1 \text{ or } \eta > 1;$$

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namely,

$$\int_{\eta_-}^{\eta_+} d\eta\, \mathcal{P}_{\mathfrak{N}(\bar{\mathbf{M}}_p)+j}[\eta;\vec{\lambda}\,|\,\bar{\mathbf{M}}_p;-+,j]\mathcal{P}_{\mathfrak{N}(\bar{\mathbf{M}}_p)+j'}[\eta;\vec{\lambda}\,|\,\bar{\mathbf{M}}_p;-+,j']W[\eta;\vec{\lambda}\,|\,\bar{\mathbf{M}}_p;-+]=0$$

$$\text{for } 0 \le j' < j \le j_{max} \quad , \quad p \le p_{max} \,. \tag{150}$$

In the following two subsections we discuss separately the X-Jacobi OPSs on the interval (–1,+1) and the RDC$\mathfrak{S}$s of the R-Jacobi polynomials forming finite EOP sequences on the infinite interval (1,∞).

7.1. Finite Net of X-Jacobi OPSs Generated Using JPWs of R-Jacobi Polynomials with Positive First Index

As mentioned above, the purpose of this subsection is to discuss the finite net of the exactly solvable RCSLEs generated using the JPWs of R-Jacobi polynomials with the positive Jacobi index λ_+.

Let us set $\vec{\lambda} = -\lambda_{0;-}, \lambda_{0;+}$, $\vec{\lambda}' = \vec{\lambda}_0 = \beta, \alpha$, $p_{max} = \lfloor \lambda_{0;-} \rfloor$, and $j_{max} = \infty$. This brings us to the net of the X-Jacobi OPSs composed of the p-WEOPs

$$\mathcal{P}_{\mathfrak{N}(\bar{\mathbf{M}}_p)+j}[\eta;-\beta,\alpha\,|\,\bar{\mathbf{M}}_p;-+,j] := \tag{151}$$

$$\begin{vmatrix} P^{(\alpha,-\beta)}_{m_1}(\eta) & P^{(\alpha,-\beta)}_{m_2}(\eta) & \ldots & (1+\eta)^{\beta} P^{(\alpha,\beta)}_{j}(\eta) \\ \frac{d}{d\eta}P^{(\alpha,-\beta)}_{m_1}(\eta) & \frac{d}{d\eta}P^{(\alpha,-\beta)}_{m_2}(\eta) & \ldots & P^{(1)}_{j+p-1}[\eta;\beta,\alpha|-+] \\ & & \ldots & \\ \frac{d^p}{d^p\eta}P^{(\alpha,-\beta)}_{m_1}(\eta) & \frac{d^p}{d^p\eta}P^{(\alpha,-\beta)}_{m_2}(\eta) & \ldots & P^{(p)}_{j}[\eta;\beta,\alpha|-+] \end{vmatrix}.$$

Note that the first p elements of the first row are represented by the R-Jacobi polynomials, while the last element is the classical Jacobi polynomial multiplied by the p-th power of the first-first degree polynomial (93) with $\vec{\sigma} = -+$. The weight function (149) takes the form:

$$W[\eta;-\alpha,\beta\,|\,\bar{\mathbf{M}}_p;-+] := \frac{\psi_0^2[\eta;\beta-p,\alpha+p]}{\left[W^{(\alpha,-\beta)}_{\mathfrak{N}(\bar{\mathbf{M}}_p)}[\eta\,|\,\bar{\mathbf{M}}_p]\right]^2} \tag{152}$$

$$\text{for } -1 < \eta < 1 \; (p \le \lfloor \lambda_{0;-} \rfloor),$$

where the polynomial Wronskian in the denominator is formed by the orthogonal R-Jacobi polynomials and therefore is the subject of the general conjectures formulated in [51] for zeros of the Wronskians of orthogonal polynomials inside the normalization interval (real zeros larger than 1 in our case). Corollary 4 assures that this JPW remains finite inside the interval (-1,+1).

Substituting (141) into (152) and comparing the resultant expression

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$$W[\eta;-\alpha,\beta\,|\,\bar{\mathbf{M}}_p;-+] := \frac{(1+\eta)^{\alpha+p}\,(1-\eta)^{\beta-p}}{\left[W^{(\alpha,-\beta)}_{\mathfrak{N}(\bar{\mathbf{M}}_p)}[\eta\,|\,\bar{\mathbf{M}}_p]\right]^2} \tag{153}$$

for $-1<\eta<1$ ($p \le \lfloor \boldsymbol{\lambda}_{o;-} \rfloor$),

with (2.36) in [22] once again confirms that our definition of the X-Jacobi OPSs apparently corresponds to the partition $\emptyset, \mu$ with $\mu_k := m_k + p - k$ (k=1,…,p).

Our observation that the seed Jacob polynomials for the X-Jacobi OPSs composed of the simple p-WPs constitutes the finite orthogonal sequence of the R-Jacobi polynomials [1,38] may be useful for deriving some nontrivial properties of the JPWs in question. In particular, based on Theorem 2.1 in [51] (summarizing Karlin and Szego's results [59]), we assert that any Wronskian of an even number of the R-Jacobi polynomials of sequential degrees may have only negative real zeros smaller than -1.

7.2. Infinite Net of Finite EOPS Sequences Generated Using JPWs of Classical Jacobi Polynomials with Positive Indexes

Finally, we come to the discussion of the infinite net of the exactly solvable RCSLEs generated using the JPWs of classical Jacobi polynomials with the positive Jacobi indexes. Conjecture 4 assures that the JPW composed of the seed polynomials in question has no zeros larger than 1, which constitutes the question of fundamental significance for this study. Below we focus solely on the RDCTs using the infinitely many PFS of type $\underset{\sim}{\mathbf{a}}$ as the seed functions, i.e., by definition $\vec{\boldsymbol{\lambda}} = \vec{\boldsymbol{\lambda}}_o$ and $p_{max} = \infty$. The corresponding eigenfunctions of the *p*-SLE (24) solved under the DBCs on the interval (1, ∞) are formed by the R-Jacobi polynomials with the Jacobi indexes $\vec{\boldsymbol{\lambda}}' = -\boldsymbol{\lambda}_{o;-}, \boldsymbol{\lambda}_{o;+}$, and their total number is equal to

$$n_{\underset{\sim}{\mathbf{c}}} = j_{max} + 1 = \lceil \boldsymbol{\lambda}_{o;-} \rceil . \tag{154}$$

This brings us to the finite net of the finite EOP sequences composed of the p-PWs

$$\mathcal{P}_{\mathfrak{N}(\bar{\mathbf{M}}_p)+j}[\eta;\lambda_{o;-},\lambda_{o;+}\,|\,\bar{\mathbf{M}}_p;-+,j] := \tag{155}$$

$$\begin{vmatrix} P^{(\lambda_{o;+},\lambda_{o;-})}_{m_1}(\eta) & P^{(\lambda_{o;+},\lambda_{o;-})}_{m_2}(\eta) & \dots & \Theta^p_1[\eta]P^{(\lambda_{o;+},-\lambda_{o;-})}_j(\eta) \\ \frac{d}{d\eta}P^{(\lambda_{o;+},\lambda_{o;-})}_{m_1}(\eta) & \frac{d}{d\eta}P^{(\lambda_{o;+},\lambda_{o;-})}_{m_2}(\eta) & \dots & P^{(1)}_{j+p-1}[\eta;-\lambda_{o;-},\lambda_{o;+}|-+] \\ & & \dots & \\ \frac{d^p}{d^p\eta}P^{(\lambda_{o;+},\lambda_{o;-})}_{m_1}(\eta) & \frac{d^p}{d^p\eta}P^{(\lambda_{o;+},\lambda_{o;-})}_{m_2}(\eta) & \dots & P^{(p)}_j[\eta;-\lambda_{o;-},\lambda_{o;+}|-+] \end{vmatrix}.$$

This time the first p elements of the first row are represented by the classical Jacobi polynomials with positive indexes while the last element is the the R-Jacobi polynomial multiplied by a constant. The weight function (149) takes the form:

$$W[\eta;\lambda_{o;-},\lambda_{o;+}\,|-+\vdots\bar{\mathbf{M}}_p]:=\frac{\psi_0^2[\eta;-\lambda_{o;-}-p,\lambda_{o;+}+p]}{W^{(\lambda_{o;+},\lambda_{o;-})}_{\mathfrak{N}(\bar{\mathbf{M}}_p)}[\eta\,|\,\bar{\mathbf{M}}_p]} \tag{156}$$

for $-1<\eta<1$,

where the Wronskian in the denominator is formed by the classical Jacobi polynomials with positive indexes.

Corollary 4 can be now reformulated as

Corollary 5: *The Wronskian of the classical Jacobi polynomials with positive indexes may not have real zeros larger than 1.*

In Appendix E we explicitly confirm this corollary for the simplest second-degree JPW Wronskian formed by the Jacobi polynomials of degrees 1 and 2.

8. Discussion

Let us first point to the most essential element of our RSLP formalism – the advanced technique for selecting the sequences of the admissible RRZTs, using PFRs below the lowest eigenvalue as the q-RTFs. Each such sequence can be then re-interpreted as the admissible RDCT. To be more precise, we laid down the mathematical grounds for this innovation in Section 7 by converting the RCSLE (67) to its prime forms (138) and (139) on the intervals (–1,+1) and (1,∞) accordingly and solving the resultant SLEs under the DBCs. The formulated SLPs allowed us to prove [19] that each RD$\mathfrak{T}$ of the PFS itself constitutes the PFS of the transformed SLE at the same energy.

In summary, we have constructed three infinite nets of the X-Jacobi DPSs composed of the simple p-WPs. The current analysis was focused on the X-DPSs containing both the X-Jacobi OPSs formed by the RDC$\mathfrak{T}$s of the classical Jacobi polynomials and finite EOP sequences formed by the RDC$\mathfrak{T}$s of the R-Jacobi polynomials (using the seed R-Jacobi polynomials with the common pair of indexes and respectively a set of the classical Jacobi polynomials with common pair of positive indexes). The constructed X-DPSs obey the second-order FPDEs, expected from Theorem 5.2 in [22].

The fourth net of the X-Jacobi DPSs is composed of the Wronskians of the Jacobi polynomials with common pair of the indexes. Since the RDCTs generating these X-DPSs cannot be decomposed into the sequence of the admissible RRZTs, we skipped their analysis in this paper. However the infinite and finite orthogonal subsets of these X-DPSs form the eigenfunctions of the RCSLEs constructed from the JRef CSLE (1) using the 'juxtaposed' [60] pairs of its eigenfunctions, provided that the partition λ with $\lambda_k := m_k + p - k$ (k=1,…,p) is composed of the even-length segments. For the X-Jacobi OPSs this assertion has been proven by Durán [22]. Comparing the weight

$$W[\eta;\alpha,\beta\,|\,\bar{\mathbf{M}}_p;++]:=\frac{(1+\eta)^{\alpha+p}\,(1-\eta)^{\beta+p}}{\left[W^{(\alpha,\beta)}_{\mathfrak{N}(\bar{\mathbf{M}}_p)}[\eta\,|\,\bar{\mathbf{M}}_p]\right]^2} \tag{157}$$

with (2.36) in [45] reveals that we deal with the partition $\lambda,\emptyset$ with $\lambda_k := m_k + p - k$ (k=1,…,p).

The 'extension of the Adler theorem [61] to the Wronskians of the R-Jacobi polynomials can be done using the argumentation put forward by us in [49] for the Wronskians of the R-Routh polynomials.

The net of the trigonometric ($|\eta|<1$) or radial ($\eta>1$) quantum-mechanical potentials exactly solvable in terms of the constructed infinite or accordingly finite EOP sequences can be obtained in following the prescriptions outlined by us in [21] for p=1.

The Liouville potentials quantized via the EOPs introduced in subsections 7.1 and 7.2 have the generic form:

$$V[\eta;-\lambda_{o;-},\lambda_{o;+}|\bar{\mathbf{M}}_p] = V_{t-PT}[\eta;\vec{\lambda}_o] + (1-\eta^2)\left\{I^o[\eta;\vec{\lambda}_o] - I^o[\eta;-\lambda_{o;-},\lambda_{o;+}|\bar{\mathbf{M}}_p]\right\} \quad \text{for } -1<\eta<1 \tag{158}$$

and

$$V[\eta;\vec{\lambda}_o|\bar{\mathbf{M}}_p] = V_{h-PT}[\eta;\vec{\lambda}_o] + (\eta^2-1)\left\{I^o[\eta;\vec{\lambda}_o] - I^o[\eta;\vec{\lambda}_o|\bar{\mathbf{M}}_p]\right\} \quad \text{for } \eta > 1 \tag{159}$$

after being expressed in terms of the variables:

$$\eta(x) = \cos x \qquad (-\pi < x < 0) \tag{160}$$

and

$$\eta(r) = \cosh r \quad (0 < r < \infty) \tag{161}$$

respectively, where the *t*-PT potential on the finite interval and the radial h-PT potential are parametrized as follows:

$$V_{t-PT}[\eta(x);\lambda_{o;+},\lambda_{o;-}] = \frac{\lambda_{o;+}^2 - 1/4}{4\sin^2 x/2} + \frac{\lambda_{o;-}^2 - 1/4}{4\cos^2 x/2} \quad (-\pi < x < 0), \tag{162}$$

and

$$V_{h-PT}(r;\lambda_{o;+},\lambda_{o;-}] = \frac{\lambda_{o;+}^2 - 1/4}{4\sinh^2 r/2} + \frac{\lambda_{o;-}^2 - 1/4}{4\cosh^2 r/2} \quad (0 < r < \infty). \tag{163}$$

As mentioned in [1], the rigorous mathematical studies [30,47] on the X-Jacobi and X-Laguerre OPSs made a few misleading references to the quantum-mechanical applications of the EOPs. To a certain extent this misinformation is traceable to the fact that the cited applications do not properly distinguish between the terms 'X-Jacobi DPS', 'X-Jacobi OPS', and 'finite EOP' sequences' (formed by the RD$\mathfrak{I}$s of the R-Jacobi polynomials), simply referring to the representatives of all the three manifolds as 'X-Jacobi polynomials'.

To be more precise, one has to distinguish between the X-Jacobi OPSs and the finite EOP sequences formed by the RDC$\mathfrak{I}$s of the three families of the Romanovski polynomials [16]; namely, the finite EOP sequences composed of the Romanovski-Bessel (R-Bessel) and Romanovski-Routh (R-Routh) polynomials analyzed by us in [62] and [49] respectively, as well as the RDC$\mathfrak{I}$s of the R-Jacobi polynomials discussed in this paper. All the associated Liouville potentials all belong to group A in Odake and Sasaki's [48] classification scheme of the translationally shape-invariant potentials (TSIPs) and as a result their eigenfunctions are expressible via the finite EOP sequences.

In the general case of the rational density function, allowing the solution of the JRef CSLE in terms of hypergeometric functions [26], the energy-dependent PF in (1) has second-order poles in the finite plane and as a result the associated Liouville potentials are quantized by the Jacobi polynomials with degree-dependent indexes. If the numerator of the given rational density function has no zeros at regular points of the JRef CSLE (or similarly of its confluent counterpart), then the associated Liouville potential turns into a TSIP of group B, with eigenfunctions expressible via the Jacobi (or

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respectively Laguerre) polynomials with at least one degree-dependent index, which have no direct relation to the theory of the EOPs.

To conclude, let us point to the crucial difference between the RDC𝔍s of the R-Jacobi polynomials and those of the R-Bessel and R-Routh polynomials analyzed by us in [62] and [49] respectively. The common feature of the latter RDCT nets is that each net is specified by a single series of Maya diagrams and as a result any finite EOP sequence allows the Wronskian representation [57]. On other hand, the complete net of the RDC𝔍s of the R-Jacobi polynomials is specified by the two series of Maya diagrams, similar to the RDC𝔍s of the classical Jacobi and classical Laguerre polynomials forming the X-Jacobi and X-Laguerre OPSs accordingly [21]. This implies that we managed to construct only a tiny manifold of the finite EOP sequences composed of the RDC𝔍s of the R-Jacobi polynomials.

For example, we can use different combinations of the PFSs $\underset{\sim}{\mathfrak{a}},k$ and $\underset{\sim}{\mathfrak{a}}',k'$ below the lowest eigenvalue to construct the PFSs of the transformed RCSLEs. We refer the reader to [21] for the scrupulous analysis of the equivalence relations between the various $\mathfrak{p}$-WPs. It should be however stressed that grouping the equivalent $\mathfrak{p}$-WPs together represents only a part of the problem. The next step would be to select the preferrable representation. For example, the RD𝔍 of the h-PT with the TF $\underset{\sim}{\mathfrak{b}},m$ seems easier to be dealt with, compared with the RDC𝔍 of this potential with the m seed functions $\underset{\sim}{\mathfrak{a}},k=1,2,...,m$, though the final results will be absolutely the same.

The additional complication comes from the fact that one has to analyze the order of $\mathfrak{p}$-WP zeros at ∓ 1 to construct the appropriate X-Jacobi DPSs. In this respect one can take advantage of Theorem 5.2 in [22] which formulates the conditions for the $\mathfrak{p}$-WPs to satisfy the FPDEs.

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Abbreviations

ChExp characteristic exponent

CSLE canonical Sturm-Liouville equation

DBC Dirichlet boundary condition

DPS differential polynomial system

DCT Darboux-Crum transformation

DC𝔍 Darboux-Crum transform

DT Darboux deformation

D𝔍 Darboux transform

EOP exceptional orthogonal polynomial

ExpDiff exponent difference

FPDE Fuchsian differential equation with polynomial coefficients

GDT generalized Darboux transformation

h-PT hyperbolic Pöschl-Teller

JPW Jacobi-polynomial Wronskian

JRef Jacobi-reference

JS Jacobi-seed

LC limit circle

LDT Liouville-Darboux transformation

LP limit point

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ODE ordinary differential equation

OPS orthogonal polynomial system

PD polynomial determinant

PDE polynomial differential equation

PF polynomial fraction

PFS principal Frobenius solution

p-SLE prime Sturm-Liouville equation

p-WP pseudo-Wronskian polynomial

p-WEOP pseudo-Wronskian exceptional orthogonal polynomial

p-W$\mathfrak{T}$ pseudo-Wronskian transform

q-RPFSs quasi-rational principal Frobenius solution

q-RS quasi-rational solution

q-RTF quasi-rational transformation function

RCSLE rational canonical Sturm-Liouville equation

RDC rational Darboux-Crum

RDCT rational Darboux-Crum transformation

RDC$\mathfrak{T}$ rational Darboux-Crum transform

RDT rational Darboux transformation

RD$\mathfrak{T}$ rational Darboux transform

restr-HRef restrictive Heun-reference

R-Jacobi Romanovski-Jacobi

R-Routh Romanovski-Routh

RRZ$\mathfrak{T}$ rational Rudjak-Zakharov transform

TF transformation function

W$\mathfrak{T}$ Wronskian transform

Appendix A. RZT of Generic CSLE

Let $\phi_\tau[\eta;\vec{\lambda}_o]$ be a nodeless solution of a CSLE

$$\left\{\frac{d^2}{d\eta^2}+I^o[\eta;\vec{\lambda}_o]+\varepsilon\,\rho[\eta]\right\}\Phi[\eta;\vec{\lambda}_o;\varepsilon]=0 \qquad \text{(A1)}$$

at the energy

$$\varepsilon=\varepsilon_\tau(\vec{\lambda}_o), \qquad \text{(A2)}$$

i.e.,

$$\left\{\frac{d^2}{d\eta^2}+I^o[\eta;\vec{\lambda}_o]+\varepsilon_\tau(\vec{\lambda}_o)\,\rho[\eta]\right\}\phi_\tau[\eta;\vec{\lambda}_o]=0. \qquad \text{(A3)}$$

We define the RZT of the given CSLE via the requirement that the function

$$*\phi_\tau[\eta;\vec{\lambda}_o]=\frac{\rho^{-1/2}[\eta]}{\phi_\tau[\eta;\vec{\lambda}_o]} \qquad \text{(A4)}$$

is the solution of the transformed CSLE:

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$$\left\{\frac{d^2}{d\eta^2}+I^o[\eta;\vec{\lambda}_o\,|\,\tau]+\varepsilon\,\rho[\eta]\right\}\Phi[\eta;\vec{\lambda}_o;\varepsilon\,|\,\tau]=0 \qquad (A5)$$

at the same energy (A2), i.e.,

$$\left\{\frac{d^2}{d\eta^2}+I^o[\eta;\vec{\lambda}_o\,|\,\tau]+\varepsilon_\tau(\vec{\lambda}_o)\rho[\eta]\right\}{}^*\phi_\tau[\eta;\vec{\lambda}_o]=0. \qquad (A6)$$

Representing both CSLEs (A3) and (A6) in the Riccati form:

$$I^o[\eta;\vec{\lambda}_o]=-ld^{\,2}\phi_\tau[\eta;\vec{\lambda}_o]-\dot{ld}\,\phi_\tau[\eta;\vec{\lambda}_o]-\varepsilon_\tau(\vec{\lambda}_o)\,\rho[\eta] \qquad (A7)$$

and

$$I^o[\eta;\vec{\lambda}_o\,|\,\tau]:=-ld^{\,2}\,{}^*\phi_\tau[\eta;\vec{\lambda}_o]-\dot{ld}\,{}^*\phi_\tau[\eta;\vec{\lambda}_o]-\varepsilon_\tau(\vec{\lambda}_o)\rho[\eta], \qquad (A8)$$

subtracting one from another, and also taking into account that the logarithmic derivatives of the TF $\phi_\tau[\eta;\vec{\lambda}_o]$ and its reciprocal (A4) are related in the elementary fashion:

$$ld{}^*\phi_\tau[\eta;\vec{\lambda}_o]=-ld\,\phi_\tau[\eta;\vec{\lambda}_o]-\tfrac{1}{2}\,ld\,\rho[\eta] \qquad (A9)$$

one finds [62]

$$I^o[\eta;\vec{\lambda}_o\,|\,\tau]=I^o[\eta;\vec{\lambda}_o]+2\sqrt{\rho[\eta]}\,\frac{d}{d\eta}\frac{ld\,\phi_\tau[\eta;\vec{\lambda}_o]}{\sqrt{\rho[\eta]}}+\mathfrak{I}\{\rho[\eta]\}, \qquad (A10)$$

where the last summand represents the so-called [19] 'universal correction' defined via the generic formula

$$\mathfrak{I}\{f[\eta]\}:=\tfrac{1}{2}\sqrt{f[\eta]}\,\frac{d}{d\eta}\frac{ld\,f[\eta]}{\sqrt{f[\eta]}}. \qquad (A11)$$

Appendix B. DCT of the Generic CSLE as a Sequence of RZTs

Let $\phi_{\tau_2}[\eta;\vec{\lambda}_o]$ be another solution of a CSLE (A1) at the energy $\varepsilon_{\tau_2}(\vec{\lambda}_o)$. Then the CSLE (A6) with $\tau\equiv\tau_1$ has the solution [62]

$$\phi_{\tau_2}[\eta;\vec{\lambda}_o\,|\,\tau_1]=\frac{W\{\phi_{\tau_{k=1,2}}[\eta;\vec{\lambda}_o]\}}{\rho^{1/2}[\eta]\phi_1[\eta;\vec{\lambda}_o]}. \qquad (A12)$$

Using this solution as the TF for the next RZT, we come to the CSLE

$$\left\{\frac{d^2}{d\eta^2}+I^o[\eta;\vec{\lambda}_o\,|\,\tau]+\varepsilon\,\rho[\eta]\right\}\Phi[\eta;\vec{\lambda}_o;\varepsilon\,|\,\tau]=0 \qquad (A13)$$

with the zero-energy free term

$$I^o[\eta;\vec{\lambda}_o|\tau_{k=1,2}]=I^o[\eta;\vec{\lambda}_o\,|\,\tau_1] \qquad (A14)$$

$$+2\sqrt{\rho[\eta]}\frac{d}{d\eta}\frac{ld\,\phi_{\tau_2}[\eta;\vec{\lambda}_o\,|\,\tau_1]}{\sqrt{\rho[\eta]}}+\wp\{\rho[\eta]\}.$$

Substituting (A12) into (A14), coupled with (A10) and (A11), then gives [62]

$$I^o[\eta;\vec{\lambda}_o|\tau_{k=1,2}]=I^o[\eta;\vec{\lambda}_o]+2\sqrt{\rho[\eta]}\frac{d}{d\eta}\frac{ld\,W\{\phi_{\tau_{k=1,2}}[\eta;\vec{\lambda}_o]\}}{\sqrt{\rho[\eta]}}. \tag{A15}$$

Let us now assume that the function [12,63]

$$\phi_\tau[\eta;\vec{\lambda}_o\,|\,\tau_{k=1,\dots,p}]=\frac{W\{\phi_{\tau_{k=1,\dots,p}}[\eta;\vec{\lambda}_o],\phi_\tau[\eta;\vec{\lambda}_o]\}}{\rho^{p/2}[\eta]W\{\phi_{\tau_{k=1,\dots,p}}[\eta;\vec{\lambda}_o]\}} \tag{A16}$$

satisfies the CSLE

$$\left\{\frac{d^2}{d\eta^2}+I^o[\eta;\vec{\lambda}_o\,|\,\tau_{k=1,\dots,p}]+\varepsilon\,\rho[\eta]\right\}\Phi[\eta;\vec{\lambda}_o;\varepsilon\,|\,\tau_{k=1,\dots,p}]=0 \tag{A17}$$

with the zero-energy free term [12]

$$I^o[\eta;\vec{\lambda}_o|\tau_{k=1,\dots,p}]=I^o[\eta;\vec{\lambda}_o] \tag{A18}$$

$$+2\sqrt{\rho[\eta]}\frac{d}{d\eta}\frac{ld\,W\{\phi_{\tau_{k=1,\dots,p}}[\eta;\vec{\lambda}_o]\}}{\sqrt{\rho[\eta]}}-p(p-2)\wp\{\rho[\eta]\}.$$

The RDT of the CSLE (A17) with the TF (A16) then results in the CSLE with the zero-energy free term defined by (A18) with p and $\tau_{k=1,\dots,p}$ replaced for p+1 and $\tau_{k=1,\dots,p+1}$ accordingly.

Theorem 9. *The function (A16) is the solution of the CSLE (A17) at the energy* $\varepsilon_{\tau_p}(\vec{\lambda}_o)$.

Proof: Suppose that both functions $\phi_{\tau_p}[\eta;\vec{\lambda}_o\,|\,\tau_{k=1,\dots,p-1}]$ and $\phi_{\tau_{p+1}}[\eta;\vec{\lambda}_o\,|\,\tau_{k=1,\dots,p-1}]$ are solutions of the CSLE (A17) with p replaced by p-1. It is also assumed that the energies are equal to $\varepsilon_{\tau_p}(\vec{\lambda}_o)$ and $\varepsilon_{\tau_{p+1}}(\vec{\lambda}_o)$ accordingly. Then, by definition of the CSLE (17), the function

$$\phi_{\tau_{p+1}}[\eta;\vec{\lambda}_o\,|\,\tau_{k=1,\dots,p}] \tag{A19}$$

$$=\frac{W\{\phi_{\tau_p}[\eta;\vec{\lambda}_o\,|\,\tau_{k=1,\dots,p-1}],\phi_{\tau_{p+1}}[\eta;\vec{\lambda}_o\,|\,\tau_{k=1,\dots,p-1}]\}}{\rho^{1/2}[\eta]\phi_{\tau_p}[\eta;\vec{\lambda}_o\,|\,\tau_{k=1,\dots,p-1}]}$$

must be its solution at the energy $\varepsilon_{\tau_{p+1}}(\vec{\lambda}_o)$. Replacing p and $\tau_{k=1,\dots,p}$ in the right-hand side of (A16) for p-1 and $\tau_{k=1,...,p-1}$ accordingly and then setting $\tau = \tau_p, \tau_{p+1}$ we can re-write (A19) as

$$\phi_{\tau_{p+1}}[\eta;\vec{\lambda}_o \,|\, \tau_{k=1,...,p}] \tag{A20}$$

$$= \frac{W\left\{W\{\phi_{\tau_{k=1,...,p}}[\eta;\vec{\lambda}_o]\}, W\{\phi_{\tau_p}[\eta;\vec{\lambda}_o \,|\, \tau_{k=1,...,p-1}], \phi_{\tau_{p+1}}[\eta;\vec{\lambda}_o]\}\right\}}{\rho^{1/2p}[\eta] W\{\phi_{\tau_{k=1,...,p-1}}[\eta;\vec{\lambda}_o]\}}.$$

Choosing m = p−1, n = p+1, n−m =2 in the general Wronskian decomposition formula in [64] then gives:

$$W\{\phi_{\tau_{k=1,...,p+1}}[\eta;\vec{\lambda}_o]\} \tag{A21}$$

$$\frac{W\left\{W\{\phi_{\tau_{k=1,...,p}}[\eta;\vec{\lambda}_o]\}, W\{\phi_{\tau_p}[\eta;\vec{\lambda}_o \,|\, \tau_{k=1,...,p-1}], \phi_{\tau_{p+1}}[\eta;\vec{\lambda}_o]\}\right\}}{W\{\phi_{\tau_{k=1,...,p-1}}[\eta;\vec{\lambda}_o]}$$

then brings us back to (A16) with $\tau = \tau_{p+1}$, which completes the proof. □

Here and in the other publications we refer to the CSLE (A17) with the zero-energy free term (A18) as 'Darboux-Crum transform' ($DC\mathfrak{T}$) of the CSLE (1) with the seed functions $\tau_{k=1,...,p}$.

Appendix C. RDC Sequences of PFSs Near a 2nd-Order Pole with an Energy-Independent ExpDiff

Let $\Phi_{\mp}[\eta;\vec{\lambda};\varepsilon|\bar{M}_p]$ be the PFS of the RCSLE (63) near the pole at ∓1. Then the functions

$$\Psi_{-}[\eta;\vec{\lambda};\varepsilon|\bar{M}_p] = (1-\eta^2)^{-1/2}\,\Phi_{-}[\eta;\vec{\lambda};\varepsilon|\bar{M}_p] \quad \text{for } |\eta|<1 \tag{A22}$$

and

$$\Psi_{+}[\eta;\vec{\lambda};\varepsilon|\bar{M}_p] = \begin{cases} (1-\eta^2)^{-1/2}\,\Phi_{+}[\eta;\vec{\lambda};\varepsilon|\bar{M}_p] & \text{for } |\eta|<1, \\ (\eta-1)^{-1/2}\,\Phi_{+}[\eta;\vec{\lambda};-\varepsilon|\bar{M}_p] & \text{for } \eta>1 \end{cases} \tag{A23}$$

are the solutions of the *p*-SLEs (129) and accordingly (130), satisfying the DBCs at the corresponding singular endpoints:

$$\lim_{\eta\to\eta_{\mp}} \Psi_{\mp}[\eta;\vec{\lambda};\varepsilon|\bar{M}_p] = 0. \tag{A24}$$

Representing the $RRZ\mathfrak{T}s$ of the PFSs (A22) and (A23),

$$\Psi[\eta;\vec{\lambda};\varepsilon|\bar{M}_{p+1};\mp] = \frac{W\{\Psi[\eta;\vec{\lambda};\varepsilon_{m_{p+1}}(\vec{\lambda})|\bar{M}_p], \Psi_{\mp}[\eta;\vec{\lambda};\varepsilon|\bar{M}_p]\}}{\rho^{1/2}[\eta]\Psi[\eta;\vec{\lambda};\varepsilon_{m_{p+1}}(\vec{\lambda})|\bar{M}_p]}, \tag{A25}$$

as

$$\Psi[\eta;\vec{\lambda};\varepsilon|\bar{M}_{p+1};\mp] = |1-\eta^2|^{1/2}\,\dot{\Psi}_{\mp}[\eta;\vec{\lambda};\varepsilon|\bar{M}_p] \tag{A26}$$

$$-|1-\eta^2|^{1/2}\, ld\, \Psi[\eta;\vec{\lambda};\varepsilon_{m_{p+1}}(\vec{\lambda})|\bar{M}_p]\Psi_{\mp}[\eta;\vec{\lambda};\varepsilon|\bar{M}_p]$$

shows that

$$\lim_{\eta\to\mp1} \Psi[\eta;\vec{\lambda};\varepsilon|\bar{M}_{p+1};\mp]=0 \tag{A27}$$

and therefore

$$\Psi[\eta;\vec{\lambda};\varepsilon|\bar{M}_{p+1};\mp]\equiv\Psi_{\mp}[\eta;\vec{\lambda};\varepsilon|\bar{M}_{p+1}] \tag{A28}$$

iff the ExpDiff for the corresponding pole of the RCSLE (67) lies within the LP range:

$$^*\lambda^{(p)}_{o;\mp}=|\lambda_{\mp}+p|>1. \tag{A29}$$

Finally let us prove that the RRZ$\mathfrak{I}$ of the PFS $\Psi_{\infty}[\eta;\vec{\lambda};\underset{\sim}{\varepsilon}|\bar{M}_p]$ of the *p*-SLE (130) near the pole at infinity vanishes in the limit $\eta\to\infty$. Taking into account that differentiating of the function η^{α} decreases the power exponent, we find that both summands in

$$\Psi[\eta;\vec{\lambda};\underset{\sim}{\varepsilon}|\bar{M}_{p+1}]=(\eta^2-1)^{1/2}\,\dot{\Psi}_{\infty}[\eta;\vec{\lambda};\underset{\sim}{\varepsilon}|\bar{M}_p] \tag{A30}$$
$$-(\eta^2-1)^{1/2}\, ld\, \Psi[\eta;\vec{\lambda};-\varepsilon_{m_{p+1}}(\vec{\lambda})|\bar{M}_p]\Psi_{\infty}[\eta;\vec{\lambda};\tilde{\varepsilon}|\bar{M}_p]$$

,

vanish at infinity which confirms that the q-RS (A30) is indeed the PFS of the transformed RCSLE near its pole at infinity.

Appendix D. Exact Solvability of the p-SLEs (138) and (139) Under the DBCs

The proof presented below constitutes the special of the general technique developed by the author [19] in support of the assertion that the RRZT of an exactly solvable RDC$\mathfrak{I}$ of the JRef CSLE results in an exactly solvable RCSLE. In context of this paper we show that the q-RSs (146) represent the complete set of the eigenfunctions of the corresponding *p*-SLE (138) or (139). This proof assures that this set of the eigenfunctions starts from the nodeless solution, which in turns confirms that all the PFSs (146) under the conditions (144) and (145) lie below of the lowest eigenvalue of the *p*-SLE in question and therefore do not have zeros inside the orthogonalization interval under consideration.

Theorem 10. *All the Dirichlet problems for the p-SLEs (138) or alternatively (139) under the conditions (144) and (145) in both cases (coupled with the upper bound (143) for the degrees of the seed Jacobi polynomials in case of the former p-SLE), have exactly the same discrete energy spectrum as the precursor p-SLE (24) solved under the DBCs on the intervals (-1,+1) or (1,∞) accordingly.*

Proof: For the *p*-SLE (139) with p=1 this assertion has been proven in [1] and below reproduce similar arguments for the finite interval. We shall come back to these arguments, while proving the general case of $p > 1$. For now let us just assume that the cited statement is correct for both *p*-SLEs (138) and (139) with p=1.

Re-writing (78) with $p=1$ as

$$\psi_{\mp+,\mathbf{m}_2}[\eta;\mp\lambda_{o;-},\lambda_{o;+}|\mathbf{m}_1]=\psi_0[\eta;\mp\lambda_{o;-}-1,\lambda_{o;+}+1]\times \tag{A31}$$

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$$\frac{W^{(\mp\lambda_{o;-},\lambda_{o;+})}_{\mathbf{m}_1+\mathbf{m}_2-1}[\eta\,|\,\mathbf{m}_1,\mathbf{m}_2]}{P^{(\mp\lambda_{o;-},\lambda_{o;+})}_{\mathbf{m}_1}(\eta)}$$

,

we find that the power exponents of $\eta \pm 1$ coincide with halves of the corresponding ExpDiffs

$$*\lambda^{(p)}_{o;\mp} = |\,\lambda_{\mp} + p\,| = \lambda_{o;\mp} \mp p > 0. \quad \text{(A32)}$$

Keeping in mind that the exponent power $(\lambda_{o;+}+1)/2$ is positive for the both intervals (-1,+1) and (1,∞), we conclude that the listed solution vanishes in the limits $\eta \to 1\mp$ and therefore represents the PFS of the corresponding *p*-SLEs (138) or (139) near this singular endpoint. Since we assume that the formulated Dirichlet problems are exactly solvable for p=1, this PFS lies below the lowest eigenvalue of the given p-SLE and therefore the JPW in the numerator of the PF in the right-hand side of (A31) may not have zeros inside the orthogonalization interval in question as far as the Jacobi indexes are restricted by the conditions (144).

Since the *p*-SLEs in question are exactly solvable for p=1 and $\bar{\mathbf{M}}_2$ is the admissible set of the polynomial seed solutions, we use the mathematical induction, assuming that the p-SLE (138) or (139) for the admissible set of the polynomial seed solutions, $\bar{\mathbf{M}}_p = \bar{\mathbf{M}}_{P-1}$, has exactly the same discrete energy spectrum as the *p*-SLE (24) with the leading coefficient function (28). Under the latter assumption one can then repeat the above arguments to prove that the JPW (65) with $\bar{M}_p = \bar{\mathbf{M}}_P$ is nodeless inside the corresponding orthogonalization interval and therefore $\bar{\mathbf{M}}_P$ is the admissible set of the polynomial seed solutions. Our final step is to prove that the given *p*-SLE with $\bar{\mathbf{M}}_p = \bar{\mathbf{M}}_P$ has exactly the same discrete energy spectrum as (24).

Suppose that the given *p*-SLE has another eigenfunction $\psi_{\mathbf{c},n}[\eta;\lambda'\,|\,\bar{\mathbf{M}}_P]$ at an energy E_n with the absolute value $|\,E_n\,| \neq \varepsilon_j(\vec{\lambda}')$ for any j $\leq$ jmax and therefore, by definition, it must obey the DBCs

$$\lim_{\eta\to\eta_{\mp}} \psi_{\mathbf{c},n}[\eta;\vec{\lambda}'\,|\,\bar{\mathbf{M}}_P] = 0\,. \quad \text{(A33)}$$

If $*\lambda^{(P)}_{o;-} > 1$ (which assures ExpDiffs $*\lambda^{(P-1)}_{o;\mp}$ lie within the LP range) then we can take advantage of the arguments presented in Appendix C to show that the RRZT with the TF

$$*\phi_{m_P}[\eta;\vec{\lambda}|\bar{\mathbf{M}}_{P-1}] = \frac{\rho^{-1/2}[\eta]}{\phi_{m_P}[\eta;\vec{\lambda}|\bar{\mathbf{M}}_{P-1}]} \quad \text{(A34)}$$

converts the extraneous eigenfunction into the eigenfunction of the *p*-SLE (138) or (139) with $\bar{\mathbf{M}}_p = \bar{\mathbf{M}}_{P-1}$. However this conclusion contradicts the assumption that the *p*-SLE in question has exactly the same energy spectrum as the *p*-SLE (24). We thus assert that q-RSs (146) represent all possible eigenfunctions of the p-SLEs (138) and (139) accordingly.

Finally, one can easily verify that the presented arguments are fully applied to the starting instance p=1, since we only need the presumption that the p-SLEs (24) with the leading coefficient functions (28) are exactly solvable under the DBCs imposed at the ends of the corresponding orthogonalization interval and that the TF of the given RRZT does not have zeros inside of this interval. □

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The direct consequence of the proven theorem is that the p-WPs (99) with $\bar{M}_p = \bar{\boldsymbol{M}}_p$ and $\vec{\sigma} = -+$ have exactly j real zeros larger than 1.

Appendix E. Xm-Jacobi DPS in the Second-Order Darboux-Crum Representation

It has been proven by the author in [65] that the JPW of degree 2 can be reduced to the second-degree Jacobi polynomial as follows

$$W_2^{(\lambda_+,\lambda_-)}[\eta\,|\,1,2] = \tfrac{1}{2}(\lambda_+ + \lambda_- + 4)P_2^{(-\lambda_+ -3,-\lambda_- -3)}(\eta). \qquad \text{(A35)}$$

At that time the author did not realized that it deals with the very special case of the equivalence theorem initially sketched in [47,48] and then scrutinized in great detail [21] (see [57] for additional comments).

In particular, this implies the Wronskian of two classical Jacobi polynomials with positive indexes can be written as

$$W_2^{(\lambda_{0;+},\lambda_{0;-})}(\eta\,|\,1,2) = \tfrac{1}{2}(\lambda_{0;+} + \lambda_{0;-} + 4)P_2^{(-\lambda_{0;+}-3,-\lambda_{0;-}-3)}(\eta). \qquad \text{(A36)}$$

Examination of the explicit formula for the second-degree Jacobi polynomial:

$$P_2^{(\lambda_+,\lambda_-)}(\eta) = \tfrac{1}{8}[(\lambda_- + 2)(\lambda_- + 1)(1+\eta)^2 + (\lambda_+ + 2)(\lambda_+ + 1)(1-\eta)^2 \qquad \text{(A37)}$$

$$-2(\lambda_0 + 2)(\lambda_1 + 2)(1-\eta^2)]$$

reveals that that the polynomial in the right-hand side of (A36) may not have a zero at $\eta = \mp 1$ since both Jacobi indexes are smaller than -3, in agreement with Theorem 1.

It can be shown that the discriminant of the polynomial (A37) is given by the simple formula:

$$\Delta(\vec{\lambda}) = \tfrac{1}{32}(\lambda_+ + \lambda_- + 3)(\lambda_- + 2)(\lambda_+ + 2) \qquad \text{(A38)}$$

and therefore the second-degree Jacobi polynomial of our interest has the negative discriminant. Firstly, this observation confirms Theorem 2 stating all the zeros of the JPW with positive indexes may have only simple zeros. Secondly, we conclude that the second-degree polynomial (A36) has a pair of complex conjugated zeros and therefore remains finite on the interval [1,∞), in line with Corollary 5.

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