Article

D-dimensional $f(R, \phi)$ gravity

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Abstract: In this work, we explore a the different forms of a new type of modified gravity, namely $f(\phi)$ gravity. We construct the Big Rip type for the energy density and the curvature of the universe. We show that dark energy is a result of the transformation of the field ϕ mass (dark matter) to energy. In addition, we provide that $\Omega_m \approx 0,050$, $\Omega_{DM} \approx 0,2$, $\Omega_{DE} \approx 0,746$, is in excellent agreement with observation data. We explore a generalized formalism of braneworld modified gravity. We also construct a new field equations, which generalize the Einstein field equations. We provide a relation between the extra dimension in 3-brane with the vacuum energy density. We show that the energy density of matter depends directly on the number of dimensions. We manage to find the value of the Gauss-Bonnet coupling $\alpha = 1/4$ which is a good agreement with the results in the literature, this correspondence creates a passage between $f(R)$ gravity and Gauss-Bonnet gravity, this comparison leads to a number of bulk dimensions equal to $D = 10^{121} + 4$.

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1. Introduction

In a homogeneous and isotropic spacetime, the Einstein field equations give rise to the Friedmann equations that describe the evolution of the Universe. In fact, the standard big-bang cosmology based on radiation and matter dominated era can be well described within the framework of general relativity [1]. Note that dark matter and dark energy cannot be explained by general relativity. Most authors think that dark matter and the dark energy is abundant in the universe and it has had a strong influence on its structure and evolution of the Universe. Recently, various models of modified gravity [2] have been proposed in order to solve the problem of dark matter and dark energy. The $f(R)$ theory is a type of modified gravity theory [3], which generalizes Einstein's general relativity. Recently, a class of extra dimensions gravity theories has been proposed. 3-Brane world [4] is a set of cosmological scenarios inspired by the ideas of the second revolution in string theory, the aim of which is to solve the famous problem of hierarchy, as randall sundrum model [5] and ADD (Arkani-Hamed, Dvali, Dimopoulos) model [6]. In these models, the observable 4-dimensional universe is a subpart of the total universe that has additional dimensions. The original higher-dimensional ideas of Kaluza and Klein in the 1920s [7], having demonstrated that Maxwell's theory of electromagnetism and general relativity could easily be found from a 5-dimensional geometric relativistic theory (4-dimensional of space and one time).

The holographic principle is a speculative conjecture within the framework of the theory of quantum gravity, proposed by Gerard 't Hooft then improved by Leonard Susskind [8]. This conjecture proposes that all the information contained in a volume of space can be described by a theory that lies on the edges of this region. which can be written as a Bekenstein bound $S \preceq A/4$ [9].

In our previous paper [10], we have chosen the scalar field ϕ with 5-dimensional bulk, which is coupled with Ricci scalar R at 4-dimensional 3-brane. In the framework

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of our model, the angle ϕR represents the transition from 4-dimensional (4d) to 5-dimensional (5d). ϕR is the rotation of 3-brane in the bulk on the 5th dimension with the parameter $x^5 = y = L\phi R$, with L is the radius of the large compact dimension. We also define the Planck mass M_5^3 at 5d by $M_5^3 = M_P^2 L^{-1}$, with M_P^2 is the Planck mass in 4d. The mass of the scalar field ϕ in 5th dimension represents dark matter in 3-brane.

The paper is organized as follows. In Section 2, we introduce the notion of passage from 4d to 5d in $f(\phi)$ gravity, we will simultaneously study matter and dark matter in the framework of $f(\phi)$ gravity. In Section 3; we study dark energy and we find equivalent results with the model of holographic dark energy, then we obtain the values of the density paramete. We study generally the holographic $f(\phi)$ gravity. Section 4 focus on in the general framework of D-dimonsional $f(R,\phi)$ gravity. In section 5 we will study a passage between Gauss-Bonnet gravity and $f(R)$ gravity. Section 6 contains an discussion.

2. 5-dimensional $f(R,\phi)$ gravity

2.1. 4d and 5d action

We start with the action in 4-dimensional $f(R,\phi) = \frac{M_P^2}{2} R e^{-\phi R}$ gravity model [10,11]:

$$I_4 = \frac{M_P^2}{2} \int d^4x \sqrt{-g} R e^{-\phi R} + I_m \tag{1}$$

this action is expressed in 3-brane M_4 (4d) by the geometric volume $Vol_{M_4} = \int_{M_4} d^4x \sqrt{-g}$.

We can then define the five-dimensional bulk (5d) as $M_5 = M_4 \times Y$, in which Y is a large compact dimension. $Vol_Y = e^{-\phi R} \int_Y dy$ corresponding to the volume of excitations of the graviton field in the compactified dimension. A conformal class $[g]$ on Riemannian manifold $M_5$is defined as the equivalence class of metrics: $g_{(5)MN} = g_{\mu\nu}\Omega^2$, Ω being a smooth function on M_5 [11]. In M_5 we define the metric determinant (a conformal transformation) by

$$g_{(5)} = g e^{-2\phi R} \tag{2}$$

we can write the action (1) with the geometrical form by

$$I_4 = Vol_{M_4}\left(\frac{1}{\phi} Vol_Y\right) \neq \frac{1}{\phi} Vol_Y(Vol_{M_4}) \tag{3}$$

This last equation shows us that the structure of 3-brane depends exactly on the behavior of the large extra dimensions. Using the conformal transformation from 4d to 5d by the transition with the rotation ϕR [10]. We describe the large compact dimension by $\phi RL = \int_0^{\phi RL} dy$. One can find the action describing the coupling between the Ricci scalar and the scalar field in 5d:

$$I_5 = \frac{M_5^3}{2} \int_{M_5} d^5x \sqrt{-g_{(5)}} \frac{1}{\phi} + I_m \tag{4}$$

This action depends only on the scalar field ϕ, i.e. the bulk contains only the field ϕ, and that the Ricci scalar exists in 3-brane. Hence, $f(\phi)$ gravity is expressed by

$$f(\phi) = \frac{M_5^3}{2\phi} \tag{5}$$

the rotation ϕR allows to see $f(R,\phi)$ gravity in 4d as a $f(\phi)$ gravity in 5d: $f(\phi)_{5d} = f(R,\phi)_{4d}$. By calculating the variation of action Eq.(4) we obtain the equation of motion of ϕ in 5d

$$\frac{\delta\phi}{\delta g_{(5)}^{MN}} + \frac{1}{2} g_{(5)MN}\phi = -\frac{1}{M_5^3}\phi^2 T_{MN} \tag{6}$$

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this equation describes both dark energy (DE), dark matter (DM) and ordinary matter (OM) by the field ϕ. $\frac{\delta\phi}{\delta g^{MN}}$ is the scalar field variation on the manifold M_5, the term $\frac{\delta\phi}{\delta g^{MN}}$ describes the density of dark energy. T_{MN} is the energy-momentum tensor of matter given by

$$T_{MN} = \frac{-2}{\sqrt{-g_{(5)}}} \frac{\delta\left(\sqrt{-g_{(5)}}L_m[g^{\mu\nu}, \Psi]\right)}{\delta g^{MN}_{(5)}}, \qquad (M, N = 0, 1, 2, 3, 5) \tag{7}$$

the energy-momentum tensor is defined by the variation of the lagrangian of matter $L_m[g^{\mu\nu}, \Psi]$ in accordance to the metric $g_{(5)MN}$ on the manifold M_5(the bulk). Ψ is the matter field on 3-brane. We define a new metric of a flat space, that is written as a direct sum of Friedmann–Lemaître–Robertson—Walker (FLRW) metric and the fifth dimension component in the homogeneous Universe in large structure, given by

$$ds^2_{(5)} = -dt^2 + a^2(t)d\mathbf{x}^2 + e^{-\phi R}dy^2 \tag{8}$$

with $a(t)$ is a scale factor. In this context we write $g^{MN}_{(5)}T_{MN} = g^{\mu\nu}T_{\mu\nu} + g^{55}_{(5)}T_{55}$. Let us consider that $g^{\mu\nu}T_{\mu\nu} = (3\omega - 1)\rho_m$ [12], where ρ_m is the energy density of matter and ω is the equation of state of matter, we chose $\omega = 0$. From Eq.(2) we have $g^{55}_{(5)} = e^{\phi R}$. We will consider that the component of the energy-momentum tensor T_{55} describes the dark matter by the relation

$$T_{55} := \rho_{DM} \tag{9}$$

which leads to

$$g^{MN}_{(5)}T_{MN} = -\rho_m + e^{\phi R}\rho_{DM} \tag{10}$$

In the following, we will consider a some specific forms of the eqution (6).

2.2. A simple solution

To describe the dark matter in $f(\phi)$ gravity, we assume that the term of DE in Eq.(6) is zero: $\frac{\delta\phi}{\delta g^{MN}_{(5)}} = 0$

$$\frac{1}{2}g_{(5)MN}\phi = -\frac{1}{M_5^3}\phi^2 T_{MN} \tag{11}$$

which leads to

$$T_{MN} = -f(\phi)g_{(5)MN} \tag{12}$$

this relation, show that $f(\phi)$ is an energy density, i.e. the distribution of matter depends on the structure of space-time of $f(\phi)$ gravity. Substituting Eq.(12) into Eq.(10) one can obtain

$$f_M(\phi) = -\frac{1}{5}e^{\phi R}\rho_{DM} + \frac{1}{5}\rho_m \tag{13}$$

or equivalently

$$f_M(\phi) = f_{DM}(\phi) + f_{OM}(\phi) \tag{14}$$

from this last equation we can define the function of dark matter f_{DM} and the function of matter f_{OM} by

$$f_{DM}(\phi) = -\frac{1}{5}e^{\phi R}\rho_{DM} \text{ and } f_{OM} = +\frac{1}{5}\rho_m \tag{15}$$

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the function f_{DM} depends directly on the scalar field ϕ. On the other hand, function f_{OM} does not depend on ϕ. Therefore, dark matter is the result of coupling between ϕ and R. Or DM is the mass of ϕ coupled with R.

3. Holographic 5d $f(\phi)$ gravity

3.1. Holographic dark energy

In 4d $f(R)$ gravity, it was shown that the field $\phi = t$ [13]. Here, since we are working on $f(\phi)$ gravity in 5d, we have that $\phi \sim t^2$ [10]. Thus, the scalar field is equivalent to the square of time, which is compatible with the accelerated expansion of the Universe [10]:

$$\phi = \frac{t^2}{6q(2q-1)} \tag{16}$$

where q is the deceleration parameter ($a \propto t^q$). The field ϕ describes the acceleration of the expansion of the Universe (dark energy). So that the equation (6) is homogeneous; the density of DE (ρ_{DE}) must inversely depend on the scalar field ϕ. To describe the dark energy, we calculate Eq.(6) in a vacuum $T_{MN} = 0$, we obtain

$$\frac{g^{MN}_{(5)}\delta\phi}{\delta g^{MN}_{(5)}} \equiv -\frac{5\phi}{2} \tag{17}$$

we introduce the function $f_{DE}(\phi)$, which represents the content of dark energy in the Universe. Using the same method of Eq.(15) in Eq.(17), we can relate the density of DE with $f_{DE}(\phi)$ by

$$f_{DE}(\phi) = +\frac{1}{5}\rho_{DE}(\phi) \tag{18}$$

according to Eq.(16), the field ϕ describes the acceleration of the expansion of the Universe if $\phi \succ 0$. Then the choice of the function f_{DE} or ρ_{DE} which will describe the dark energy depends on ϕ. For that we assume

$$\rho_{DE} := -\frac{M_5^3}{\phi^2}\frac{g^{MN}_{(5)}\delta\phi}{\delta g^{MN}_{(5)}} \tag{19}$$

the fonction f_{DE} (18) depends directly on the scalar field ϕ; since $\rho_{DE} = \rho_{DE}(\phi)$. the choice of metric (8) give us following relation $\frac{g^{MN}_{(5)}\delta\phi}{\delta g^{MN}_{(5)}} = -\frac{3\delta\phi}{2H\delta t}$, which leads to

$$\rho_{DE} = \frac{3M_5^3}{2H}\frac{\dot{\phi}}{\phi^2} \tag{20}$$

substituting Eq.(16) into Eq.(20) and we use the power-law-type expansion, whose Hubble parameter is given as follows $a(t) \propto t^q \Longrightarrow H = \frac{q}{t}$, we find

$$\rho_{DE} = 3c^2 M_5^3 H^2 \tag{21}$$

We can also give the parameter:

$$c^2 = \frac{3(2q-1)}{q^2} \tag{22}$$

The Eq.(21) is equivalent to the density of DE found in other models, among these models: the model of holographic dark energy [14].

- *In the matter-dominated era* $q = \frac{2}{3}$, one can obtain with $c = \frac{3}{2}$. According to Miao Li [14]; for $c = 1,5 \succ 1$ in Eq.(21), the second law of thermodynamics is not violated, while in a situation without any other component of energy, space–time is not de Sitter. We introduce the DE density parameter $\Omega_{DE} = \rho_{DE}/\rho_c$ where the critical

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density $\rho_c = 3M_5^3H^2$ and L the length scale of the extra dimensions, from Eq.(21) one obtains the following equation

$$c = \sqrt{\Omega_{DE}} \tag{23}$$

this equation verifies the condition of the holographic dark energy obtained by M.R. Setare [13]. Therefore, Eq.(21) is a special case of the condition $c \preceq \sqrt{\Omega_{DE}}$, which describes a phantom model of dark energy [15] by $\omega_{DE} \preceq -1$. For that, we use the relation between c and Ω_{DE} given by the model of holographic dark energy: $\Omega_{DE} = \frac{c^2}{R_h^2 H^2}$. If we compare this relation with Eq.(23) we obtain the following equation

$$HR_h = 1 \tag{24}$$

with R_h is the future event horizon is given by $R_h = a\int_t^{\infty}\frac{dt}{a} = a\int_a^{\infty}\frac{da}{Ha^2}$. we have to differentiate between R_h and L.

3.2. Density parameter in $f(\phi)$ gravity

In this section we study the equation (6) to find the percentages of content of DE, DM, OM in the Universe. we multiply Eq.(6) by $g_{(5)}^{MN}$ we find

$$-\frac{M_5^3 g_{(5)}^{MN}\delta\phi}{\phi^2\delta g_{(5)}^{MN}} + \frac{5M_5^3}{2\phi} = (3\omega - 1)\rho_m + e^{\phi R}\rho_{DM} \tag{25}$$

using Eqs.(13) (19) into Eq.(25) one can obtain

$$f(\phi) = (3\omega - 1)\rho_m + e^{\phi R}\rho_{DM} - \rho_{DE} \tag{26}$$

- In the radiation-dominated era:

We take the small values of the angle ϕR in this era. From Eq.(26) we obtain

$$f(\phi) = \rho_{DM} - \rho_{DE} \tag{27}$$

this last equation can be written this way $\rho_{DE} = \frac{6}{7}\rho_{DM}$. It can be seen clearly that the DM density is very important than the DE density in the radiation-dominated era. And if compare this result with the observation data [16,17]; we see that the density of DM decreases and that the density of DE increases. We have shown that the DM is the mass of the field ϕ (15), also that DE is the energy of ϕ since it depends on the time square (16). This shows that the DE is a result of the transformation of the field ϕ mass (dark matter) to energy since the transformation of mass into energy is possible with the einstein's equation $E = m$.

- In the matter-dominated era :

$$f(\phi) = e^{\phi R}\rho_{DM} - \rho_m - \rho_{DE} \tag{28}$$

in this era, we have the presence of DE, DM and OM. Consider now *the dark-energy-dominated era*. By assuming an acceleration which tends towards infinity ($\phi \longrightarrow \infty$) Eqs.(5) and (26). In our previous paper we find this result $e^{\phi R}\rho_{DM}(\phi) = 4\rho_m(\phi)$ [10] in dark-energy-dominated era. Substituting this result in Eq.(26)

$$f(\phi) = -\rho_{DE} \tag{29}$$

This equation shows us that the gravity in the far future directly depends on DE density. Which means that there is complete disappearance of DM and OM in the far future ($\phi \longrightarrow \infty$). this result is equivalent with the Big Rip scenario [18]. From Eq.(5), Eq.(16) and Eq.(21) we obtain $f(\phi) = \frac{q}{3}\rho_{DE}$, using the density parameter, Eq.(26) can be expressed as

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$$\left(\frac{q}{3}+1\right)\Omega_{DE} = (3\omega - 1)\Omega_m + e^{\phi R}\Omega_{DM} \tag{30}$$

since we have chosen ϕR as the angle of the transition rotation between 4d and 5d. We propose that the 5th dimension is perpendicular on 3-brane, in this case we take $\phi R = \frac{\pi}{2}$.We suppose that the change of Ω_{DM} relative to Ω_m is minimal, then using $\Omega_{DM} \approx 4\Omega_m$ [10] in *the matter-dominated era*, Eq.(30) leading to

$$\Omega_m \approx 0,05018\,;\ \Omega_{DM} \approx 0,2\,;\ \Omega_{DE} \approx 0,74625 \tag{31}$$

This result is in excellent agreement with new observation data [16,17]. In our previous paper [10], we just found the values of Ω_m and Ω_{DM} but in this paper we also obtained Ω_{DE}. Next, we will study the holographic $f(\phi)$ gravity.

3.3. Holographic $f(\phi)$ gravity

Note that Eq.(5), Eq.(16) and Eq.(21) together imply the relation between $f(\phi)$ and the density of DE : $f(\phi) = \frac{q}{3}\rho_{DE}$, or equivalently

$$f(\phi) = \frac{5q}{3} f_{DE}(\phi) \tag{32}$$

this equation shows us that gravity in the Universe depends on dark energy. On the other hand, since we have already shown that the DE in our model is a holographic dark energy. Hence, braneworld $f(\phi)$ gravity is holographic gravity. Note also that, conformal transformation (2) in (1) give $f(R,\phi)_{4d} = R_{4d}$. Therefore, $f(R,\phi)$ is Ricci scalare transformation in Einstein-Hilbert action. The expression $f(\phi)_{5d} = f(R,\phi)_{4d}$, show that the gravity on the 3-brane is a conformal transformation of the gravity on the bulk [19]. From Eq.(20)we obtain $\rho_{DE} = -\frac{3}{H}\dot{f}(\phi)$. The dark energy the conservation law in dark energy dominated universe given by $\dot{\rho}_{DE} + 3H(\rho_{DE} + p_{DE}) = 0$, substituting Eq.(32) into the conservation law, we find $p_{DE} = \left(9 - \frac{3}{q}\right)\frac{\dot{f}(\phi)}{3H}$. Let us consider the dark energy equation of state $\omega_{DE} = p_{DE}/\rho_{DE}$

$$\omega_{DE} = -1 + \frac{1}{3q} \tag{33}$$

we use Eq.(22) and Eq.(23) we obtain

$$\omega_{DE} = -1 + \frac{\Omega_{DE}}{9 \pm 3\sqrt{3(3-\Omega_{DE})}} \tag{34}$$

using (31) this last equation leading to the two solutions $\omega^-_{DE} = -0,378$ and $\omega^+_{DE} = -0,956$. According to Alessandro Melchiorri and all [20] we have this condition: $-1,38 \prec \omega_{DE} \prec -0,82$. The solution ω^-_{DE} is not valid according to the condition of ω_{DE}. However, we have the second solution ω^+_{DE}; which is to check the interval condition of ω_{DE}. This result is in excellent agreement with observation data [21,22]: indicate that $\omega_{DE} \prec -0,76$. But there is a problem, in this case, we have $\omega^+_{DE} \succ -1$ which corresponds with $c \succ \sqrt{\Omega_{DE}}$, which is incompatible with the result (23). However, we can compare this result with the transition of the DE equation of state: from $\omega_{DE} \succ -1$ to $\omega_{DE} \prec -1$ in an interacting holographic dark energy model [23]. By comparing this last result with Eq.(32); the factor $\frac{5q}{3}$ represents the transition of the DE equation of state. Hence, one can find a justification for the problem of the transition of the ω_{DE}: The measure of ω_{DE} in $f_{DE}(\phi)$ gravity framework is equivalent to $\omega_{DE} \preceq -1$ or $c \preceq \sqrt{\Omega_{DE}}$. Additionally, for $\omega_{DE} \succ -1$ or $c \succ \sqrt{\Omega_{DE}}$ is a property that could serve as the $f(\phi)$ gravity framework. Therefore, our model presents the interaction of DE and DM as a unification by the transition rotation ϕR between 4d and 5d. We have shown that the interaction between DM and DE is done by the transformation of the field ϕ mass of to DE. The transition from $f_{DE}(\phi)$ gravity in 5th dimension to $f(\phi)$ gravity in 5d, creates a

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transition on the DE equation of state. This model [23] justifies our work; the interaction between dark matter and dark energy influence the transition of the equation of state of DE. This transition is an observable characteristic of the interaction between DE and DM in addition to its influence on the small l CMB spectrum argued in [24]. The observation of ω_{DE} crossing -1 behavior in the future, can give information about the interaction between DE and DM.

4. D-dimensional $f(R,\phi)$ gravity

4.1. Formalism

In the following, we propose to study 4-dimensional 3-brane M_4 into D-dimensional bulk $\mathbf{M}_D$: $\mathbf{M}_D = M_4 \times Y^{D-4}$. The geometric volume in M_D is $Vol_{M_D} = \int_{M_D} d^D x \sqrt{-g_{(D)}}$, with $g_{(D)} = g_{(4)} e^{-2(D-4)\phi R}$ is the metric determinant at M_D. The standard form volume in large compact dimension Y is $Vol_Y = e^{-\phi R} \int_0^{L\phi R} dy$. The Planck mass M_p is $M_p^2 = M_D^{D-2} L^{D-4}$, with M_D is the Planck mass in D-dimensional. In $\mathbf{M}_D$, the action I_4 can be expressed as

$$I_4 = \frac{M_D^{D-2}}{2} \int_{M_4} d^4 x \sqrt{-g_{(4)}} R (L\phi R)^{D-4} \frac{1}{(\phi R)^{D-4}} e^{-(D-4)\phi R} + I_m, \tag{35}$$

which leads to

$$I = \frac{M_D^{D-2}}{2} \int_{M_4} d^4 x \sqrt{-g_{(4)}} \frac{R}{(\phi R)^{D-4}} e^{-(D-4)\phi R} \left(\int_0^{L\phi R} dy \right)^{D-4} + I_m \tag{36}$$

therefore, we get the general form of $f(R,\phi)$ gravity in D-dimensional as $f_D(R,\phi) = \frac{M_D^{D-2}}{2} \phi^{4-D} R^{5-D}$, defined by the following action

$$I_D = \frac{M_D^{D-2}}{2} \int_{M_D} d^D x \sqrt{-g_{(D)}} \phi^{4-D} R^{5-D} + I_m \tag{37}$$

we can see that the fields (ϕ, R) describe a exceptional dimensions, are $D = (4,5)$. Additionally, Ricci scalar R describe the gravity at 4D, and the field ϕ describe gravity at 5D [10]. According to the action (37) we can write I_4 and I_5 as

$$I_4 = \frac{M_p^2}{2} \int_{M_4} d^4 x \sqrt{-g_{(4)}} R + I_m \tag{38}$$

$$I_5 = \frac{M_5^3}{2} \int_{M_5} d^5 x \sqrt{-g_{(5)}} \frac{1}{\phi} + I_m. \tag{39}$$

Only $D = (4,5)$ which describe the fields (ϕ, R) without a coupling. For all the dimensions $D \neq (4,5)$ we will always have a coupling between ϕ and R. By calculating the variation of action (37) one can obtain

$$\delta I_D = \frac{M_D^{D-2}}{2} \int_{M_D} d^D x \delta \left(\sqrt{-g_{(D)}} \right) \phi^{4-D} R^{5-D} + \sqrt{-g_{(D)}} \delta \left(\phi^{4-D} R^{5-D} \right) + \delta I_m \tag{40}$$

Let us consider that $\delta(\sqrt{-g_{(D)}}) = -\frac{1}{2} \sqrt{-g_{(D)}} g_{(D)MN} \delta g_{(D)}^{MN}$, which leads to

$$\delta I_D = \frac{M_D^{D-2}}{2} \int_{M_D} d^D x \delta g_{(D)}^{MN} \sqrt{-g_{(D)}} \left(\frac{\delta (\phi^{4-D} R^{5-D})}{\delta g_{(D)}^{MN}} - \frac{g_{(D)MN}}{2} \phi^{4-D} R^{5-D} - \frac{T_{MN}^{(D)}}{M_D^{D-2}} \right) \tag{41}$$

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T_{MN} is the energy-momentum tensor of matter (dark matter and ordinary matter) given by $T^{(D)}_{MN} = \frac{-2}{\sqrt{-g_{(D)}}} \frac{\delta\left(\sqrt{-g_{(D)}} L_m[g^{\mu\nu}, \Psi]\right)}{\delta g^{MN}_{(D)}}$, $(M, N = 0, 1, 2, ..., D)$. The energy-momentum tensor is defined by the variation of the lagrangian of matter $L_m[g^{\mu\nu}, \Psi]$ in accordance to the metric $g_{(D)MN}$ on the bulk $\mathbf{M}_D$. Ψ is the matter field on 3-brane. Therefore, we obtain the field equations:

$$\frac{\delta\left(\phi^{4-D} R^{5-D}\right)}{\delta g^{MN}_{(D)}} - \frac{1}{2} g_{(d)MN} \phi^{4-D} R^{5-D} = \frac{1}{M_D^{D-2}} T^{(d)}_{MN} \tag{42}$$

we rewrite the field equations in $D = 4$ we obtain the equation of motion of the field R: $R_{MN} - \frac{1}{2} g_{MN} R = \frac{1}{M_p^2} T_{MN}$, this equation describes Einstein's gravity. In $D = 5$ we obtain the equation of motion of the field ϕ: $\frac{\delta\phi}{\delta g^{MN}_{(5)}} + \frac{1}{2} g_{(5)MN} \phi = -\frac{1}{M_5^3} \phi^2 T_{MN}$, this equation describes dark matter and dark energy [10]. We notice, from these two field equations, we can represent the gravity in 3-brane by two dual field equations.

4.2. Vacuum energy density

In this section we consider a approach to to resolve the field equations. Eq.(42) is written as

$$(4-D) \frac{g^{MN}_{(D)} \delta\phi}{\phi \delta g^{MN}_{(D)}} + (5-D) \frac{g^{MN}_{(D)} \delta R}{R \delta g^{MN}_{(D)}} = \frac{g^{MN}_{(D)} T^{(D)}_{MN}}{\phi^{4-D} R^{5-D} M_D^{D-2}} + \frac{D}{2} \tag{43}$$

We define a metric of a flat space, that is written as a direct sum of Friedmann-Lemaître-Robertson-Walker (FLRW) metric and the fifth dimension component given by the metric determinant at $\mathbf{M}_D$ in Eq.(37), We begin with the parametrization of D-dimensional metric

$$ds^2_{(5)} = -dt^2 + a^2(t) d\mathbf{x}^2 + \sum_{k=5}^{D-4} e^{-\phi R} dy_k^2 \tag{44}$$

$a(t)$ is a scale factor. This metric describes the homogeneous Universe in large structure. We then multiply $g^{MN}_{(5)}$ to T_{MN}, one can obtain

$$T_D \equiv g^{MN}_{(D)} T^{(D)}_{MN} = g^{\mu\nu} T_{\mu\nu} + \sum_{K=5}^{D} g^{KK}_{(D)} T_{KK} \tag{45}$$

with $g^{KK}_{(D)} = e^{-2\phi R}$. In our previous paper [10], we have described the density of dark matter ρ_{DM} as the component T_{55} of the energy-momentum tensor of matter in 5th dimension ($\rho_5 = T_{55}$). The 3-brane contains the field R and the 5D bulk contains the field ϕ. The dimensions $D \neq (4, 5)$ doesn't have a proper fields, are a mixture (coupling) between ϕ and R (37). Therefore, the mass density of champ ϕ is equivalent with T_{55} in 5th dimension. Since the mass of ϕ represents dark matter, then, we can describe ρ_5 only in in the 5-dimensional bulk M_5 by the component T_{55}. Eq.(45) can be expressed as

$$T_D = -\rho_m + e^{-2\phi R} \rho_5 + e^{-2\phi R} \sum_{K=6}^{D} \rho_{DK} \tag{46}$$

with $g^{\mu\nu} T_{\mu\nu} \equiv -\rho_m$ and $\rho_{DK} = g^{KK}_{(D)} T_{KK}$ are the coupling densities for $K = (6, ..., D)$. If $D \preceq 5$ we will have $\rho_{DK} = 0$. Eq.(46) can help us to test the number of dimensions D in the Universe. Next, we will define a series of densities: $\rho_D = \sum_{K=5}^{D} \rho_{DK}$. If there is a difference between the densities ρ_5 and ρ_D, hence, we can test some fluctuations in the density of dark matter by observation data. Additionally, the density ρ_D can be represented by several hypotheses: If the dimension of the bulk $\mathbf{M}_D$ increases with time, we can say that ρ_D represents the dark energy density. But it is difficult to accept the

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idea of the evolution of the dimensions of the Universe. If we assume that there are only 5D large extra dimensions, and dimensions $(D-5)$ are extra dimensions in 3-brane, like kaluza klein compactification. We propose in this case, that ρ_D represents the vacuum energy density [25]. The idea of ρ_D is the vacuum energy density, which seems a little logical. In that case, therefore, density represents hidden dimensions in the Universe. We can solve the equation (43), one can obtain

$$T_D = D\left(\frac{C}{2}\frac{M_D^{D-2}}{\sqrt{-g_{(D)}}} - f_D(R,\phi)\right) \tag{47}$$

with C is an integration constant.

This result shows us that the density of matter always depends on the number of dimensions D of $\mathbf{M}_D$. Therefore, the distribution of matter in 3-brane depends on D. Additionally, we notice that the function $f_D(R,\phi)$ represents the energy density. In the vacuum of the bulk M_D, Eq.(47) become $f_D(R,\phi) = \frac{C}{2}\frac{M_D^{D-2}}{\sqrt{-g_{(D)}}}$.

5. Gauss-Bonnet gravity form $f(R,\phi)$ gravity

In recent developments of Einstein-Gauss-Bonnet (EGB) gravity has obviously played crucial roles [26]. offers a a new 4-dimensional gravitational theory with only two dynamical degrees of freedom by taking the $D \longrightarrow 4$ limit of the Einstein-Gauss-Bonnet gravity in $D \succ 4$ dimensions [28]. which is in contradiction with Lovelock theorem, As is well-known, the Gauss-Bonnet (GB) term in 4 dimensions is a total derivative and thus does not contribute to the equations of motion. An intriguing idea of [26] is to multiply the GB term by the factor $1/(D-4)$ before taking the limit. It was shown that, at the level of equations of motion under a concrete ansatz of the metric, the divergent factor $1/(D-4)$is canceled by the vanishing GB contributions yielding finite nontrivial effects. Despite the sin-gular limit, it was conjectured that the $D \longrightarrow 4$ limit should have only two dofs, based on the fact that the number of dofs of the D-dimensional EGB gravity is $D(D-3)/2$.

In what follows is to study the densities ρ_5 and ρ_D within the framework of Gauss-Bonnet gravity. Eq.(46) can be expressed in the vacuum of 3-brane as

$$f_D(R,\phi) = \frac{1}{D(D-4)}\partial_\phi\left[(\rho_{DM}+\rho_D)e^{-2\phi R}\right] \tag{48}$$

The term $1/(D-4)$ has already been proposed by [26], they multiplied the GB term by $1/(D-4)$ to describe the the Einstein-Gauss-Bonnet in the limit $D \longrightarrow 4$. To create an equivalent $f_D(R,\phi)$ model with EGB gravity we take

$$\alpha = \frac{1}{D} \tag{49}$$

where α is a finite non-vanishing dimensionless GaussBonnet coupling. In the case where $D=4$, the GaussBonnet coupling is $\alpha = 1/4$. This value of the GaussBonnet coupling agrees with the results of other models [27]. From this result we can create a passage between $f(R)$ gravity and Gauss-Bonnet gravity. The densities ρ_5 and ρ_D do not depend on the field ϕ. Therefore, the action (3) becomes

$$I_D = \int_{M_D} d^D x\sqrt{-g_{(D)}}\frac{-2\alpha R}{(D-4)}e^{-2\phi R}(\rho_5+\rho_D) + I_m \tag{50}$$

showing in this case the properties similar to EGB gravity; this expression is equivalent to the scalar field ϕ coupled with GB gravity [29], by the change of Gauss-Bonnet coupling $\alpha \longrightarrow \frac{\alpha}{(D-4)}$ of EGB gravity [26,28] in $D=4$. We choose a maximally Symmetric Space-time $R \sim D(D-1) = 1/\phi_0$ [29]. To past from $f(R)$ gravity to GB gravity, we take $G_\phi \longleftrightarrow -G$, where

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$$G = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \text{ and } G_\phi = -2Re^{-2\phi R}(\rho_5 + \rho_D) \tag{51}$$

G_ϕ equivalent to the form proposed by [10], but with an additional factor. We take a maximally Symmetric Space-time where $R^2 = \frac{4D^2}{(D-2)^2}\Lambda^2$, $R_{\mu\nu}R^{\mu\nu} = \frac{4D}{(D-2)^2}\Lambda^2$ and $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = \frac{8D^2}{(D-1)^2(D-2)^2}\Lambda^2$, where Λ is the cosmological constant, one can obtain

$$G = \frac{2R}{(D-2)}\left[\frac{2D\Lambda}{(D-1)^2} + (D-4)\Lambda\right] \tag{52}$$

we compare the last value of G with the value of G_ϕ, we get

$$e^{2\phi R} = (D-2) \text{ and } \rho_5 + \rho_D = \frac{2D\Lambda}{(D-1)^2} + (D-4)\Lambda \tag{53}$$

we can see from Eq.(53), the passage between $f(R)$ gravity and GB gravity related the densities $\rho_5 + \rho_D$ with the cosmological constant i.e. the bond between $f(R)$ gravity and GB gravity implies that ρ_5 is not absolutely a density of dark matter since it depends on Λ, since Λ represents the density of dark energy. For $D = 4$ one of these two densities will be zero. From Eq.(46) we know that ρ_D is a slice density with $D \succeq 5$. This shows that

$$\rho_5 = \frac{2D\Lambda}{(D-1)^2} \text{ and } \rho_D = (D-4)\Lambda \tag{54}$$

consequently, the relationship between $f(R)$ gravity and GB gravity implies that the density ρ_5 is not the density of the dark matter but it is the density of the dark energy. The density ρ_D zero in our 3-brane but on the bulk this density will take place. For $D = 4$ we can find the relation between the scalar field and the Ricci scalar $\phi \approx 0,35R^{-1}$. Then, we consider $\rho_{vac} \approx 10^{74}Gev^2$ is the quantum vacuum density found by the quantum mechanics. And $\rho_\Lambda \approx 10^{-47}Gev^2$ is the vacuum density found by the ΛCDM model. If we consider $\rho_{vac} \equiv \rho_5$, in this case, it is impossible to determine the number of dimensions in the bulk, but if we choose $\rho_{vac} \equiv \rho_D$, in this case, we can find a well-determined value of the number of bulk dimensions

$$D = 10^{121} + 4 \tag{55}$$

according to this value, the 4-dimensional space-time is immersed in a $(10^{121} + 4)$-dimonsional bulk. This number of strange dimensions, comes the attempt to connect $f(R)$ gravity with GB gravity.

6. Conclusion

In summary, we have studied dark matter and dark energy in the context of $f(\phi)$ gravity. We have also found that the DE in 5d $f(\phi)$ gravity is equivalent to the model of holographic dark energy. This study indicates that the scalar field ϕ mass (DM) transforms into DE. We also show that $\Omega_m \approx 0,05018$; $\Omega_{DM} \approx 0,2$; $\Omega_{DE} \approx 0,74625$, this result is in excellent agreement with observation data. The main result of this paper is that the transition rotation ϕR between in 4d and 5d creates a transition from $f_{DE}(\phi)$ gravity in 5th dimension to $f(\phi)$ gravity in 5d. That solves the problem of the transition of the ω_{DE}. We investigated the generalized formalism of D-dimensional braneworld modified gravity. We have obtained new field equations that generalize the Einstein field equations. We have discussed the geometrical properties of solutions of these equations. Then, we studied the relation between an extra dimension in 3-brane with the vacuum energy density. Finally, we have shown that the energy density of matter depends directly on the number of dimensions D. We compared our model with EGB gravity, we found in this context a value Gauss-Bonnet coupling which is in agreement with the results of the other works.

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