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Article

On Some Unification Theorems: Yang-Baxter Systems; Johnson-Tzitzeica Theorem

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Abstract: The Yang–Baxter equation emerged from a theoretical physics article written by the Nobel laureate C.N. Yang and from statistical mechanics work of R.J. Baxter. Attempts to unify the algebra structures and the coalgebra structures have eventually led to new solutions of the Yang–Baxter equation. But, in order to capture the information encapsulated in modules over algebras and comodules over coalgebras, we will need some kind of extensions of the Yang–Baxter equation, called Yang–Baxter systems. Also, the Tzitzeica–Johnson theorem is generalized in the current paper.

Keywords: Yang–Baxter equations; Yang–Baxter systems; associative algebras; modules

MSC: 16T05; 16T25; 53C05; 51N20; 01A60; 00A05 (Primary); 17C05; 17C50 (Secondary)

1. Introduction

The Yang–Baxter equation emerged from a theoretical physics article written by the Nobel laureate C.N. Yang [1] and from statistical mechanics work of R.J. Baxter. [2,3]. In Knot Theory, invariants of links can be obtained from “enhanced” Yang–Baxter operators [5]. For example, the Jones polynomial [6], the HOMFLY polynomial [16,33] and the Kauffman polynomial [19], are produced from “enhanced” Yang–Baxter operators.

Attempts to unify the algebra structures and the coalgebra structures have eventually led to new solutions of the Yang–Baxter equation. The only invariant which can be obtained from these Yang–Baxter operators is the Alexander polynomial of knots [23]. In order to capture the information encapsulated in modules over algebras and comodules over coalgebras, we will need some kind of extensions of the Yang–Baxter equation, called Yang–Baxter systems.

Also, the Tzitzeica–Johnson theorem is generalized in the current paper. Several conclusions and explanations are given at the end of the paper.

2. Yang–Baxter Equations and Yang–Baxter Systems

As usually, we work over a field k , and our tensor products are defined over k . Let $I = I_V : V \rightarrow V$ be the identity map of the k -space V . For $R : V \otimes V \rightarrow V \otimes V$ a k -linear map, let $R^{12} = R \otimes I$, $R^{23} = I \otimes R : V \otimes V \otimes V \rightarrow V \otimes V \otimes V$.

Let R^{13} be the linear map acting on the first and third component of $V \otimes V \otimes V$.

Definition 2.1. A **Yang–Baxter operator** is an invertible k -linear map $R : V \otimes V \rightarrow V \otimes V$, which satisfies the braid condition (the Yang–Baxter equation):

$$R^{12} \circ R^{23} \circ R^{12} = R^{23} \circ R^{12} \circ R^{23}. \quad (1)$$

Remark 2.2. The quantum Yang–Baxter equation is the following:

$$R^{12} \circ R^{13} \circ R^{23} = R^{23} \circ R^{13} \circ R^{12}. \quad (2)$$

Remark 2.3. It is well-known that (1) and (2) are equivalent.

Remark 2.4. If A is a k -algebra, then for all non-zero $r, s \in k$, the linear map

$$R_{r,s}^A : A \otimes A \rightarrow A \otimes A, \quad a \otimes b \mapsto sab \otimes 1 + r1 \otimes ab - sa \otimes b \quad (3)$$

is a Yang-Baxter operator [15].

Definition 2.5. Let V, V', V'' be k -spaces.

Let $R : V \otimes V' \rightarrow V \otimes V'$, $S : V \otimes V'' \rightarrow V \otimes V''$ and $T : V' \otimes V'' \rightarrow V' \otimes V''$ be k -maps. A Yang-Baxter commutator is a map $[R, S, T] : V \otimes V' \otimes V'' \rightarrow V \otimes V' \otimes V''$, defined by

$$[R, S, T] = R_{12} \circ S_{13} \circ T_{23} - T_{23} \circ S_{13} \circ R_{12}. \quad (4)$$

There are several types of Yang-Baxter systems, and in this section we consider the following example and its implications.

Definition 2.6. For V and V' k -spaces, we consider the following maps

$$W : V \otimes V \rightarrow V \otimes V, \quad Z : V' \otimes V' \rightarrow V' \otimes V', \quad X : V \otimes V' \rightarrow V \otimes V'$$

A WXZ-system verifies the following equations:

$$[W, W, W] = 0, \quad (5)$$

$$[Z, Z, Z] = 0, \quad (6)$$

$$[W, X, X] = 0, \quad (7)$$

$$[X, X, Z] = 0. \quad (8)$$

There are many applications of WXZ-systems. They can be used to construct dually-paired bialgebras of the FRT type, thus leading to quantum doubles.

Given a WXZ-system as in Definition 2.6 one can construct a Yang-Baxter operator on $V \oplus V'$, provided the map X is invertible. This is a special case of a *gluing procedure* described in ([22] [Theorem 2.7]) (cf. [22] [Example 2.11]).

Entwining structures were introduced in order to recapture the symmetry structure of non-commutative (coalgebra) principal bundles or coalgebra-Galois extensions.

Theorem 2.7 ([14]). Let A be an algebra and let C be a coalgebra. For any $s, r, t, p \in k$ define linear maps

$$W : A \otimes A \rightarrow A \otimes A, \quad a \otimes b \mapsto sba \otimes 1 + r1 \otimes ba - sb \otimes a,$$

$$Z : C \otimes C \rightarrow C \otimes C, \quad c \otimes d \mapsto t\epsilon(c) \sum d_{(1)} \otimes d_{(2)} + p\epsilon(d) \sum c_{(1)} \otimes c_{(2)} - pd \otimes c.$$

Let $X : A \otimes C \rightarrow A \otimes C$ be a linear map such that $X \circ (\iota \otimes Id_C) = \iota \otimes Id_C$ and $(Id_A \otimes \epsilon) \circ X = Id_A \otimes \epsilon$. Then W, X, Z is a Yang-Baxter system if and only if A is entwined with C by the map $\psi := X \circ \tau_{C,A}$.

3. Unifications Using Yang-Baxter Systems

In this section we propose another kind of Yang-Baxter system.

We unify the following theorems:

- (i) For an A -bimodule M , $A \times M$ becomes an algebra.
- (ii) For a C -bicomodule N , $C \times N$ becomes a coalgebra.

Theorem 3.1. Let V and V' be vector spaces, and

$$W : V \otimes V \rightarrow V \otimes V, \quad X : V \otimes V' \rightarrow V \otimes V', \quad Y : V' \otimes V \rightarrow V' \otimes V$$

such that the following equations are satisfied:

$$[W, W, W] = 0, \quad (9)$$

$$[W, X, X] = 0, \quad (10)$$

$$[Y, Y, W] = 0, \quad (11)$$

$$[X, W, Y] = 0. \quad (12)$$

Then the linear map $R : (V \oplus V') \otimes (V \oplus V') \rightarrow (V \oplus V') \otimes (V \oplus V')$, given by $R|_{V \otimes V} = W$, $R|_{V \otimes V'} = X$, $R|_{V' \otimes V} = Y$, and $R|_{V' \otimes V'} = 0$, has the property $[R, R, R] = 0$.

Proof. The proof is based on observations on the decomposition of $(V \oplus V')^{\otimes 3}$ into direct summands of tensor products. $\square$

Remark 3.2. If A is an associative algebra, $W : A \otimes A \rightarrow A \otimes A$, $a \otimes b \mapsto ab \otimes 1 + 1 \otimes ab - b \otimes a$, M an A -bimodule, $X : A \otimes M \rightarrow A \otimes M$, $a \otimes m \mapsto 1 \otimes am$, and $Y : M \otimes A \rightarrow M \otimes A$, $n \otimes b \mapsto nb \otimes 1$, then we have the conditions of the above theorem fulfilled.

Remark 3.3. Let C be a coalgebra, $W : C \otimes C \rightarrow C \otimes C$, $c \otimes d \mapsto c_1 \otimes c_2 \varepsilon(d) + \varepsilon(c) d_1 \otimes d_2 - d \otimes c$, N a C -bicomodule, $X : C \otimes N \rightarrow C \otimes N$, $c \otimes n \mapsto \varepsilon(c) n_{-1} \otimes n_0$, and $Y : N \otimes C \rightarrow N \otimes C$, $n \otimes d \mapsto n_0 \otimes n_1 \varepsilon(d)$, then we have the conditions of the above theorem fulfilled.

4. On the Tzitzeica-Johnson Theorem

Let us start with the Tzitzeica-Johnson's problem (see, for example, [24]).

We consider three circles of radius r , the intersection points of pairs of circles are denoted by A , B , C , and the common point of intersection of the three circles is denoted by O (see the Figure 1 below).

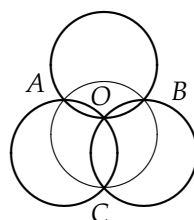


Figure 1. The Tzitzeica-Johnson's theorem (Tzitzeica - 1908, Johnson - 1916). Under the above assumptions, there exists a circle with radius r , passing through the points A , B and C .

We now propose a new theorem. We now start with one circle of radius r (see the Figure 2).



Figure 2. We start with a circle of radius r .

Related to our initial circle, we consider 3 tangent circles of radius R .

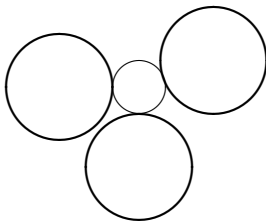


Figure 3. Three circles of radius R , and a common tangent circle of radius r .

We now consider three circles tangent to the three circles of radius R :

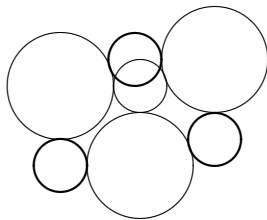


Figure 4. Three circles of radius R , and a other tangent circles of radius r .

At this moment, we can present our **Theorem**: under the above assumptions, there exists a circle of radius R , tangent to the three bold circles.

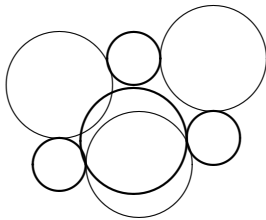


Figure 5. Conclusion: There exists a fourth circle of radius R , which is tangent to the three exterior circles of radius r .

The proof of our theorem is based on reducing the above figure to the Tzitzeica-Johnson's theorem for circles of radius $r + R$.

5. Conclusions and Further Comments

Some of the results of this paper were presented at the 14-th International Workshop on Differential Geometry and Its Applications, hosted by the Petroleum Gas University from Ploiesti, between July 9-th and July 11-th, 2019. Several talks given on that occasion were dedicated to the memory of St. Papadima (see [12,26,32,34]). It was during DGA 14, when we observed that some results from [32] can be extended for the virtual braid groups from [12].

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