Article

Scalar Field Kantowski-Sachs Solutions in Teleparallel $f(t)$ Gravity

Alexandre Landry *

Posted Date: 17 December 2024

doi: 10.20944/preprints202412.1308.v1

Keywords: Teleparallel Gravity; Field Equations; Kantowski-Sachs spacetimes; Scalar Field Source solutions; Frame-based Approach; Time-dependent Spacetimes

Article

Scalar Field Kantowski-Sachs Solutions in Teleparallel $F(T)$ Gravity

A. Landry

Department of Mathematics and Statistics, Dalhousie University, Halifax, Nova Scotia, Canada, B3H 3J5;
a.landry@dal.ca

Abstract: In this paper we investigate time-dependent Kantowski-Sachs spherically symmetric teleparallel $F(T)$ gravity with a scalar field source. We begin by setting the exact field equations to be solved and solve conservation laws for possible scalar field potential $V(\phi)$ solutions. Then we proceed to find new non-trivial teleparallel $F(T)$ solutions by using power-law and exponential ansatz for each potential cases arising from conservation laws such as linear, quadratic, logarithmic to name a few. We find a general formula allowing to compute all possible new teleparallel $F(T)$ solutions applicable for any scalar field potential and ansatz. Then we apply this formula and find a large number of exact and approximate new teleparallel $F(T)$ solutions for several types of cases. Some new $F(T)$ solution classes may be relevant for future cosmological applications, especially concerning the dark matter, the dark energy quintessence, phantom energy leading to the Big Rip event and quintom models of physical processes.

Keywords: teleparallel gravity; field equations; kantowski-sachs spacetimes; scalar field source solutions; frame-based approach; time-dependent spacetimes

---

1. Introduction

The teleparallel $F(T)$-type theory of gravity is a very promising alternative theory to General Relativity (GR) [1–3]. This theory is characterized by spacetime torsion-dependent geometry, which is depending on the coframe $\mathbf{h}^a$ (and its derivatives) and the spin-connection $\omega^a{}_{bc}$. The teleparallel gravity in general uses a frame basis approach instead of a metric tensor. The role of symmetry is not as in pseudo-Riemannian geometry, where symmetry depends on the existence of Killing Vectors (KVs). The Riemannian geometry in GR is fully defined in terms of metric and spacetime curvature, but it is different for the teleparallel $F(T)$-type gravity.

The frame based approach development for finding the spacetime symmetries has been explored [4–6]. A complication may arise from the possible non-trivial linear isotropy group: a Lorentz frame transformations Lie group will leave the associated tensors of the geometry invariant. A new approach was introduced for symmetry determination of any geometry based on an independent frame and spin-connection with the torsion and curvature tensors as geometric objects [7]. In this situation, the spin-connection is an independent object and all geometry with a zero curvature and a zero non-metricity tensors is defined as a *teleparallel geometry* and the approach is based on a particular class of invariantly defined symmetry frames. For a non-trivial linear isotropy group, the symmetries are defined by a set of inhomogeneous differential equations (DEs) for coframes and spin-connection in an orthonormal gauge $g_{ab} = diag[-1,1,1,1]$. In the literature, the spin-connection $\omega^a{}_{bc}$ will be defined in terms of an arbitrary Lorentz transformation $\Lambda^a{}_b$ and obtained from the zero curvature requirement (see refs.[7,8] and references within). The fundamental requirements of a symmetry are [9]:

$$\mathcal{L}_{\mathbf{X}}\mathbf{h}^a = \lambda^a{}_b\mathbf{h}^b, \tag{1a}$$

$$\mathcal{L}_{\mathbf{X}}\omega^a{}_{bc} = 0, \tag{1b}$$

where $\mathbf{h}^a$ is the orthonormal coframe basis, $\mathcal{L}_{\mathbf{X}}$ is the Lie derivative with respect to the KV $\mathbf{X}$, $\lambda^a{}_b$ is an invariant Lie algebra generator of Lorentz transformations $\Lambda^a{}_b$. For a pure teleparallel geometry, the zero Riemann curvature criteria needs to be satisfied [8]:

$$\begin{aligned} R^a{}_{b\mu\nu} =& \partial_\mu \omega^a{}_{b\nu} - \partial_\nu \omega^a{}_{b\mu} + \omega^a{}_{e\mu}\omega^e{}_{b\nu} - \omega^a{}_{e\nu}\omega^e{}_{b\mu} = 0, \\ \Rightarrow & \quad \omega^a{}_{b\mu} = \Lambda^a{}_c \partial_\mu \Lambda_b{}^c. \end{aligned} \tag{2}$$

The eqn (2) solution yields to the teleparallel spin-connection defined in terms of Lorentz transformation $\Lambda^a{}_b$.

A well-known subclass of teleparallel theory equivalent to GR is the Teleparallel Equivalent to General Relativity (TEGR), based on a torsion scalar T constructed from the torsion tensor [1]. The most common generalization of TEGR is the teleparallel $F(T)$-type gravity, where F is a function of the torsion scalar T [10–12]. By the covariant approach to $F(T)$-type gravity, the teleparallel geometry is gauge invariant by imposing the zero curvature and zero non-metricity spin-connection conditions. This same fundamental quantity is zero for all proper frames and non-zero for all other frames [1,3,13]. The resulting teleparallel gravity theory is locally invariant with covariant field equations (FEs) under Lorentz definition [14]. A proper frame is not invariantly defined in terms of the spin-connection (a non-tensorial quantity) and some problems may potentially arise by using this type of frame to determine symmetries. However, some considerations have also been developed, used and/or adapted for theories close to teleparallel $F(T)$ gravity. We can think of the New General Relativity (NGR) where the action integral is essentially described by a linear combination of three terms corresponding to the three irreducible torsion tensor component individual contributions (see refs. [15–17] and references within). Some previous elements were also considered and used for the symmetric teleparallel $F(Q)$-type gravity, a theory in development with some potential (see refs. [18–21] and references within). Or again, some of these elements are useful for improving the study of the geometrical trinity of gravity and especially intermediate theories like $F(T,Q)$-type, $F(R,Q)$-type, $F(R,T)$-type and others (see refs. [22–28] and references within). So, there are a non-exhaustive number of possible approaches, but perhaps the best approach is to go first with the teleparallel $F(T)$-type gravity for the current developments.

There are a very large number of papers in the literature on spherically symmetric spacetimes and solutions in teleparallel $F(T)$ gravity [29–45]. The main features can be summarized by power-law $F(T)$ solutions obtained from power-law ansatz frame components (see [29–32] and references within). Most of these papers use the Weitzenback gauge (proper frames trivially satisfy the antisymmetric FEs), but extra degrees of freedom (DoF) problem may arise by imposing the null spin-connection in a such case. This specific condition applies to only symmetric parts of FEs and the presented $F(T)$ solutions are usually limited to power-law for a big coframe expression. Therefore, this type of problem may result to the potential scenario of fully valid $F(T)$ solutions satisfying the FEs on a proper frame, but becoming invalid and no longer satisfying the FEs when we perform a frame changing from proper to non-proper frames. These symmetric FEs and the solutions are similar between the different gauges, however a frame changing is necessary for solving the possible extra DoF issue. When a non-proper frame is used, the non-zero spin-connections are solutions to the non-trivial antisymmetric parts of the FEs, and then all DoFs are completely covered by the FEs. This method is used a number of recent papers on Teleparallel spherically symmetric geometries and some possible class of solutions [33,46–48]. The general FEs are defined in an orthonormal gauge assuming a diagonal frame and a non-trivial spin-connection, leading to well-defined non-trivial antisymmetric and symmetric parts of FEs without extra DoF. Some of these papers studied the Kantowski-Sachs (KS) geometry case and found the vacuum and perfect fluids $F(T)$ solutions by using the power-law and exponential ansatz [33,47,48]. Therefore, we need to find additional classes of solutions for the KS spacetime, especially concerning the scalar field sources necessary for the study of the dark matter, dark energy quintessence, phantom energy and quintom cosmological models. The KS spacetime fourth symmetry is defined by the radial-coordinate derivative ∂_r leading to time-coordinate dependence for coframes, spin-connections and FEs [33,47,48].

In the literature, there are some works on KS spacetimes and solutions in GR and some specific $f(\overset{\circ}{R})$-type theory of gravity [49–51]. These papers deal with some $F(\overset{\circ}{R})$ solutions and proceed to a detailed and dynamical study concerning critical points, limits on physical quantities, asymptotes and also the evolution of curvature to only name these features. There are several other papers on more elaborated KS spacetime models, but they are not made in the framework of teleparallel gravity and they are essentially focusing on $F(\overset{\circ}{R})$-type gravity. For KS teleparallel $F(T)$ theory, there are a small number of recent works [52,53]. There are recently some generalizations of KS models and solutions for $F(T,B)$ gravity, the LRS Bianchi III Universe and even the $F(T,R)$-type and teleparallel $F(Q)$-type gravity. However, these papers focus essentially on solutions and possible equilibrium points of these cosmological systems [54–59]. Therefore, their approaches do not really provide new classes of teleparallel $F(T)$-type solutions. All of these works imply that KS spacetimes geometries are relevant and made for more complete and precise cosmological solutions. Recently, there are some works on teleparallel KS spacetimes from Paliathanasis which may also lead to scalar field and quantized cosmological solutions [60,61]. However, quantization will not be considered in the current paper, because we only aim to find $F(T)$ solutions for KS spacetimes with a non-quantized scalar field source.

A first physical motivation for this type of study concerns the dark energy quintessence models [62–66]. This new type of models was first developed by Paul J. Steinhardt et al. in the late 1990s [62–64]. There have been various phenomenological studies performed via the GR frameworks in order to obtain a model explaining the accelerating universe expansion (see refs. [62–69] and several other ones). In these models, a perfect linear fluid of negative pressure is assumed ($P = \alpha_Q\,\rho$ where $-1 < \alpha_Q < -\frac{1}{3}$ models), supported and induced by a some scalar field. From this starting point, there have been various dynamical and other studies on the possibilities of scalar fields and possible potentials that most faithfully describe the dark energy quintessence process and thus the accelerating universe expansion [66,67]. Even recently, there are increasingly advanced studies on this very important subject in cosmology and specifically for observational constraints [68,69]. However, several years before first Paul Steinhardt's works on quintessence processes, there had been attempts at models of accelerated universe expansion through the use of scalar field-based models and their symmetries explaining accelerating universe expansion [70,71]. The same precursor models therefore allowed all of P. Steinhardt's work on the dark energy quintessence, generalizations and other subsequent studies to emerge and arrive at the current models.

Among the set of recent models are especially the teleparallel gravity based dark energy quintessence models and the extensions ($F(T)$-type, $F(T,B)$-type, scalar-torsion based in particular [72–81]. However, these papers have mainly concentrated on dynamics and especially stability studies on FEs using simple and predefined $F(T)$ and $F(T,B)$ functions as superpositions of power-law and/or logarithm terms. Even if these latter types of functions are often used in teleparallel cosmology, this somewhat simplistic approach does not truly and fundamentally solve the FEs of teleparallel $F(T)$ gravity and/or its extensions. These studies mainly only study primarily the impacts on some physical parameters of teleparallel gravity. At the same time, this type of approach makes the teleparallel $F(T)$ exact and correct solutions of FEs that can really model the dark energy quintessence processes go under the radar. There is therefore room to go further by really solving the teleparallel $F(T)$ gravity FEs for scalar field sources, just as was the case in refs. [47,48] for teleparallel $F(T)$ solutions for perfect fluids sources. Most recently, there have been detailed studies of Teleparallel Robertson-Walker (TRW) $F(T)$ and $F(T,B)$ solutions with perfect fluids and scalar field sources allowing a larger number of possible solutions for teleparallel dark energy quintessence models [82,83]. In this manner, we will find the most fundamental solutions for teleparallel KS spacetimes allowing to really study the best and fundamental possible models explaining the dark energy quintessence by using the frameworks and FEs of teleparallel $F(T)$ gravity.

In addition, the first and primarily dark energy quintessence model studies also had led to the scenarios of cosmological models involving strong negative pressure dark energy perfect fluids (where $\alpha_Q < -1$) [84,85]. This is also a new physical motivation for studying more extreme scenarios of strong and fast accelerating universe expansion such as phantom (or negative) dark energy models [86–89]. Even in teleparallel gravity, some studies have also been considered in phantom dark energy models using the same approaches as for dark energy quintessence processes studies [75,76,90–92]. These works still here focus on dynamic and stability studies using very specific teleparallel $F(T)$ functions (and/or $F(T,B)$) to study the evolution of some specific physical parameters without really solving directly the FEs of teleparallel gravity. Obviously, there has been less work on this type of scenario, because it is more hypothetical. This type of model often involves nonlinear fluids, but the determining feature is mostly the energy conditions violation of such a cosmological system (i.e. $P + \rho \ngeq 0$ situation), which makes that hypothetical. The recent works in teleparallel $F(T)$ and $F(T,B)$ types gravity for TRW spacetimes provide a larger amount of materials and new solutions to further study the phantom energy models in addition to quintessence process [82,83]. There is also a way and reason here to go further by directly solving the teleparallel $F(T)$ gravity FEs with scalar field source, especially for the KS spacetimes.

Furthermore, the works just mentioned show the existence of at least three possible types of dark energy: quintessence, cosmological constant, phantom energy. There are also the quintom dark energy models defined as a mixture of quintessence dark energy ($-1 < \alpha_Q < -\frac{1}{3}$) and phantom dark energy ($\alpha_Q < -1$) having as intermediate limit the cosmological constant [93–97]. This type of hybrid model is often defined by spacetime geometry and two scalar fields: the first field can describe quintessence and the second phantom energy or one field for the unified process and the other for coupling between the two types of dark energy [93,94,97]. Some works specifically address this problem through the oscillation between the quintessence and phantom states [95]. However in teleparallel gravity, there have been very few works dealing with this type of more complex physical problem [98]. This is also an additional reason favoring the present approach to solve the teleparallel $F(T)$ gravity FEs with a scalar field source. The results arising from the current work will provide much more materials and tools for better developing the teleparallel quintom dark energy models in the future. There are also anisotropic cosmological models that can be added to the list of possibilities emerging from the present approach [99]. There is no lack of possibilities, but here we focus on the KS teleparallel $F(T)$ solutions with scalar fields to open up these possibilities for future development.

For this paper, we assume a time-coordinate dependent spherically symmetric teleparallel geometry, in particular Kantowski-Sachs teleparallel geometry, in an orthonormal gauge as defined and used in refs [33,47,48]. We will find $F(T)$ solutions for a power-law defined scalar field and then for an exponential scalar field source. After a summary of the teleparallel FEs and Kantowski-Sachs class of geometries in section 2, we will find in section 3 the power-law scalar field exact and approximated $F(T)$ solutions. We then repeat the exercise in sect 4 with an exponential scalar field source for exact and approximated $F(T)$ solutions. In both cases, we will use a power-law and an exponential coframe component ansatz for a better comparison between $F(T)$ solutions. This paper has also common features, aims and similar structure with some recent papers on perfect fluids teleparallel $F(T)$ solutions (see refs. [46–48] for details).

The notation is defined as: coordinate indices $\mu, \nu, \ldots$, tangent space indices $a, b, \ldots$ (see ref [7]), spacetime coordinates x^μ, frame fields $\mathbf{h}_a$ and its dual one-forms $\mathbf{h}^a$, vierbein $h_a^{\ \mu}$ or $h^a_{\ \mu}$, spacetime metric $g_{\mu\nu}$, Minkowski tangent space metric η_{ab}, spin-connection one-form $\omega^a_{\ b} = \omega^a_{\ bc}\mathbf{h}^c$, curvature tensor $R^a_{\ bcd}$, torsion tensor $T^a_{\ bc}$. The derivatives with respect to (w.r.t.) t is $F_t = F'$ (with a prime).

2. Summary of Teleparallel Gravity and Scalar field Kantowski-Sachs Field Equations

2.1. Teleparallel $F(T)$ Gravity Theory

The teleparallel $F(T)$-type gravity action integral is [1–3,33,46–48]:

$$S_{F(T)} = \int d^4\,x \left[\frac{h}{2\kappa}\,F(T) + \mathcal{L}_{Source}\right], \tag{3}$$

where h is the coframe determinant and κ is the coupling constant. We apply the least-action principle to the eqn (3), the symmetric and antisymmetric parts of FEs are [33,46–48]:

$$\begin{aligned}
\kappa\,\Theta_{(ab)} &= F_T\,(T)\,\overset{\circ}{G}_{ab} + F_{TT}\,(T)\;S_{(ab)}^{\ \ \ \mu}\,\partial_\mu T + \frac{g_{ab}}{2}\,\left[F\,(T) - T\,F_T\,(T)\right], && \text{(4a)}\\
0 &= F_{TT}\,(T)\;S_{[ab]}^{\ \ \ \mu}\,\partial_\mu T, && \text{(4b)}
\end{aligned}$$

with $\overset{\circ}{G}_{ab}$ the Einstein tensor, $\Theta_{(ab)}$ the energy-momentum, T the torsion scalar, g_{ab} the gauge metric, $S_{ab}^{\ \ \ \mu}$ the superpotential (torsion dependent) and κ the coupling constant. The canonical energy-momentum is obtained from $\mathcal{L}_{Matter}$ term of eqn (3) as:

$$\Theta_a^{\ \mu} = \frac{1}{h}\frac{\delta\mathcal{L}_{Source}}{\delta h^a_{\ \mu}}. \tag{5}$$

The eqn (5) antisymmetric and symmetric parts are [33]:

$$\Theta_{[ab]} = 0, \qquad \Theta_{(ab)} = T_{ab}, \tag{6}$$

where T_{ab} is the symmetric part of energy-momentum tensor. The eqn (6) is valid only when the matter field interacts with the metric $g_{\mu\nu}$ defined from the coframe $h^a_{\ \mu}$ and the gauge g_{ab}, and is not directly coupled to the $F(T)$ gravity. This consideration is valid when the hypermomentum is zero (i.e. $\mathfrak{T}^{\mu\nu} = 0$) as discussed in refs. [31,46–48]. The hypermomentum is defined from eqns (4a) and (4b) as [31]:

$$\mathfrak{T}_{ab} = \kappa\Theta_{ab} - F_T\,(T)\,\overset{\circ}{G}_{ab} - F_{TT}\,(T)\;S_{ab}^{\ \ \ \mu}\,\partial_\mu T - \frac{g_{ab}}{2}\,\left[F\,(T) - T\,F_T\,(T)\right]. \tag{7}$$

The conservation of energy-momentum in teleparallel gravity for $\mathfrak{T}^{\mu\nu} = 0$ case states that $\Theta_a^{\ \mu}$ satisfies the relation [1,2]:

$$\overset{\circ}{\nabla}_\nu\left(\Theta^{\mu\nu}\right) = 0, \tag{8}$$

with $\overset{\circ}{\nabla}_\nu$ the covariant derivative and $\Theta^{\mu\nu}$ the conserved energy-momentum tensor. This eqn (8) is also the GR conservation of energy-momentum expression. Satisfying the eqn (8) is automatically required because the $\mathfrak{T}^{\mu\nu} = 0$ condition (null hypermomentum condition). But eqn (8) also imposes the symmetry of $\Theta^{\mu\nu}$ and hence the condition stated by eqns (6). The general hypermomentum $\mathfrak{T}^{\mu\nu}$ definition for any teleparallel gravity theory is exactly [31,100–102]:

$$\mathfrak{T}_\sigma^{\ \mu\nu} = \frac{2}{h}\frac{\delta\,\left(h\,\mathcal{L}_{Source}\right)}{\delta\,\Gamma^\sigma_{\ \mu\nu}}, \tag{9}$$

where $\Gamma^{\sigma}_{\mu\nu}$ is the Weitzenbock connection. The non-zero hypermomentum (i.e. $\mathfrak{T}^{\mu\nu} \neq 0$) conservation law will be defined as [31,100–102]:

$$\overset{\circ}{\nabla}_{\nu}\left(h\,\mathfrak{T}_{\sigma}^{\;\mu\nu}\right) = 0. \tag{10}$$

This last expression is trivially satisfied for the null hypermomentum situation (i.e., $\mathfrak{T}^{\mu\nu} = 0$).

2.2. Teleparallel Kantowski-Sachs Geometry and Antisymmetric Field Equations

The orthonormal time-dependent Kantowski-Sachs resulting vierbein is [33,47,48]:

$$h^{a}_{\;\mu} = Diag\left[1,\, A_2(t),\, A_3(t),\, A_3(t)\sin(\theta)\right], \tag{11}$$

where we are able to choose new coordinate such that $A_1(t) = 1$ without any lost of generality. This will allow us to find cosmological-like solutions.

Another possible orthonormal coframe expression is by taking the choice $A_3(t) = t$ and then solve for $A_1(t)$ and $A_2(t)$ [48]:

$$h^{a}_{\;\mu} = Diag\left[A_1(t),\, A_2(t),\, t,\, t\,\sin(\theta)\right], \tag{12}$$

In refs. [33,47], we find that the pure vacuum KS coframe solutions implies that $A_3 = t$. This is another justification for the eqn (12) coframe expression. However, eqn (12) is not as practical and appropriate as eqn (11) for cosmological-like solutions.

The spin-connection ω_{abc} non-null components for time-dependent spacetimes are expressed by using eqn (11) coframe components as $\omega_{abc} = \omega_{abc}(\psi(t), \chi(t))$ where χ and ψ are arbitrary functions ($\chi' = \chi_t$ and $\psi' = \psi_t$) and $a \neq b$ indexes (see refs [33,47,48] for detailed discussion and calculation). From this consideration, the antisymmetric part of the $F(T)$ FEs are defined from eqn (11) coframe expression [33,47,48]:

$$\frac{F_{TT}(T)\,T'\cosh(\psi)\cos(\chi)}{A_3} = 0 \quad \text{and} \quad \frac{F_{TT}(T)\,T'\sinh(\psi)\sin(\chi)}{A_3} = 0. \tag{13}$$

The eqns (13) lead to $\psi = 0$ and $\chi = \frac{\pi}{2}$ (also $\frac{3\pi}{2}$ as second solution) for non-constant T situation. By using eqn (13) solutions, we find that the only non-zero ω_{abc} components are [33,47,48]:

$$\omega_{234} = -\omega_{243} = \delta, \quad \omega_{344} = -\frac{\cos(\theta)}{A_3\sin(\theta)}, \tag{14}$$

where $\delta = \pm 1$. Fundamentally, eqns (11) (or eqn (12)) and (14) are also solutions of eqns (1a) and (1b) by using the Cartan-Karlhede (CK) algorithm method and also the eqn (2) for zero curvature criteria (see refs. [7,8,33] for details and justifications).

2.3. Scalar Field Source Conservation Laws

For the scalar field source action integral, the $\mathcal{L}_{Source}$ term will be described by the Lagrangian density [2,73,74,78,79,82,83]:

$$\mathcal{L}_{Source} = \frac{h}{2}\overset{\circ}{\nabla}_{\nu}\phi\,\overset{\circ}{\nabla}^{\nu}\phi - h\,V(\phi)\,. \tag{15}$$

We use in eqn (16) the covariant derivative $\overset{\circ}{\nabla}_{\nu}$ for satisfying the GR conservation law defined by eqn (8).

For a time-dependent $\phi = \phi(t)$ scalar field source and a cosmological-like spacetime (i.e., $A_1 = 1$), eqn (15) will simplify as:

$$\mathcal{L}_{Source} = h\frac{\phi'^2}{2} - h\,V\left(\phi\right). \tag{16}$$

where $\phi' = \partial_t\,\phi$. By applying the least-action principle to the eqn (16), we find that the energy-momentum tensor T_{ab} is [46–48,104,105]:

$$T_{ab} = \left(P_\phi + \rho_\phi\right)\,u_a\,u_b + g_{ab}\,P_\phi, \tag{17}$$

where P_ϕ and ρ_ϕ are respectively the pressure and density equivalent for the scalar field and $u_a = (-1, 0, 0, 0)$. The definition of P_ϕ and ρ_ϕ are [82]:

$$P_\phi = \frac{\phi'^2}{2} - V\left(\phi\right) \quad \text{and} \quad \rho_\phi = \frac{\phi'^2}{2} + V\left(\phi\right), \tag{18}$$

where $\phi' = \phi_t$ and $V = V\left(\phi\right)$ is the scalar field potential. Then the conservation law for scalar field with density and pressure defined by eqns (18) for time-dependent spacetimes is [33,47,48]:

$$\phi'\,\left(\ln(A_2\,A_3^2)\right)' + \phi'' + \frac{dV}{d\phi} = 0, \tag{19}$$

The eqn (19) is the most general scalar field conservation law equation for any A_2 and A_3 ansatz components with any $\phi(t)$ scalar field expression. For coming steps, we will first solve the eqn (19) for possible potential to study in this investigation. Then we will solve the FEs to find the corresponding classes of teleparallel $F(T)$ solutions for each potential case satisfying the eqn (19). We will do these derivation steps for power-law and exponential scalar field $\phi(t)$ definitions.

To complete the discussion on physical implications, from eqns (18) and to make the link with a linear perfect fluids EoS equivalent $P_\phi = \alpha_Q\rho_\phi$, we need to find the quintessence coefficient index α_Q for making the parallel between ϕ and the quintessence dark energy process. This phenomena is physically another dark energy possible state explaining the universe accelerating expansion by using an associated scalar field ϕ, so-called the quintessence scalar field [62–64,69]. Usually, the α_Q index parameter will be $-1 < \alpha_Q < -\frac{1}{3}$ where $\alpha_Q = -1$ is described by the cosmological constant, the first and fundamental dark energy state. The α_Q index will be defined from eqns (18):

$$\alpha_Q = \frac{P_\phi}{\rho_\phi} = \frac{\phi'^2 - 2V\left(\phi\right)}{\phi'^2 + 2V\left(\phi\right)}. \tag{20}$$

By using this eqn (20), we can find the quintessence perfect fluid equivalent for every new teleparallel $F(T)$ solutions, any potential $V\left(\phi\right)$ and any ansatz. For phantom energy processes, it is required in this case that $\alpha_Q < -1$ and leads to an uncontrolled accelerating universe expansion going to the Big Rip at the end of physical process [86–88]. The same considerations will also be applicable for quintom physical processes [93–97]. This for better making the new teleparallel $F(T)$ solutions classification in terms of dark energy quintessence, phantom and quintom processes.

2.4. *Symmetric and Unified Field Equations*

The torsion scalar and the symmetric FEs components for $\chi = \frac{\pi}{2}$ ($\delta = +1$) case in eqns (13) and (14) are [33,47,48]:

$$T = 2\left(\ln(A_3)\right)'\left(\left(\ln(A_3)\right)' + 2\left(\ln(A_2)\right)'\right) - \frac{2}{A_3^2}, \tag{21a}$$

$$B' = -\left(\ln\left(A_2\,A_3^2\right)\right)' + \frac{\frac{1}{A_3^2} - \left(\ln\left(\frac{A_2}{A_3}\right)\right)''}{\left(\ln\left(\frac{A_2}{A_3}\right)\right)'}, \tag{21b}$$

$$\kappa\,\phi'^2 + 2\kappa\,V(\phi) = -F(T) + 2\left(T + \frac{2}{A_3^2}\right)F_T(T), \tag{21c}$$

$$-\kappa\phi'^2 = \left[\left(\ln(A_3)\right)'\left(B' - \left(\ln\left(\frac{A_2}{A_3}\right)\right)'\right) + \left(\ln(A_3)\right)''\right]F_T(T), \tag{21d}$$

where $F_T(T) \neq$ constant and $B' = \partial_t\left(\ln F_T(T)\right)$. Comparing with the version of Kantowski-Sachs $F(T)$-gravity FEs in the literature [52,53], the FEs are different. For $\delta = -1$ FEs set, there are some small minor differences for some terms in eqns (21d) to (21c), mainly some different signs at very specific terms. For the rest, the general form of the eqns (21a) to (21c) remains identical, regardless of δ. However, we can simplify these eqns (21a)–(21d) by adding eqns (21c) and (21d) and then by substituting eqns (21a) and (21b):

$$2\kappa\,V(\phi(T)) = -F(T) + \left[\frac{3}{2}\left(T + \frac{2}{A_3^2}\right) + \frac{\left(\ln(A_3)\right)'}{\left(\ln\left(\frac{A_2}{A_3}\right)\right)'}\left(\frac{1}{A_3^2} - \left(\ln\left(\frac{A_2}{A_3}\right)\right)''\right) + \left(\ln(A_3)\right)''\right]F_T(T). \tag{22}$$

The eqn (22) will be the DE to solve for $F(T)$. We will substitute the right ansatz and use again eqn (21a) as a characteristic equation as in ref [47]. In addition, we will have to calculate for each used ansatz the corresponding $V(\phi(T))$ by using the eqn (19) for conservation laws. The eqn (21a) characteristic equation will also be useful for finding $\phi(T)$ scalar field and then the $V(\phi(T))$ functions.

3. Power-Law Scalar Field Solutions

In this section, we will set a power-law scalar field $\phi(t) = p_0\,t^p$ where p_0 is a constant and p is a real power. The conservation law defined by eqn (19) becomes by field expression inversion as $t(\phi) = \left(\frac{\phi}{p_0}\right)^{1/p}$:

$$0 = p\,p_0^{1/p}\left(\ln(A_2\,A_3^2)\right)'\,\phi^{1-1/p} + p(p-1)\,p_0^{2/p}\,\phi^{1-2/p} + \frac{dV}{d\phi}, \tag{23}$$

where $\left(\ln(A_2\,A_3^2)\right)'$ is depending on ϕ and the used ansatz for A_2 and A_3 components. There are a number of possible ansatz:

1. **Power-law:** This is the most simple, used and well-known ansatz. It is defined as [47]:

$$A_2 = t^b \quad \text{and} \quad A_3 = c_0\,t^c, \tag{24}$$

where c_0 is a constant. Then we find that $\left(\ln(A_2\,A_3^2)\right)' = \frac{(b+2c)}{t(\phi)} = (b+2c)\,p_0^{1/p}\,\phi^{-1/p}$ and eqn (23) becomes:

$$0 = p\,p_0^{2/p}\,(b+2c+p-1)\,\phi^{1-2/p} + \frac{dV}{d\phi},$$

$$\Rightarrow\ V(\phi) = \phi_0 - \frac{p^2\,p_0^{2/p}}{2(p-1)}\,(b+2c+p-1)\,\phi^{2-2/p}. \tag{25}$$

The α_Q-index from eqn (20) is by substituting eqn (25):

$$\alpha_Q = -1 + \frac{p^2\, p_0^{2/p}\, \phi^{2-2/p}}{\phi_0 - \frac{p^2\, p_0^{2/p}}{2}\left(\frac{b+2c}{p-1}\right)\phi^{2-2/p}}. \tag{26}$$

There are special cases:

- $\mathbf{p \gg 1}$: Eqn (25) can be approximated as:

  $$V(\phi) \approx \phi_0 - \frac{p^2}{2}\,\phi^2. \tag{27}$$

  Then eqn (26) will be for this case:

  $$\alpha_Q \approx -1 + \frac{p^2\,\phi^2}{\phi_0 - \frac{p(b+2c)}{2}\,\phi^2} \approx -1 - \frac{2p}{b+2c}. \tag{28}$$

  where we have the physical situations:

  - $-\frac{1}{3} < \frac{\mathbf{p}}{\mathbf{b+2c}} < \mathbf{0}$: dark energy quintessence process. If $p > 0$, then $b + 2c < 0$.
  - $\frac{\mathbf{p}}{\mathbf{b+2c}} > \mathbf{0}$: phantom energy process. If $p > 0$, then $b + 2c > 0$.

  If $b + 2c$ is not large, eqn (28) will rather be as $\alpha_Q = -1 + \frac{p^2}{\phi_0}\,\phi^2$ without any constraint on $b + 2c$. This last result can be directly found by using eqn (27) and then the eqn (20) α_Q definition.
- $\mathbf{p = 1}$: Eqn (25) simplifies and then becomes:

  $$\begin{aligned} 0 =& p_0^2\,(b+2c)\,\phi^{-1} + \frac{dV}{d\phi}, \\ \Rightarrow&\ V(\phi) = \phi_0 - p_0^2\,(b+2c)\,\ln\phi. \end{aligned} \tag{29}$$

  Then eqn (20) is by substituting eqn (29):

  $$\alpha_Q = -1 + \frac{p_0^2}{\frac{p_0^2}{2} + \phi_0 - p_0^2\,(b+2c)\,\ln\phi}. \tag{30}$$

2. **Exponential:** This case is defined by [47]:

$$A_2 = e^{bt} \quad \text{and} \quad A_3 = c_0\, e^{ct}. \tag{31}$$

The eqn (31) form leads to $\left(\ln(A_2\,A_3^2)\right)' = (b+2c)$ and eqn (23) becomes in this case:

$$\begin{aligned} 0 =& p\, p_0^{1/p}\,(b+2c)\,\phi^{1-1/p} + p(p-1)\,p_0^{2/p}\,\phi^{1-2/p} + \frac{dV}{d\phi}, \\ \Rightarrow&\ V(\phi) = \phi_0 - \frac{p^2\,(b+2c)}{(2p-1)}\,p_0^{1/p}\,\phi^{2-1/p} - \frac{p^2}{2}\,p_0^{2/p}\,\phi^{2-2/p}. \end{aligned} \tag{32}$$

Then eqn (20) will be:

$$\alpha_Q = -1 + \frac{p^2 p_0^{2/p}\phi^{2-2/p}}{\phi_0 - \frac{p^2\,(b+2c)}{(2p-1)}\,p_0^{1/p}\,\phi^{2-1/p}}. \tag{33}$$

The previous special cases become:

- $\mathbf{p \gg 1}$: Eqns (32) and (33) will be exactly the eqns (27) and (28).
- $\mathbf{p = 1}$: Eqn (32) yields a linear potential:

  $$\begin{aligned} 0 =& p_0\,(b+2c) + \frac{dV}{d\phi}, \\ \Rightarrow&\ V(\phi) = \phi_0 - p_0\,(b+2c)\,\phi, \end{aligned} \tag{34}$$

and then eqn (20) becomes with the eqn (34) potential:

$$\alpha_Q = -1 + \frac{p_0^2}{\frac{p_0^2}{2} + \phi_0 - p_0\,(b+2c)\,\phi}. \tag{35}$$

We also note the scalar potential $V(\phi)$ solutions to eqn (23) obtained for a power-law scalar field confirm those found in one of the first study on the dark energy quintessence [66]. This was just confirmed by the separate use of the eqns (24) and (31) ansatzes, which shows the rightness and relevance of the potential solutions compared to the first models studied in the past literature.

3.1. Power-Law Ansatz Solutions

The eqn (21a) in terms of eqns (24) leads to the characteristic equation (see ref [47]):

$$0 = 2c\,(c+2b)\;t^{-2} - \frac{2}{c_0^2}\,t^{-2c} - T, \tag{36}$$

where $c \neq 0$ ($c = 0$ subcase leads to GR solutions). Then eqn (22) becomes under the eqn (24) ansatz and by replacing $V\left(\phi(T)\right) = \phi_0 + V(T)$ as:

$$\begin{aligned} 2\kappa\,\phi_0 + 2\kappa\,V(T) &= -\,F(T) + \left[\frac{3}{2}T + \frac{(3b-2c)}{c_0^2\,(b-c)}t^{-2c}(T)\right]\,F_T(T), \\ &= -\,F(T) + A(T)\,F_T(T), \end{aligned} \tag{37}$$

where the function $A(T)$ is defined as

$$A(T) = \frac{3}{2}T + \frac{(3b-2c)}{c_0^2\,(b-c)}t^{-2c}(T). \tag{38}$$

with $t(T)$ as eqn (36) solution. From this eqn (36), there are several possible solutions to eqn (37) depending on values of c as in ref [47]. Each value of c will lead to a specific $V(T)$ and $A(T)$ functions, and then we will solve eqn (37) with this form. In general, the eqn (37) solution will be under the form:

$$F(T) = -\,2\kappa\,\phi_0 + \exp\left[\int_T \frac{dT'}{A(T')}\right]\left[F_0 + 2\kappa\,\int_T dT'\,\frac{V(T')}{A(T')}\,\exp\left[-\int_{T'}\frac{dT''}{A(T'')}\right]\right]. \tag{39}$$

where $A(T)$ is eqn (38). Eqn (39) is the general formula applicable for any subcases and will be used to find all new teleparallel $F(T)$ solutions in the current paper.

The subcases are:

1. $\mathbf{c = -2b}$ ($b \neq 0$): Eqn (36) simplifies as [47]:

$$t(T) = \left(\frac{c_0^2}{2}(-T)\right)^{1/4b}. \tag{40}$$

Then eqn (38) simplifies to $A(T) = \frac{T}{3}$. The scalar field will be $\phi(T) = p_0\,\left(\frac{c_0^2}{2}(-T)\right)^{p/4b}$ and we will use this expression for the subcases:

   (a) **General:** Eqn (25) for scalar field potential is:

$$V(T) = -\frac{V_p}{2}(-T)^{(p-1)/2b}, \tag{41}$$

   where $V_p = \frac{p^2\,p_0^2\,(p-1-3b)}{2^{(p-1)/2b}\,(p-1)}\,c_0^{(p-1)/b}$. We obtain that $V_p = 0$ for $p = 3b+1$ and $V(T)$ is undefined for $p = 1$. Then by substituting the simplifed $A(T)$, eqns (40) and (41) into eqn (39), the teleparallel $F(T)$ solution will be:

$$F(T) = -2\kappa\,\phi_0 - \frac{6\kappa\,b\,V_p}{(p-1-6b)}\,(-T)^{(p-1)/2b} + F_0\,T^3, \tag{42}$$

where F_0 is an integration constant and $p \neq 1$ and $p \neq 1 + 6b$.

(b) **p ≫ 1 case:** By using the same scalar field $\phi(T)$, eqn (27) yields as potential:

$$V(T) = -\frac{V_\infty}{2}\,(-T)^{p/2b}, \tag{43}$$

where $V_\infty = \frac{p^2\,p_0^2\,c_0^{p/b}}{2^{p/2b}}$. Then by using the same $A(T)$, eqns (40) and (43), the eqn (39) is:

$$F(T) \approx -2\kappa\,\phi_0 - \frac{6\kappa\,b\,V_\infty}{p}\,(-T)^{p/2b} + F_0\,T^3 \approx -\frac{6\kappa\,b\,p\,p_0^2\,c_0^{p/b}}{2^{p/2b}}\,(-T)^{p/2b}, \tag{44}$$

where $p \to \infty$ and $T \leq 0$ in some situations.

(c) **p = 1 case:** Still by using the same scalar field $\phi(T)$, eqn (29) becomes:

$$V(T) = \tilde{\phi}_0 - \phi_0 + \frac{3p_0^2}{4}\,\ln(-T), \tag{45}$$

where $\tilde{\phi}_0 = \phi_0 + \frac{3p_0^2}{4}\,\ln\left(\frac{p_0^{4b}\,c_0^2}{2}\right)$. Then eqn (39) becomes by using the same $A(T)$, eqns (40) and (45):

$$F(T) = -2\kappa\,\tilde{\phi}_0 - \frac{\kappa\,p_0^2}{2} - \frac{3\kappa\,p_0^2}{2}\,\ln(-T) + F_0\,T^3, \tag{46}$$

where $T \leq 0$.

By comparing eqns (42), (44) and (46), we see that the homogeneous parts are very similar and their respective differences are only from the $V(T)$ parts (particular solution).

2. $\mathbf{c = 1}$: Eqn (36) becomes [47]:

$$\begin{aligned} 0 &= \left(2(1+2b) - \frac{2}{c_0^2}\right)\,t^{-2} - T, \\ \Rightarrow\; t^{-2}(T) &= \frac{T}{2\left(1 + 2b - \frac{1}{c_0^2}\right)}. \end{aligned} \tag{47}$$

From eqn (47), we find as eqn (38):

$$A(T) = \left[\frac{3}{2} + \frac{(3b-2)}{2c_0^2\,(b-1)\left(1 + 2b - c_0^{-2}\right)}\right]\,T = C\,T, \tag{48}$$

where $C = \frac{3}{2} + \frac{(3b-2)}{2c_0^2\,(b-1)\left(1+2b-c_0^{-2}\right)}$. The scalar field for eqn (47) is:

$$\phi(T) = \;p_0\,2^{p/2}\left(1 + 2b - c_0^{-2}\right)^{p/2}\,T^{-p/2}. \tag{49}$$

The potentials $V(T)$ and $F(T)$ solutions are for the subcases:

(a) **General:** Eqn (25) becomes:

$$V(T) = -\frac{V_p}{2}\,T^{1-p}, \tag{50}$$

where $p \neq 1$ and $V_p = \frac{p^2\,p_0^2\,2^{p-1}}{(p-1)}\,(b+p+1)\left(1 + 2b - c_0^{-2}\right)^{p-1}$. Eqn (50) will be constant for $b = -\frac{1}{2} + \frac{1}{2c_0^2}$ and $b = -p - 1$. By substituting eqns (48) and (50) into eqn (39), we find as solution:

$$F(T) = -\,2\kappa\,\phi_0 + \frac{\kappa\,V_p}{(1 + C(p-1))}\,T^{1-p} + F_0\,T^{1/C}, \tag{51}$$

where $p \neq 1$.

(b) **Eqn** (27) **potential:** This equation in terms of eqns (49) is ($p \gg 1$):

$$V(T) = -\frac{V_\infty}{2}\, T^{-p}, \tag{52}$$

where $V_\infty = p^2\, p_0^2\, 2^p \left(1 + 2b - \frac{1}{c_0^2}\right)^p$. By substituting eqns (48) and (52) into eqn (39), we find as solution:

$$F(T) = -\,2\kappa\,\phi_0 + F_0\, T^{1/C} + \frac{\kappa\, V_\infty}{[1 + p\, C]}\, T^{-p} \approx -2\kappa\,\phi_0 + F_0\, T^{1/C}. \tag{53}$$

(c) **Eqn** (29) **potential:** This equation in terms of eqns (49) is ($p = 1$):

$$V(T) = \tilde{\phi}_0 - \phi_0 + \frac{p_0^2}{2}\,(b+2)\,\ln\,(T)\,. \tag{54}$$

where $b \neq -2$ and $\tilde{\phi}_0 = \phi_0 - \frac{p_0^2}{2}\,(b+2)\,\ln\left[2p_0^2\left(1 + 2b - \frac{1}{c_0^2}\right)\right]$. By substituting eqns (48) and (54) into eqn (39), we find as solution:

$$F(T) = -\,2\kappa\,\tilde{\phi}_0 + F_0\, T^{1/C} - \kappa\, p_0^2\,(b+2)\,[\ln\,(T) + C]\,. \tag{55}$$

3. $\mathbf{c = -1}$: Eqn (36) becomes [47]:

$$\begin{aligned} 0 &= t^4 + \frac{c_0^2\, T}{2}\, t^2 - c_0^2(1-2b), \\ \Rightarrow\ t^2(T) &= \frac{c_0^2}{4}\left[-T + \delta_1\sqrt{T^2 + 16(1-2b)\, c_0^{-2}}\right], \end{aligned} \tag{56}$$

where $\delta_1 = \pm 1$. The eqn (38) for eqn (56) is:

$$\begin{aligned} A(T) =& \frac{3}{2}T + \frac{(3b+2)}{4\,(b+1)}\left[-T + \delta_1\sqrt{T^2 + 16(1-2b)\, c_0^{-2}}\right] \\ =& \left(\frac{3}{2} - C_1\right) T + \delta_1\, C_1\sqrt{T^2 + C_2}, \end{aligned} \tag{57}$$

where $C_1 = \frac{(3b+2)}{4\,(b+1)}$ and $C_2 = 16(1-2b)\, c_0^{-2}$. The scalar field for eqn (56) is:

$$\phi(T) = \frac{p_0\, c_0^p}{2^p}\left[-T + \delta_1\sqrt{T^2 + 16(1-2b)\, c_0^{-2}}\right]^{p/2} = \frac{p_0\, c_0^p}{2^p}\left[-T + \delta_1\sqrt{T^2 + C_2}\right]^{p/2}. \tag{58}$$

The potentials $V(T)$ and $F(T)$ solutions are for the subcases:

(a) **General:** Eqn (25) becomes:

$$V(T) = -\frac{V_p}{2}\left[-T + \delta_1\sqrt{T^2 + C_2}\right]^{p-1}, \tag{59}$$

where $p \neq 1$ and $V_p = \frac{p^2\, p_0^2\, c_0^{2p-2}}{2^{2p-2}\,(p-1)}\,(b + p - 3)$. Eqn (59) will be constant for $b = 3 - p$. By substituting eqns (57) and (59) into eqn (39), we find as solution:

$$\begin{aligned} F(T) =& -\,2\kappa\,\phi_0 + \left[(2C_1 - 3)T - 2C_1\delta_1\sqrt{T^2 + C_2}\right]^{\frac{2(3-2C_1)}{3(3-4C_1)}}\left[T + \sqrt{T^2 + C_2}\right]^{-\frac{4C_1\delta_1}{3(3-4C_1)}} \\ & \times\left[F_0 + 2\kappa\, V_p\int_T dT'\left[-T' + \delta_1\sqrt{T'^2 + C_2}\right]^{p-1}\right. \\ & \left.\times\left[(2C_1 - 3)T' - 2C_1\delta_1\sqrt{T'^2 + C_2}\right]^{-\frac{2(3-2C_1)}{3(3-4C_1)} - 1}\left[T' + \sqrt{T'^2 + C_2}\right]^{\frac{4C_1\delta_1}{3(3-4C_1)}}\right]. \end{aligned} \tag{60}$$

Eqn (60) is difficult to solve under this current form. However, the following special case solutions are possible:

- $C_2 = 0$ ($b = \frac{1}{2}$ and $C_1 = \frac{7}{12}$) and $\delta_1 = -1$:

$$F(T) = -\,2\kappa\,\phi_0 + F_0\,T^3 - \frac{3\kappa V_p(-2)^p}{(p-4)}\,T^{p-1}. \tag{61}$$

- $C_1 = \frac{3}{2}$ and $\delta_1 = +1$ ($b = -\frac{4}{3}$ and $C_2 = \frac{176}{3c_0^2}$): There are three possible solutions:
  - $p \neq \frac{1}{3}$:

$$F(T) = -\,2\kappa\,\phi_0 + F_0\left[T + \sqrt{T^2 + C_2}\right]^{\frac{2\delta_1}{3}} + \frac{2\kappa\,V_p}{(3p-1)}\,\left[-\delta_1\,T + \sqrt{T^2 + C_2}\right]^{p-1}. \tag{62}$$

  - $p = \frac{1}{3}$ and $\delta_1 = +1$:

$$F(T) = -\,2\kappa\,\phi_0 + \left[T + \sqrt{T^2 + C_2}\right]^{\frac{2}{3}}\left[F_0 - \frac{2\kappa\,V_p}{3\,C_2^{\frac{2}{3}}}\,\ln\left[T + \sqrt{T^2 + C_2}\right]\right]. \tag{63}$$

  - $p = \frac{1}{3}$ and $\delta_1 = -1$:

$$F(T) = -\,2\kappa\,\phi_0 + \left[T + \sqrt{T^2 + C_2}\right]^{\frac{2}{3}}\left[F_0 + \frac{2\kappa\,V_p}{3\,(-1)^{\frac{2}{3}}}\,\ln\left[T + \sqrt{T^2 + C_2}\right]\right]. \tag{64}$$

- $C_1 \to \frac{3}{4}$: By setting $C_1 = \frac{3}{4} + \epsilon$ where $\epsilon \ll 1$, we find:

$$F(T) = -\,2\kappa\,\phi_0 + F_0 + \frac{4\kappa\,V_p}{3C_2\,(p+1)(p-1)}\,\left[-T + \delta_1\sqrt{T^2 + C_2}\right]^p\left[T + \delta_1 p\sqrt{T^2 + C_2}\right]. \tag{65}$$

(b) **Eqn (27) potential:** This equation in terms of eqns (58) is ($p \gg 1$):

$$V(T) = -\frac{V_\infty}{2}\,\left[-T + \delta_1\,\sqrt{T^2 + C_2}\right]^p, \tag{66}$$

where $V_\infty = \frac{p^2\,p_0^2\,c_0^{2p}}{2^{2p}}$. By substituting eqns (57) and (66) into eqn (39), we find as solution:

$$\begin{aligned} F(T) = -\,2\kappa\,\phi_0 + &\left[(2C_1 - 3)T - 2C_1\delta_1\sqrt{T^2 + C_2}\right]^{\frac{2(3-2C_1)}{3(3-4C_1)}}\,\left[T + \sqrt{T^2 + C_2}\right]^{-\frac{4C_1\delta_1}{3(3-4C_1)}} \\ &\times\left[F_0 + 2\kappa\,V_\infty\,\int_T dT'\,\left[-T + \delta_1\,\sqrt{T^2 + C_2}\right]^p\right. \\ &\times\left.\left[(2C_1 - 3)T - 2C_1\delta_1\sqrt{T^2 + C_2}\right]^{-\frac{2(3-2C_1)}{3(3-4C_1)}-1}\,\left[T + \sqrt{T^2 + C_2}\right]^{\frac{4C_1\delta_1}{3(3-4C_1)}}\right]. \end{aligned} \tag{67}$$

Once again, eqn (67) do not yield to a general solution. But there are specific case leading to analytical $F(T)$ solutions:

- $C_2 = 0$ ($b = \frac{1}{2}$ and $C_1 = \frac{7}{12}$) and $\delta_1 = -1$:

$$F(T) = -\,2\kappa\,\phi_0 + F_0\,T^3 - \frac{3\kappa\,V_\infty(-2)^p}{p}\,T^p. \tag{68}$$

- $C_1 = \frac{3}{2}$ ($b = -\frac{4}{3}$ and $C_2 = \frac{176}{3c_0^2}$):

$$F(T) = -\,2\kappa\,\phi_0 + F_0\left[T + \sqrt{T^2 + C_2}\right]^{\frac{2\delta_1}{3}} + \frac{2\kappa\,V_\infty}{3p}\,\left[-T + \delta_1\sqrt{T^2 + C_2}\right]^p. \tag{69}$$

- $C_1 \;\to\; \frac{3}{4}$: By setting $C_1 = \frac{3}{4} + \epsilon$ where $\epsilon \ll 1$, we find:

$$F(T) = -\,2\kappa\,\phi_0 + \left[F_0 + \frac{4\kappa\,V_\infty}{3C_2\,p}\,\left[-T + \delta_1\sqrt{T^2 + C_2}\right]^p\left(C_2 + T\left[T - \delta_1\sqrt{T^2 + C_2}\right]\right)\right]. \tag{70}$$

(c) **Eqn (29) potential:** This equation in terms of eqns (58) is ($p = 1$):

$$V(T) = \tilde{\phi}_0 - \phi_0 - \frac{p_0^2}{2}\,(b-2)\,\ln\,\left[-T + \delta_1\,\sqrt{T^2 + C_2}\right]. \tag{71}$$

where $b \neq 2$ and $\tilde{\phi}_0 = \phi_0 - p_0^2\,(b-2)\,\ln\,\left[\frac{p_0\,c_0}{2}\right]$. By substituting eqns (57) and (71) into eqn (39), we find as solution:

$$\begin{aligned}F(T) = &-\,2\kappa\,\tilde{\phi}_0 + \left[(2C_1 - 3)T - 2C_1\delta_1\sqrt{T^2 + C_2}\right]^{\frac{2(3-2C_1)}{3(3-4C_1)}}\,\left[T + \sqrt{T^2 + C_2}\right]^{-\frac{4C_1\delta_1}{3(3-4C_1)}}\\ &\times\left[F_0 + 2\kappa\,p_0^2\,(b-2)\int_T dT'\,\ln\,\left[-T + \delta_1\,\sqrt{T^2 + C_2}\right]\right.\\ &\times\left.\left[(2C_1 - 3)T - 2C_1\delta_1\sqrt{T^2 + C_2}\right]^{-\frac{2(3-2C_1)}{3(3-4C_1)}-1}\,\left[T + \sqrt{T^2 + C_2}\right]^{\frac{4C_1\delta_1}{3(3-4C_1)}}\right].\end{aligned} \tag{72}$$

- $C_2 = 0$ ($b = \frac{1}{2}$ and $C_1 = \frac{7}{12}$) and $\delta_1 = -1$:

$$F(T) = -\,2\kappa\,\tilde{\phi}_0 + F_0\,T^{\frac{2}{(3-4C_1)}} - \frac{\kappa\,p_0^2\,(b-2)}{2}\,\left(4C_1 - 3 - 2\ln\left(-2T\right)\right). \tag{73}$$

- $C_1 = \frac{3}{2}$ ($b = -\frac{4}{3}$ and $C_2 = \frac{176}{3c_0^2}$):

$$F(T) = -\,2\kappa\,\tilde{\phi}_0 + F_0\left[T + \sqrt{T^2 + C_2}\right]^{\frac{2\delta_1}{3}} + \kappa\,p_0^2\,(b-2)\,\left(\ln\,\left[-T + \delta_1\sqrt{T^2 + C_2}\right] - \frac{3}{2}\right). \tag{74}$$

- $C_1 \;\to\; \frac{3}{4}$: By setting $C_1 = \frac{3}{4} + \epsilon$ where $\epsilon \ll 1$, we find:

$$\begin{aligned}F(T) = &-\,2\kappa\,\tilde{\phi}_0 + F_0 - \frac{\kappa\,p_0^2}{3}\,(b-2)\,\left[\delta_1\ln\,\left[T + \sqrt{T^2 + C_2}\right] - \ln^2\,\left[-T + \delta_1\sqrt{T^2 + C_2}\right]\right.\\ &\left.+\frac{2T}{C_2}\left[-T + \delta_1\sqrt{T^2 + C_2}\right]\left(\ln\,\left[-T + \delta_1\sqrt{T^2 + C_2}\right] - \frac{1}{2}\right)\right].\end{aligned} \tag{75}$$

4. **c = 2**: Eqn (36) becomes [47]:

$$\begin{aligned}0 &= t^{-4} - 4c_0^2\,(1+b)\,t^{-2} + \frac{c_0^2\,T}{2},\\ \Rightarrow\; t^{-2}(T) &= 2c_0^2\left[(1+b) + \delta_1\,\sqrt{(1+b)^2 - \frac{T}{8c_0^2}}\right],\end{aligned} \tag{76}$$

where $\delta_1 = \pm 1$. The eqn (38) for eqn (76) is:

$$\begin{aligned}A(T) =&\frac{3}{2}T + \frac{4(3b-4)(1+b)^2\,c_0^2}{(b-2)}\,\left[1 + \delta_1\sqrt{1 - \frac{T}{8c_0^2(1+b)^2}}\right]^2,\\ =&\frac{1}{(b-2)}\left[-T + C_1\left(1 + \delta_1\,\sqrt{1 - \frac{T}{C_2}}\right)\right],\end{aligned} \tag{77}$$

where $C_1 = (3b-4)C_2$ and $C_2 = 8c_0^2(1+b)^2$ and $b \neq 2$. The scalar field for eqn (76) is:

$$\phi(T) = \frac{p_0}{2^{p/2} c_0^p (1+b)^{p/2}} \left[1 + \delta_1 \sqrt{1 - \frac{T}{C_2}}\right]^{-p/2}. \tag{78}$$

The potentials $V(T)$ and $F(T)$ solutions are for the subcases:

(a) **General:** Eqn (25) becomes:

$$V(T) = -\frac{V_p}{2} \left[1 + \delta_1 \sqrt{1 - \frac{T}{C_2}}\right]^{1-p}, \tag{79}$$

where $p \neq 1$ and $V_p = \frac{p^2\, p_0^2}{2^{p-1}\,(p-1)\, c_0^{2p-2}}\, (b+p+3)\, (1+b)^{1-p}$. Eqn (79) will be constant for $b = -p-3$. By substituting eqns (77) and (79) into eqn (39), we find as solution:

$$\begin{aligned} F(T) = &-2\kappa\,\phi_0 + \left[-T + C_1\left(1 + \delta_1\sqrt{1 - \frac{T}{C_2}}\right)\right]^{(2-b)} \left[\frac{-C_2\,T}{C_1\,(C_1 - 2C_2) + C_2\,T}\right]^{\frac{(b-2)C_1}{2\left(C_1 - 2C_2\right)}} \\ &\times \left[\frac{T + (C_1 - 2C_2)\left(1 + \delta_1\sqrt{1 - \frac{T}{C_2}}\right)}{T + (C_1 - 2C_2)\left(1 - \delta_1\sqrt{1 - \frac{T}{C_2}}\right)}\right]^{\frac{\delta_1\,(b-2)C_1}{2\left(C_1 - 2C_2\right)}} \\ &\times \left[F_0\, C_2^{(2-b)/2} - \kappa\, V_p\,(b-2) \int_T dT' \left[1 + \delta_1\sqrt{1 - \frac{T'}{C_2}}\right]^{1-p}\right. \\ &\times \left[-T' + C_1\left(1 + \delta_1\sqrt{1 - \frac{T'}{C_2}}\right)\right]^{(b-3)} \left[-\frac{C_2\,T'}{C_1\,(C_1 - 2C_2) + C_2\,T'}\right]^{-\frac{(b-2)C_1}{2\left(C_1 - 2C_2\right)}} \\ &\times \left.\left[\frac{T' + (C_1 - 2C_2)\left(1 + \delta_1\sqrt{1 - \frac{T'}{C_2}}\right)}{T' + (C_1 - 2C_2)\left(1 - \delta_1\sqrt{1 - \frac{T'}{C_2}}\right)}\right]^{-\frac{\delta_1\,(b-2)C_1}{2\left(C_1 - 2C_2\right)}}\right]. \end{aligned} \tag{80}$$

There is no general solution for this eqn (80). However, there are solutions for the specific cases:

- $C_1 = 0$ and $\delta_1 = 1$:

$$F(T) = -2\kappa\,\phi_0 + F_0\,(-T)^{(2-b)} - \kappa\, V_p\,(2)^{1-p}\, {}_3F_2\left(b-2, \frac{p}{2}, \frac{p-1}{2}; p, b-1; \frac{T}{C_2}\right). \tag{81}$$

- $C_1 = 0$ and $\delta_1 = -1$:

$$\begin{aligned} F(T) = &-2\kappa\,\phi_0 + F_0\,(-T)^{(2-b)} - \kappa\, V_p\, \frac{(b-2)(-2C_2)^{p-1}}{(b-p-1)}\,(-T)^{1-p} \\ &\times\ {}_3F_2\left(b-p-1, \frac{2-p}{2}, \frac{1-p}{2}; b-p, 2-p; \frac{T}{C_2}\right). \end{aligned} \tag{82}$$

- $C_1 = 2C_2$:

$$\begin{aligned} F(T) = &-2\kappa\,\phi_0 + \left[-\frac{T}{C_2} + 2\left(1 + \delta_1\sqrt{1 - \frac{T}{C_2}}\right)\right]^{(2-b)} \left[F_0 - \frac{\kappa\, V_p\,(b-2)}{C_2}\right. \\ &\times \left.\int_T dT' \left[1 + \delta_1\sqrt{1 - \frac{T'}{C_2}}\right]^{1-p} \left[-\frac{T'}{C_2} + 2\left(1 + \delta_1\sqrt{1 - \frac{T'}{C_2}}\right)\right]^{(b-3)}\right]. \end{aligned} \tag{83}$$

Eqn (83) solutions are possible only for the subcases:

- $b = 3$ and $\delta_1 = 1$:

$$F(T) = -\,2\kappa\,\phi_0 + \left[-\frac{T}{C_2} + 2\left(1 + \sqrt{1 - \frac{T}{C_2}}\right)\right]^{-1} \left[F_0 - \frac{\kappa\,V_p}{C_2}\,(2)^{1-p} \times\; T\,{}_3F_2\left(1, \frac{p}{2}, \frac{p-1}{2}; 2, p; \frac{T}{C_2}\right)\right]. \tag{84}$$

- $b = 3$ and $\delta_1 = -1$:

$$F(T) = -\,2\kappa\,\phi_0 + \left[-\frac{T}{C_2} + 2\left(1 - \sqrt{1 - \frac{T}{C_2}}\right)\right]^{-1} \left[F_0 + \frac{2^{p-1}\kappa\,V_p}{C_2^{2-p}(p-2)} \times\; T^{2-p}\,{}_2F_1\left(\frac{2-p}{2}, \frac{1-p}{2}; 3-p; \frac{T}{C_2}\right)\right]. \tag{85}$$

- $b = 4$ and $\delta_1 = 1$:

$$F(T) = -\,2\kappa\,\phi_0 + \left[-\frac{T}{C_2} + 2\left(1 + \sqrt{1 - \frac{T}{C_2}}\right)\right]^{-2} \left[F_0 - \frac{\kappa\,V_p}{2^{p-1}}\left[-\left(\frac{T}{C_2}\right)^2 \times\; {}_3F_2\left(2, \frac{p}{2}, \frac{p-1}{2}; 3, p; \frac{T}{C_2}\right) + 8\left(\frac{T}{C_2}\right)\,{}_3F_2\left(1, \frac{p-2}{2}, \frac{p-1}{2}; 2, p-1; \frac{T}{C_2}\right)\right]\right]. \tag{86}$$

- $b = 4$ and $\delta_1 = -1$:

$$F(T) = -\,2\kappa\,\phi_0 + \left[-\frac{T}{C_2} + 2\left(1 - \sqrt{1 - \frac{T}{C_2}}\right)\right]^{-2} \left[F_0 - \frac{8\kappa\,V_p}{(p-3)}\left(\frac{T}{2C_2}\right)^{(3-p)} \times \left[{}_3F_2\left(3-p, \frac{2-p}{2}, \frac{1-p}{2}; 4-p, 2-p; \frac{T}{C_2}\right) + {}_2F_1\left(\frac{2-p}{2}, \frac{3-p}{2}; 4-p; \frac{T}{C_2}\right)\right]\right]. \tag{87}$$

(b) **Eqn (27) potential:** This equation in terms of eqns (78) is ($p \gg 1$):

$$V(T) = -\frac{V_\infty}{2}\left[1 + \delta_1\sqrt{1 - \frac{T}{C_2}}\right]^{-p}, \tag{88}$$

where $V_\infty = \frac{p^2\,p_0^2\,(1+b)^{-p}}{2^p c_0^{2p}}$. By substituting eqns (76), (77) and (88) into eqn (39), we find as solution:

$$\begin{aligned} F(T) = &-\,2\kappa\,\phi_0 + \left[-T + C_1\left(1 + \delta_1\sqrt{1 - \frac{T}{C_2}}\right)\right]^{(2-b)} \left[\frac{-C_2\,T}{C_1\,(C_1 - 2C_2) + C_2\,T}\right]^{\frac{(b-2)C_1}{2(C_1-2C_2)}} \\ &\times \left[\frac{T + (C_1 - 2C_2)\left(1 + \delta_1\sqrt{1 - \frac{T}{C_2}}\right)}{T + (C_1 - 2C_2)\left(1 - \delta_1\sqrt{1 - \frac{T}{C_2}}\right)}\right]^{\frac{\delta_1\,(b-2)C_1}{2(C_1-2C_2)}} \\ &\times \Bigg[F_0\,C_2^{(2-b)/2} - \kappa\,V_\infty\,(b-2)\int_T dT'\left[1 + \delta_1\sqrt{1 - \frac{T'}{C_2}}\right]^{-p} \\ &\times \left[-T' + C_1\left(1 + \delta_1\sqrt{1 - \frac{T'}{C_2}}\right)\right]^{(b-3)} \left[-\frac{C_2\,T'}{C_1\,(C_1 - 2C_2) + C_2\,T'}\right]^{-\frac{(b-2)C_1}{2(C_1-2C_2)}} \\ &\times \left[\frac{T' + (C_1 - 2C_2)\left(1 + \delta_1\sqrt{1 - \frac{T'}{C_2}}\right)}{T' + (C_1 - 2C_2)\left(1 - \delta_1\sqrt{1 - \frac{T'}{C_2}}\right)}\right]^{-\frac{\delta_1\,(b-2)C_1}{2(C_1-2C_2)}}\Bigg]. \end{aligned} \tag{89}$$

There is no general solution for this eqn (89). There are solutions for the cases:

- $C_1 = 0$ and $\delta_1 = 1$:

$$F(T) \approx -2\kappa\,\phi_0 + \left[F_0\,(-T)^{(2-b)} + \frac{\kappa\,V_\infty}{2^p}\,{}_3F_2\left(b-2, \frac{p}{2}, \frac{p}{2}; b-1, p; \frac{T}{C_2}\right)\right]. \tag{90}$$

- $C_1 = 0$ and $\delta_1 = -1$:

$$\begin{aligned} F(T) \approx -2\kappa\,\phi_0 + \Bigg[F_0\,(-T)^{(2-b)} \\ - \frac{\kappa\,V_\infty(-2C_2)^p}{p}\,(b-2)\,(-T)^{-p}\,{}_3F_2\left(-p, -\frac{p}{2}, -\frac{p}{2}; -p, p; \frac{T}{C_2}\right)\Bigg]. \end{aligned} \tag{91}$$

- $C_1 = 2C_2$:

$$\begin{aligned} F(T) = -2\kappa\,\phi_0 + \left[-T + 2C_2\left(1 + \delta_1\sqrt{1 - \frac{T}{C_2}}\right)\right]^{(2-b)}\Bigg[F_0 - \kappa\,V_\infty\,(b-2) \\ \times \int_T dT'\left[1 + \delta_1\sqrt{1 - \frac{T'}{C_2}}\right]^{-p}\left[-T' + 2C_2\left(1 + \delta_1\sqrt{1 - \frac{T'}{C_2}}\right)\right]^{(b-3)}\Bigg]. \end{aligned} \tag{92}$$

  Eqn (92) solutions are possible only for the subcases:

  - $b = 3$ and $\delta_1 = 1$:

$$\begin{aligned} F(T) \approx -2\kappa\,\phi_0 + \left[-\frac{T}{C_2} + 2\left(1 + \sqrt{1 - \frac{T}{C_2}}\right)\right]^{-1} \\ \times \left[F_0 - \frac{\kappa\,V_p}{C_2\,2^p}\,T\,{}_3F_2\left(1, \frac{p}{2}, \frac{p}{2}; 2, p; \frac{T}{C_2}\right)\right]. \end{aligned} \tag{93}$$

  - $b = 3$ and $\delta_1 = -1$:

$$\begin{aligned} F(T) \approx -2\kappa\,\phi_0 + \left[-\frac{T}{C_2} + 2\left(1 - \sqrt{1 - \frac{T}{C_2}}\right)\right]^{-1} \\ \times \left[F_0 + \frac{(2C_2)^p\,\kappa\,V_p}{p}\,T^{-p}\,{}_2F_1\left(-\frac{p}{2}, -\frac{p}{2}; -p; \frac{T}{C_2}\right)\right]. \end{aligned} \tag{94}$$

  - $b = 4$ and $\delta_1 = 1$:

$$\begin{aligned} F(T) \approx -2\kappa\,\phi_0 + \left[-\frac{T}{C_2} + 2\left(1 + \sqrt{1 - \frac{T}{C_2}}\right)\right]^{-2}\Bigg[F_0 - \frac{\kappa\,V_p}{2^p}\Bigg[-\left(\frac{T}{C_2}\right)^2 \\ \times {}_3F_2\left(2, \frac{p}{2}, \frac{p}{2}; 3, p; \frac{T}{C_2}\right) + 8\left(\frac{T}{C_2}\right)\,{}_3F_2\left(1, \frac{p}{2}, \frac{p}{2}; 2, p; \frac{T}{C_2}\right)\Bigg]\Bigg]. \end{aligned} \tag{95}$$

  - $b = 4$ and $\delta_1 = -1$:

$$\begin{aligned} F(T) \approx -2\kappa\,\phi_0 + \left[-\frac{T}{C_2} + 2\left(1 - \sqrt{1 - \frac{T}{C_2}}\right)\right]^{-2}\Bigg[F_0 - \frac{2\kappa\,V_p}{p}\left(\frac{T}{2C_2}\right)^{-p} \\ \times {}_2F_1\left(-\frac{p}{2}, -\frac{p}{2}; -p; \frac{T}{C_2}\right)\Bigg]. \end{aligned} \tag{96}$$

(c) **Eqn (29) potential:** This equation in terms of eqns (78) is ($p=1$):

$$V(T)=\tilde{\phi}_0-\phi_0+\frac{p_0^2}{2}\,(b+4)\,\ln\left[1+\delta_1\sqrt{1-\frac{T}{C_2}}\right]. \tag{97}$$

where $b\neq -1,\,-4$ and $\tilde{\phi}_0=\phi_0-p_0^2\,(b+4)\,\ln\left[\frac{p_0}{2^{1/2}c_0\sqrt{1+b}}\right]$. By substituting eqns (76), (77) and (97) into eqn (39), we find as solution:

$$\begin{aligned}F(T)=&-2\kappa\,\tilde{\phi}_0+\left[-T+C_1\left(1+\delta_1\sqrt{1-\frac{T}{C_2}}\right)\right]^{(2-b)}\left[\frac{-C_2\,T}{C_1\,(C_1-2C_2)+C_2\,T}\right]^{\frac{(b-2)C_1}{2(C_1-2C_2)}}\\&\times\left[\frac{T+(C_1-2C_2)\left(1+\delta_1\sqrt{1-\frac{T}{C_2}}\right)}{T+(C_1-2C_2)\left(1-\delta_1\sqrt{1-\frac{T}{C_2}}\right)}\right]^{\frac{\delta_1\,(b-2)C_1}{2(C_1-2C_2)}}\\&\times\left[F_0\,C_2^{(2-b)/2}+\frac{\kappa\,p_0^2\,(b+4)\,(b-2)}{C_1}\int_T dT'\,\ln\left[1+\delta_1\sqrt{1-\frac{T'}{C_2}}\right]\right.\\&\times\left[-T'+C_1\left(1+\delta_1\sqrt{1-\frac{T'}{C_2}}\right)\right]^{(b-3)}\left[-\frac{C_2\,T'}{C_1\,(C_1-2C_2)+C_2\,T'}\right]^{-\frac{(b-2)C_1}{2(C_1-2C_2)}}\\&\left.\times\left[\frac{T'+(C_1-2C_2)\left(1+\delta_1\sqrt{1-\frac{T'}{C_2}}\right)}{T'+(C_1-2C_2)\left(1-\delta_1\sqrt{1-\frac{T'}{C_2}}\right)}\right]^{-\frac{\delta_1\,(b-2)C_1}{2(C_1-2C_2)}}\right].\end{aligned} \tag{98}$$

There is no general solution for this eqn (98). There are solutions for the cases:

- $C_1=2C_2$:

$$\begin{aligned}F(T)=&-2\kappa\,\tilde{\phi}_0+\left[-T+2C_2\left(1+\delta_1\sqrt{1-\frac{T}{C_2}}\right)\right]^{(2-b)}\left[F_0+\frac{\kappa\,p_0^2\,(b+4)\,(b-2)}{2C_2}\right.\\&\left.\times\int_T dT'\,\ln\left[1+\delta_1\sqrt{1-\frac{T'}{C_2}}\right]\left[-T'+2C_2\left(1+\delta_1\sqrt{1-\frac{T'}{C_2}}\right)\right]^{(b-3)}\right].\end{aligned} \tag{99}$$

  The simplest solutions of eqn (98) are:

  - $b=3$:

$$\begin{aligned}F(T)=&-2\kappa\,\tilde{\phi}_0+\left[-T+2C_2\left(1+\sqrt{1-\frac{T}{C_2}}\right)\right]^{-1}\left[F_0-\frac{7\kappa\,p_0^2}{2C_2}\right.\\&\left.\times\left[C_2\left(1+\frac{T}{2C_2}\right)+\delta_1\,C_2\sqrt{1-\frac{T}{C_2}}-T\,\ln\left[1+\delta_1\sqrt{1-\frac{T}{C_2}}\right]\right]\right].\end{aligned} \tag{100}$$

  - $b=4$:

$$\begin{aligned}F(T)=&-2\kappa\,\tilde{\phi}_0+\left[-T+2C_2\left(1+\sqrt{1-\frac{T}{C_2}}\right)\right]^{-2}\left[F_0+\frac{8\kappa\,p_0^2}{C_2}\right.\\&\times\left[\left[-\frac{4\delta_1\,C_2^2}{3}\left(1-\frac{T}{C_2}\right)^{3/2}-3\,T^2+12C_2\,T-8C_2^2\right]\ln\left[1+\delta_1\sqrt{1-\frac{T}{C_2}}\right]\right.\\&\left.\left.+\frac{\delta_1\,C_2}{18}\,(2C_2-5\,T)\sqrt{1-\frac{T}{C_2}}+\frac{T^2}{8}-\frac{C_2}{3}\,T+\frac{C_2^2}{9}\right]\right].\end{aligned} \tag{101}$$

For other values of b, c and/or subcases, the most of eqn (39) integral cases will not lead to a closed and analytic $F(T)$ solutions as in ref. [47]. All these previous teleparallel $F(T)$ solutions are new results.

doi:10.20944/preprints202412.1308.v1

3.2. Exponential Ansatz Solutions

The eqn (21a) in terms of eqns (31) leads to the characteristic equation:

$$e^{-2ct(T)} = \frac{c_0^2}{2}\left(T_0 - T\right), \tag{102}$$

where $c \neq 0$ and $T_0 = 2c\,(c+2b)$ ($c = 0$ subcase leads to GR solutions). By substituting eqn (102) inside eqn (22), we find exactly eqn (37) DE form with

$$A(T) = \frac{c}{2(b-c)}\left[\left(\frac{3b}{2} - c\right)\,T_0 - T\right] = \frac{c}{2(b-c)}(T_1 - T), \tag{103}$$

where $T_1 = c\,(3b-2c)\,(c+2b)$ and the solution is also described by eqn (39). The subcases are:

1. $\mathbf{c = -2b}$: $T_0 = 0$ and eqn (102) simplifies:

$$e^{4bt(T)} = \frac{c_0^2}{2}\left(-T\right), \tag{104}$$

where $T \leq 0$. Then eqn (103) simplifies to $A(T) = \frac{T}{3}$, the scalar field will be $\phi(T) = \frac{p_0}{(4b)^p}\left(\ln\left(\frac{c_0^2}{2}\,(-T)\right)\right)^p$. The potentials $V(T)$ and $F(T)$ solutions are for the subcases:

(a) **General:** Eqn (32) becomes:

$$V(T) = \frac{V_{1p}}{2}\left(\ln\left(\frac{c_0^2}{2}(-T)\right)\right)^{2p-1} - \frac{V_{2p}}{2}\left(\ln\left(\frac{c_0^2}{2}\,(-T)\right)\right)^{2p-2}, \tag{105}$$

where $V_{1p} = \frac{3p^2\,p_0^2}{2(2p-1)(4b)^{2p-2}}$ and $V_{2p} = \frac{p^2\,p_0^2}{(4b)^{2p-2}}$. By substituting $A(T)$ and eqn (105) into eqn (39), we find as solution:

$$F(T) = -2\kappa\,\phi_0 + F_0\,T^3 + 3\kappa\,T^3\left[V_{1p}\,\mathcal{N}_{2p-1}\left(-\frac{c_0^2}{2}, T\right) - V_{2p}\,\mathcal{N}_{2p-2}\left(-\frac{c_0^2}{2}, T\right)\right], \tag{106}$$

where F_0 is an integration constant, $p \neq \frac{1}{2}$ and $\mathcal{N}_k\left(-\frac{c_0^2}{2}, T\right)$ is new special function class defined as:

$$\mathcal{N}_k\left(B_1, x\right) = \int dx\,\frac{\left(\ln\left(B_1\,x\right)\right)^k}{x^4}. \tag{107}$$

Table 1. Some values of eqn (107) $\mathcal{N}_k\left(B_1, x\right)$ special functions.

k	$\mathcal{N}_k\left(B_1, x\right)$
0	$-\frac{1}{3x^3}$
1	$-\frac{1}{3x^3}\left[\ln(A\,x) + \frac{1}{3}\right]$
2	$-\frac{1}{3x^3}\left[\ln^2(A\,x) + \frac{2}{3}\ln(A\,x) + \frac{2}{9}\right]$
3	$-\frac{1}{3x^3}\left[\ln^3(A\,x) + \ln^2(A\,x) + \frac{2}{3}\ln(A\,x) + \frac{2}{9}\right]$
4	$-\frac{\ln^4(A\,x)}{3x^3} - \frac{4}{9x^3}\left[\ln^3(A\,x) + \ln^2(A\,x) + \frac{2}{3}\ln(A\,x) + \frac{2}{9}\right]$
5	$-\frac{1}{3x^3}\left[\ln^5(A\,x) + \frac{5}{3}\ln^4(A\,x)\right] - \frac{20}{27x^3}\left[\ln^3(A\,x) + \ln^2(A\,x) + \frac{2}{3}\ln(A\,x) + \frac{2}{9}\right]$
6	$-\frac{1}{3x^3}\left[\ln^6(A\,x) + 2\ln^5(A\,x) + \frac{10}{3}\ln^4(A\,x)\right] - \frac{40}{27x^3}\left[\ln^3(A\,x) + \ln^2(A\,x) + \frac{2}{3}\ln(A\,x) + \frac{2}{9}\right]$

(b) **Eqn (27) potential:** This equation in terms of eqns (104) scalar field is ($p \gg 1$):

$$V(T) = -\frac{V_\infty}{2}\left(\ln\left(\frac{c_0^2}{2}\,(-T)\right)\right)^{2p}, \tag{108}$$

where $V_\infty = \frac{p^2\,p_0^2}{(4b)^{2p}}$. By substituting $A(T) = \frac{T}{3}$ and (108) into eqn (39), we find as solution:

$$F(T) = -\,2\kappa\,\phi_0 + F_0\,T^3 - 3\kappa\,V_\infty\,T^3\,\mathcal{N}_{2p}\left(-\frac{c_0^2}{2}, T\right), \tag{109}$$

where $\mathcal{N}_{2p}\left(-\frac{c_0^2}{2}, T\right)$ is defined by eqn (107).

(c) **Eqn** (34) **potential:** This equation in terms of eqns (104) scalar field is ($p = 1$):

$$V(T) = \tilde{\phi}_0 - \phi_0 + \frac{3p_0^2}{4}\,\ln(-T), \tag{110}$$

where $\tilde{\phi}_0 = \phi_0 + \frac{3p_0^2}{4}\,\ln\left(\frac{c_0^2}{2}\right)$. By substituting $A(T) = \frac{T}{3}$ and (110) into eqn (39), we find as solution:

$$F(T) = -\,2\kappa\,\tilde{\phi}_0 + F_0\,T^3 + \frac{9\kappa\,p_0^2}{2}\,T^3\,\mathcal{N}_1\left(-1, T\right), \tag{111}$$

where $\mathcal{N}_1\left(-1, T\right)$ is defined by eqn (107).

2. $\mathbf{c} \neq -\mathbf{2b}$ (General case): From eqn (102), we find that the scalar field is:

$$\phi(T) = \frac{p_0}{(-2c)^p}\,\left[\ln\left(\frac{c_0^2}{2}\,(T_0 - T)\right)\right]^p. \tag{112}$$

For potentials $V(T)$ and $F(T)$ solutions, we find for the subcases:

(a) **General:** Eqn (32) becomes:

$$V(T) = -\,\frac{V_{1p}}{2}\,\left[\ln\left(-\frac{c_0^2}{2}\,(T - T_0)\right)\right]^{2p-1} - \frac{V_{2p}}{2}\,\left[\ln\left(-\frac{c_0^2}{2}\,(T - T_0)\right)\right]^{2p-2}, \tag{113}$$

where $V_{1p} = \frac{2p^2\,p_0^2(b+2c)}{(2p-1)(-2c)^{2p-1}}$ and $V_{2p} = \frac{p^2\,p_0^2}{(-2c)^{2p-2}}$. By substituting eqns (103) and (113) into eqn (39), we find as solution:

$$\begin{aligned} F(T) = &-\,2\kappa\,\phi_0 + F_0\,\left(T - T_1\right)^{2(c-b)/c} - \frac{2\kappa\,(c-b)}{c}\,\left(T - T_1\right)^{2(c-b)/c} \\ &\times \left[V_{1p}\,\mathcal{Q}_{2p-1}\left(-\frac{c_0^2}{2}, T_0, T_1, T\right) + V_{2p}\,\mathcal{Q}_{2p-2}\left(-\frac{c_0^2}{2}, T_0, T_1, T\right)\right], \end{aligned} \tag{114}$$

where $\mathcal{Q}_{2p}\left(B_1, B_2, B_3, x\right)$ is defined as:

$$\mathcal{Q}_k\left(B_1, B_2, B_3, x\right) = \int dx\,\left(\ln\left(B_1\,(x - B_2)\right)\right)^k\,(x - B_3)^{2b/c-3}. \tag{115}$$

The eqn (115) is a generalization of eqn (107) special function class.

Table 2. Some values of eqn (115) for $\mathcal{Q}_k\left(B_1, B_2, B_3, x\right)$ special functions.

k	b/c	$\mathcal{Q}_k\left(B_1, B_2, B_3, x\right)$
0	all	$\frac{c}{2(b-c)}\left(x-B_3\right)^{2(b-c)/c}$
1	0	$\frac{2}{\left(B_2-B_3\right)^2}\left[\frac{\left(B_2-B_3\right)}{\left(x-B_3\right)}-\ln\left(B_1\left(x-B_3\right)\right)+\frac{\left(x-B_2\right)\left(B_2-2B_3+x\right)}{\left(x-B_3\right)^2}\ln\left(B_1\left(x-B_2\right)\right)\right]$
1	1	$dilog\left(\frac{x-B_3}{B_2-B_3}\right)+\ln(B_1\left(x-B_2\right))\ln\left(\frac{x-B_3}{B_2-B_3}\right)$
1	$\frac{3}{2}$	$(\ln(B_1\left(x-B_2\right))-1)\left(x-B_2\right)$
1	2	$\frac{\left(x-B_2\right)}{2}\left[\left(B_2-2B_3+x\right)\ln(B_1\left(x-B_2\right))-\frac{x}{2}-\frac{3B_2}{2}+2B_3\right]$
2	1	$\ln(B_1\left(x-B_2\right))^2\ln\left(\frac{x-B_3}{B_2-B_3}\right)+2\ln(B_1\left(x-B_2\right))\,\mathrm{polylog}\left(2,\frac{-x+B_2}{B_2-B_3}\right)-2\mathrm{polylog}\left(3,\frac{-x+B_2}{B_2-B_3}\right)$
2	$\frac{3}{2}$	$\left(\ln(B_1\left(x-B_2\right))^2-2\ln(B_1\left(x-B_2\right))+2\right)\left(x-B_2\right)$
2	2	$\frac{\left(x-B_2\right)}{2}\left[\left(B_2-2B_3+x\right)\ln(B_1\left(x-B_2\right))^2+\left(-x-3B_2+4B_3\right)\ln(B_1\left(x-B_2\right))+\frac{x}{2}+\frac{7B_2}{2}-4B_3\right]$

(b) **Eqn (27) potential:** This equation in terms of eqns (112) scalar field is ($p \gg 1$):

$$V(T)=-\frac{V_\infty}{2}\left[\ln\left(-\frac{c_0^2}{2}\left(T-T_0\right)\right)\right]^{2p}, \tag{116}$$

where $V_\infty=\frac{p_0^2\,p^2}{(-2c)^{2p}}$. By substituting eqns (103) and (116) into eqn (39), we find as solution:

$$F(T)=-2\kappa\,\phi_0+\left(T-T_1\right)^{2(c-b)/c}\left[F_0-\frac{2\kappa\left(c-b\right)V_\infty}{c}\,\mathcal{Q}_{2p}\left(-\frac{c_0^2}{2}, T_0, T_1, T\right)\right], \tag{117}$$

where $\mathcal{Q}_{2p}\left(-\frac{c_0^2}{2}, T_0, T_1, T\right)$ is defined by eqn (115).

(c) **Eqn (34) potential:** This equation in terms of eqns (112) scalar field is ($p=1$):

$$V(T)=\tilde{\phi}_0-\phi_0+\frac{p_0^2\left(b+2c\right)}{2c}\ln\left(T_0-T\right), \tag{118}$$

where $\tilde{\phi}_0=\phi_0+\frac{p_0^2\left(b+2c\right)}{2c}\ln\left(\frac{c_0^2}{2}\right)$. By substituting eqns (118) into eqn (39), we find as solution:

$$F(T)=-2\kappa\,\tilde{\phi}_0+\left(T-T_1\right)^{2(c-b)/c}\left[F_0+\frac{2\kappa\,p_0^2\left(c-b\right)\left(b+2c\right)}{c^2}\,\mathcal{Q}_1\left(-1, T_0, T_1, T\right)\right], \tag{119}$$

where $\mathcal{Q}_1\left(-1, T_0, T_1, T\right)$ is defined by eqn (115).

All these teleparallel $F(T)$ solutions in this section are new. However, there are in principle other additional subcases leading to new $F(T)$ solutions.

4. Exponential Scalar Field Solutions

In this section, we will set an exponential scalar field $\phi(t)=p_0\,\exp\left(p\,t\right)$ where p_0 is a constant and p is the exponential coefficient. The conservation law defined by eqn (19) becomes as:

$$0=p\,\left(\ln(A_2\,A_3^2)\right)'\,\phi+p^2\,\phi+\frac{dV}{d\phi}, \tag{120}$$

where $\left(\ln(A_2\,A_3^2)\right)'$ is still depending on ϕ and the used ansatz for A_2 and A_3 components as for eqn (23) This eqn(120) is at first glance really simpler than eqn (23). The possible ansatz are:

1. **Power-law:** By substituting eqn (24) into eqn (120), we find that:

$$\begin{aligned} 0 =& p\,(b+2c)\,\frac{\phi}{\ln\left(\phi/p_0\right)} + p^2\,\phi + \frac{dV}{d\phi}, \\ \Rightarrow\; V\left(\phi\right) =& \phi_0 - \frac{p^2}{2}\,\phi^2 + p\,(b+2c)\,p_0^2\,Ei\left(-2\ln\left(\frac{\phi}{p_0}\right)\right). \end{aligned} \tag{121}$$

   Eqn (121) by the last integral term is a different kind of potential and this last can be considered as an additional interaction term. The eqn (20) for α_Q is by substituting eqn (121):

$$\alpha_Q = -1 + \frac{p^2\phi^2}{\phi_0 + p\,(b+2c)\,p_0^2\,Ei\left(-2\ln\left(\frac{\phi}{p_0}\right)\right)}. \tag{122}$$

   $\mathbf{p \gg 1:}$ Eqns (121) and (122) become exactly eqns (27) and (28) where the $p\,(b+2c)\,p_0^2\,Ei\left(-2\ln\left(\frac{\phi}{p_0}\right)\right)$ term is negligible in this case.

2. **Exponential:** This is the most simple and naturally adapted ansatz for an pure exponential scalar field. By substituting eqn (31) into eqn (120), the conservation law becomes:

$$\begin{aligned} 0 =& p\,(b+2c+p)\,\phi + \frac{dV}{d\phi}, \\ \Rightarrow\; V\left(\phi\right) =& \phi_0 - \frac{p}{2}\,(b+2c+p)\,\phi^2, \end{aligned} \tag{123}$$

   and then eqn (20) will be:

$$\alpha_Q = -1 + \frac{p^2\phi^2}{\phi_0 - \frac{p}{2}\,(b+2c)\,\phi^2}. \tag{124}$$

   - $\mathbf{p \gg 1:}$ Eqns (123) and (124) become eqns (27) and (28) with the same physical process scenarios.
   - $\mathbf{p\,(b+2c+p) < 0:}$ Eqn (123) is a simple harmonic oscillator (SHO) potential of angular frequency $\omega_\phi^2 = -p\,(b+2c+p)$. In a such case, the scalar field will be:

$$\phi(t) = C_1\,\cos\left(\omega_\phi\,t\right) + C_2\,\sin\left(\omega_\phi\,t\right). \tag{125}$$

     The eqn (125) is an oscillating scalar field, but it is not really relevant as a scalar field source for usual cosmological solutions. Beyond that, there are two situations leading to this case:

     - $p < 0$ and $p > -b - 2c$.
     - $p > 0$ and $p < -b - 2c$.

     This quadratic $V\left(\phi\right)$ potential case can be easily used for dark energy quintom oscillating models [93–95]. The eqn (125) scalar field definition can be only relevant for this type of physical models.
   - $\mathbf{p = 0}$ **and/or** $\mathbf{p = -b - 2c}$**:** Eqn (123) leads to a constant scalar field $V\left(\phi\right) = \phi_0$.

As for power-law potentials obtained in sect 3, the eqn (120) solutions are going to the same direction than the first quintessence process studies [66]. This situation is also confirmed separately by eqns (24) and (31) ansatz approaches in this section.

4.1. Power-Law Ansatz Solutions

By using eqns (36) and (37) with the eqn (24) ansatz, we obtain once again the eqn (37) DE form with $A(T)$ defined by eqn (38). By using eqn (121) and setting $V\left(\phi(T)\right) = \phi_0 + V(T)$ in eqn (37) potential, the general solution will be described again by eqn (39). We will find new $F(T)$ solutions by computing eqn (39) for each $V(T)$ potential cases:

1. $\mathbf{c=-2b}$: By using $A(T)=\frac{T}{3}$ and substituting eqn (40) for $t(T)$ solution, the scalar field is $\phi(T)=p_0\,\exp\left(p\,\left(\frac{c_0^2}{2}(-T)\right)^{1/4b}\right)$ and eqn (121) for $V(T)$ potential will be:

$$V(T)=-\frac{p^2\,p_0^2}{2}\,\exp\left(2p\,\left(\frac{c_0^2}{2}(-T)\right)^{1/4b}\right)-3p\,b\,p_0^2\,Ei\left(-2p\,\left(\frac{c_0^2}{2}(-T)\right)^{1/4b}\right). \tag{126}$$

By substituting $A(T)$ and eqn (126) into eqn (39), we find as solution:

$$\begin{aligned}F(T)=&-2\kappa\,\phi_0+F_0\,T^3-\frac{\kappa\,p^2\,p_0^2}{(12b-1)}\left[\left(2p\left(-\frac{c_0^2}{2}T\right)^{\frac{1}{4b}}\right)^{6b}\exp\left(-p\left(-\frac{c_0^2}{2}T\right)^{\frac{1}{4b}}\right)\right.\\&\left.\times\,WhittakerM\left(-6b,-6b+\frac{1}{2},2p\left(-\frac{c_0^2}{2}T\right)^{\frac{1}{4b}}\right)+(1-12b)\exp\left(-2p\left(-\frac{c_0^2}{2}T\right)^{\frac{1}{4b}}\right)\right]\\&-\frac{6\kappa\,p\,p_0^2}{\left(b-\frac{1}{12}\right)}\left[2p\,b^2\left(-\frac{c_0^2}{2}T\right)^{\frac{1}{4b}}\,{}_3F_3\left(1,1,1-12b;2,2,-12b+2;-2p\left(-\frac{c_0^2}{2}T\right)^{\frac{1}{4b}}\right)\right.\\&\left.-\left(b-\frac{1}{12}\right)\left(b\ln\left(2p\left(-\frac{c_0^2}{2}\right)^{\frac{1}{4b}}\right)+\gamma b+\frac{\ln(T)}{4}+\frac{1}{12}\right)\right],\end{aligned} \tag{127}$$

where $WhittakerM(B_1,B_2,x)=\exp\left(-\frac{x}{2}\right)\,x^{\frac{1}{2}+B_2}\,{}_1F_1\left(\frac{1}{2}+B_2-B_1;1+2B_2;x\right)$ is a special function.

$\mathbf{p\gg 1:}$ By using $A(T)=\frac{T}{3}$ and substituting eqn (40) into eqn (27) potential, $V(T)$ becomes:

$$V(T)=-\frac{p^2\,p_0^2}{2}\,\exp\left(2p\,\left(\frac{c_0^2}{2}(-T)\right)^{1/4b}\right). \tag{128}$$

By substituting eqn (128) into eqn (39), we find as solution:

$$\begin{aligned}F(T)\approx&-2\kappa\,\phi_0+F_0\,T^3-\frac{\kappa\,p^2\,p_0^2}{(12b-1)}\left[\left(2p\left(-\frac{c_0^2}{2}T\right)^{\frac{1}{4b}}\right)^{6b}\exp\left(-p\left(-\frac{c_0^2}{2}T\right)^{\frac{1}{4b}}\right)\right.\\&\left.\times\,WhittakerM\left(-6b,-6b+\frac{1}{2},2p\left(-\frac{c_0^2}{2}T\right)^{\frac{1}{4b}}\right)\right].\end{aligned} \tag{129}$$

2. $\mathbf{c=1}$: By using eqn (48) and substituting eqn (47), we find that:

$$\phi(T)=p_0\,\exp\left(\frac{\sqrt{2}p\left(1+2b-\frac{1}{c_0^2}\right)^{1/2}}{\sqrt{T}}\right), \tag{130}$$

and eqn (121) potential becomes:

$$V(T)=-\frac{p^2\,p_0^2}{2}\,\exp\left(\frac{T_2}{\sqrt{T}}\right)+p\,(b+2)\,p_0^2\,Ei\left(-\frac{T_2}{\sqrt{T}}\right), \tag{131}$$

where $T_2=2\sqrt{2}p\left(1+2b-\frac{1}{c_0^2}\right)^{1/2}$. By substituting eqn (131) into eqn (39), we find that:

$$\begin{aligned}F(T)=&-2\kappa\,\phi_0+F_0\,T^{1/C}-\frac{\kappa\,p_0^2}{C}\left[p^2\,T^{1/C}\int_T dT'\,T'^{-1/C-1}\,\exp\left(\frac{T_2}{\sqrt{T'}}\right)\right.\\&\left.-2p\,(b+2)\,T^{1/C}\int_T dT'\,T'^{-1/C-1}\,Ei\left(-\frac{T_2}{\sqrt{T'}}\right)\right].\end{aligned} \tag{132}$$

Eqn (132) solutions are possible only for specific values of C ($C\neq 0$) as for example:

- $C = 1$:

$$F(T) = -\,2\kappa\,\phi_0 + F_0\,T - \kappa\,p_0^2\Bigg[\frac{2p^2}{T_2^2}\,\sqrt{T}\left(\sqrt{T}-T_2\right)\,\exp\left(\frac{T_2}{\sqrt{T}}\right) + \frac{2p\,(b+2)}{T_2^2}\left[\left(\sqrt{T}T_2+T\right)\exp\left(-\frac{T_2}{\sqrt{T}}\right)+Ei\left(-\frac{T_2}{\sqrt{T}}\right)T_2^2\right]\Bigg]. \tag{133}$$

- $C = -1$:

$$F(T) = -\,2\kappa\,\phi_0 + F_0\,T^{-1} + \kappa\,p_0^2\Bigg[p^2\,T^{-1}\,\left(Ei_1\left(-\frac{T_2}{\sqrt{T}}\right)T_2^2+\exp\left(\frac{T_2}{\sqrt{T}}\right)\left(T_2\sqrt{T}+T\right)\right) - p\,(b+2)\,\left(\frac{\left(-T_2\sqrt{T}+T\right)\exp\left(-\frac{T_2}{\sqrt{T}}\right)}{T}+\frac{(T_2^2-2T)\,Ei_1\left(\frac{T_2}{\sqrt{T}}\right)}{T}\right)\Bigg]. \tag{134}$$

- $C = 2$:

$$F(T) = -\,2\kappa\,\phi_0 + F_0\,T^{1/2} + \kappa\,p_0^2\left[\left(p^2-2p\,(b+2)\right)\,\frac{\sqrt{T}}{T_2}\exp\left(\frac{T_2}{\sqrt{T}}\right)-2p\,(b+2)\,Ei\left(-\frac{T_2}{\sqrt{T}}\right)\right]. \tag{135}$$

- $C = -2$:

$$F(T) = -\,2\kappa\,\phi_0 + F_0\,T^{-1/2} + \kappa\,p_0^2\Bigg[p^2\,\left(\exp\left(\frac{T_2}{\sqrt{T}}\right)+\frac{T_2}{\sqrt{T}}\,Ei_1\left(-\frac{T_2}{\sqrt{T}}\right)\right) - 2p\,(b+2)\left(\exp\left(-\frac{T_2}{\sqrt{T}}\right)-\left(\frac{T_2}{\sqrt{T}}+1\right)Ei_1\left(\frac{T_2}{\sqrt{T}}\right)\right)\Bigg]. \tag{136}$$

There are several possible values of C yielding to analytical $F(T)$ solutions.
$\mathbf{p \gg 1}$: By using eqn (48) and substituting eqn (47), we find as eqn (27) potential:

$$V(T) = -\,\frac{p^2\,p_0^2}{2}\,\exp\left(\frac{T_2}{\sqrt{T}}\right). \tag{137}$$

By substituting eqn (137) into eqn (39), we find that:

$$F(T) = -\,2\kappa\,\phi_0 + T^{1/C}\left[F_0 - \frac{\kappa\,p^2\,p_0^2}{C}\int_T dT'\,T'^{-1/C-1}\,\exp\left(\frac{T_2}{\sqrt{T'}}\right)\right]. \tag{138}$$

As for eqn (132), there is no general solution, but we can solve some cases for $T_2 \gg 1$:

- $C = 1$:

$$F(T) = -\,2\kappa\,\phi_0 + F_0\,T - 2\kappa\,p^2\,p_0^2\,\frac{T}{T_2^2}\left(1-\frac{T_2}{\sqrt{T}}\right)\exp\left(\frac{T_2}{\sqrt{T}}\right). \tag{139}$$

- $C = -1$:

$$F(T) = -\,2\kappa\,\phi_0 + F_0\,T^{-1} + \kappa\,p^2\,p_0^2\left[\frac{T_2^2}{T}\,Ei_1\left(-\frac{T_2}{\sqrt{T}}\right)+\left(\frac{T_2}{\sqrt{T}}+1\right)\right]. \tag{140}$$

- $C = 2$:

$$F(T) = -\,2\kappa\,\phi_0 + F_0\,T^{1/2} + \kappa\,p^2\,p_0^2\,\frac{T^{1/2}}{T_2}\,\exp\left(\frac{T_2}{\sqrt{T}}\right). \tag{141}$$

- $C = -2$:

$$F(T) = -\,2\kappa\,\phi_0 + F_0\,T^{-1/2} + \kappa\,p^2\,p_0^2\left[\frac{T_2}{\sqrt{T}}\,Ei_1\left(-\frac{T_2}{\sqrt{T}}\right)+\exp\left(\frac{T_2}{\sqrt{T}}\right)\right]. \tag{142}$$

There are several other possible $F(T)$ solutions arising from eqn (138).

3. $\mathbf{c = -1}$: By using eqn (57) and substituting eqn (56), we find that:

$$\phi(T) = p_0 \exp\left(\frac{c_0\, p}{2}\left[-T + \delta_1\sqrt{T^2 + C_2}\right]^{1/2}\right), \tag{143}$$

and eqn (121) potential becomes:

$$\begin{aligned} V(T) = &- \frac{p^2\, p_0^2}{2} \exp\left(c_0\, p\left[-T + \delta_1\sqrt{T^2 + C_2}\right]^{1/2}\right) \\ &+ p\,(b-2)\, p_0^2\, Ei\left(-c_0\, p\left[-T + \delta_1\sqrt{T^2 + C_2}\right]^{1/2}\right). \end{aligned} \tag{144}$$

By substituting eqn (144) into eqn (39), we find that:

$$\begin{aligned} F(T) = &-2\kappa\,\phi_0 + \left[(2C_1 - 3)T - 2C_1\delta_1\sqrt{T^2 + C_2}\right]^{\frac{2(3-2C_1)}{3(3-4C_1)}} \left[T + \sqrt{T^2 + C_2}\right]^{-\frac{4C_1\delta_1}{3(3-4C_1)}} \\ &\times \left[F_0 + 2\kappa\, p_0^2 \int_T dT' \left[p^2 \exp\left(c_0\, p\left[-T' + \delta_1\sqrt{T'^2 + C_2}\right]^{1/2}\right)\right.\right. \\ &- 2p\,(b-2)\, Ei\left(-c_0\, p\left[-T' + \delta_1\sqrt{T'^2 + C_2}\right]^{1/2}\right)\Big] \\ &\times \left[(2C_1 - 3)T' - 2C_1\delta_1\sqrt{T'^2 + C_2}\right]^{-\frac{2(3-2C_1)}{3(3-4C_1)} - 1} \left[T' + \sqrt{T'^2 + C_2}\right]^{\frac{4C_1\delta_1}{3(3-4C_1)}}\Bigg]. \end{aligned} \tag{145}$$

As in sect 3.1, there is no general solution for eqn (145). However, there are specific cases yielding to analytical solutions:

- $C_2 = 0$ and $\delta_1 = -1$:

$$\begin{aligned} F(T) = &-2\kappa\,\phi_0 + F_0\, T^{\frac{2}{(3-4C_1)}} + 2\kappa\, p_0^2\left[\frac{p^2}{(4C_1 - 3)}\, T^{\frac{2}{(3-4C_1)}}\, \mathcal{R}\left(p, C_1, -2c_0^2\, T\right)\right. \\ &+ p\,(b-2)\left(\frac{4c_0 p}{(4C_1 + 1)}\sqrt{-2T}\, {}_3F_3\left(1, 1, \frac{4C_1 + 1}{4C_1 - 3}; 2, 2, \frac{8C_1 - 2}{4C_1 - 3}; -c_0 p\sqrt{-2T}\right)\right. \\ &- \left.\left.\left(\Psi\left(\frac{4}{4C_1 - 3}\right) + \gamma - \Psi\left(\frac{1 + 4C_1}{4C_1 - 3}\right) + \ln\left(c_0 p\sqrt{-2T}\right)\right)\right)\right], \end{aligned} \tag{146}$$

  where $\Psi(x)$ is the Digamma function and $\mathcal{R}\,(p, C_1, x)$ is defined by:

$$\mathcal{R}\,(p, C_1, x) = \int_0^x dx'\, x'^{-\frac{2}{(3-4C_1)} - 1} \exp\left(p\, x'^{1/2}\right), \tag{147}$$

  where $p \neq 0$ and $C_1 \neq \frac{3}{4}$.

Table 3. Some values of eqn (147) for$\mathcal{R}\left(p, C_1, x\right)$ special function.

C_1	$\mathcal{R}\left(p, C_1, x\right)$
0	$\frac{9\left(e^{p\sqrt{x}}\left(-p\sqrt{x}-\frac{1}{3}\right)\left(-p\sqrt{x}\right)^{\frac{2}{3}}+p^2 x\left(\Gamma\left(\frac{2}{3}\right)-\Gamma\left(\frac{2}{3},-p\sqrt{x}\right)\right)\right)}{2x^{\frac{2}{3}}\left(-p\sqrt{x}\right)^{\frac{2}{3}}}$
1	$\frac{2e^{p\sqrt{x}}\left(x^{\frac{3}{2}}p^3-3p^2x+6p\sqrt{x}-6\right)}{p^4}$
-1	$\frac{7\left(-e^{p\sqrt{x}}\left(-p\sqrt{x}\right)^{\frac{3}{7}}+\sqrt{x}\,p\left(\Gamma\left(\frac{3}{7}\right)-\Gamma\left(\frac{3}{7},-p\sqrt{x}\right)\right)\right)}{2x^{\frac{2}{7}}\left(-p\sqrt{x}\right)^{\frac{3}{7}}}$
$\frac{5}{4}$	$\frac{2e^{p\sqrt{x}}\left(p\sqrt{x}-1\right)}{p^2}$
2	$\frac{2x^{\frac{2}{5}}\left(\Gamma\left(\frac{4}{5}\right)-\Gamma\left(\frac{4}{5},-p\sqrt{x}\right)\right)}{\left(-p\sqrt{x}\right)^{\frac{4}{5}}}$
-2	$\frac{11\left(-e^{p\sqrt{x}}\left(-p\sqrt{x}\right)^{\frac{7}{11}}+p\sqrt{x}\left(\Gamma\left(\frac{7}{11}\right)-\Gamma\left(\frac{7}{11},-p\sqrt{x}\right)\right)\right)}{2x^{\frac{2}{11}}\left(-p\sqrt{x}\right)^{\frac{7}{11}}}$

- Other values of C_1 and/or C_2: No analytical and/or closed form of $F(T)$ solution.

$\mathbf{p \gg 1:}$ By using eqn (57) and substituting eqn (56), we find as eqn (27) potential:

$$V(T) = -\frac{p^2\, p_0^2}{2}\,\exp\left(c_0\, p\left[-T+\delta_1\sqrt{T^2+C_2}\right]^{1/2}\right). \tag{148}$$

By substituting eqn (148) into eqn (39), we find that:

$$\begin{aligned} F(T) = & -2\kappa\,\phi_0 + \left[(2C_1-3)T - 2C_1\delta_1\sqrt{T^2+C_2}\right]^{\frac{2(3-2C_1)}{3(3-4C_1)}}\left[T+\sqrt{T^2+C_2}\right]^{-\frac{4C_1\delta_1}{3(3-4C_1)}} \\ & \times\left[F_0 + 2\kappa\, p^2\, p_0^2\int_T dT'\,\exp\left(c_0\, p\left[-T'+\delta_1\sqrt{T'^2+C_2}\right]^{1/2}\right)\right. \\ & \left.\times\left[(2C_1-3)T' - 2C_1\delta_1\sqrt{T'^2+C_2}\right]^{-\frac{2(3-2C_1)}{3(3-4C_1)}-1}\left[T'+\sqrt{T'^2+C_2}\right]^{\frac{4C_1\delta_1}{3(3-4C_1)}}\right]. \end{aligned} \tag{149}$$

Once again, there is no general solution for eqn (149), except for $C_2 = 0$ and $\delta_1 = -1$:

$$F(T) \approx -2\kappa\,\phi_0 + T^{\frac{2}{(3-4C_1)}}\left[F_0 + \frac{2\kappa\, p^2\, p_0^2}{(4C_1-3)}\,\mathcal{R}\left(p, C_1, -2c_0^2\, T\right)\right], \tag{150}$$

where $\mathcal{R}\left(p, C_1, -2c_0^2\, T\right)$ is described by eqn (147).

4. $\mathbf{c = 2}$: By using eqn (77) and substituting eqn (76), we find that:

$$\phi(T) = p_0\,\exp\left(\frac{p(1+b)}{\sqrt{2}c_0}\left[1+\delta_1\sqrt{1-\frac{T}{C_2}}\right]^{-1/2}\right), \tag{151}$$

and eqn (121) potential becomes:

$$\begin{aligned} V(T) = & -\frac{p^2\, p_0^2}{2}\,\exp\left(\frac{2p(1+b)}{\sqrt{2}c_0}\left[1+\delta_1\sqrt{1-\frac{T}{C_2}}\right]^{-1/2}\right) \\ & + p\,(b+4)\,p_0^2\, Ei\left(-\frac{2p(1+b)}{\sqrt{2}c_0}\left[1+\delta_1\sqrt{1-\frac{T}{C_2}}\right]^{-1/2}\right). \end{aligned} \tag{152}$$

By substituting eqn (152) into eqn (39), we find that:

$$\begin{aligned}
F(T) =& -2\kappa\,\phi_0 + \left[-T + C_1\left(1+\delta_1\sqrt{1-\frac{T}{C_2}}\right)\right]^{(2-b)} \left[-\frac{C_2\,T}{C_1\,(C_1-2C_2)+C_2\,T}\right]^{\frac{(b-2)C_1}{2(C_1-2C_2)}} \\
& \times \left[\frac{T+(C_1-2C_2)\left(1+\delta_1\sqrt{1-\frac{T}{C_2}}\right)}{T+(C_1-2C_2)\left(1-\delta_1\sqrt{1-\frac{T}{C_2}}\right)}\right]^{\frac{\delta_1\,(b-2)C_1}{2(C_1-2C_2)}} \\
& \times \Bigg[F_0 + \kappa\,(b-2)\,p_0^2 \int_T dT' \Bigg[-p^2\,\exp\left(\frac{\sqrt{2}p(1+b)}{c_0}\left[1+\delta_1\sqrt{1-\frac{T'}{C_2}}\right]^{-1/2}\right) \\
& + 2p\,(b+4)\,Ei\left(-\frac{\sqrt{2}p(1+b)}{c_0}\left[1+\delta_1\sqrt{1-\frac{T'}{C_2}}\right]^{-1/2}\right)\Bigg] \\
& \times \left[-T' + C_1\left(1+\delta_1\sqrt{1-\frac{T'}{C_2}}\right)\right]^{(b-3)} \left[-\frac{C_2\,T'}{C_1\,(C_1-2C_2)+C_2\,T'}\right]^{-\frac{(b-2)C_1}{2(C_1-2C_2)}} \\
& \times \left[\frac{T'+(C_1-2C_2)\left(1+\delta_1\sqrt{1-\frac{T'}{C_2}}\right)}{T'+(C_1-2C_2)\left(1-\delta_1\sqrt{1-\frac{T'}{C_2}}\right)}\right]^{-\frac{\delta_1\,(b-2)C_1}{2(C_1-2C_2)}}\Bigg]. \qquad (153)
\end{aligned}$$

There is no general $F(T)$ solution for eqn (153). However, there are analytical $F(T)$ solutions for the subcases:

- $C_1 = 0$:

$$\begin{aligned}
F(T) =& -2\kappa\,\phi_0 + (-T)^{(2-b)}\Bigg[F_0 + \kappa\,(b-2)\,p_0^2 \\
& \times \Bigg[-p^2\int_T dT'\,(-T')^{(b-3)}\,\exp\left(\frac{\sqrt{2}p(1+b)}{c_0}\left[1+\delta_1\sqrt{1-\frac{T'}{C_2}}\right]^{-1/2}\right) \\
& + 2p\,(b+4)\int_T dT'\,(-T')^{(b-3)}\,Ei\left(-\frac{\sqrt{2}p(1+b)}{c_0}\left[1+\delta_1\sqrt{1-\frac{T'}{C_2}}\right]^{-1/2}\right)\Bigg]\Bigg]. \qquad (154)
\end{aligned}$$

There are solutions for specific values of b such as:

doi:10.20944/preprints202412.1308.v1

- $b = 1$:

$$\begin{aligned}
F(T) = &-2\kappa\,\phi_0 + (-T)\Bigg[F_0 - \kappa\,p_0^2\Bigg[-\frac{1}{4\sqrt{1+\delta_1\sqrt{\frac{C_2-T}{C_2}}}\,c_0 C_2\left(-\delta_1+\sqrt{\frac{C_2-T}{C_2}}\right)} \\
&\times\Bigg[-\sqrt{1+\delta_1\sqrt{\frac{C_2-T}{C_2}}}\,p^3\left(-\delta_1+\sqrt{\frac{C_2-T}{C_2}}\right) \\
&\times\Bigg(\mathrm{e}^{\frac{2p}{c_0}}\,\mathrm{Ei}_1\left(-\frac{2p\left(\sqrt{2}-\sqrt{1+\delta_1\sqrt{\frac{C_2-T}{C_2}}}\right)}{c_0\sqrt{1+\delta_1\sqrt{\frac{C_2-T}{C_2}}}}\right) \\
&-\mathrm{e}^{-\frac{2p}{c_0}}\,\mathrm{Ei}_1\left(-\frac{2p\left(\sqrt{2}+\sqrt{1+\delta_1\sqrt{\frac{C_2-T}{C_2}}}\right)}{c_0\sqrt{1+\delta_1\sqrt{\frac{C_2-T}{C_2}}}}\right)\Bigg)+\Bigg(\Bigg(\left(p^2+\frac{c_0^2}{2}\right)\sqrt{\frac{C_2-T}{C_2}} \\
&+\delta_1\left(p^2-\frac{c_0^2}{2}\right)\Bigg)\sqrt{1+\delta_1\sqrt{\frac{C_2-T}{C_2}}}-c_0\sqrt{2}\,p\left(-\delta_1+\sqrt{\frac{C_2-T}{C_2}}\right)\Bigg) \\
&\times\exp\left(\frac{2\sqrt{2}\,p}{c_0\sqrt{1+\delta_1\sqrt{\frac{C_2-T}{C_2}}}}\right)c_0\Bigg]+\frac{5}{2\left(1+\delta_1\sqrt{\frac{C_2-T}{C_2}}\right)^{\frac{3}{2}}C_2^2 p\left(\delta_1\sqrt{\frac{C_2-T}{C_2}}-1\right)} \\
&\times\Bigg[\Bigg(\frac{T}{2}\mathrm{Ei}_1\left(\frac{2p\left(\sqrt{2}-\sqrt{1+\delta_1\sqrt{\frac{C_2-T}{C_2}}}\right)}{c_0\sqrt{1+\delta_1\sqrt{\frac{C_2-T}{C_2}}}}\right)\mathrm{e}^{-\frac{2p}{c_0}} \\
&+\frac{T}{2}\mathrm{Ei}_1\left(\frac{2p\left(\sqrt{2}+\sqrt{1+\delta_1\sqrt{\frac{C_2-T}{C_2}}}\right)}{c_0\sqrt{1+\delta_1\sqrt{\frac{C_2-T}{C_2}}}}\right)\mathrm{e}^{\frac{2p}{c_0}} \\
&-4C_2\,\mathrm{Ei}_1\left(\frac{2\sqrt{2}\,p}{c_0\sqrt{1+\delta_1\sqrt{\frac{C_2-T}{C_2}}}}\right)\Bigg)\sqrt{1+\delta_1\sqrt{\frac{C_2-T}{C_2}}}\,p^2 \\
&+\frac{Tc_0}{4}\left(2\sqrt{2}\,p+c_0\sqrt{1+\delta_1\sqrt{\frac{C_2-T}{C_2}}}\right)\exp\left(-\frac{2\sqrt{2}\,p}{c_0\sqrt{1+\delta_1\sqrt{\frac{C_2-T}{C_2}}}}\right)\Bigg]\Bigg]\Bigg].
\end{aligned} \tag{155}$$

doi:10.20944/preprints202412.1308.v1

- $b = 3$:

$$
\begin{aligned}
F(T) = {} & -2\kappa\,\phi_0 - T^{-1}\Bigg[F_0 + \kappa\,p_0^2\Bigg[-\frac{p^2}{3c_0^4}\Bigg[-512\left(p^2 - \frac{3c_0^2}{8}\right)p^2 C_2 \mathrm{Ei}_1\left(-\frac{4\sqrt{2}\,p}{c_0\sqrt{1+\delta_1\sqrt{\frac{C_2-T}{C_2}}}}\right) \\
& + \Bigg(-64pC_2\sqrt{2}\left(\frac{c_0^2}{16}\delta_1\sqrt{\frac{C_2-T}{C_2}} + p^2 - \frac{5c_0^2}{16}\right)\sqrt{1+\delta_1\sqrt{\frac{C_2-T}{C_2}}} \\
& + c_0\Bigg(-16p^2C_2\left(1+\delta_1\sqrt{\frac{C_2-T}{C_2}}\right) + 3Tc_0^2\Bigg)\Bigg)c_0\exp\left(\frac{4\sqrt{2}\,p}{c_0\sqrt{1+\delta_1\sqrt{\frac{C_2-T}{C_2}}}}\right)\Bigg] \\
& + \frac{7p}{3c_0^4}\Bigg[2\left(96C_2p^2c_0^2 - 128p^4C_2 - 3c_0^4T\right)\mathrm{Ei}_1\left(\frac{4\sqrt{2}\,p}{c_0\sqrt{1+\delta_1\sqrt{\frac{C_2-T}{C_2}}}}\right) \\
& + c_0\Bigg(32pC_2\sqrt{2}\left(\frac{c_0^2\delta_1}{16}\sqrt{\frac{C_2-T}{C_2}} + p^2 - \frac{11c_0^2}{16}\right)\sqrt{1+\delta_1\sqrt{\frac{C_2-T}{C_2}}} \\
& + \frac{3c_0}{2}\left(2C_2\delta_1\left(-\frac{8p^2}{3} + c_0^2\right)\sqrt{\frac{C_2-T}{C_2}} + (T+2C_2)\,c_0^2 - \frac{16p^2C_2}{3}\right)\Bigg) \\
& \times\,\exp\left(-\frac{4\sqrt{2}\,p}{c_0\sqrt{1+\delta_1\sqrt{\frac{C_2-T}{C_2}}}}\right)\Bigg]\Bigg]\Bigg]. \qquad (156)
\end{aligned}
$$

- $b = 4$:

$$
\begin{aligned}
F(T) = &-2\kappa\,\phi_0 + (-T)^{-2}\Bigg[F_0 + 2\kappa\, p_0^2\Bigg[-\frac{p^2}{126c_0^8}\Bigg[-78125p^4C_2^2\left(p^4-\frac{84}{25}p^2c_0^2+\frac{168}{125}c_0^4\right)\\
&\times \mathrm{Ei}_1\left(-\frac{5\sqrt{2}\,p}{c_0\sqrt{1+\delta_1\sqrt{\frac{C_2-T}{C_2}}}}\right) - 63\Bigg(-\frac{1}{21}\Bigg(25\Bigg(\frac{3c_0^2\delta_1}{5}\Bigg(\left(T-\frac{32C_2}{15}\right)c_0^4+20C_2p^2c_0^2\\
&-\frac{125p^4C_2}{18}\Bigg)\sqrt{\frac{C_2-T}{C_2}} - \frac{c_0^6}{25}\left(39T+32C_2\right)+p^2\left(T-128C_2\right)c_0^4+\frac{2075p^4C_2c_0^2}{6}\\
&-\frac{625p^6C_2}{6}\Bigg)pC_2\sqrt{2}\sqrt{1+\delta_1\sqrt{\frac{C_2-T}{C_2}}}\Bigg)+c_0\Bigg(-\frac{25\delta_1}{21}\Bigg(\left(T-\frac{76C_2}{5}\right)c_0^4+65C_2p^2c_0^2\\
&-\frac{125p^4C_2}{6}\Bigg)p^2C_2\sqrt{\frac{C_2-T}{C_2}}+T^2c_0^6+\frac{45p^2C_2\left(T+\frac{76C_2}{27}\right)c_0^4}{7}-\frac{125p^4C_2\left(T+26C_2\right)c_0^2}{42}\\
&+\frac{3125p^6C_2^2}{126}\Bigg)\Bigg)c_0\exp\left(\frac{5\sqrt{2}\,p}{c_0\sqrt{1+\delta_1\sqrt{\frac{C_2-T}{C_2}}}}\right)\Bigg]\\
&+\frac{p}{63c_0^8}\Bigg[\left(-210000p^4c_0^4C_2^2+350000c_0^2p^6C_2^2-78125C_2^2p^8+504T^2c_0^8\right)\\
&\times\mathrm{Ei}_1\left(\frac{5\sqrt{2}\,p}{c_0\sqrt{1+\delta_1\sqrt{\frac{C_2-T}{C_2}}}}\right)+252\Bigg(\frac{1}{4}\Bigg(-\frac{25pC_2}{21}\sqrt{2}\Bigg(\frac{3\delta_1}{5}c_0^2\Bigg(\left(T-\frac{116C_2}{15}\right)c_0^4\\
&+\frac{250C_2p^2c_0^2}{9}-\frac{125p^4C_2}{18}\Bigg)\sqrt{\frac{C_2-T}{C_2}}+\frac{\left(-67T-116C_2\right)c_0^6}{25}+p^2\left(T-\frac{790C_2}{3}\right)c_0^4\\
&+\frac{925p^4C_2c_0^2}{2}-\frac{625p^6C_2}{6}\Bigg)\sqrt{1+\delta_1\sqrt{\frac{C_2-T}{C_2}}}-c_0\Bigg(-\frac{25\delta_1}{21}\Bigg(\frac{28\left(-T-2C_2\right)c_0^6}{25}\\
&+p^2\left(T-\frac{188C_2}{5}\right)c_0^4+\frac{265p^4C_2c_0^2}{3}-\frac{125p^6C_2}{6}\Bigg)C_2\sqrt{\frac{C_2-T}{C_2}}+\left(T^2+\frac{8C_2^2}{3}\right)c_0^6\\
&+\frac{205p^2C_2\left(T+\frac{188C_2}{41}\right)c_0^4}{21}-\frac{125p^4C_2\left(T+\frac{106C_2}{3}\right)c_0^2}{42}+\frac{3125p^6C_2^2}{126}\Bigg)\Bigg)\\
&\times\exp\left(-\frac{5\sqrt{2}\,p}{c_0\sqrt{1+\delta_1\sqrt{\frac{C_2-T}{C_2}}}}\right)\Bigg)c_0\Bigg]\Bigg]\Bigg].
\end{aligned}
\tag{157}
$$

There are other values of b yielding to analytical $F(T)$ solutions.

- $C_1 = 2C_2$:

$$\begin{aligned} F(T) = &-2\kappa\,\phi_0 + \left[-T + 2C_2\left(1 + \delta_1\sqrt{1 - \frac{T}{C_2}}\right)\right]^{(2-b)} \Bigg[F_0 + \kappa\,(b-2)\,p_0^2 \\ &\times \int_T dT' \Bigg[-p^2 \exp\left(\frac{\sqrt{2}p(1+b)}{c_0}\left[1 + \delta_1\sqrt{1 - \frac{T'}{C_2}}\right]^{-1/2}\right) \\ &+ 2p\,(b+4)\,Ei\left(-\frac{\sqrt{2}p(1+b)}{c_0}\left[1 + \delta_1\sqrt{1 - \frac{T'}{C_2}}\right]^{-1/2}\right)\Bigg] \\ &\times \left[-T' + 2C_2\left(1 + \delta_1\sqrt{1 - \frac{T'}{C_2}}\right)\right]^{(b-3)}\Bigg]. \end{aligned} \tag{158}$$

There are solutions for specific values of δ_1 and b:

  - $b = 3$:

$$\begin{aligned} F(T) = &-2\kappa\,\phi_0 + \left[-T + 2C_2\left(1 + \delta_1\sqrt{1 - \frac{T}{C_2}}\right)\right]^{-1} \Bigg[F_0 + \kappa\,p_0^2 \\ &\times \Bigg[-\frac{p^2}{3c_0^4}\Bigg[-512p^2\left(p^2 - \frac{3c_0^2}{8}\right) C_2\mathrm{Ei}_1\left(-\frac{4\sqrt{2}\,p}{c_0\sqrt{1 + \delta_1\sqrt{\frac{C_2-T}{C_2}}}}\right) \\ &+ \exp\left(\frac{4\sqrt{2}\,p}{c_0\sqrt{1 + \delta_1\sqrt{\frac{C_2-T}{C_2}}}}\right)\Bigg(-64p\sqrt{2}\,C_2\left(\frac{c_0^2\delta_1}{16}\sqrt{\frac{C_2 - T}{C_2}} + p^2 - \frac{5c_0^2}{16}\right) \\ &\times \sqrt{1 + \delta_1\sqrt{\frac{C_2 - T}{C_2}}} + c_0\left(-16p^2C_2\left(1 + \delta_1\sqrt{\frac{C_2 - T}{C_2}}\right) + 3Tc_0^2\right)\Bigg)c_0\Bigg] \\ &+ \frac{7p}{3c_0^4}\Bigg[2\left(96C_2c_0^2p^2 - 128C_2p^4 - 3Tc_0^4\right)\,\mathrm{Ei}_1\left(\frac{4\sqrt{2}\,p}{c_0\sqrt{1 + \delta_1\sqrt{\frac{C_2-T}{C_2}}}}\right) \\ &+ \Bigg(\Bigg(32p\sqrt{2}C_2\left(\frac{c_0^2\delta_1}{16}\sqrt{\frac{C_2 - T}{C_2}} + p^2 - \frac{11c_0^2}{16}\right)\sqrt{1 + \delta_1\sqrt{\frac{C_2 - T}{C_2}}} \\ &+ \frac{3c_0}{2}\left(2C_2\delta_1\left(-\frac{8p^2}{3} + c_0^2\right)\sqrt{\frac{C_2 - T}{C_2}} + (T + 2C_2)\,c_0^2 - \frac{16p^2C_2}{3}\right)\Bigg) \\ &\times \exp\left(-\frac{4\sqrt{2}\,p}{c_0\sqrt{1 + \delta_1\sqrt{\frac{C_2-T}{C_2}}}}\right)\Bigg)c_0\Bigg]\Bigg]\Bigg]. \end{aligned} \tag{159}$$

doi:10.20944/preprints202412.1308.v1

- $b = 4$:

$$
\begin{aligned}
F(T) = &-2\kappa\,\phi_0 + \left[-T + 2C_2\left(1+\delta_1\sqrt{1-\frac{T}{C_2}}\right)\right]^{-2}\Bigg[F_0 + 2\kappa\,p_0^2 \\
&\times\Bigg[-\frac{p^2}{126c_0^8}\Bigg[-78125p^6C_2^2\left(p^2-\frac{28c_0^2}{25}\right)\mathrm{Ei}_1\left(-\frac{5\sqrt{2}\,p}{c_0\sqrt{1+\delta_1\sqrt{\frac{C_2-T}{C_2}}}}\right) \\
&-63\,\exp\left(\frac{5\sqrt{2}\,p}{c_0\sqrt{1+\delta_1\sqrt{\frac{C_2-T}{C_2}}}}\right)c_0\Bigg(-\frac{25}{21}p\sqrt{2}\,C_2\Bigg(\frac{3\delta_1c_0^2}{5}\Bigg(\left(T-\frac{4C_2}{15}\right)c_0^4 \\
&+\frac{40c_0^2p^2C_2}{9}-\frac{125p^4C_2}{18}\Bigg)\sqrt{\frac{C_2-T}{C_2}}+\frac{(17T-4C_2)\,c_0^6}{25}+p^2\left(T+\frac{8C_2}{3}\right)c_0^4 \\
&+\frac{225p^4C_2c_0^2}{2}-\frac{625p^6C_2}{6}\Bigg)\sqrt{1+\delta_1\sqrt{\frac{C_2-T}{C_2}}}+\Bigg(-\frac{25}{21}C_2\delta_1\Bigg(\frac{56\,(T-C_2)\,c_0^6}{25} \\
&+p^2\left(T+\frac{8C_2}{5}\right)c_0^4+\frac{55p^4C_2c_0^2}{3}-\frac{125p^6C_2}{6}\Bigg)\sqrt{\frac{C_2-T}{C_2}}+\left(T^2-4TC_2+\frac{8}{3}C_2^2\right) \\
&\times\,c_0^6-\frac{5p^2C_2\,(8C_2+T)\,c_0^4}{21}-\frac{125p^4\left(T+\frac{22C_2}{3}\right)C_2c_0^2}{42}+\frac{3125p^6C_2^2}{126}\Bigg)c_0\Bigg)\Bigg] \\
&+\frac{p}{378c_0^8}\Bigg[\Bigg(8064C_2^2c_0^8\delta_1\left(\frac{C_2-T}{C_2}\right)^{3/2}+\left(8064C_2^2-12096TC_2+3024T^2\right)c_0^8 \\
&+700000C_2^2p^6c_0^2-468750p^8C_2^2\Bigg)\mathrm{Ei}_1\left(\frac{5\sqrt{2}\,p}{c_0\sqrt{1+\delta_1\sqrt{\frac{C_2-T}{C_2}}}}\right)+1512c_0 \\
&\times\Bigg(\Bigg(-\frac{25p}{84}\Bigg(\frac{3}{5}\Bigg(\left(T+\frac{44C_2}{45}\right)c_0^4+\frac{190c_0^2p^2C_2}{27}-\frac{125p^4C_2}{18}\Bigg)c_0^2\delta_1\sqrt{\frac{C_2-T}{C_2}} \\
&+\left(\frac{23T}{75}+\frac{44C_2}{75}\right)c_0^6+p^2\left(T+\frac{38C_2}{9}\right)c_0^4+\frac{2725p^4C_2c_0^2}{18}-\frac{625p^6C_2}{6}\Bigg) \\
&\times\,\sqrt{2}\,C_2\sqrt{1+\delta_1\sqrt{\frac{C_2-T}{C_2}}}-\frac{c_0}{4}\Bigg(-\frac{25}{21}C_2\delta_1\Bigg(\left(\frac{28T}{15}-\frac{56C_2}{75}\right)c_0^6 \\
&+p^2\left(T+\frac{52C_2}{15}\right)c_0^4+\frac{235p^4C_2c_0^2}{9}-\frac{125p^6C_2}{6}\Bigg)\sqrt{\frac{C_2-T}{C_2}} \\
&+\left(T^2-\frac{8}{3}TC_2+\frac{8}{9}C_2^2\right)c_0^6+\frac{55p^2\left(T-\frac{52C_2}{11}\right)C_2c_0^4}{63}-\frac{125p^4\left(T+\frac{94C_2}{9}\right)C_2c_0^2}{42} \\
&+\frac{3125p^6C_2^2}{126}\Bigg)\Bigg)\,\exp\left(-\frac{5\sqrt{2}\,p}{c_0\sqrt{1+\delta_1\sqrt{\frac{C_2-T}{C_2}}}}\right)\Bigg)\Bigg]\Bigg]\Bigg].
\end{aligned}
\tag{160}
$$

There are other values of b yielding to analytical $F(T)$ solutions.

$\mathbf{p \gg 1:}$ By using eqn (77) and substituting eqn (76), we find as eqn (27) potential:

$$
V(T) = -\frac{p^2\,p_0^2}{2}\,\exp\left(\frac{2p(1+b)}{\sqrt{2}c_0}\left[1+\delta_1\sqrt{1-\frac{T}{C_2}}\right]^{-1/2}\right).
\tag{161}
$$

By substituting eqn (161) into eqn (39), we find that:

$$
\begin{aligned}
F(T) = &-2\kappa\,\phi_0 + \left[-T + C_1\left(1+\delta_1\sqrt{1-\frac{T}{C_2}}\right)\right]^{(2-b)}\,\left[-\frac{C_2\,T}{C_1\,(C_1-2C_2)+C_2\,T}\right]^{\frac{(b-2)C_1}{2\left(C_1-2C_2\right)}} \\
&\times\left[\frac{T+(C_1-2C_2)\left(1+\delta_1\sqrt{1-\frac{T}{C_2}}\right)}{T+(C_1-2C_2)\left(1-\delta_1\sqrt{1-\frac{T}{C_2}}\right)}\right]^{\frac{\delta_1\,(b-2)C_1}{2\left(C_1-2C_2\right)}} \\
&\times\left[F_0-\kappa\,p^2\,p_0^2\int_T dT'\,\exp\left(\frac{2p(1+b)}{\sqrt{2}c_0}\,\left[1+\delta_1\,\sqrt{1-\frac{T'}{C_2}}\right]^{-1/2}\right)\right. \\
&\times\left[-T'+C_1\left(1+\delta_1\sqrt{1-\frac{T'}{C_2}}\right)\right]^{(b-3)}\,\left[-\frac{C_2\,T'}{C_1\,(C_1-2C_2)+C_2\,T'}\right]^{-\frac{(b-2)C_1}{2\left(C_1-2C_2\right)}} \\
&\left.\times\left[\frac{T'+(C_1-2C_2)\left(1+\delta_1\sqrt{1-\frac{T'}{C_2}}\right)}{T'+(C_1-2C_2)\left(1-\delta_1\sqrt{1-\frac{T'}{C_2}}\right)}\right]^{-\frac{\delta_1\,(b-2)C_1}{2\left(C_1-2C_2\right)}}\right]. \qquad (162)
\end{aligned}
$$

There is no general $F(T)$ solution for eqn (162). However, there are analytical $F(T)$ solution for the subcases:

- $C_1 = 0$:

$$
\begin{aligned}
F(T) = &-2\kappa\,\phi_0 + (-T)^{(2-b)}\left[F_0-\kappa\,p^2\,p_0^2\right. \\
&\left.\times\int_T dT'\,\exp\left(\frac{\sqrt{2}p(1+b)}{c_0}\,\left[1+\delta_1\sqrt{1-\frac{T'}{C_2}}\right]^{-1/2}\right)\,(-T')^{(b-3)}\right]. \qquad (163)
\end{aligned}
$$

  There are solutions for specific values of b:

  - $b = 1$:

$$
\begin{aligned}
F(T) \approx &-2\kappa\,\phi_0 - T\left[F_0-\frac{\kappa\,p_0^2\,p^2}{4\,c_0C_2}\left[p\exp\left(\frac{2p}{c_0}\right)\mathrm{Ei}_1\left(-\frac{2p\left(\sqrt{2}-\sqrt{1+\delta_1\sqrt{\frac{C_2-T}{C_2}}}\right)}{c_0\sqrt{1+\delta_1\sqrt{\frac{C_2-T}{C_2}}}}\right)\right.\right. \\
&\left.\left.+\frac{\left(1+\delta_1\sqrt{\frac{C_2-T}{C_2}}\right)}{\left(1-\delta_1\sqrt{\frac{C_2-T}{C_2}}\right)}\exp\left(\frac{2\sqrt{2}\,p}{c_0\sqrt{1+\delta_1\sqrt{\frac{C_2-T}{C_2}}}}\right)c_0\right]\right]. \qquad (164)
\end{aligned}
$$

  - $b = 3$:

$$
F(T) \approx -2\kappa\,\phi_0 - \left[\frac{F_0}{T}+\frac{64\kappa\,p_0^2p^5C_2}{3c_0^3}\,\frac{\sqrt{2}}{T}\sqrt{1+\delta_1\sqrt{\frac{C_2-T}{C_2}}}\exp\left(\frac{4\sqrt{2}\,p}{c_0\sqrt{1+\delta_1\sqrt{\frac{C_2-T}{C_2}}}}\right)\right]. \qquad (165)
$$

  - $b = 4$:

$$
F(T) \approx -2\kappa\,\phi_0 + \left[\frac{F_0}{T^2}+\frac{15625\kappa\,p_0^2\,p^9C_2^2}{63\sqrt{2}c_0^7\,T^2}\,\sqrt{1+\delta_1\sqrt{\frac{C_2-T}{C_2}}}\,\exp\left(\frac{5\sqrt{2}\,p}{c_0\sqrt{1+\delta_1\sqrt{\frac{C_2-T}{C_2}}}}\right)\right]. \qquad (166)
$$

- $C_1 = 2C_2$:

$$\begin{aligned} F(T) = &- 2\kappa\,\phi_0 + \left[-T + C_1\left(1+\delta_1\sqrt{1-\frac{T}{C_2}}\right)\right]^{(2-b)}\left[\frac{-C_2\,T}{C_1\,(C_1-2C_2)+C_2\,T}\right]^{\frac{(b-2)C_1}{2(C_1-2C_2)}} \\ &\times \left[\frac{T+(C_1-2C_2)\left(1+\delta_1\sqrt{1-\frac{T}{C_2}}\right)}{T+(C_1-2C_2)\left(1-\delta_1\sqrt{1-\frac{T}{C_2}}\right)}\right]^{\frac{\delta_1\,(b-2)C_1}{2(C_1-2C_2)}} \\ &\times \Bigg[F_0 - \kappa\,p^2\,p_0^2\int_T dT'\,\exp\left(\frac{\sqrt{2}p(1+b)}{c_0}\left[1+\delta_1\sqrt{1-\frac{T'}{C_2}}\right]^{-1/2}\right) \\ &\times \left[-T' + C_1\left(1+\delta_1\sqrt{1-\frac{T'}{C_2}}\right)\right]^{(b-3)}\left[-\frac{C_2\,T'}{C_1\,(C_1-2C_2)+C_2\,T'}\right]^{-\frac{(b-2)C_1}{2(C_1-2C_2)}} \\ &\times \left[\frac{T'+(C_1-2C_2)\left(1+\delta_1\sqrt{1-\frac{T'}{C_2}}\right)}{T'+(C_1-2C_2)\left(1-\delta_1\sqrt{1-\frac{T'}{C_2}}\right)}\right]^{-\frac{\delta_1\,(b-2)C_1}{2(C_1-2C_2)}}\Bigg]. \end{aligned} \tag{167}$$

  There are solutions for specific values of δ_1 and b:
  - $b = 3$:

$$\begin{aligned} F(T) \approx &- 2\kappa\,\phi_0 + \left[-T + C_1\left(1+\delta_1\sqrt{1-\frac{T}{C_2}}\right)\right]^{-1} \\ &\times \left[F_0 + \frac{64\kappa\,p^5\,p_0^2}{3c_0^3}\sqrt{2}\,C_2\sqrt{1+\delta_1\sqrt{\frac{C_2-T}{C_2}}}\exp\left(\frac{4\sqrt{2}\,p}{c_0\sqrt{1+\delta_1\sqrt{\frac{C_2-T}{C_2}}}}\right)\right]. \end{aligned} \tag{168}$$

  - $b = 4$:

$$\begin{aligned} F(T) \approx &- 2\kappa\,\phi_0 + \left[-T + C_1\left(1+\delta_1\sqrt{1-\frac{T}{C_2}}\right)\right]^{-2} \\ &\times \left[F_0 + \frac{15625\kappa\,p^9\,p_0^2}{126\sqrt{2}c_0^7}\,C_2^2\sqrt{1+\delta_1\sqrt{\frac{C_2-T}{C_2}}}\exp\left(\frac{5\sqrt{2}\,p}{c_0\sqrt{1+\delta_1\sqrt{\frac{C_2-T}{C_2}}}}\right)\right]. \end{aligned} \tag{169}$$

There are several other cases yielding to new analytical $F(T)$ solutions. All the previous teleparallel $F(T)$ solutions are new.

4.2. Exponential Ansatz Solutions

By using eqns (102) and (103) with the eqn (31) ansatz, we substitute eqn (123) into eqn (37) for again finding the $F(T)$ solution defined by eqn (39) formula. We will compute eqn (39) results the following potential cases:

1. $\mathbf{c} = -\mathbf{2b}$: By using $A(T) = \frac{T}{3}$ and eqn (104), we find that $\phi(T) = p_0\left(\frac{c_0^2}{2}\right)^{p/4b}(-T)^{p/4b}$ and the eqn (123) $V(T)$ potential is:

$$V(T) = -\frac{p\,(p-3b)\,p_0^2}{2}\left(\frac{c_0^2}{2}\right)^{p/2b}(-T)^{p/2b}, \tag{170}$$

where $p \neq 3b$. By substituting eqn (170) into eqn (39), the $F(T)$ solution is:

$$F(T) = -\,2\kappa\,\phi_0 + F_0\,T^3 - \frac{6\kappa\,p\,b\,(p-3b)\,p_0^2}{(p-6b)}\left(\frac{c_0^2}{2}\right)^{p/2b}(-T)^{p/2b}. \tag{171}$$

$\mathbf{p \gg 1}$: Under this limit, the eqn (27) becomes:

$$V(T) = -\frac{p^2\,p_0^2}{2}\left(\frac{c_0^2}{2}\right)^{p/2b}(-T)^{p/2b}. \tag{172}$$

By substituting eqn (172) into eqn (39), we find as solution:

$$F(T) \approx -\,6\kappa\,p\,b\,p_0^2\left(\frac{c_0^2}{2}\right)^{p/2b}(-T)^{p/2b}. \tag{173}$$

2. **$\mathbf{c \neq -2b}$** (General case): From eqns (102) and (103), we find that:

$$\phi(T) = p_0\left(\frac{c_0^2}{2}\right)^{-p/2c}(T_0-T)^{-p/2c}, \tag{174}$$

and eqn (123) potential becomes:

$$V(T) = -\frac{p\,(b+2c+p)\,p_0^2}{2}\left(\frac{2}{c_0^2}\right)^{p/c}(T_0-T)^{-p/c}, \tag{175}$$

where $p \neq -b-2c$. By substituting eqn (175) into eqn (39), the $F(T)$ solution is:

$$F(T) = -\,2\kappa\,\phi_0 + (T-T_1)^{2(c-b)/c}\left[F_0 + \frac{2\kappa\,p\,p_0^2\,(b-c)(b+2c+p)}{c}\left(-\frac{2}{c_0^2}\right)^{p/c}\right.$$
$$\left.\times\int_T dT'\;(T'-T_0)^{-p/c}\;(T'-T_1)^{(2b-3c)/c}\right]. \tag{176}$$

There is no general solution to eqn (176), but there are $F(T)$ solutions for the following cases:

- $T_1 = T_0$:

$$F(T) = -\,2\kappa\,\phi_0 + F_0\,(T-T_0)^{2-2b/c} + 2\kappa\,p\,p_0^2\,\frac{(b-c)(b+2c+p)}{(2b-p-2c)}\left(\frac{-2}{c_0^2\,(T-T_0)}\right)^{p/c}. \tag{177}$$

- $b = \frac{3c}{2}$:

$$F(T) = -\,2\kappa\,\phi_0 + (T-T_1)^{-1}\left[F_0 + \kappa\,p\,p_0^2\,\frac{c(7c+2p)}{2\,(c-p)}\left(-\frac{2}{c_0^2}\right)^{p/c}(T-T_0)^{1-p/c}\right]. \tag{178}$$

- $b = 2c$:

$$F(T) = -\,2\kappa\,\phi_0 + (T-T_1)^{-2}\left[F_0 + 2\kappa\,p\,p_0^2\,c(4c+p)\left(-\frac{2}{c_0^2}\right)^{p/c}\right.$$
$$\left.\times\,\frac{(T-T_0)^{\frac{-p+c}{c}}\,\left((T+T_0-2T_1)\,c - p\,(T-T_1)\right)}{2c^2-3cp+p^2}\right]. \tag{179}$$

- $b = 3c$:

$$F(T) = -\,2\kappa\,\phi_0 + (T - T_1)^{-4}\left[F_0 + \frac{24\kappa\,p\,p_0^2\,c(3c + 2c + p)}{(24c^4 - 50c^3 p + 35c^2 p^2 - 10c\,p^3 + p^4)}\left(-\frac{2}{c_0^2}\right)^{p/c}\right.$$
$$\times\,(T - T_0)^{\frac{-p+c}{c}}\left(\left(2T_1^2 + (-2T - 2T_0)\,T_1 + T^2 + T_0^2\right)(T + T_0 - 2T_1)\,c^3\right.$$
$$-\frac{11}{6}\left(\frac{26T_1^2}{11} + \left(-\frac{31T}{11} - \frac{21T_0}{11}\right)T_1 + T^2 + \frac{9TT_0}{11} + \frac{6T_0^2}{11}\right)p\,(T - T_1)\,c^2$$
$$\left.\left.+\left(T + \frac{T_0}{2} - \frac{3T_1}{2}\right)p^2\,(T - T_1)^2\,c - \frac{p^3\,(T - T_1)^3}{6}\right)\right]. \tag{180}$$

There are several other possible $F(T)$ solutions for $b \geq \frac{3c}{2}$.

$\mathbf{p \gg 1:}$ Under this limit, the eqn (27) becomes:

$$V(T) = -\frac{p^2\,p_0^2}{2}\left(\frac{2}{c_0^2}\right)^{p/c}(T_0 - T)^{-p/c}. \tag{181}$$

By substituting eqn (175) into eqn (39), we find as solution:

$$F(T) = -\,2\kappa\,\phi_0 + (T - T_1)^{2(c-b)/c}\left[F_0 - \frac{2\kappa\,p^2\,p_0^2(c - b)}{c}\left(-\frac{2}{c_0^2}\right)^{p/c}\right.$$
$$\left.\times\int_T dT'\,\left(T' - T_0\right)^{-p/c}\,\left(T' - T_1\right)^{(2b-3c)/c}\right]. \tag{182}$$

There is no general solution to eqn (182), but there are $F(T)$ solution for the following cases:

- $T_1 = T_0$:

$$F(T) \approx -\,2\kappa\,\phi_0 + F_0\,(T - T_0)^{2-2b/c} - 2\kappa\,p\,p_0^2\,(b - c)\left(-\frac{2}{c_0^2}\right)^{p/c}(T - T_0)^{-p/c}. \tag{183}$$

  For $c > 0$ and $p \to \infty$, eqn (183) becomes:

$$F(T)\;\to\;-2\kappa\,\phi_0 + F_0\,(T - T_0)^{2-2b/c}. \tag{184}$$

  For $c < 0$ and $p \to \infty$, eqn (183) becomes:

$$F(T)\;\to\;-2\kappa\,p\,p_0^2\,(b + |c|)\left(-\frac{c_0^2}{2}\right)^{p/|c|}(T - T_0)^{p/|c|}\;\to\;\infty. \tag{185}$$

- $b = \frac{3c}{2}$:

$$F(T) \approx -\,2\kappa\,\phi_0 + F_0\,(T - T_1)^{-1} - \kappa\,p\,p_0^2\,c\left(-\frac{2}{c_0^2}\right)^{p/c}(T - T_0)^{-p/c}\,(T - T_1)^{-1}. \tag{186}$$

- $b = 2c$:

$$F(T) \approx -\,2\kappa\,\phi_0 + F_0\,(T - T_1)^{-2} - 2\kappa\,p\,p_0^2\,c\left(-\frac{2}{c_0^2}\right)^{p/c}(T - T_0)^{-\frac{p}{c}}\,(T - T_1)^{-1}. \tag{187}$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 17 December 2024 doi:10.20944/preprints202412.1308.v1

- $b = 3c$:

$$F(T) \approx -2\kappa\,\phi_0 + F_0\,(T - T_1)^{-4} - 4\kappa\,p\,p_0^2\,c\left(-\frac{2}{c_0^2}\right)^{p/c}\,(T - T_0)^{-\frac{p}{c}}\,(T - T_1)^{-1}. \tag{188}$$

- For $c > 0$ and $p \to \infty$, we find that all cases go to the eqn (184) limit. For the $c < 0$ and $p \to \infty$ limit, we obtain that:

$$F(T) \sim\,(T - T_0)^{\frac{p}{|c|}}\,(T - T_1)^{-1}. \tag{189}$$

All the previous teleparallel $F(T)$ solutions are new. However, there are other possible subcases leading to additional teleparallel $F(T)$ solutions.

5. Discussion and Conclusions

The primary aim was to find the teleparallel $F(T)$ solutions coming from the FEs precisely described by the eqns (21a) to (21d) with two usual types of scalar field $\phi(t)$: power-law and exponential. In both cases and depending on the values of p in the scalar field, we solved the laws conservation and the FEs for various types of potentials $V(T)$. We first obtain that the FEs transform by various substitutions into the single eqn (37), this for any type of scalar field and ansatz. From this last equation, the possible teleparallel $F(T)$ solutions all boil down to eqn (39) using the function $A(T)$ described by eqn (38). All new $F(T)$ solutions obtained in sections 3.1 through the end of section 4.2 were found by applying the eqn (39) for various types of scalar potentials. For the A_2 and A_3 coframe components, the power-law ansatz has been used in sections 3.1 and 4.1 and the application has been restricted to the values of $c = -2b$, 1, -1 and 2. Most of other values of c do not result in closed analytic functions and/or are not analytically solvable. In sections 3.2 and 4.2, the exponential coframe ansatz has been used to study a well-known case of infinite sum of power terms. At the same time, this ansatz is much simpler to treat and only brings out two types of cases instead of an infinity: general and $c = -2b$. For the rest, each type of case treated here leads to its own class of $F(T)$ solutions with its particularities.

For the power-law $\phi(t)$ treated in section 3, we first studied in section 3.1 via the power-law coframe ansatz the cases of potentials described by eqns (25), (27) and (29), i.e. the general case, $p \gg 1$ and $p = 1$. This was intended to study the effects of different scalar potential types associated with gravitational sources. The $p \gg 1$ potential cases can be useful for phantom energy models studies involving fast accelerating universe expansion. Often physical phenomena describing by a scalar field are characterized by its associated potential and teleparallel $F(T)$-gravity cosmology is no exception to the rule. In this section 3.1, we find general and exact $F(T)$ solutions for the $c = -2b$ and $c = 1$ cases. For the $c = -1$ and $c = 2$ cases, even if there is no real general and analytic large coverage solution, however there are analytical $F(T)$ solutions for specific cases depending on the parameters C_1 and C_2 specially defined to simply solve these cases. Therefore, larger values of c subcases can be useful for studies on teleparallel phantom energy and quintom models. In section 3.2 via the exponential ansatz coframe and for the potentials described by eqns (32) and (34), we obtain purely analytic $F(T)$solutions, but this required the introduction and definition of new special functions depending on the parameters of the scalar field and the ansatz used to achieve them. Even in cosmology and teleparallel gravity, special functions are a necessary evil to achieve analytic solutions. Other types of approaches such as Quantum Gravity theories or fundamental particle interactions with gravity regularly require special functions to express certain new solutions (See for example refs. [106,107]). It would not be surprising if special functions were to be used even more in future work in teleparallel gravity, more specifically for solutions involving more complex and sophisticated source terms to be solved.

By considering exponential $\phi(t)$ in section 4, we first have in section 4.1 studied the power-law coframe ansatz and for the potentials described in eqns (27) and (121). We obtain teleparallel $F(T)$ solutions that are a bit more complex compared to the results obtained in section 3.1, but with some clearly visible similarities. Again, the $c = -2b$ and $c = 1$ cases lead to analytical $F(T)$ solutions. However, we only obtain $F(T)$ solutions for very specific subcases in the $c = -1$ and $c = 2$ situations. This is not surprising compared to the results of section 3.1. For section 4.2 with the exponential ansatz coframe and the potentials described by eqns (27) and (123), we obtain essentially purely analytical and easily usable $F(T)$ solutions. These latter solutions confirm at the same time some new simple solutions obtained in section 3. Let us not forget that the exponential ansatz and scalar field are cases fundamentally expressed by an infinite sum of terms in powers. This last fact also explains some common points in the newly obtained $F(T)$ solutions. In comparison, the new simplest $F(T)$ solutions obtained in this paper are also similar to the teleparallel cosmology solutions obtained in the recent literature (see for example refs [72,80,82]). This finding is also an additional argument in favor of the new solutions rightness obtained in the present paper.

Beyond the new solutions, this approach aimed to obtain purely analytical teleparallel $F(T)$ solutions that could be used in the future to study complex cosmological models in more detail using teleparallel $F(T)$-gravity tools. One need only think in particular of the quintessence Dark Energy, the cosmological Phantom Energy (negative energy) or even cosmological quintom models. In these latter cases, one will be able in the near future to study models involving perfect fluids and scalar fields at the same time, in order to represent reality more faithfully. In such a case, we will use the common (or very similar) $F(T)$ solutions of the present paper and of some recent papers concerning teleparallel $F(T)$ solutions in perfect fluids [47,48]. This will ultimately lead to complete and realistic cosmological models that can fully describe and explain the dark energy quintessence process ($-1 < \alpha_Q < -\frac{1}{3}$), a realistic and most probable scenario according to the recent literature [62–64,69]. After that, one could just as well study with a similar approach some physical models of the type Cosmological Phantom Energy ($\alpha_Q < -1$) leading to the extreme scenario of the acceleration of uncontrolled universe expansion leading to the Big Rip [86–88]. We can also add the quintom dark energy models, because this is by definition a mix of the quintessence and phantom models [93–98]. Like the fundamental scalar field associated with the Dark Energy quintessence process, one could obtain for the different models involving Phantom Energy such fundamental scalar fields. Perhaps the cosmological process of uncontrolled universe acceleration leading to the Big Rip would in fact be explainable by a fundamental scalar field. The same questions will also be relevant and useful for future quintom models studies in teleparallel gravity. The questions will then be: what type(s) of $V(\phi(T))$ exactly; what are the most appropriate teleparallel $F(T)$ solutions; what are the coframes, spin-connections and other conditions. All these questions and hypotheses deserve to be answered seriously and tactfully in future works by using the teleparallel gravity framework and toolkit.

Acknowledgments: AL is supported by an Atlantic Association of Research in Mathematical Sciences (AARMS) fellowship. Thanks to A.A. Coley for useful and constructive comments. Thanks also to R.J. van den Hoogen for relevant discussions on the topic.

Abbreviations

The following abbreviations are used in this manuscript:

AL	Alexandre Landry
CK	Cartan-Karlhede
DE	Differential Equation
EoS	Equation of State
Eqn	Equation
FE	Field Equation
GR	General Relativity
KV	Killing Vector
NGR	New General Relativity
SHO	Simple Harmonic Oscillator
TEGR	Teleparallel Equivalent of General Relativity

References

1. Aldrovandi, R. & Pereira, J.G. *Teleparallel Gravity, An Introduction*, Springer, 2013.
2. Bahamonde, S., Dialektopoulos, K., Escamilla-Rivera, C., Farrugia, G., Gakis, V., Hendry, M., Hohmann, M., Said, J.L., Mifsud, J. & Di Valentino, E., Teleparallel Gravity: From Theory to Cosmology, *Report Progress Physics* **2023**, *86*, 026901 [arXiv:2106.13793 [gr-qc]].
3. Krssak, M., van den Hoogen, R., Pereira, J., Boehmer, C. & Coley, A., Teleparallel Theories of Gravity: Illuminating a Fully Invariant Approach, *Classical And Quantum Gravity* **2019**, *36*, 183001, [arXiv:1810.12932 [gr-qc]].
4. Chinea, F., Symmetries in tetrad theories, *Classical and Quantum Gravity* **1988**, *5*, 135.
5. Estabrook, F. & Wahlquist, H., Moving frame formulations of 4-geometries having isometries, *Classical and Quantum Gravity* **1996**, *13*, 1333.

6. Papadopoulos, G. & Grammenos, T., Locally homogeneous spaces, induced Killing vector fields and applications to Bianchi prototypes, *Journal of Mathematical Physics* **2012**, *53*, 072502 , [arXiv:1106.3897 [gr-qc]].
7. McNutt, D.D., Coley, A.A. & van den Hoogen, R.J., A frame based approach to computing symmetries with non-trivial isotropy groups, *Journal of Mathematical Physics* **2023**, *64*, 032503, [arXiv:2302.11493 [gr-qc]].
8. Coley, A. A., van den Hoogen, R. J. & McNutt, D.D., Symmetry and Equivalence in Teleparallel Gravity, *Journal of Mathematical Physics* **2020**, *61*, 072503, [arXiv:1911.03893 [gr-qc]].
9. Olver, P. *Equivalence, invariants and symmetry*, Cambridge University Press, 1995.
10. Ferraro, R. & Fiorini, F., Modified teleparallel gravity: Inflation without an inflation, *Physical Review D* **2007**, *75*, 084031, [arXiv:gr-qc/0610067].
11. Ferraro, R. & Fiorini, F. On Born-Infeld Gravity in Weitzenbock spacetime, *Physical Review D* **2008**, *78*, 124019, [arXiv:0812.1981 [gr-qc]].
12. Linder, E., Einstein's Other Gravity and the Acceleration of the Universe, *Physical Review D* **2010**, *81*, 127301, [Erratum: *Physical Review D*, **2010**, *82*, 109902], [arXiv:1005.3039 [gr-qc]].
13. Lucas, T.G., Obukhov, Y. & Pereira, J.G., Regularizing role of teleparallelism, *Physical Review D* **2009**, *80*, 064043, [arXiv:0909.2418 [gr-qc]].
14. Krssak, M. & Pereira, J.G., Spin Connection and Renormalization of Teleparallel Action, *The European Physical Journal C* **2015**, *75*, 519, [arXiv:1504.07683 [gr-qc]].
15. Hayashi, K. & Shirafuji, T., New general relativity, *Physical Review D* **1979**, *19*, 3524.
16. Jimenez, J.B. & Dialektopoulos, K.F., Non-Linear Obstructions for Consistent New General Relativity, *Journal of Cosmology and Astroparticle Physics* **2020** 01, 018, [arXiv:1907.10038 [gr-qc]].
17. Bahamonde, S., Blixt, D., Dialektopoulos, K.F. & Hell A., Revisiting Stability in New General Relativity, **2024**, preprint: [arXiv:2404.02972 [gr-qc]].
18. Heisenberg, L., Review on $f(Q)$ Gravity, **2023**, preprint: [arXiv:2309.15958 [gr-qc]].
19. Heisenberg, L., Hohmann, M. & Kuhn, S., Cosmological teleparallel perturbations, **2023**, preprint: [arXiv:2311.05495 [gr-qc]].
20. Flathmann, K. & Hohmann, M., Parametrized post-Newtonian limit of generalized scalar-nonmetricity theories of gravity, *Physical Review D* **2022**, *105*, 044002, [arXiv:2311.02806 [gr-qc]].
21. Hohmann, M., General covariant symmetric teleparallel cosmology, *Physical Review D* **2021**, *104*, 124077, [arXiv:2109.01525 [gr-qc]].
22. Jimenez, J.B., Heisenberg, L. & Koivisto, T.S., The Geometrical Trinity of Gravity, *Universe* **2019**, *5*(7), 173, [arXiv:1903.06830 [gr-qc]].
23. Nakayama, Y., Geometrical trinity of unimodular gravity, *Classical and Quantum Gravity* **2023**, *40*, 125005, [arXiv:2209.09462 [gr-qc]].
24. Xu, Y., Li, G., Harko, T. & Liang, S.-D., $f(Q,T)$ gravity, *The European Physical Journal C* **2019**, *79*, 708, [arXiv:1908.04760 [gr-qc]].
25. Maurya, D.C., Yesmakhanova, K., Myrzakulov, R. & Nugmanova, G., Myrzakulov. $F(T,Q)$ gravity: cosmological implications and constraints, **2024**, preprint: [arXiv:2404.09698 [gr-qc]].
26. Maurya, D.C., Yesmakhanova, K., Myrzakulov, R. & Nugmanova, G., Myrzakulov, FLRW Cosmology in Myrzakulov $F(R,Q)$ Gravity, **2024**, preprint: [arXiv:2403.11604 [gr-qc]].
27. Maurya, D.C. & Myrzakulov, R., Exact Cosmology in Myrzakulov Gravity, **2024**, preprint: [arXiv:2402.02123 [gr-qc]].
28. Harko, T., Lobo, F.S.N., Nojiri, S. & Odintsov, S.D., $f(R,T)$ gravity, *Physical Review D* **2011**, *84*, 024020, [arXiv:1104.2669 [gr-qc]].
29. Golovnev, A. & Guzman, M.-J., Approaches to spherically symmetric solutions in $f(T)$-gravity, *Universe* **2021**, *7* (5), 121, [arXiv:2103.16970 [gr-qc]].
30. Golovnev, A., Issues of Lorentz-invariance in $f(T)$-gravity and calculations for spherically symmetric solutions, *Classical and Quantum Gravity* **2021**, *38*, 197001, [arXiv:2105.08586 [gr-qc]].
31. Golovnev, A. & Guzman, M.-J., Bianchi identities in $f(T)$-gravity: Paving the way to confrontation with astrophysics, *Physics Letter B* **2020**, *810*, 135806, [arXiv:2006.08507 [gr-qc]].
32. DeBenedictis, A., Ilijić, S. & Sossich, M., On spherically symmetric vacuum solutions and horizons in covariant $f(T)$ gravity theory, *Physical Review D* **2022**, *105*, 084020, [arXiv:2202.08958 [gr-qc]].
33. Coley, A.A., Landry, A., van den Hoogen, R.J. & McNutt, D.D., Spherically symmetric teleparallel geometries, *The European Physical Journal C* **2024**, *84*, 334, [arXiv:2402.07238 [gr-qc]].

34. Coley, A.A., Landry, A., van den Hoogen, R.J. & McNutt, D.D., Generalized Teleparallel de Sitter geometries, *The European Physical Journal C* **2023**, *83*, 977, [arXiv:2307.12930 [gr-qc]].
35. Bahamonde, S. & Camci, U., Exact Spherically Symmetric Solutions in Modified Teleparallel gravity, *Symmetry* **2019**, *11*, 1462, [arXiv:1911.03965 [gr-qc]].
36. Awad, A., Golovnev, A., Guzman, M.-J. & El Hanafy, W., Revisiting diagonal tetrads: New Black Hole solutions in $f(T)$-gravity, *The European Physical Journal C* **2022**, *82*, 972, [arXiv:2207.00059 [gr-qc]].
37. Bahamonde, S., Golovnev, A., Guzmán, M.-J., Said, J.L. & Pfeifer, C., Black Holes in $f(T,B)$ Gravity: Exact and Perturbed Solutions, *Journal of Cosmology and Astroparticle Physics* **2022**, *01* 037, [arXiv:2110.04087 [gr-qc]].
38. Bahamonde, S., Faraji, S., Hackmann, E. & Pfeifer, C., Thick accretion disk configurations in the Born-Infeld teleparallel gravity, *Physical Review D* **2022**, *106*, 084046, [arXiv:2209.00020 [gr-qc]].
39. Nashed, G.G.L., Quadratic and cubic spherically symmetric black holes in the modified teleparallel equivalent of general relativity: Energy and thermodynamics, *Classical and Quantum Gravity* **2021**, *38*, 125004, [arXiv:2105.05688 [gr-qc]].
40. Pfeifer, C. & Schuster, S., Static spherically symmetric black holes in weak $f(T)$-gravity, *Universe* **2021**, *7*, 153, [arXiv:2104.00116 [gr-qc]].
41. El Hanafy, W. & Nashed, G.G.L., Exact Teleparallel Gravity of Binary Black Holes, *Astrophysical Space Science* **2016**, *361*, 68, [arXiv:1507.07377 [gr-qc]].
42. Aftergood, J. & DeBenedictis, A., Matter Conditions for Regular Black Holes in $f(T)$ Gravity, *Physical Review D* **2014**, *90*, 124006, [arXiv:1409.4084 [gr-qc]].
43. Bahamonde, S., Doneva, D.D., Ducobu, L., Pfeifer, C. & Yazadjiev, S.S., Spontaneous Scalarization of Black Holes in Gauss-Bonnet Teleparallel Gravity, *Physical Review D* **2023**, *107*, 10, 104013, [arXiv:2212.07653 [gr-qc]].
44. Bahamonde, S., Ducobu, L. & Pfeifer, C., Scalarized Black Holes in Teleparallel Gravity, *Journal of Cosmology and Astroparticle Physics* **2022**, *04*(04), 018, [arXiv:2201.11445 [gr-qc]].
45. Calza, M. & Sebastiani, L., A class of static spherically symmetric solutions in $f(T)$-gravity, *The European Physical Journal C* **2024**, *84*, 476, [arXiv:2309.04536 [gr-qc]].
46. Landry, A., Static spherically symmetric perfect fluid solutions in teleparallel F(T) gravity, *Axioms* **2024**, *13* (5), 333, [arXiv:2405.09257 [gr-qc]].
47. Landry, A., Kantowski-Sachs spherically symmetric solutions in teleparallel $F(T)$ gravity, *Symmetry* **2024** *16* (8), 953, [arXiv:2406.18659 [gr-qc]].
48. van den Hoogen, R.J. & Forance, H., Teleparallel Geometry with Spherical Symmetry: The diagonal and proper frames, *Journal of Cosmology and Astrophysics* **2024**, 11, 033, [arXiv:2408.13342 [gr-qc]].
49. Leon, G. & Roque, A.A., qualitative analysis of Kantowski-Sachs metric in a generic class of $f(R)$ models, *Journal of Cosmology and Astroparticle Physics*, **2014**, *05*, 032, [arXiv:1308.5921 [gr-qc]].
50. Shaikh, A.A. & Chakraborty, D., Curvature properties of Kantowski-Sachs metric, *Journal of Geometry and Physics*, *160*, **2021**, 103970.
51. Oliveira-Neto, G., Canedo, D.L. & Monerat, G.A., An anisotropic Kantowski-Sachs universe with radiation, dust and a phantom fluid, *Brazilian Journal of Physics* **2022**, *52*, 130, [arXiv:2109.12229 [gr-qc]].
52. Rodrigues, M. E., Kpadonou, A.V., Rahaman, F., Oliveira, P.J. & Houndjo, M.J.S., Bianchi type-I, type-III and Kantowski-Sachs solutions in $f(T)$ gravity, *Astrophysics and Space Science* **2015**, *357*, 129, [arXiv:1408.2689 [gr-qc]].
53. Amir, M.J. & Yussouf, M., Kantowski-Sachs Universe Models in $f(T)$ Theory of Gravity, *International Journal of Theoretical Physics* **2015**, *54*, 2798, [arXiv:1502.00777 [gr-qc]].
54. Leon, G. & Paliathanasis, A., Anisotropic spacetimes in $f(T,B)$ theory II: Kantowski-Sachs Universe, *The European Physical Journal Plus* **2022**, *137*, 855, [arXiv:2207.08570 [gr-qc]].
55. Leon, G. & Paliathanasis, A., Anisotropic spacetimes in $f(T,B)$ theory III: LRS Bianchi III Universe, *The European Physical Journal Plus* **2022**, *137*, 927, [arXiv:2207.08571 [gr-qc]].
56. Vinutha, T., Niharika, K. & Sri Kavya, K., The Study of Kantowski-Sachs Perfect Fluid Cosmological Model in Modified Gravity, *Astrophysics* **2023**, *66*, 64, [arXiv:2301.01163 [gr-qc]].
57. Samanta, G.C., Kantowski-Sachs Universe Filled with Perfect Fluid in $f(R,T)$ Theory of Gravity, *International Journal of Theoretical Physics* **2013**, *52*, 2647.

58. Dimakis, N., Roumeliotis, M., Paliathanasis, A. & Christodoulakis, T., Anisotropic Solutions in Symmetric Teleparallel $f(Q)$-theory: Kantowski-Sachs and Bianchi III LRS Cosmologies, *The European Physical Journal C* **2023**, *83*, 794, [arXiv:2304.04419 [gr-qc]].
59. Millano, A.D., Dialektopoulos, K., Dimakis, N., Giacomini, A., Shababi, H., Halder, A. & Paliathanasis, A., Kantowski-Sachs and Bianchi III dynamics in $f(Q)$-gravity, *Physical Review D* **2024**, *109*, 124044, [arXiv:2403.06922 [gr-qc]].
60. Paliathanasis, A., Classical and Quantum Cosmological Solutions in Teleparallel Dark Energy with Anisotropic Background Geometry, *Symmetry* **2022**, *14* (10), 1974, [arXiv:2209.08817 [gr-qc]].
61. Paliathanasis, A., Kantowski-Sachs cosmology in scalar-torsion theory, *The European Physical Journal C* **2023**, *83*, 213, [arXiv:2302.09608 [gr-qc]].
62. Zlatev, I., Wang, L. & Steinhardt, P., Quintessence, Cosmic Coincidence, and the Cosmological Constant, *Physical Review Letters*, **1999**, *82* (5), 896, [arXiv:astro-ph/9807002 [astro-ph]].
63. Steinhardt, P., Wang, L. & Zlatev, I., Cosmological tracking solutions, *Physical Review D*, **1999**, *59* (12), 123504, [arXiv:astro-ph/9812313 [astro-ph]].
64. Caldwell, R.R., Dave, R. & Steinhardt, P., Cosmological Imprint of an Energy Component with General Equation of State, *Physical Review Letters*, **1998**, *80*, 1582, [arXiv:astro-ph/9708069 [astro-ph]].
65. Carroll, S.M., Quintessence and the Rest of the World, *Physical Review Letters*, **1998**, *81*, 3067, [arXiv:astro-ph/9806099 [astro-ph]].
66. Doran, M., Lilley, M., Schwindt, J. & Wetterich, C., Quintessence and the Separation of CMB Peaks, *Astrophysical Journal*, **2001**, *559*, 501, [arXiv:astro-ph/0012139v2 [astro-ph]]
67. Zeng, X.-X., Chen, D.-Y., Li, L.-F., Holographic thermalization and gravitational collapse in the spacetime dominated by quintessence dark energy, *Physical Review D*, **2015**, *91*, 046005, [arXiv:1408.6632 [hep-th]].
68. Chakraborty, S., Mishra, S. & Chakraborty, S., Dynamical system analysis of quintessence dark energy model, *International Journal of Geometric Methods in Modern Physics*, **2025**, *22*, 2450250, [arXiv:2406.10692 [gr-qc]]
69. Shlivko, D., Steinhardt, P.J., Assessing observational constraints on dark energy, *Physics Letters B*, **2024**, *855*, 138826, [arXiv:2405.03933 [astro-ph]]
70. Wetterich, C., Cosmology and the Fate of Dilatation Symmetry, *Nuclear Physics B*, **1988**, *302*, 668, [arXiv:1711.03844 [hep-th]].
71. Ratra, B. & Peebles, P.J.E., Cosmological consequences of a rolling homogeneous scalar field, *Physical Review D*, **1988**, *37*, 3406.
72. Duchaniya, L.K., Gandhi, K. & Mishra, B., Attractor behavior of $f(T)$ modified gravity and the cosmic acceleration, *Physics of the Dark Universe* **2024**, *44*, 101464, [arXiv:2303.09076 [gr-qc]].
73. Hohmann, M., Teleparallel gravity, *Modified and Quantum Gravity*, **2022**, 145, Springer, [arXiv:2207.06438 [gr-qc]].
74. Hohmann, M., Scalar-torsion theories of gravity III: analogue of scalar-tensor gravity and conformal invariants, *Physical Review D* **2018**, *98*, 064004, [arXiv:1801.06531 [gr-qc]].
75. Zubair, M., Waheed, S., Fayyaz, M.A. & Ahmad, I., Energy Constraints and Phenomenon of Cosmic Evolution in $f(T,B)$ Framework, *European Physical Journal Plus* **2018**, *133*, 452, [arXiv:1807.07399 [gr-qc]].
76. Bhattacharjee, S., Constraining $f(T,B)$ teleparallel gravity from energy conditions, *New Astronomy* **2021**, *83*, 101495, [arXiv:2004.12060 [gr-qc]].
77. Trivedi, O., Khlopov, M., Said, J.L. & Nunes, R., Cosmological singularities in $f(T,\phi)$ gravity, *European Physical Journal C* **2023**, *83*, 1017, [arXiv:2310.20222 [gr-qc]].
78. Paliathanasis, A. & Leon, G., $f(T,B)$ gravity in a Friedmann–Lemaître–Robertson–Walker universe with nonzero spatial curvature, *Mathematical Methods of Applied Science* **2023**, *46*, 3905, [arXiv:2201.12189 [gr-qc]].
79. Paliathanasis, A. & Leon, G., Cosmological evolution in $f(T,B)$ gravity, *European Physical Journal Plus* **2021**, *136*, 1092, [arXiv:2106.01137 [gr-qc]].
80. Kofinas, G., Leon, G. & Saridakis, E.N., Dynamical behavior in $f(T,T_G)$ cosmology, *Classical and Quantum Gravity* **2014**, *31*, 175011, [arXiv:1404.7100 [gr-qc]].
81. Farrugia, G., Said, J.L. & Finch, A., Gravitoelectromagnetism, Solar System Test and Weak-Field Solutions in $f(T,B)$ Gravity with Observational Constraints, *Universe* **2020**, *6* (2), 34, [arXiv:2002.08183 [gr-qc]].
82. Coley, A.A., Landry, A. and Gholami, F., Teleparallel Robertson-Walker Geometries and Applications, *Universe* **2023**, *9*, 454, [arXiv:2310.14378 [gr-qc]].

83. Gholami, F. & Landry, A., Cosmological solutions in teleparallel $F(T, B)$ gravity, **2024**, preprint: [arXiv:2411.18455 [gr-qc]].
84. Chiba, T., Okabe, T., Yamaguchi, M., Kinetically Driven Quintessence, *Physical Review D*, **2000**, *62*, 023511, [arXiv:astro-ph/9912463 [astro-ph]].
85. Carroll, S.M., Hoffman, M., Trodden, M., Can the dark energy equation-of-state parameter w be less than -1?, *Physical Review D*, **2003**, *68*, 023509, [arXiv:astro-ph/0301273 [astro-ph]].
86. Caldwell, R.R., A phantom menace? Cosmological consequences of a dark energy component with super-negative equation of state, *Physics Letters B*, **2002**, *545*, 23, [arXiv:astro-ph/9908168 [astro-ph]].
87. Farnes, J.S., A Unifying Theory of Dark Energy and Dark Matter: Negative Masses and Matter Creation within a Modified ΛCDM Framework, *Astronomy & Astrophysics*, **2018**, *620*, A92, [arXiv:1712.07962 [physics.gen-ph]].
88. Baum, L. & Frampton, P.H., Turnaround in Cyclic Cosmology, *Physical Review Letters*, **2007**, *98*, 071301, [arXiv:hep-th/0610213 [hep-th]].
89. Hu, W., Crossing the Phantom Divide: Dark Energy Internal Degrees of Freedom, *Physical Review D*, **2005**, *71*, 047301, [arXiv:astro-ph/0410680v2 [astro-ph]].
90. Karimzadeh, S. & Shojaee, R., Phantom-Like Behavior in Modified Teleparallel Gravity, *Advances in High Energy Physics*, **2019**, 4026856, [arXiv:1902.04406 [physics.gen-ph]].
91. Pati, L., Kadam, S.A., Tripathy, S.K. & Mishra, B., Rip cosmological models in extended symmetric teleparallel gravity, *Physics of the Dark Universe*, **2022**, *35*, 100925, [arXiv:2112.00271 [gr-qc]].
92. Kucukakca, Y., Akbarieh, A.R. & Ashrafi, S., Exact solutions in teleparallel dark energy model, *Chinese Journal of Physics*, **2023**, *82*, 47.
93. Cai, Y.-F., Saridakis, E.N., Setare, M.R. & Xia, J.-Q., Quintom Cosmology: Theoretical implications and observations, *Physics Report*, **2010**, *493*, 1, [arXiv:0909.2776 [hep-th]].
94. Guo, Z.-K., Piao, Y.-S., Zhang, X. & Zhang, Y.-Z., Cosmological evolution of a quintom model of dark energy, *Physics Letters B*, **2005**, *608*, 177, [arXiv:astro-ph/0410654 [astro-ph]].
95. Feng, B., Li, M., Piao, Y.-S., Zhang, X., Oscillating quintom and the recurrent universe, *Physics Letters B*, **2006**, *634*, 101, [arXiv:astro-ph/0407432 [astro-ph]].
96. Mishra, S. & Chakraborty, S., Dynamical system analysis of quintom dark energy model, *European Physical Journal C*, **2018**, *78*, 917, [arXiv:1811.08279 [gr-qc]].
97. Tot, J., Coley, A.A., Yildrim, B. & Leon, G., The dynamics of scalar-field Quintom cosmological models, *Physics of the Dark Universe*, **2023**, *39*, 101155, [arXiv:2204.06538 [gr-qc]].
98. Bahamonde, S., Marciu, M. & Rudra, P., Generalised teleparallel quintom dark energy non-minimally coupled with the scalar torsion and a boundary term, *Journal of Cosmology and Astroparticle Physics*, **2018**, *04*, 056, [arXiv:1802.09155 [gr-qc]].
99. Tajahmad, B., Exact solutions of an anisotropic universe in a modified teleparallel gravity model via the Noether and B.N.S. approaches, *International Journal of Geometric Methods in Modern Physics*, **2024**, *21*, 2450230, [arXiv:1610.08707 [gr-qc]].
100. Iosifidis, D., Cosmological Hyperfluids, Torsion and Non-metricity, *European Physical Journal C* **2020**, *80*, 1042, [arXiv:2003.07384 [gr-qc]].
101. Heisenberg, L., Hohmann, M. & Kuhn, S., Homogeneous and isotropic cosmology in general teleparallel gravity, *European Physical Journal C* **2023**, *83*, 315, [arXiv:2212.14324 [gr-qc]].
102. Heisenberg, L. & Hohmann, M., Gauge-invariant cosmological perturbations in general teleparallel gravity, *European Physical Journal C* **2024**, *84*, 462, [arXiv:2311.05597 [gr-qc]].
103. Bohmer, C.G. & d'Alfonso del Sordo, A., Cosmological fluids with boundary term couplings, *General Relativity and Gravitation* **2024**, *56*, 75, [arXiv:2404.05301 [gr-qc]].
104. Hawking, S.W. & Ellis, G.F.R., *The Large Scale Structure of Space-Time*, Cambridge University Press, 2010.
105. Coley, A.A., *Dynamical systems and cosmology*, Kluwer Academic, Dordrecht, 2003.
106. Landry, A. & Hammad, F., Landau levels in a gravitational field: The Schwarzschild spacetime case, *Universe* **2021**, *7*, 144, [arXiv:1909.01827 [gr-qc]].
107. Hammad, F. & Landry, A., Landau levels in a gravitational field: The Levi-Civita and Kerr spacetimes case, *European Physical Journal Plus* **2020**, *135*, 90, [arXiv:1910.01899 [gr-qc]].