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Posted Date: 2 September 2025

doi: 10.20944/preprints202509.0197.v1

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Phase States, a Fully Separable Basis in the n-Qubit Computational Space

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Abstract

The article examines the Pegg–Barnett phase eigenstates and their embedding into an n-qubit computational framework using a Fock–binary mapping under condition that the truncated Hilbert space dimension is a power of two. In this setting, the phase eigenstates form a complete orthonormal basis of fully separable states, with each state factorizing into independent single-qubit components. As a result, all bipartite entanglement entropies vanish, and the reduced density matrices obtained by tracing over any subset of qubits remain pure. We further analyze the effects of dimension reduction by tracing out qubits: individual basis states remain pure under this process, while superpositions generally become mixed unless their coefficients satisfy a specific matching condition. These features establish phase states as a unique entanglement-free basis with potential applications in digitized quantum fields, qubit-based representations of bosonic modes, and controlled information processing where the management of entropy and correlations is crucial.

Keywords: quantum information; Pegg–Barnett phase states; separable product states; entanglement entropy; von Neumann entropy; partial trace; dimension reduction

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1. Introduction

In the early twentieth century, quantum mechanics altered our understanding of the physical world surrounding us by describing phenomena that classical physics was not capable of, such as entanglement and quantum superposition. These principles have given rise to a new field, quantum information theory. This discipline seeks to understand how information can be represented, transmitted and processed when the underlying system is governed by the laws of quantum mechanics. At its core, quantum information is built on the concept of the quantum bit, or qubit. Unlike a classical bit, which can take values of only 0 or 1, a qubit can exist in a superposition of both states simultaneously. Mathematically, this is represented as a linearcombination of these basis states. The other main pillar is entanglement, a phenomenon in which the states of two or more qubits become inseparably correlated, regardless of the distance between them. Entanglement challenges classical intuition and supports many quantum information protocols, such as quantum teleportation and quantum key distribution.

In this study, we use the tools of quantum mechanics to shed light on the role of phase states in quantum information theory. Nevertheless studying the phase properties of quantum systems is arguably one of the most controversial subjects in physics. The reason for that is the lack of a well defined Hermitian phase operator. Subsequent to the work of Dirac [1] which first addressed this issue many attempts has been made to characterize the phase as a quantum observable. The difficulties to achieve this goal were made quite explicit in the work of Susskind and Glogower [2]. Pegg and Barnett [3] showed how to construct an Hermitian phase operator, the eigenstates of which, in an appropriate limit, generate the correct phase statistics for arbitrary states. Our goal here, however, is not to go into the details of the phase properties of quantum system, but to show a unique and surprising feature of the phase eigenstates used by Pegg and Barnett to construct their Hermitian phase operator.

2. Phase Eigenstates

The Pegg-Barnett Hermitian phase operator, defined on the truncated $N+1$ dimensional Hilbert space, is expressed in the following form

$$\phi_{PB} = \sum_{m=0}^{N} \phi_m |\phi_m\rangle\langle\phi_m|, \tag{1}$$

where $|\phi_m\rangle$ are the eigenstates of the phase operator in the number state representation (Fock space)

$$|\phi_m\rangle = \frac{1}{\sqrt{N+1}} \sum_{l=0}^{N} e^{il\phi_m}|l\rangle. \tag{2}$$

In order to constitute a basis these states must form an orthonormal set. This condition using Eq. (2) leads to the restriction that $e^{-i(N+1)(\phi_m-\phi_n)} = 1$ for $m, n = 0, \ldots, N$, which in turn requires that

$$\phi_m = \phi_0 + \frac{m2\pi}{N+1} \quad \text{for } m = 0, \ldots, N, \tag{3}$$

and thus $\phi_0 \le \phi_m < \phi_0 + 2\pi$.

3. Qubit Embedding

Now let us find how to embed quantum information represented by the sequence of qubits into a Fock space, where the phase states exist. If N is the largest number of photons which can be stored in the truncated Hilbert space then

$$n = \log_2(N+1) \tag{4}$$

is the number of qubits which can be mapped to this state. Thus to be able to fully digitize the truncated Hilbert space n must be an integer number. When this condition is met, it is quite straightforward to embed the basis vectors specifying the qubits into the Fock space, just by using the decimal value of the bits to find the corresponding Fock state, e.g.

$$|000\rangle = |0\rangle, \quad |001\rangle = |1\rangle, \quad |010\rangle = |2\rangle, \quad |011\rangle = |3\rangle, \quad \ldots \tag{5}$$

Now having this mapping we can work backwards, and find the representation of the phase states in the form of qubits. After a bit of algebra we will obtain the following expression

$$|\phi_m\rangle = \frac{1}{2^{n/2}}(|0\rangle + (-1)^m|1\rangle) \otimes \ldots \otimes \left(|0\rangle + (-1)^{m/2^s}|1\rangle\right) \otimes \ldots \otimes \left(|0\rangle + (-1)^{m/2^{(n-1)}}|1\rangle\right) \tag{6}$$

where the states are labeled the same way as earlier $m = 0, 1, \ldots N$, and the information carrying qubits are addressed using

$$s = 0, 1, \ldots, n-1. \tag{7}$$

From this representation it is immediately evident that these states have great potential in Quantum information technology. In the next sections we will investigate these possibilities.

4. Bipartite Entanglement Entropy of the Phase Eigenstates

Now that we have an orthonormal set of $N+1$ states all information that can be stored in the truncated Hilbert space can be decomposed into a linear combination of these basis states. Furthermore, all basis state consist of n number of qubits. First, let us investigate how much entanglement the basis states carry. To answer this question we apply bipartition. A bipartition of a basis state is a partition which divides the system into two parts A and B, containing k and l qubits respectively with $k+l=n$. While the von Neumann entropy of the whole state is zero, since it is a pure state, the entropy of its subsystems could be greater than zero, which would indicate that the subsystems are entangled.

 doi:10.20944/preprints202509.0197.v1

An example of such entangled state could be any one of the 2 qubit Bell states, which are maximally entangled pure states that cannot be separated into bipartite pure states.

How about the phase states, let us calculate the bipartite von Neumann entanglement entropy S for these states [4,5]. This is defined with respect to bipartition as the von Neumann entropy of either of its reduced states, since they are of the same value. For a pure state, $\rho_{AB} = |\phi_m\rangle\langle\phi_m|_{AB}$, it is given by

$$\mathcal{S}(\rho_A) = -\operatorname{Tr}[\rho_A \log \rho_A] = -\operatorname{Tr}[\rho_B \log \rho_B] = \mathcal{S}(\rho_B), \tag{8}$$

where $\rho_A = \operatorname{Tr}_B[\rho_{AB}]$ and $\rho_B = \operatorname{Tr}_A[\rho_{AB}]$ are the reduced density matrices for each partition. Many entanglement measures (distillable entanglement, entanglement cost, entanglement of formation, relative entropy of entanglement, squashed entanglement) reduce to the von Neumann entanglement entropy when evaluated on pure states. Using Eq. (6) the density matrix representing a phase state is

$$\begin{aligned}|\phi_m\rangle\langle\phi_m| \;=\;& \frac{1}{2}(|0\rangle + (-1)^m|1\rangle)(\langle 0| + ((-1)^m)^*\langle 1|) \otimes \ldots \\ & \otimes\frac{1}{2}\Big(|0\rangle + (-1)^{m/2^s}|1\rangle\Big)\Big(\langle 0| + \Big((-1)^{m/2^s}\Big)^*\langle 1|\Big) \otimes \ldots \\ & \otimes\frac{1}{2}\Big(|0\rangle + (-1)^{m/2^{(n-1)}}|1\rangle\Big)\Big(\langle 0| + \Big((-1)^{m/2^{(n-1)}}\Big)^*\langle 1|\Big),\end{aligned} \tag{9}$$

where $(\ldots)^*$ denotes the complex conjugate. To set up a bipartition of this system we can now select a set of qubits and designate them as system A, while the rest of the qubits will form system B. The selected qubits do not need to be sequential, e.g. $A = \{q_1, q_2, q_4, \ldots, q_{n-1}\}$ and $B = \{q_0, q_3, q_5, \ldots, q_{n-2}\}$. Furthermore from Eq. (9) it follows that the density matrix of a phase state is the tensor product of the density matrices of the single qubits it is built up from.

$$|\phi_m\rangle\langle\phi_m| = |q_0\rangle\langle q_0| \otimes \ldots \otimes |q_s\rangle\langle q_s| \otimes \ldots \otimes |q_{n-1}\rangle\langle q_{n-1}|, \tag{10}$$

where the density matrix of a single qubit is given by

$$|q_s\rangle\langle q_s| = \frac{1}{2}\Big(|0\rangle + (-1)^{m/2^s}|1\rangle\Big)\Big(\langle 0| + \Big((-1)^{m/2^s}\Big)^*\langle 1|\Big). \tag{11}$$

Executing the partial trace required by the calculation of the von Neumann entanglement entropy on the system with density matrix in the form described by Eq. (9) is quite simple. The only thing one must do is to trace over the qubits which belong to the designated bipartition A or B. Tracing over a single qubit in the form of Eq. (11) is also an easy task

$$\operatorname{Tr}_{\{0,1\}}[|q_s\rangle\langle q_s|] = \langle 0|q_s\rangle\langle q_s|0\rangle + \langle 1|q_s\rangle\langle q_s|1\rangle = 1. \tag{12}$$

Which means that the reduced density matrix, $\rho_A = \operatorname{Tr}_B[\rho_{AB}]$ or $\rho_B = \operatorname{Tr}_A[\rho_{AB}]$ that we obtained after bipartitioning the original one and executing a trace over one part of the system still describes a pure state, and thus

$$\mathcal{S}(\rho_A) = \mathcal{S}(\rho_B) = 0. \tag{13}$$

This is quite a strong result, since it indicates that phase eigenstates when they get "digitized" form an orthonormal basis with no entanglement encoded in its basis vectors. If we compare this to the Bell states, those are maximally entangled pure states which form a basis in the 2-qubit computational space, and the phase states are fully separable (not entangled) pure states which form a basis in the n-qubit computational space.

5. Example: How Entropy Is Generated Using Phase Eigenstates

To learn more about the properties of phase eigenstates when used as a computational basis let us study how entropy is generated via partial tracing of pure state density matrices. First let us rewrite

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$\left|\phi_m^{(n)}\right\rangle$ (here in the notation we inserted the number of qubits (n) the phase state was encoded in, since during the tracing we will reduce this dimension with the aid of this notation we can keep track of the process), using Eq. (6) we can write

$$\left|\phi_m^{(n)}\right\rangle = \bigotimes_{s=0}^{n-1} \frac{1}{\sqrt{2}}\Big(|0\rangle + (-1)^{\frac{m}{2^s}}|1\rangle\Big). \tag{14}$$

Since the phase eigenstates form an orthonormal set any state can be expressed as their linear combination

$$|\psi^{(n)}\rangle = \sum_{m=0}^{N} c_m \left|\phi_m^{(n)}\right\rangle, \tag{15}$$

which in turn provides the following density matrix

$$|\psi^{(n)}\rangle\langle\psi^{(n)}| = \left(\sum_{l=0}^{N} c_l \left|\phi_l^{(n)}\right\rangle\right)\left(\sum_{m=0}^{N} c_m^* \left\langle\phi_m^{(n)}\right|\right) == \sum_{l,m=0}^{N} c_l c_m^* \left|\phi_l^{(n)}\right\rangle\left\langle\phi_m^{(n)}\right|. \tag{16}$$

The most important part of this expression is $\left|\phi_l^{(n)}\right\rangle\left\langle\phi_m^{(n)}\right|$ where the dimension reduction will take place. Let us take a closer look at this expression

$$\left|\phi_l^{(n)}\right\rangle\left\langle\phi_m^{(n)}\right| = \bigotimes_{s=0}^{n-1} \frac{1}{2}\Big(|0\rangle + (-1)^{\frac{l}{2^s}}|1\rangle\Big)\Big(\langle 0| + \Big((-1)^{\frac{m}{2^s}}\Big)^*\langle 1|\Big). \tag{17}$$

When we reduce the dimension of our computational space we will trace over these expressions, more precisely if we take away for example the last qubit q_{n-1} representing our state $|\psi\rangle\langle\psi|$ then we get

$$\mathrm{Tr}_{q_{n-1}}\{|\psi^{(n)}\rangle\langle\psi^{(n)}|\} = \sum_{l,m=0}^{N} c_l c_m^* \,\mathrm{Tr}_{q_{n-1}}\left\{\left|\phi_l^{(n)}\right\rangle\left\langle\phi_m^{(n)}\right|\right\}. \tag{18}$$

After a bit of algebra this reduces to

$$\mathrm{Tr}_{q_{n-1}}\left\{\left|\phi_l^{(n)}\right\rangle\left\langle\phi_m^{(n)}\right|\right\} = \frac{1}{2}\left(1 + e^{i\pi\frac{l-m}{2^{n-1}}}\right)\bigotimes_{s=0}^{n-2}\frac{1}{2}\Big(|0\rangle + (-1)^{\frac{l}{2^s}}|1\rangle\Big)\Big(\langle 0| + \Big((-1)^{\frac{m}{2^s}}\Big)^*\langle 1|\Big) \tag{19}$$

The true power of using phase states becomes apparent when we establish the connection with the dimension reduced phase eigenstate basis

$$\mathrm{Tr}_{q_{n-1}}\left\{\left|\phi_l^{(n)}\right\rangle\left\langle\phi_m^{(n)}\right|\right\} = \frac{1}{2}\left(1 + e^{i\pi\frac{l-m}{2^{n-1}}}\right)\left|\phi_{l'}^{(n-1)}\right\rangle\left\langle\phi_{m'}^{(n-1)}\right|, \tag{20}$$

with

$$l' = \begin{cases} l & \text{if} \quad l < 2^{n-1} \\ l - 2^{n-1} & \text{if} \quad l \geqslant 2^{n-1} \end{cases} \qquad m' = \begin{cases} m & \text{if} \quad m < 2^{n-1} \\ m - 2^{n-1} & \text{if} \quad m \geqslant 2^{n-1} \end{cases}, \tag{21}$$

which basically means that by reducing the number of qubits by 1 we fold the phase eigenstate basis onto itself without disturbing any ordering. That also explains why a phase eigenstate when undergoes repeated dimension reduction still remains a pure state until the last qubit of information

$$\mathrm{Tr}_{q_{n-1}}\left\{\left|\phi_m^{(n)}\right\rangle\left\langle\phi_m^{(n)}\right|\right\} = \left|\phi_{m'}^{(n-1)}\right\rangle\left\langle\phi_{m'}^{(n-1)}\right|. \tag{22}$$

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Now we return to our initial question, how entropy gets generated in this system for that we substitute Eq. (20) into Eq. (18) and obtain

$$\mathrm{Tr}_{q_{n-1}}\{|\psi^{(n)}\rangle\langle\psi^{(n)}|\} = \sum_{l',m'=0}^{2^{n-1}-1} c_{l'} c^*_{m'} \frac{1}{2}\left(1 + e^{i\frac{l'-m'}{2^{n-1}}}\right) \left|\phi^{(n-1)}_{l'}\right\rangle\left\langle\phi^{(n-1)}_{m'}\right|. \tag{23}$$

But for $\mathrm{Tr}_{q_{n-1}}\{|\psi^{(n)}\rangle\langle\psi^{(n)}|\}$ to remain a pure state $\left|\psi^{(n-1)}\right\rangle\left\langle\psi^{(n-1)}\right|$ we must have

$$\left|\psi^{(n-1)}\right\rangle\left\langle\psi^{(n-1)}\right| = \sum_{l',m'=0}^{2^{n-1}-1} b_{l'} b^*_{m'} \left|\phi^{(n-1)}_{l'}\right\rangle\left\langle\phi^{(n-1)}_{m'}\right|, \tag{24}$$

thus

$$b_{l'} b^*_{m'} = c_{l'} c^*_{m'} \frac{1}{2}\left(1 + e^{i\pi\frac{l'-m'}{2^{n-1}}}\right), \tag{25}$$

which leads to the condition

$$\left(\frac{e^{-i\frac{\pi}{2^n}l'} b_{l'}}{c_{l'}}\right)\left(\frac{b^*_{m'} e^{i\frac{\pi}{2^n}m'}}{c^*_{m'}}\right) = \cos\left(\frac{\pi}{2^n}(l'-m')\right). \tag{26}$$

Which means that in order to not generate entropy during the reduction of our computational space for each pair of $c_{l'}$ and $c^*_{m'}$ coefficients we must find a matching pair of $b_{l'}$ and $b^*_{m'}$ parameters which satisfy the above condition.

6. Conclusion and Possible Links to Other Areas

In summary, under the truncation $N + 1 = 2^n$, Pegg–Barnett phase eigenstates map to a remarkable orthonormal n-qubit basis of fully separable product states, each decomposing into single-qubit terms with binary-determined phases. This results in zero bipartite von Neumann entanglement entropy across all partitions, contrasting sharply with entangled bases like Bell states. Detailed analysis of partial tracing reveals a “folding” mechanism: individual basis states stay pure during dimension reduction, while superpositions generate entropy unless coefficients meet a specific phase-matching condition. These features position phase states as an ideal, correlation-free framework for quantum information encoding and manipulation in digitized field modes and multi-layered algorithms, advancing quantum computing protocols and opening avenues for further exploration in entanglement management, with rigorous mathematical formulation ensuring reproducibility.

Results presented in this article might be linked to other disciplines of physics. Namely, Bekenstein in his pioneering work [6,7] established that the maximum entropy which can be stored within a finite region of space does not scale with the region’s volume but is constrained by a specific upper limit, proportional to the surface area of the enclosing boundary. In Planck units, the entropy bound relates to the maximum number of qubits that can be stored within a given spatial region. The existence of this universal limit was also supported by Hawking’s discovery of black hole radiation [8,9] showing that black holes carry entropy equal to one quarter of their horizon area. The Bekenstein bound strengthened by Bekenstein–Hawking entropy exposes that the fundamental degrees of freedom of spacetime obeys an inherently holographic scaling, with information measured in qubits residing effectively on surfaces rather than in volumes. This recognition was generalized in the holographic principle indicating that the complete physics of a finite spatial region can be described on its boundary [10,11]. As a concrete manifestation of this principle the AdS/CFT correspondence assumes a connection between a non-gravitational conformal field theory on the boundary and the full gravitational dynamics in the bulk [12]. Within the framework of this correspondence, the entanglement has the leading role. So one might ask, what if the phase states are the building blocks of our physical reality and the entanglement might be the reason for the formation of structures. While the Bekenstein bound sustains

 doi:10.20944/preprints202509.0197.v1

a limit of informational capacity of spacetime the holographic principle imposes the encoding of this information on boundaries.

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