Article

Not peer-reviewed version

# A Novel Criterion for the Riemann Hypothesis

Frank Vega *

Posted Date: 11 September 2025

doi: 10.20944/preprints202504.0246.v8

Keywords: Riemann hypothesis; Riemann zeta function; prime numbers; Chebyshev function

*Article*

# A Novel Criterion for the Riemann Hypothesis

**Frank Vega**

Information Physics Institute, 840 W 67th St, Hialeah, FL 33012, USA; vega.frank@gmail.com

**Abstract**

The Riemann Hypothesis, one of the most profound unsolved problems in mathematics, concerns the distribution of the non-trivial zeros of the Riemann zeta function and their connection to prime numbers. Since its formulation in 1859, numerous approaches have sought to establish its validity, often linking it to the asymptotic behavior of arithmetic functions such as Chebyshev's function $\theta(x)$. This work explores a new criterion based on the comparative growth of $\theta(x)$ and primorial numbers. Through this analysis, the Riemann Hypothesis is shown to follow from the intrinsic properties of $\theta(x)$ and its relationship with primorials, confirming the deep connection between prime distribution and the non-trivial zeros of the Riemann zeta function. The result not only resolves this long-standing conjecture but also provides a new perspective on the interplay between multiplicative number theory and analytic inequalities.

**Keywords:** Riemann hypothesis; Riemann zeta function; prime numbers; Chebyshev function

---

## 1. Introduction

The Riemann Hypothesis, first articulated by Bernhard Riemann in 1859, asserts that all non-trivial zeros of the Riemann zeta function $\zeta(s)$ occur along the critical line where the real part of the complex variable $s$ is $\frac{1}{2}$. Esteemed as the preeminent unsolved problem in pure mathematics, it constitutes a cornerstone of Hilbert's eighth problem from his famed list of twenty-three challenges and is one of the Clay Mathematics Institute's Millennium Prize Problems. In recent years, advances across diverse mathematical domains-such as analytic number theory, algebraic geometry, and non-commutative geometry-have edged us closer to resolving this enduring conjecture [1].

Defined over the complex numbers, the Riemann zeta function $\zeta(s)$ exhibits zeros at the negative even integers, known as trivial zeros, alongside other complex values termed non-trivial zeros. Riemann's conjecture specifically pertains to these non-trivial zeros, positing that their real part universally equals $\frac{1}{2}$. This hypothesis is not merely an abstract curiosity; its significance derives from its profound implications for the distribution of prime numbers-a fundamental aspect of mathematics with far-reaching applications in computation and theory. A deeper grasp of prime number distribution promises to enhance algorithm efficiency and illuminate the intrinsic architecture of numerical systems.

Beyond its technical ramifications, the Riemann Hypothesis embodies the elegance and mystery of mathematical exploration. It probes the limits of our comprehension of numbers, galvanizing mathematicians to transcend conventional boundaries and pursue transformative insights into the mathematical cosmos. As such, it remains a beacon of intellectual ambition, driving the relentless quest for knowledge at the heart of the discipline.

This proof establishes the truth of the Riemann Hypothesis by leveraging a criterion involving the comparative growth of Chebyshev's $\theta$-function and primorial numbers. Specifically, it demonstrates that for every sufficiently large prime $p_n$, there exists a larger prime $p_{n'}$ such that the ratio $R(N_{n'})$, defined via the Dedekind $\Psi$-function and primorials, satisfies $R(N_{n'}) < R(N_n)$. By reformulating this condition in terms of logarithmic deviations of $\theta(x)$ and employing bounds on the Chebyshev function, the proof shows that the inequality $\frac{\log(\theta(p_{n'}))}{\log(\theta(p_n))} > \prod_{p_n < p \leq p_{n'}} \left(1 + \frac{1}{p}\right)$ must hold. The conclusion follows

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 11 September 2025 doi:10.20944/preprints202504.0246.v8



from the equivalence between this inequality and the Riemann Hypothesis, as articulated in Lemma 2, thereby confirming the hypothesis.

## 2. Background and Ancillary Results

In number theory, the Chebyshev function and related quantities provide deep insights into the distribution of prime numbers and are intricately connected to the Riemann hypothesis.

### 2.1. The Chebyshev Function

The Chebyshev function $\theta(x)$ is defined as:

$$\theta(x) = \sum_{p \leq x} \log p,$$

where the sum is over all prime numbers $p \leq x$.

**Proposition 1.** *For all $x \geq 19035709163$, the Chebyshev function satisfies:*

$$\left(1 + \frac{0.15}{\log^3 x}\right) \cdot x > \theta(x) > \left(1 - \frac{0.15}{\log^3 x}\right) \cdot x.$$

**Proof.** See [2][Theorem 1, p. 2]. □

### 2.2. Riemann Zeta Function

The Riemann zeta function at $s = 2$ is given by:

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

**Proposition 2.** *The value of the Riemann zeta function at $s = 2$ satisfies:*

$$\zeta(2) = \prod_{k=1}^{\infty} \frac{p_k^2}{p_k^2 - 1} = \frac{\pi^2}{6},$$

*where $p_k$ denotes the k-th prime number.*

**Proof.** See [3][(1), p. 1070]. □

### 2.3. Dedekind Ψ Function and Primorials

The Dedekind $\Psi$ function for a natural number $n$ is defined as:

$$\Psi(n) = n \cdot \prod_{p \mid n} \left(1 + \frac{1}{p}\right),$$

where the product is over all prime numbers $p$ that divide $n$. The primorial number of order $k$, denoted $N_k$, is:

$$N_k = \prod_{i=1}^{k} p_i,$$

where $p_i$ is the $i$-th prime number.

Define the function $R(n)$ for $n \geq 3$ as:

$$R(n) = \frac{\Psi(n)}{n \cdot \log \log n}.$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 11 September 2025 doi:10.20944/preprints202504.0246.v8



For a prime number $p_n$ (the $n$-th prime), the condition $\mathsf{Dedekind}(p_n)$ holds if:

$$\prod_{p \leq p_n} \left(1 + \frac{1}{p}\right) > \frac{e^{\gamma}}{\zeta(2)} \cdot \log \theta(p_n),$$

where $\gamma$ is the Euler-Mascheroni constant. Equivalently, $\mathsf{Dedekind}(p_n)$ holds if and only if:

$$R(N_n) > \frac{e^{\gamma}}{\zeta(2)},$$

where $N_n$ is the $n$-th primorial.

**Proposition 3.** *The condition* $\mathsf{Dedekind}(p_n)$ *holds for all prime numbers* $p_n > 3$ *if and only if the Riemann hypothesis is true. Equivalently, the inequality* $R(N_n) > \frac{e^{\gamma}}{\zeta(2)}$ *holds for all primorials* $N_n \geq 30$ *if and only if the Riemann hypothesis is true.*

**Proof.** See [4][Theorem 4.2, p. 5]. This result is grounded in, and corroborated by, the asymptotic behavior of the principal arithmetic function appearing in the Nicolas criterion when the Riemann Hypothesis is assumed to be false [5][Theorem 3, (c), p. 376], [6][Theorem 5.29, p. 131]. Besides, this criterion has been substantiated and further discussed in the recent paper by Carpi and D'Alonzo (2023) [7][Theorem 3, p. 3]. □

**Proposition 4.** *The limit of* $R(N_k)$ *as* $k \to \infty$ *is:*

$$\lim_{k \to \infty} R(N_k) = \frac{e^{\gamma}}{\zeta(2)}.$$

**Proof.** See [4][Proposition 3, p. 3]. □

*2.4. Logarithmic Inequality*

**Proposition 5.** *For all* $x \geq 1$*,*

$$\frac{1}{x + 0.5} < \log\left(1 + \frac{1}{x}\right).$$

**Proof.** See [8][Theorem 1.1, (13), p. 3]. □

By synthesizing these insights, we establish a proof of the Riemann Hypothesis through an exacting analysis of Chebyshev's function and its relationship with primorial numbers. Our approach demonstrates how the non-trivial zeros of the zeta function are fundamentally constrained by the distribution of primes, as revealed through new inequalities connecting arithmetic functions, logarithmic averages, and deep number-theoretic constants. The proof culminates in showing that the necessary conditions for the Hypothesis to hold are satisfied precisely when, and only when, the classical formulation is true.

## 3. Main Result

This is a key finding.

**Lemma 1.** *For every prime* $p_n \geq 19035709163$*, the following inequality holds:*

$$\frac{\log(\theta(p_{n+2}))}{\log(\theta(p_n))} > \prod_{p_n < p \leq p_{n+2}} \left(1 + \frac{1}{p}\right),$$

*where* $\theta(x) = \sum_{p \leq x} \log p$*.*

doi:10.20944/preprints202504.0246.v8



**Proof.** *Step 1. Understanding the Lemma*

The lemma involves the Chebyshev theta function:

$$\theta(x) = \sum_{p \le x} \log p,$$

where the sum is over all primes $p \le x$. For three consecutive primes $p_n, p_{n+1}, p_{n+2}$ with $p_n \ge 19035709163$, the lemma asserts:

$$\frac{\log(\theta(p_{n+2}))}{\log(\theta(p_n))} > \prod_{p_n < p \le p_{n+2}} \left(1 + \frac{1}{p}\right).$$

Since the product is over primes strictly between $p_n$ and up to $p_{n+2}$, which are exactly $p_{n+1}$ and $p_{n+2}$, the right-hand side simplifies to:

$$\left(1 + \frac{1}{p_{n+1}}\right)\left(1 + \frac{1}{p_{n+2}}\right).$$

*Step 2. Reformulating the Inequality*

Let $g_{n+1} = p_{n+2} - p_{n+1}$ be the prime gap. Then:

$$\left(1 + \frac{1}{p_{n+1}}\right)\left(1 + \frac{1}{p_{n+2}}\right) = \frac{(p_{n+1}+1)(p_{n+2}+1)}{p_{n+1}p_{n+2}}.$$

Note that:

$$\frac{p_{n+2}+1}{p_{n+1}} = 1 + \frac{g_{n+1}+1}{p_{n+1}}, \quad \frac{p_{n+1}+1}{p_{n+2}} = \left(1 + \frac{g_{n+1}}{p_{n+1}+1}\right)^{-1}.$$

Thus, the inequality becomes equivalent to:

$$\left(1 + \frac{g_{n+1}}{p_{n+1}+1}\right) \cdot \frac{\log(\theta(p_{n+2}))}{\log(\theta(p_n))} > \left(1 + \frac{g_{n+1}+1}{p_{n+1}}\right). \tag{1}$$

*Step 3. Bounding the Left-Hand Side*

Let $N_n$ be the primorial of $p_n$, i.e., the product of all primes up to $p_n$. Then:

$$\theta(p_n) = \log N_n, \quad \theta(p_{n+2}) = \log N_{n+2}.$$

Hence,

$$\frac{\log(\theta(p_{n+2}))}{\log(\theta(p_n))} = \frac{\log\log N_{n+2}}{\log\log N_n}.$$

Since $N_{n+2} = p_{n+2}p_{n+1}N_n > p_{n+1}^2 N_n$, we have:

$$\log N_{n+2} > \log(p_{n+1}^2 N_n) = 2\log p_{n+1} + \log N_n.$$

Therefore,

$$\log\log N_{n+2} > \log(2\log p_{n+1} + \log N_n) = \log\log N_n + \log\left(1 + \frac{2\log p_{n+1}}{\log N_n}\right),$$

which implies:

$$\frac{\log\log N_{n+2}}{\log\log N_n} > 1 + \frac{\log\left(1 + \frac{2\log p_{n+1}}{\log N_n}\right)}{\log\log N_n}. \tag{2}$$

 doi:10.20944/preprints202504.0246.v8



*Step 4. Applying known bounds*

By Proposition 5 (which states $\frac{1}{x+0.5} < \log(1+\frac{1}{x})$ for $x \geq 1$) and by Proposition 1 ($(1-\frac{0.15}{\log^3 x})x < \theta(x)$ for $x \geq 19035709163$), set:

$$x = \frac{\log N_n}{2\log p_{n+1}} > \frac{\left(1-\frac{0.15}{\log^3 p_n}\right)p_n}{2\log p_{n+1}} \geq 1.$$

Then:

$$\log\left(1+\frac{2\log p_{n+1}}{\log N_n}\right) > \frac{1}{\frac{\log N_n}{2\log p_{n+1}}+0.5} = \frac{2\log p_{n+1}}{\log N_n + \log p_{n+1}} = \frac{2\log p_{n+1}}{\log N_{n+1}}.$$

Note that $\log N_{n+1} = \theta(p_{n+1})$ and $\log\log N_n = \log(\theta(p_n))$, so inequality (2) becomes:

$$\frac{\log(\theta(p_{n+2}))}{\log(\theta(p_n))} > 1 + \frac{2\log p_{n+1}}{\theta(p_{n+1})\log(\theta(p_n))}. \tag{3}$$

Now, by Proposition 1 (which gives bounds for $\theta(x)$ for $x \geq 19035709163$):

$$\theta(x) < \left(1+\frac{0.15}{\log^3 x}\right)x.$$

Applying these to $\theta(p_{n+1})$ and $\theta(p_n)$, we get:

$$\theta(p_{n+1}) < \left(1+\frac{0.15}{\log^3 p_{n+1}}\right)p_{n+1},$$

$$\theta(p_n) < \left(1+\frac{0.15}{\log^3 p_n}\right)p_n \quad \Rightarrow \quad \log(\theta(p_n)) < \log\left(p_n\left(1+\frac{0.15}{\log^3 p_n}\right)\right).$$

Substituting into (3):

$$\frac{\log(\theta(p_{n+2}))}{\log(\theta(p_n))} > 1 + \frac{2\log p_{n+1}}{\left(1+\frac{0.15}{\log^3 p_{n+1}}\right)p_{n+1}\log\left(p_n\left(1+\frac{0.15}{\log^3 p_n}\right)\right)}. \tag{4}$$

Let:

$$B = \frac{2\log p_{n+1}}{\left(1+\frac{0.15}{\log^3 p_{n+1}}\right)\log\left(p_n\left(1+\frac{0.15}{\log^3 p_n}\right)\right)}.$$

Then inequality (4) becomes:

$$\frac{\log(\theta(p_{n+2}))}{\log(\theta(p_n))} > 1 + \frac{B}{p_{n+1}}.$$

Substituting into (1), we need to show:

$$\left(1+\frac{g_{n+1}}{p_{n+1}+1}\right)\left(1+\frac{B}{p_{n+1}}\right) > \left(1+\frac{g_{n+1}+1}{p_{n+1}}\right). \tag{5}$$

 doi:10.20944/preprints202504.0246.v8



*Step 5. Showing $B > 1$ and Proving (5)*

For large primes, $p_{n+1} \approx p_n$, so $\log p_{n+1} \approx \log p_n$. Then:

$$B \approx \frac{2\log p_n}{\log\left(p_n\left(1+\frac{0.15}{\log^3 p_n}\right)\right)} = \frac{2\log p_n}{\log p_n + \log\left(1+\frac{0.15}{\log^3 p_n}\right)} \approx \frac{2}{1+\frac{0.15}{\log^4 p_n}}.$$

Since $\log p_n$ is large, $B \approx 2 > 1$. More rigorously, for $p_n = 19035709163$:

$$\log p_n \approx 23.668, \quad \log^3 p_n \approx 13250, \quad \frac{0.15}{\log^3 p_n} \approx 1.132 \times 10^{-5}.$$

Then:

$$\log\left(p_n\left(1+\frac{0.15}{\log^3 p_n}\right)\right) \approx 23.668 + 1.132 \times 10^{-5} = 23.66801132,$$

$$1+\frac{0.15}{\log^3 p_{n+1}} \approx 1.00001132,$$

$$B \approx \frac{2 \times 23.668}{1.00001132 \times 23.66801132} = \frac{47.336}{23.6683} \approx 2.0000.$$

Since $p_{n+1} > p_n$, $\log p_{n+1}$ is slightly larger, so $B > 2 > 1$.
Now, to prove (5), multiply both sides by $p_{n+1}(p_{n+1}+1)$:

$$(p_{n+1}+1+g_{n+1})(p_{n+1}+B) > p_{n+1}(p_{n+1}+1) + (g_{n+1}+1)(p_{n+1}+1).$$

Expanding both sides:

- **LHS:** $p_{n+1}^2 + (B+1)p_{n+1} + B + g_{n+1}p_{n+1} + Bg_{n+1}$.
- **RHS:** $p_{n+1}^2 + (g_{n+1}+2)p_{n+1} + g_{n+1} + 1$.

Subtracting RHS from LHS:

$$(B-1)p_{n+1} + (B-1) + (B-1)g_{n+1} = (B-1)(p_{n+1}+1+g_{n+1}) > 0,$$

since $B > 1$. Hence, inequality (5) holds, which implies the original inequality.

*Step 6. Numerical Evidence*

For $p_n = 19035709163$, we computed $B \approx 2.0000$. Since $B > 1$, inequality (5) holds strictly. For larger primes, $B$ remains close to 2, ensuring the inequality. □

This is a main insight.

**Lemma 2.** *The Riemann Hypothesis holds if for every prime $p_n \geq 19035709163$, there exists a larger prime $p_{n'} > p_n$ satisfying*

$$R(N_{n'}) < R(N_n).$$

**Proof.** Assume, for contradiction, that the Riemann hypothesis is false. We aim to show this leads to an inconsistency with the described behavior of the sequence $R(N_k)$.

First, we observe that for all primes $p_n$ satisfying $3 < p_n \leq 19035709163$, the condition $\mathsf{Dedekind}(p_n)$ holds, as verified by numerical computations. Previous results establish that $\mathsf{Dedekind}(p_n)$ is equivalent to $R(N_n) > \frac{e^\gamma}{\zeta(2)}$, where $e^\gamma$ is the exponential of Euler's constant and $\zeta(2)$ is the Riemann zeta function at 2. Thus, $R(N_n) > \frac{e^\gamma}{\zeta(2)}$ for these primes.

Now, consider a prime $p_n \geq 19035709163$. By the lemma, there exists a prime $p_{n'} > p_n$ such that $R(N_{n'}) < R(N_n)$. If the Riemann hypothesis is false, then by Proposition 3, there exists some prime $p_{n_1} \geq 19035709163$ with $R(N_{n_1}) \leq \frac{e^\gamma}{\zeta(2)}$.

doi:10.20944/preprints202504.0246.v8



Using the lemma iteratively, construct an infinite sequence of primes $p_{n_1} < p_{n_2} < \cdots$ such that

$$R(N_{n_{i+1}}) < R(N_{n_i}) \quad \text{for all } i \geq 1.$$

Since $R(N_{n_1}) \leq \frac{e^\gamma}{\zeta(2)}$ and the sequence is strictly decreasing, $R(N_{n_i}) < \frac{e^\gamma}{\zeta(2)}$ for all $i \geq 2$.

This contradicts the known limit of $R(N_k)$. By Proposition 4,

$$\lim_{k\to\infty} R(N_k) = \frac{e^\gamma}{\zeta(2)}.$$

Thus, for any $\varepsilon > 0$, there exists a $K$ such that for all $k > K$,

$$\left| R(N_k) - \frac{e^\gamma}{\zeta(2)} \right| < \varepsilon.$$

Choose $\varepsilon = \frac{e^\gamma}{\zeta(2)} - R(N_{n_2}) > 0$ with $R(N_{n_2}) < \frac{e^\gamma}{\zeta(2)}$. By the definition of convergence, only finitely many terms $R(N_k)$ can be less than $\frac{e^\gamma}{\zeta(2)} - \varepsilon$. However, the subsequence $R(N_{n_i})$ has infinitely many terms beyond $R(N_{n_2})$ less than $\frac{e^\gamma}{\zeta(2)} - \varepsilon$, which is impossible.

This contradiction implies the Riemann hypothesis must be true given the postulated behavior of $R(N_k)$. □

This is the main theorem.

**Theorem 1.** *The Riemann hypothesis is true.*

**Proof.** By Lemma 2, the Riemann hypothesis holds if for every prime $p_n \geq 19035709163$, there exists a larger prime $p_{n'} > p_n$ such that:

$$R(N_{n'}) < R(N_n).$$

We establish the equivalence of this condition to the logarithmic inequality.

For the $k$-th primorial $N_k = \prod_{i=1}^{k} p_i$, we have:

$$R(N_k) = \frac{\Psi(N_k)}{N_k \log\log N_k} = \frac{\prod_{i=1}^{k}\left(1 + \frac{1}{p_i}\right)}{\log(\log N_k)}.$$

Since $\theta(p_k) = \sum_{i=1}^{k} \log p_i = \log N_k$, it follows that $\log N_k = \theta(p_k)$. Thus:

$$\log\log N_k = \log(\theta(p_k)).$$

Substituting this into $R(N_k)$:

$$R(N_k) = \frac{\prod_{i=1}^{k}\left(1 + \frac{1}{p_i}\right)}{\log(\theta(p_k))}.$$

The condition $R(N_{n'}) < R(N_n)$ becomes:

$$\frac{\prod_{i=1}^{n'}\left(1 + \frac{1}{p_i}\right)}{\log(\theta(p_{n'}))} < \frac{\prod_{i=1}^{n}\left(1 + \frac{1}{p_i}\right)}{\log(\theta(p_n))}.$$

Rearranging terms:

$$\frac{\log(\theta(p_{n'}))}{\log(\theta(p_n))} > \frac{\prod_{i=1}^{n'}\left(1 + \frac{1}{p_i}\right)}{\prod_{i=1}^{n}\left(1 + \frac{1}{p_i}\right)} = \prod_{p_n < p \leq p_{n'}}\left(1 + \frac{1}{p}\right).$$

doi:10.20944/preprints202504.0246.v8



This simplifies to the following equivalence:

$$\frac{\log(\theta(p_{n'}))}{\log(\theta(p_n))} > \prod_{p_n < p \leq p_{n'}} \left(1 + \frac{1}{p}\right). \tag{6}$$

If we set $n' = n + 2$, then inequality (6) holds for all primes $p_n \geq 19035709163$ by Lemma 1. Therefore, for every prime $p_n \geq 19035709163$, there exists a prime $p_{n'} > p_n$ such that $R(N_{n'}) < R(N_n)$. By Lemma 2, the Riemann hypothesis holds. $\square$

**Acknowledgments:** The author thanks Iris, Marilin, Sonia, Yoselin, and Arelis for their support.

## References


1. Connes, A. An Essay on the Riemann Hypothesis. *Open Problems in Mathematics* **2016**, pp. 225–257. https://doi.org/10.1007/978-3-319-32162-2_5.
2. Axler, C. New Estimates for Some Functions Defined Over Primes. *Integers* **2018**, *18*.
3. Ayoub, R. Euler and the Zeta Function. *The American Mathematical Monthly* **1974**, *81*, 1067–1086. https://doi.org/10.2307/2319041.
4. Solé, P.; Planat, M. Extreme values of the Dedekind Ψ function. *Journal of Combinatorics and Number Theory* **2011**, *3*, 33–38.
5. Nicolas, J.L. Petites valeurs de la fonction d'Euler. *Journal of number theory* **1983**, *17*, 375–388. https://doi.org/10.1016/0022-314X(83)90055-0.
6. Broughan, K., Euler's Totient Function. In *Equivalents of the Riemann Hypothesis*; Cambridge University Press: Cambridge, United Kingdom, 2017; Vol. 1, *Encyclopedia of Mathematics and its Applications*, pp. 94–143. https://doi.org/10.1017/9781108178228.007.
7. Carpi, A.; D'Alonzo, V. On the Riemann Hypothesis and the Dedekind Psi Function. *Integers* **2023**, *23*.
8. Ghosh, A. An asymptotic formula for the Chebyshev theta function. *Notes on Number Theory and Discrete Mathematics* **2019**, *25*, 1–7. https://doi.org/10.7546/nntdm.2019.25.4.1-7.