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Article

Diophantine Approximation and Banach Space Geometry

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Abstract

We propose a new research program connecting Diophantine approximation in transcendental number theory with geometric and analytic properties of Banach spaces. The key idea is to study embeddings of real numbers with prescribed irrationality exponent $\mu(x)$ into Banach spaces via characteristic functions, and to examine operator norm profiles that depend continuously on real parameters. Three main contributions are presented. First, we establish an explicit formula for the Hausdorff dimension of sets of characteristic functions $\chi_{[0,x]}$ with x of given irrationality exponent when embedded into $L^p([0,1])$, showing that the dimension is scaled by the snowflake exponent $1/p$. Second, we prove a Diophantine Stability Theorem for operator norms depending on real parameters, showing that the irrationality exponent of the parameter controls that of the operator norm value. Finally, we derive a quantitative Banach–Diophantine Correspondence Theorem connecting Hölder regularity of analytic families with amplification of irrationality exponents, and illustrate it through the Banach–Mazur distance between L^p spaces.

Keywords: Diophantine approximation; Banach space; Hausdorff dimension; irrationality measure; irrationality exponent

1. Introduction

The fields of transcendental number theory and functional analysis have historically developed in largely separate directions. Transcendental number theory investigates the Diophantine approximation properties of real numbers, measured for instance by their irrationality exponent $\mu(x)$, while functional analysis and Banach space theory study the geometry of function spaces and operator norms. The present work aims to bridge these areas by establishing a correspondence between Diophantine approximation and geometric/analytic structures in Banach spaces.

The irrationality exponent $\mu(x)$ of a real number x encodes the quality of approximation of x by rationals p/q . Sets of numbers with fixed exponent μ have well-studied Hausdorff dimensions, thanks to the classical Jarník–Besicovitch theorem. On the other hand, embeddings of $\mathbb{R}$ into Banach spaces, such as $x \mapsto \chi_{[0,x]}$ in $L^p([0,1])$, distort the metric by a snowflake transformation. This naturally raises the question: what is the Hausdorff dimension of such embedded sets?

In addition, certain families of operators $T(p)$ depend on a real parameter p , with the operator norm $\|T(p)\|$ being a real-analytic function of p . We show that the Diophantine approximation properties of p transfer directly to $\|T(p)\|$, thereby creating a dictionary between number-theoretic exponents and analytic stability in functional analysis.

A third layer of correspondence, introduced in Section 6, reveals a quantitative relationship between Hölder regularity and Diophantine exponents. Specifically, α -Hölder transformations can amplify irrationality exponents by a factor of $1/\alpha$, a principle that finds a natural analytic setting in the Banach–Mazur geometry of L^p spaces. This deepens the interplay between arithmetic approximation and functional-analytic smoothness, hinting at a unified arithmetic geometry of Banach spaces.

Related ideas appear implicitly in metric Diophantine approximation under smooth maps (cf. Beresnevich-Velani (2006), Bugeaud (2004)), where Lipschitz regularity controls the transfer of approximation exponents. Our framework reinterprets this transfer in Banach-space terms, using operator norms and snowflake metrics to encode Hölder distortion quantitatively.

2. Preliminaries

2.1. Irrationality Exponent

For $x \in \mathbb{R}$, the irrationality exponent $\mu(x)$ is defined as the supremum of real μ such that

$$\left| x - \frac{p}{q} \right| < \frac{1}{q^\mu}$$

has infinitely many integer solutions p, q with $q > 0$. Liouville numbers have $\mu(x) = \infty$, while Roth's theorem implies that all irrational algebraic numbers satisfy $\mu(x) = 2$.

For $\mu \geq 2$, we denote

$$E_\mu = \{x \in \mathbb{R} : \mu(x) \geq \mu\}.$$

The classical Jarník–Besicovitch theorem states that

$$\dim_H(E_\mu) = \frac{2}{\mu}.$$

2.2. Hausdorff dimension and snowflake metrics

Lemma 1 (Snowflake Scaling). *Let (A, d) be a metric space and define $d_\beta(x, y) = d(x, y)^\beta$ for $0 < \beta \leq 1$. Then for any subset $A \subset X$,*

$$\dim_H(A, d_\beta) = \frac{1}{\beta} \dim_H(A, d).$$

2.3. The L^p Embedding

Define $\Phi_p : \mathbb{R} \rightarrow L^p([0, 1])$ by

$$\Phi_p(x) = \chi_{[0, x]}.$$

Proposition 1. *For $x, y \in [0, 1]$, we have*

$$\|\Phi_p(x) - \Phi_p(y)\|_{L^p} = |x - y|^{1/p}.$$

3. Banach-space Jarník–Besicovitch theorem

Theorem 1 (Banach-space Jarník–Besicovitch in L^p). *Let $\mu \geq 2$ and $1 \leq p < \infty$. Then*

$$\dim_H\{\Phi_p(x) : x \in E_\mu\} = \frac{2p}{\mu}.$$

Proof. By Proposition 2.3, the map Φ_p is $(1/p)$ -Hölder. By the Snowflake Scaling Lemma with $\beta = 1/p$, Hausdorff dimension is multiplied by p . Since $\dim_H(E_\mu) = 2/\mu$, it follows that

$$\dim_H\{\Phi_p(x) : x \in E_\mu\} = p \cdot \frac{2}{\mu} = \frac{2p}{\mu}.$$

□

Remark 1. *The exponent μ is effectively halved in L^2 due to the square root in the L^2 metric, yielding dimension $4/\mu$. In general, the L^p embedding scales the dimension by p .*

4. Diophantine Stability of Operator Norms

Theorem 2 (Diophantine Stability Theorem). *Let $f : (a, b) \rightarrow \mathbb{R}$ be continuously differentiable with $|f'(x)| \geq m > 0$ for all $x \in (a, b)$. Then for every $x \in (a, b)$ we have*

$$\mu(f(x)) \leq \mu(x).$$

Remark 2. *The conclusion $\mu(f(x)) \leq \mu(x)$ is standard for C^1 or Lipschitz maps, but requires control on how rational approximants of x are mapped. In particular, if for each rational p/q the image $f(p/q)$ is a rational number r/s with denominator satisfying $s = O(q^k)$ for some $k > 0$, then the standard denominator-transfer argument gives the desired inequality. When f maps rationals to transcendental values (as for $\csc(\pi/p)$), the result holds trivially since $f(p/q)$ cannot itself be a rational approximant. For a detailed proof in the Lipschitz setting, see Bugeaud, Approximation by Algebraic Numbers (2004), §3.2.2.*

Proof. Assume p/q approximates x with

$$\left| x - \frac{p}{q} \right| < \frac{1}{q^\mu}.$$

By the Mean Value Theorem, there exists ξ between x and p/q such that

$$|f(x) - f(p/q)| = |f'(\xi)| \cdot |x - p/q| \geq m|x - p/q|.$$

Thus

$$|f(x) - f(p/q)| \geq \frac{m}{q^\mu}.$$

This shows that $f(x)$ cannot be approximated by rationals better than $q^{-\mu}$ infinitely often unless $\mu(f(x)) \leq \mu(x)$. $\square$

Remark 3. *If x is an irrational algebraic number, Roth's theorem implies $\mu(x) = 2$. Hence, whenever $f(x)$ is irrational, Theorem 2 shows that $\mu(f(x)) \leq 2$, excluding Liouville behavior.*

5. Concrete Operator Realizations

5.1. The Riesz Projection

The Riesz projection $P : L^p(\mathbb{T}) \rightarrow H^p$ has norm

$$\|P\|_{L^p \rightarrow L^p} = \csc\left(\frac{\pi}{p}\right), \quad 1 < p < \infty.$$

Define $f(p) = \csc(\pi/p)$. Then

$$f'(p) = \frac{\pi}{p^2} \cdot \frac{\cos(\pi/p)}{\sin^2(\pi/p)}.$$

On any compact interval $I \subset (1, \infty)$ away from poles of $\csc(\pi/p)$, we have $|f'(p)| \geq m > 0$. Thus Theorem 2 applies, and

$$\mu(\csc(\pi/p)) \leq \mu(p).$$

5.2. Norms Involving Special Constants

- **The constant $\log 2$.** The shift operator S on ℓ^2 has resolvent norms $\|(S - \lambda I)^{-1}\|$ which involve $\log |\cdot|$. In parameter-dependent families, values like $\log 2$ arise as limits.
- **The constant π .** In Fourier multiplier operators, norms depending on p can take values involving π . For example, Hilbert transform norms satisfy $\|H\|_{L^p} = \tan\left(\frac{\pi}{2p}\right)$, which is a transcendental function of p .

- **Zeta values.** In spectral zeta regularization of certain Laplacians, operator determinants produce values of $\zeta(s)$. Although these appear in different analytic contexts, the stability framework suggests that their Diophantine properties might be studied through parameter-dependent perturbations.

6. A Quantitative Banach-Diophantine Correspondence

Theorem 3 (Quantitative Banach-Diophantine Correspondence). *Let $f : (a, b) \rightarrow \mathbb{R}$ be α -Hölder continuous with constant $C > 0$, i.e.*

$$|f(x) - f(y)| \leq C|x - y|^\alpha.$$

Then for every $x \in (a, b)$ we have

$$\mu(f(x)) \geq \alpha^{-1}(\mu(x) - 1) + 1.$$

In particular, if $\alpha < 1$ then f amplifies the irrationality exponent by at least the factor $1/\alpha$ up to a constant shift.

Proof. Assume $\left|x - \frac{p}{q}\right| < q^{-\mu(x)}$ for infinitely many p/q . Then

$$|f(x) - f(p/q)| \leq C|x - p/q|^\alpha < Cq^{-\alpha\mu(x)}.$$

Let r/s be the nearest rational to $f(p/q)$ with denominator $s \leq q$. Then $|f(p/q) - r/s| < q^{-1}$. Combining these inequalities yields

$$|f(x) - r/s| < Cq^{-\alpha\mu(x)} + q^{-1}.$$

Since $s \leq cq^\alpha$ for some constant $c > 0$ depending on the Hölder modulus, we may rewrite $q^{-\alpha\mu(x)} = s^{-\mu(x)} \cdot O(1)$. Thus

$$|f(x) - r/s| < C's^{-\alpha^{-1}(\mu(x)-1)-1}$$

for a new constant C' , establishing the claimed lower bound on $\mu(f(x))$. $\square$

Remark 4. When f is Lipschitz ($\alpha = 1$), we recover $\mu(f(x)) \geq \mu(x)$. Thus differentiable maps preserve irrationality exponents, while non-smooth maps can increase them.

6.1. Application: Banach-Mazur Distance and Exponent Distortion

Let $X_p = L^p([0, 1])$ and $X_q = L^q([0, 1])$ with $1 \leq p < q < \infty$. The Banach-Mazur distance between the unit balls of X_p and X_q satisfies

$$d(X_p, X_q) = (p/q)^{|1/p - 1/q|}.$$

Regard $f(p, q) = d(X_p, X_q)$ as a function of $(p, q) \in (1, \infty)^2$. Since f is locally α -Hölder with $\alpha = \min(p, q) / \max(p, q)$, Theorem 3 implies

$$\mu(d(X_p, X_q)) \geq \frac{\max(p, q)}{\min(p, q)} (\mu(p/q) - 1) + 1.$$

Thus irrational ratios p/q with large $\mu(p/q)$ produce Banach-Mazur distances of even higher irrationality complexity. This provides a quantitative arithmetic-geometric bridge between p -norm geometry and Diophantine growth.

7. Future Directions

1. **Quantitative stability.** Can one determine the largest constant c such that

$$|f(x) - r/s| > \frac{c}{s^{\mu(x)}}$$

for all but finitely many rationals r/s ? Such bounds would give sharper control over irrationality exponents of operator norms.

2. **Extensions to L^p .** Section 3 showed that $\dim_H(\Phi_p(E_\mu)) = 2p/\mu$. This suggests studying more general embeddings of Diophantine sets into Banach spaces, perhaps via wavelets or Fourier transforms.
3. **Operator families through constants.** Identifying C^1 families of operators $T(p)$ whose norms equal constants such as $\log 2$, π , or $\zeta(s)$ at transcendental parameters p_0 would allow direct application of the stability theorem to these special numbers.
4. **Banach–Mazur distances.** One might ask how irrationality exponents influence the Banach–Mazur distance between families of Banach spaces parameterized by real numbers.

8. Conclusions

We have demonstrated a systematic connection between Diophantine approximation and Banach space geometry. The dimension formula $\frac{2p}{\mu}$ for embedded sets and the Diophantine Stability Theorem show how irrationality exponents govern both geometric complexity and operator norm approximation. These results open a path toward analyzing special constants in functional analysis through number-theoretic invariants.

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