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Mancho Manev *

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*Article*

# Pair of Associated $\eta$-Ricci–Bourguignon Almost Solitons with Vertical Potential on Sasaki-Like Almost Contact Complex Riemannian Manifolds

**Mancho Manev**

Department of Algebra and Geometry, Faculty of Mathematics and Informatics, University of Plovdiv Paisii Hilendarski, 24 Tzar Asen St, 4000 Plovdiv, Bulgaria; mmanev@uni-plovdiv.bg; Tel.: +359-889-521-244

**Abstract:** The manifolds studied are almost contact complex Riemannian manifolds, known also as almost contact B-metric manifolds. They are equipped with a pair of pseudo-Riemannian metrics that are mutually associated to each other using an almost contact structure. Furthermore, the structural endomorphism acts as an anti-isometry for these metrics, called B-metrics, if its action is restricted to the contact distribution of the manifold. In this paper, some curvature properties of a special class of these manifolds, called Sasaki-like, are studied. Such a manifold is defined by the condition that its complex cone is a holomorphic complex Riemannian manifold (also called a Kähler–Norden manifold). Each of the two B-metrics on the considered manifold is specialised here as an $\eta$-Ricci-Bourguignon almost soliton, where $\eta$ is the contact form, i.e. has an additional curvature property such that the metric is a self-similar solution of a special intrinsic geometric flow. Almost solitons are generalizations of solitons because their defining condition uses functions rather than constants as coefficients. The introduced (almost) solitons are a generalization of some well-known (almost) solitons (such as those of Ricci, Schouten, Einstein). The soliton potential is chosen to be collinear with the Reeb vector field and is therefore called vertical. The special case of the soliton potential being solenoidal (i.e., divergence-free) with respect to each of the B-metrics is also considered. The resulting manifolds equipped with the pair of associated $\eta$-Ricci-Bourguignon almost solitons are characterized geometrically. An example of arbitrary dimension is constructed and the properties obtained in the theoretical part are confirmed.

**Keywords:** $\eta$-Ricci–Bourguignon almost soliton; almost contact B-metric manifold; almost contact complex Riemannian manifold; Sasaki-like manifold; solenoidal vector field

**MSC:** 53C25; 53D15; 53C50; 53C44; 53D35; 70G45

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## 1. Introduction

The concept of *Ricci-Bourguignon flow* is well known and was introduced by J. P. Bourguignon in 1981 [1].

Let a smooth manifold $\mathcal{M}$ be equipped with a time-dependent family $g(t)$ of (pseudo-)Riemannian metrics. The corresponding Ricci tensor and scalar curvature are denoted by $\rho(t)$ and $\tau(t)$, respectively. The family $g(t)$ is said to evolve by a Ricci-Bourguignon flow if the following evolution equation is satisfied

$$\frac{\partial}{\partial t}g = -2(\rho - \ell\tau g), \qquad g(0) = g_0,$$

where $\ell$ is a real constant. For some specific values of $\ell$, other known geometric flows are obtained: the Ricci flow for $\ell = 0$, the Einstein flow for $\ell = \frac{1}{2}$, the traceless Ricci flow for $\ell = \frac{1}{m}$, and the Schouten flow for $\ell = \frac{1}{2(m-1)}$, where $m$ is the dimension of the manifold [2,3].

It is known that solitons of this internal geometric flow on $\mathcal{M}$ are its fixed points or self-similar solutions. The *Ricci-Bourguignon soliton* (abbreviated RB soliton) is defined by the following equation [4,5]

$$\rho + \frac{1}{2}\mathcal{L}_{\vartheta}g + (\lambda + \ell\tau)g = 0, \tag{1}$$

where $\mathcal{L}_{\vartheta}g$ is the Lie derivative of $g$ along the vector field $\vartheta$, called the soliton potential, and $\lambda$ is the soliton constant. The RB soliton is called *expanding* if $\lambda > 0$, *steady* if $\lambda = 0$ and *shrinking* if $\lambda < 0$.

If $\lambda$ is a differentiable function on $\mathcal{M}$, then (1) defines an *RB almost soliton* [5].

A *trivial* RB (almost) soliton is called one whose potential $\vartheta$ is a Killing vector field, i.e. $\mathcal{L}_{\vartheta}g$ is zero.

Other recent studies by other authors on RB solitons are published in [6–10], and on almost contact B-metric manifolds are [11–14].

We study almost contact complex Riemannian manifolds (abbreviated accR manifolds), also known as almost contact B-metric manifolds. Remarkably, they are equipped with a pair of pseudo-Riemannian metrics (known as B-metrics) $g$ and $\tilde{g}$ that are mutually related using the almost contact structure. This fact gives us reason in [15] to introduce a generalization of the RB almost soliton in (1) by

$$\rho + \frac{1}{2}\mathcal{L}_{\vartheta}g + \frac{1}{2}\mathcal{L}_{\vartheta}\tilde{g} + (\lambda + \ell\tau)g + (\tilde{\lambda} + \ell\tilde{\tau})\tilde{g} = 0. \tag{2}$$

In this paper we study another idea, different from (2), for including both B-metrics in the definition of RB almost solitons. Namely, to generate a RB almost soliton from each of the two B-metrics by involving the contact form $\eta$, using the well-known concept of $\eta$ almost solitons of the studied type.

For definiteness of the Lie derivative of the metric, we specialize the accR manifolds into the type called Sasaki-like accR manifolds. They are defined geometrically by the condition that the complex cone of such a manifold is a holomorphic complex Riemannian manifold (also called a Kähler-Norden manifold).

The soliton potential is chosen to be in some special positions relative to the structures. One is that it be vertical, i.e. pointwise collinear with the Reeb vector field, and the second is that the Lie derivative of any B-metric with respect to the potential be the same metric multiplied by a function.

## 2. accR Manifolds

We consider a smooth manifold $\mathcal{M}$ of dimension $2n+1$. It is equipped with an almost contact structure $(\varphi, \xi, \eta)$ and a B-metric $g$. This means that $\varphi$ is an endomorphism in the Lie algebra $\mathfrak{X}(\mathcal{M})$ of vector fields on $\mathcal{M}$, $\xi$ is a Reeb vector field, $\eta$ is its dual contact form and $g$ is a pseudo-Riemannian metric of signature $(n+1, n)$ such that

$$\begin{gathered}\varphi\xi = 0, \qquad \varphi^2 = -\iota + \eta \otimes \xi, \qquad \eta \circ \varphi = 0, \qquad \eta(\xi) = 1, \\ g(\varphi x, \varphi y) = -g(x,y) + \eta(x)\eta(y),\end{gathered} \tag{3}$$

where $\iota$ denotes the identity in $\mathfrak{X}(\mathcal{M})$ [16].

In the latter equality and further, $x$, $y$, $z$, $w$ will stand for arbitrary elements of $\mathfrak{X}(\mathcal{M})$ or vectors in the tangent space $T_p\mathcal{M}$ of $\mathcal{M}$ at an arbitrary point $p$ of $\mathcal{M}$.

The following identities follow directly from (3) and together with (3) are widely used further on

$$g(\varphi x, y) = g(x, \varphi y), \qquad g(x,\xi) = \eta(x), \qquad g(\xi,\xi) = 1, \qquad \eta(\nabla_x \xi) = 0,$$

where $\nabla$ denotes the Levi-Civita connection of $g$.

On such a manifold $\mathcal{M}$ there exists an associated metric $\tilde{g}$ of $g$, which is also a B-metric and is defined by

$$\tilde{g}(x,y) = g(x,\varphi y) + \eta(x)\eta(y). \tag{4}$$

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Then the considered manifold is called an *almost contact B-metric manifold* or *almost contact complex Riemannian manifold* (abbreviated, an *accR manifold*) and we use $(\mathcal{M}, \varphi, \xi, \eta, g, \tilde{g})$ to denote it.

The studied manifolds are classified in [16], where the Ganchev–Mihova–Gribachev classification is introduced. It consists of 11 basic classes $\mathcal{F}_i$, $i \in \{1, 2, \dots, 11\}$, determined by conditions for the (0,3)-tensor $F$ defined by

$$F(x, y, z) = g\big((\nabla_x \varphi)y, z\big).$$

It has the following basic properties:

$$\begin{aligned} F(x, y, z) = F(x, z, y) = F(x, \varphi y, \varphi z) + \eta(y) F(x, \xi, z) + \eta(z) F(x, y, \xi), \\ F(x, \varphi y, \xi) = (\nabla_x \eta)(y) = g(\nabla_x \xi, y). \end{aligned} \tag{5}$$

The Lee forms of $(\mathcal{M}, \varphi, \xi, \eta, g, \tilde{g})$ are the following 1-forms associated with $F$:

$$\theta(z) = g^{ij} F(e_i, e_j, z), \quad \theta^*(z) = g^{ij} F(e_i, \varphi e_j, z), \quad \omega(z) = F(\xi, \xi, z),$$

where $\left(g^{ij}\right)$ is the inverse matrix of the matrix $\left(g_{ij}\right)$ of $g$ with respect to a basis $\{e_i; \xi\}$ $(i = 1, 2, \dots, 2n)$ of $T_p M$.

### 2.1. Sasaki-Like accR Manifolds

In [17], the construction of the complex cone of such a manifold was used with the requirement that it be a Kähler-Norden manifold. Thus, an interesting class of accR manifolds was introduced. The manifolds contained in it are called *Sasaki-like accR manifolds* and are defined by the condition

$$F(x, y, z) = g(\varphi x, \varphi y)\eta(z) + g(\varphi x, \varphi z)\eta(y). \tag{6}$$

As a consequence of the last equality, the Lee forms of Sasaki-like accR manifolds are

$$\theta = -2n\,\eta, \qquad \theta^* = \omega = 0. \tag{7}$$

Since due to (5) and (6) the contact form $\eta$ is closed in this case, then $\theta$ in (7) is also of this type. Moreover, the following identities are valid for Sasaki-like accR manifolds [17]

$$\nabla_x \xi = -\varphi x, \qquad \rho(x, \xi) = 2n\,\eta(x), \qquad \rho(\xi, \xi) = 2n, \tag{8}$$

where $\rho$ denotes the Ricci tensor for $g$.

The class of Sasaki-like accR manifolds is a subclass of the basic class $\mathcal{F}_4$ of the Ganchev–Mihova–Gribachev classification, which does not intersect with the special class $\mathcal{F}_0$ of cosymplectic accR manifolds defined by $F = 0$. The definition condition of $\mathcal{F}_4$ is

$$F(x, y, z) = -\alpha\{g(\varphi x, \varphi y)\eta(z) + g(\varphi x, \varphi z)\eta(y)\},$$

where we use the notation $\alpha = \frac{\theta(\xi)}{2n}$ for brevity. Furthermore, since $\theta$ is a closed 1-form on every Sasaki-like accR manifold, then their class is contained in the subclass $\mathcal{F}_4^0$ of $\mathcal{F}_4$ defined by the condition $\mathrm{d}\theta = 0$.

**Lemma 1.** *Let $(\mathcal{M}, \varphi, \xi, \eta, g, \tilde{g})$ be a Sasaki-like accR manifold with Ricci tensors $\rho$ and $\tilde{\rho}$ for associated B-metrics $g$ and $\tilde{g}$, respectively. Then, $\rho$ and $\tilde{\rho}$ coincide, i.e.*

$$\tilde{\rho}(y, z) = \rho(y, z). \tag{9}$$

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**Proof.** For the manifolds of $\mathcal{F}_4^0$, the following relation between the corresponding curvature tensors for $g$ and $\tilde{g}$ has been obtained in [18]:

$$\begin{aligned}\tilde{R}(x,y)z = R(x,y)z + \alpha^2\{[g(y,\varphi z) - g(\varphi y,\varphi z)]\varphi x - [g(x,\varphi z) - g(\varphi x,\varphi z)]\varphi y\} \\ - \{\mathrm{d}\alpha(\xi) - \alpha^2\}\{g(\varphi y,\varphi z)\eta(x) - g(\varphi x,\varphi z)\eta(y)\}\xi \\ + \{\mathrm{d}\alpha(\xi) + \alpha^2\}\{g(y,\varphi z)\eta(x) - g(x,\varphi z)\eta(y)\}\xi.\end{aligned}$$

According to (7), Sasaki-like accR manifolds have $\alpha = -1$ and then the above formula is specialized to the following form:

$$\begin{aligned}\tilde{R}(x,y)z = R(x,y)z + \{g(y,\varphi z) - g(\varphi y,\varphi z)\}\varphi x + \{g(y,\varphi z) + g(\varphi y,\varphi z)\}\eta(x)\xi \\ - \{g(x,\varphi z) - g(\varphi x,\varphi z)\}\varphi y - \{g(x,\varphi z) + g(\varphi x,\varphi z)\}\eta(y)\xi.\end{aligned} \tag{10}$$

We apply (4) to (10) and get

$$\begin{aligned}\tilde{R}(x,y,z,w) = R(x,y,z,\varphi w) + R(x,y,z,\xi)\eta(w) \\ + \{g(y,\varphi z) - g(\varphi y,\varphi z)\}g(\varphi x,\varphi w) \\ - \{g(x,\varphi z) - g(\varphi x,\varphi z)\}g(\varphi y,\varphi w) \\ + \{g(y,\varphi z) + g(\varphi y,\varphi z)\}\eta(x)\eta(w) \\ - \{g(x,\varphi z) + g(\varphi x,\varphi z)\}\eta(y)\eta(w).\end{aligned}$$

Contracting the last equality for $x = e_i$ and $w = e_j$ by $\tilde{g}^{ij} = -\varphi^j_s g^{is} + \xi^i\xi^j$, which follows from (4), we obtain that (9) is true. □

Similarly, we contract (9) and obtain a formula for the scalar curvature $\tilde{\tau}$ concerning $\tilde{g}$ and the associated quantity of $\tau$ regarding $\varphi$, defined by $\tau^* = g^{ij}\rho(e_i,\varphi e_j)$. This formula has the form $\tilde{\tau} = -\tau^* + \rho(\xi,\xi)$, which together with the last result in (8) for Sasaki-like accR manifolds leads to

$$\tilde{\tau} = -\tau^* + 2n. \tag{11}$$

The same result is also obtained as a consequence of the following formula for an arbitrary $\mathcal{F}_4^0$-manifold, given in [18]

$$\tilde{\tau} = -\tau^* + 2n\Big\{\mathrm{d}\alpha(\xi) + \alpha^2\Big\}.$$

### 2.2. *Almost Einstein-Like accR Manifolds*

In [19], the notion of an Einstein-like manifold was introduced for accR manifolds and it was studied for Ricci-like solitons. An accR manifold $(\mathcal{M},\varphi,\xi,\eta,g,\tilde{g})$ is said to be *Einstein-like* if its Ricci tensor $\rho$ satisfies the condition

$$\rho = a\,g + b\,\tilde{g} + c\,\eta\otimes\eta \tag{12}$$

for some triplet of constants $(a,b,c)$. In particular, $(\mathcal{M},\varphi,\xi,\eta,g,\tilde{g})$ is called an *$\eta$-Einstein manifold* if $b = 0$, or an *Einstein manifold* if $b = c = 0$. If $a$, $b$, $c$ in (12) are functions on $\mathcal{M}$, then the manifold is called *almost Einstein-like*, *almost $\eta$-Einstein* and *almost Einstein*, respectively [20].

An (almost) Einstein-like manifold is said to be *proper* if it cannot be (almost) $\eta$-Einstein manifold or (almost) Einstein manifold.

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## 3. $\eta$-RB Almost Solitons

The idea of generalizing a soliton with an additional 1-form to an $\eta$-soliton is known. We apply this idea to RB almost solitons defined by (1), and the contact form $\eta$. Thus, we obtain the so-called *$\eta$-Ricci-Bourguignon almost soliton* (in short, $\eta$-RB almost soliton) induced by the metric $g$ as follows

$$\rho + \frac{1}{2}\mathcal{L}_{\vartheta} g + (\lambda + \ell\tau)g + \mu\, \eta \otimes \eta = 0, \tag{13}$$

where $\mu$ is also a function on $\mathcal{M}$ [6]. We denote this almost soliton by $(g; \vartheta; \lambda, \mu, \ell)$.

Obviously, an $\eta$-RB almost soliton with $\mu = 0$ is an RB almost soliton, otherwise it is called *proper* $\eta$-RB almost soliton.

Again, in the case where $\lambda$ and $\mu$ are constants on the manifold, *solitons* of the corresponding kind are said to be given.

In the present paper, we explore an idea of including both B-metrics in the definition of RB almost solitons, but in a different way than (2). In addition to the pair of B-metrics, we also have the structure 1-form $\eta$, so $\eta \otimes \eta$ is included in $g$ and $\tilde{g}$ as their restriction on the vertical distribution span$(\xi)$.

Similarly to (13), we also have an $\eta$-RB almost soliton induced by the other B-metric $\tilde{g}$, defined as follows

$$\tilde{\rho} + \frac{1}{2}\mathcal{L}_{\vartheta} \tilde{g} + (\tilde{\lambda} + \tilde{\ell}\tilde{\tau})\tilde{g} + \tilde{\mu}\, \eta \otimes \eta = 0, \tag{14}$$

where $\tilde{\mu}$ is also a function on $\mathcal{M}$. We denote this almost soliton by $(\tilde{g}; \vartheta; \tilde{\lambda}, \tilde{\mu}, \tilde{\ell})$.

Clearly, an $\eta$-RB almost soliton with respect to $\tilde{g}$ with $\tilde{\mu} = 0$ is an RB almost soliton with respect to $\tilde{g}$, otherwise it is called *proper*.

If $\tilde{\lambda}$ and $\tilde{\mu}$ are constants on the manifold, then *solitons* of the corresponding kind are said to be given.

**Definition 1.** *An accR manifold $(\mathcal{M}, \varphi, \xi, \eta, g, \tilde{g})$ is said to be equipped with a pair of associated $\eta$-RB almost solitons with potential vector field $\vartheta$ if the corresponding Ricci tensors $\rho$, $\tilde{\rho}$ and scalar curvatures $\tau$, $\tilde{\tau}$ satisfy (13) and (14), respectively.*

### *3.1. The Potential is Vertical Vector Field*

Let the potential vector field $\vartheta$ be pointwise collinear with $\xi$, i.e. $\vartheta = k\xi$ is valid, where $k$ is a non-vanishing differentiable function on $\mathcal{M}$. Obviously, $k = \eta(\vartheta)$ holds and therefore $\vartheta$ belongs to the vertical distribution $\mathcal{H}^{\perp} = \text{span}(\xi)$, which is orthogonal to the contact distribution $\mathcal{H} = \ker(\eta)$ with respect to both $g$ and $\tilde{g}$.

**Theorem 1.** *Let $(\mathcal{M}, \varphi, \xi, \eta, g, \tilde{g})$ be a $(2n+1)$-dimensional Sasaki-like accR manifold that is equipped with a pair of associated $\eta$-RB almost solitons $(g; \vartheta; \lambda, \mu, \ell)$ and $(\tilde{g}; \vartheta; \tilde{\lambda}, \tilde{\mu}, \tilde{\ell})$, where $\vartheta$ is a vertical potential vector field with potential function k that does not vanish anywhere.*

*Then the manifold is proper almost Einstein-like with a triplet of functions $(a, b, c) = (-k, k, 2n)$ for both the Ricci tensors, i.e. the following expressions are valid*

$$\rho = \tilde{\rho} = -k(g - \tilde{g}) + 2n\, \eta \otimes \eta. \tag{15}$$

*The scalar curvatures for the two B-metrics are of the form*

$$\tau = 2n(1-k), \tag{16}$$

$$\tilde{\tau} = 2n(1+k). \tag{17}$$

*Additionally, the following conditions are true for the functions used:*

$$\mathrm{d}k = \mathrm{d}k(\xi)\eta, \tag{18}$$

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$$\mathrm{d}k(\xi) = -\mu - k - 2n = -\tilde{\mu} + k - 2n, \tag{19}$$

$$\lambda = (1 + 2n\,\ell)k - 2n\,\ell, \qquad \tilde{\lambda} = -(1 + 2n\,\tilde{\ell})k - 2n\,\tilde{\ell}, \tag{20}$$

$$\mu - \tilde{\mu} = -2k. \tag{21}$$

**Proof.** According to [21], for a vertical potential $\vartheta$ we have the expression

$$\mathcal{L}_\vartheta g = h - 2k(\tilde{g} - \eta \otimes \eta), \tag{22}$$

where $h$ is used for brevity and is defined by $h(x, y) = \mathrm{d}k(x)\eta(y) + \mathrm{d}k(y)\eta(x)$.

Similarly, since for a Sasaki-like accR manifold the following expression is true [22],

$$\tilde{\nabla}_x \xi = -\varphi x, \tag{23}$$

we have the corresponding expression given in [15]

$$\mathcal{L}_\vartheta \tilde{g} = h + 2k(g - \eta \otimes \eta). \tag{24}$$

Then we substitute (22) and (24) into (13) and (14) respectively and obtain the following form of the Ricci tensors for the pair of B-metrics:

$$\rho = -(\lambda + \ell\tau)g + k\tilde{g} - (\mu + k)\eta \otimes \eta - \frac{1}{2}h, \tag{25}$$

$$\tilde{\rho} = -kg - (\tilde{\lambda} + \tilde{\ell}\tilde{\tau})\tilde{g} - (\tilde{\mu} - k)\eta \otimes \eta - \frac{1}{2}h. \tag{26}$$

Applying (25) and (26) to the arguments $(\xi, \xi)$ and taking into account (9) and the last equality of (8), we get

$$\lambda + \ell\tau + \mu + \mathrm{d}k(\xi) + 2n = 0,$$

$$\tilde{\lambda} + \tilde{\ell}\tilde{\tau} + \tilde{\mu} + \mathrm{d}k(\xi) + 2n = 0.$$

The last two equalities imply the following property

$$\lambda + \ell\tau + \mu = \tilde{\lambda} + \tilde{\ell}\tilde{\tau} + \tilde{\mu}. \tag{27}$$

The following equality is obtained by combining (9), (25) and (26):

$$(\lambda + \ell\tau - k)g - (\tilde{\lambda} + \tilde{\ell}\tilde{\tau} + k)\tilde{g} + (\mu - \tilde{\mu} + 2k)\eta \otimes \eta = 0. \tag{28}$$

By virtue of (27), (4) and the last formula in (3), the result in (28) takes the following form:

$$(\lambda + \ell\tau - k)g(\varphi x, \varphi y) + (\tilde{\lambda} + \tilde{\ell}\tilde{\tau} + k)g(x, \varphi y) = 0. \tag{29}$$

Replacing $y$ with $\varphi y$ in (29) gives the following equation:

$$(\tilde{\lambda} + \tilde{\ell}\tilde{\tau} + k)g(\varphi x, \varphi y) - (\lambda + \ell\tau - k)g(x, \varphi y) = 0. \tag{30}$$

Then the system of equations (29) and (30) has a solution for arbitrary $x$ and $y$ if and only if the following equalities are satisfied:

$$\lambda + \ell\tau = k, \qquad \tilde{\lambda} + \tilde{\ell}\tilde{\tau} = -k. \tag{31}$$

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Combining (27) and (31), we obtain (21). Then, due to (31) and (21), the expressions of the Ricci tensors in (25) and (26) simplify as follows

$$\rho = \tilde{\rho} = -k(g - \tilde{g}) - (\mu + k)\eta \otimes \eta - \frac{1}{2}h. \tag{32}$$

Now we contract the expression of $\rho_{ij}$ in (32) by $g^{ij}$ and obtain

$$\tau = -\mu - (2n+1)k - \mathrm{d}k(\xi). \tag{33}$$

Then we apply tensor contraction to the same expression of $\rho_{ij}$ by $\varphi^j_s g^{is}$ and the result is

$$\tau^* = -2n\,k. \tag{34}$$

After that, we take the trace of $\tilde{\rho}_{ij}$ in (32) by $\tilde{g}^{ij} = -\varphi^j_s g^{is} + \xi^i \xi^j$ and get

$$\tilde{\tau} = -\tilde{\mu} + (2n+1)k - \mathrm{d}k(\xi). \tag{35}$$

Subtracting (33) from (35) and using (21), we get the following relation between the two scalar curvatures:

$$\tilde{\tau} = \tau + 4n\,k.$$

By virtue of (11), (21), (34), and (35), we obtain the expressions in (19).

Replacing (19) into (33) and (35), we obtain the results for the scalar curvatures in (16) and (17).

The second formula in (8) implies that $\rho(\varphi x, \xi) = 0$ and using (32) we obtain $\rho(\varphi x, \xi) = -\frac{1}{2}\mathrm{d}k(\varphi x)$. Therefore, $\mathrm{d}k \circ \varphi$ vanishes, i.e. $k$ is a constant on $\mathcal{H}$, and then we have (18). As a result, $h$ is expressed as

$$h = 2\mathrm{d}k(\xi)\eta \otimes \eta \tag{36}$$

and (32) is specialized in the form of (15), taking into account (19). Hence, according to (12), $(\mathcal{M}, \varphi, \xi, \eta, g, \tilde{g})$ is almost Einstein-like as in (15) considering $\rho$ or $\tilde{\rho}$ and it is not possible to be in particular neither almost $\eta$-Einstein nor almost Einstein.

Finally, using (16) and (17) in (31), we obtain the expressions in (20) of the soliton functions $\lambda$ and $\tilde{\lambda}$ in terms of the potential function $k$ and the soliton constants $\ell$ and $\tilde{\ell}$. □

Note that if the pair $(g; \vartheta; \lambda, \mu, \ell)$ and $(\tilde{g}; \vartheta; \tilde{\lambda}, \tilde{\mu}, \tilde{\ell})$ consists in particular of RB almost solitons (i.e. $\mu = \tilde{\mu} = 0$), then due to (21) it follows that $k = 0$, which is not permissible. However, the case is permissible when one of the two $(g; \vartheta; \lambda, \mu, \ell)$ and $(\tilde{g}; \vartheta; \tilde{\lambda}, \tilde{\mu}, \tilde{\ell})$ is an RB almost soliton, and the other is a proper $\eta$-RB almost soliton (i.e. $\mu = 0$ and $\tilde{\mu} \neq 0$ or vice versa).

#### 3.1.1. The potential is vertical and solenoidal

A vector field $\vartheta$ is called *solenoidal* (also known as incompressible or transverse) if its divergence vanishes, i.e. $\operatorname{div} \vartheta = 0$.

On accR manifolds a vector field can be solenoidal with respect to $g$ or with respect to $\tilde{g}$, i.e. $\operatorname{div}_g \vartheta = 0$ or $\operatorname{div}_{\tilde{g}} = 0$ hold.

Since $\operatorname{div}_g \vartheta = g^{ij} g(\nabla_{e_i}\vartheta, e_j) = \frac{1}{2}\operatorname{tr}_g(\mathcal{L}_\vartheta g)$ is true, then if $\vartheta$ is vertical, due to (22) and (36), we have $\operatorname{div}_g \vartheta = \mathrm{d}k(\xi)$. Similarly, $\operatorname{div}_{\tilde{g}} \vartheta = \tilde{g}^{ij}\tilde{g}(\tilde{\nabla}_{e_i}\vartheta, e_j) = \frac{1}{2}\operatorname{tr}_{\tilde{g}}(\mathcal{L}_\vartheta \tilde{g})$ holds and due to (24) and (36), we obtain $\operatorname{div}_{\tilde{g}} \vartheta = \mathrm{d}k(\xi)$. Therefore, the vertical vector field $\vartheta$ on an accR manifold is solenoidal with respect to both B-metrics if and only if $k$ is a constant, taking into account (18).

**Corollary 1.** *Let the conventions of Theorem 1 be given. If, in addition, the potential $\vartheta$ is solenoidal with respect to both B-metrics, then $(\mathcal{M}, \varphi, \xi, \eta, g, \tilde{g})$ is proper Einstein-like, as well as $(g; \vartheta; \lambda, \mu, \ell)$ and $(\tilde{g}; \vartheta; \tilde{\lambda}, \tilde{\mu}, \tilde{\ell})$ are $\eta$-RB solitons, where*

$$\mu = -k - 2n, \qquad \tilde{\mu} = k - 2n. \tag{37}$$

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**Proof.** In this case, since $k$ is a constant, the manifold is Einstein-like due to (15). It is not possible to be in particular neither $\eta$-Einstein nor Einstein. Then, considering (19), the soliton functions $\mu$ and $\tilde{\mu}$ become the constants given in (37). Moreover, the soliton functions $\lambda$ and $\tilde{\lambda}$ are the constants given in (20). Therefore, the considered $\eta$-RB almost solitons become $\eta$-RB solitons. □

3.1.2. Example of an $\eta$-RB Almost Soliton with a Vertical Potential

Now let us consider a Sasaki-like accR manifold of arbitrary dimension and with an Einstein metric $\tilde{g}$, as in [17]. A contact homothetic transformation of the metric is applied by $g = p\bar{g} + q\tilde{\bar{g}} + (1 - p - q)\eta \otimes \eta$ for $p, q \in \mathbb{R}$, $(p, q) \neq (0, 0)$. It is shown there that the resulting accR manifold is again Sasaki-like and the corresponding Ricci tensor has the form

$$\rho = \frac{2n}{p^2 + q^2}\Big\{pg - q\tilde{g} + \Big(p^2 + q^2 - p + q\Big)\eta \otimes \eta\Big\}. \tag{38}$$

Consequently, the scalar curvatures with respect to B-metrics $g$ and $\tilde{g}$ are as follows

$$\tau = 2n\bigg\{1 + \frac{2np}{p^2 + q^2}\bigg\}, \qquad \tilde{\tau} = 2n\bigg\{1 - \frac{2nq}{p^2 + q^2}\bigg\}.$$

Now, to obtain (15) from (38), we need to solve the system of the following equations to determine $k$ in terms of the parameters $p$ and $q$:

$$\frac{2np}{p^2 + q^2} = -k, \qquad \frac{2nq}{p^2 + q^2} = -k, \qquad \frac{2n}{p^2 + q^2}(p^2 + q^2 - p + q) = 2n. \tag{39}$$

Obviously it only makes sense when $p = q$. Then from (39) we can determine $k = -\frac{n}{p} = \text{const}$ for $p \neq 0$. Therefore, (19) and (21) imply (37) and the values of $\mu$ and $\tilde{\mu}$ for $p \neq 0$ are the following:

$$\mu = -n\bigg(2 - \frac{1}{p}\bigg), \qquad \tilde{\mu} = -n\bigg(2 + \frac{1}{p}\bigg). \tag{40}$$

Then we obtain that the vertical potential defined by $\vartheta = -\frac{n}{p}\xi$ is solenoidal with respect to each of the two B-metrics because $\nabla\vartheta = \tilde{\nabla}\vartheta = \frac{n}{p}\varphi$, using the first equality in (8) and its counterpart (23). But, because of $\mathcal{L}_\vartheta g = \mathcal{L}_\vartheta \tilde{g} = \frac{2n}{p}(\tilde{g} - \eta \otimes \eta) \neq 0$, it is not trivial.

Taking into account (20), we have the following expressions for $\lambda$ and $\tilde{\lambda}$ for $p \neq 0$:

$$\lambda = -\frac{n}{p}\{1 + 2(n + p)\ell\}, \qquad \tilde{\lambda} = \frac{n}{p}\{1 + 2(n - p)\tilde{\ell}\}. \tag{41}$$

Clearly, they are constants.

In conclusion, we constructed a pair of associated $\eta$-RB solitons $(g; \vartheta; \lambda, \mu, \ell)$ and $(\tilde{g}; \vartheta; \tilde{\lambda}, \tilde{\mu}, \tilde{\ell})$ with a potential that is a pointwise constant vertical and solenoidal with respect to $g$ and $\tilde{g}$. This example supports Theorem 1 and Corollary 1.

Note that according to (40), $(g; \vartheta; \lambda, \mu, \ell)$ and $(\tilde{g}; \vartheta; \tilde{\lambda}, \tilde{\mu}, \tilde{\ell})$ can be separately RB solitons for $p = \frac{1}{2}$ and $p = -\frac{1}{2}$ respectively, but not simultaneously.

Taking into account (41), we classify the constructed $\eta$-RB solitons according to the sign of the soliton constants, choosing the values of $\ell$ and $\tilde{\ell}$, respectively.

The $\eta$-RB soliton $(g; \vartheta; \lambda, \mu, \ell)$ is steady if and only if we choose

$$\ell = -\frac{1}{2(n + p)}, \qquad p \neq -n$$

Analogously, $(\tilde{g}; \vartheta; \tilde{\lambda}, \tilde{\mu}, \tilde{\ell})$ is steady if and only if we set

$$\tilde{\ell} = -\frac{1}{2(n - p)}, \qquad p \neq n.$$

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The $\eta$-RB soliton $(g;\vartheta;\lambda,\mu,\ell)$ is shrinking if and only if we choose

$$\ell > -\frac{1}{2(n+p)} \quad \text{for} \quad p \in (-\infty;-n) \cup (0;+\infty) \quad \text{or} \quad \ell < -\frac{1}{2(n+p)} \quad \text{for} \quad p \in (-n;0).$$

Analogously, $(\tilde{g};\vartheta;\tilde{\lambda},\tilde{\mu},\tilde{\ell})$ is shrinking if and only if we set

$$\tilde{\ell} > -\frac{1}{2(n-p)} \quad \text{for} \quad p \in (-\infty;0) \cup (n;+\infty) \quad \text{or} \quad \tilde{\ell} < -\frac{1}{2(n-p)} \quad \text{for} \quad p \in (0;n).$$

The $\eta$-RB soliton $(g;\vartheta;\lambda,\mu,\ell)$ is expanding if and only if we choose

$$\ell < -\frac{1}{2(n+p)} \quad \text{for} \;; p \in (-\infty;-n) \cup (0;+\infty) \quad \text{or} \quad \ell > -\frac{1}{2(n+p)} \quad \text{for} \quad p \in (-n;0).$$

Analogously, $(\tilde{g};\vartheta;\tilde{\lambda},\tilde{\mu},\tilde{\ell})$ is expanding if and only if we set

$$\tilde{\ell} < -\frac{1}{2(n-p)} \quad \text{for} \quad p \in (-\infty;0) \cup (n;+\infty) \quad \text{or} \quad \tilde{\ell} > -\frac{1}{2(n-p)} \quad \text{for} \quad p \in (0;n).$$

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