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Article

On (m, w) -Generalized Group Inverse in a Banach Algebra

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Abstract

By combining the sum of a weighted group inverse and a quasinilpotent element, we introduce an extension of the m -weak and generalized group inverses in a Banach $*$ -algebra. We develop various properties and representations of this weighted generalized inverse. Furthermore, we apply this new generalized inverse to complex matrices and solve a minimization problem.

Keywords: weak group inverse; generalized group inverse; m -generalized group inverse; weighted weak group inverse; weighted generalized Drazin inverse; complex matrix; Banach $*$ -algebra

MSC: 16U50; 15A09; 16W10

1. Introduction

An involution of a Banach algebra $\mathcal{A}$ is an anti-automorphism whose square is the identity map. 1. A Banach algebra $\mathcal{A}$ with involution $*$ is called a Banach $*$ -algebra. The involution $*$ is proper if $x^*x = 0 \implies x = 0$ for any $x \in \mathcal{A}$, e.g., in a Rickart $*$ -algebra, the involution is always proper. Let $\mathbb{C}^{n \times n}$ be the Banach algebra of all $n \times n$ complex matrices, with conjugate transpose $*$ as the involution. Then the involution $*$ is proper. Throughout the paper, all Banach algebras are complex with a proper involution $*$.

An element $a \in \mathcal{A}$ has group inverse provided that there exists $x \in \mathcal{A}$ such that

$$xa^2 = a, ax^2 = x, ax = xa.$$

Such x is unique if exists, denoted by $a^\#$, and called the group inverse of a . Evidently, a square complex matrix A has group inverse if and only if $\text{rank}(A) = \text{rank}(A^2)$.

Following Wang and Chen (see [28]), an element $a \in R$ has weak group inverse if there exist $x \in R$ and $n \in \mathbb{N}$ such that

$$ax^2 = x, (a^*a^2x)^* = a^*a^2x, xa^{n+1} = a^n.$$

If such x exists, it is unique, and denote it by $a^\oplus$. In 2014, Prasad and Mohana extended core inverse and introduced core-EP inverse for a complex matrix (see [26]). A matrix $A \in \mathbb{C}^{n \times n}$ has core-EP inverse X if and only if

$$XAX = X, \mathcal{R}(X) = \mathcal{R}(X^*) = \mathcal{R}(A^k),$$

where $k = \text{ind}(A)$ is the Drazin index of A and $\mathcal{R}(X)$ is the range space of X . Such X is unique, and we denote it by $A^\oplus$. Evidently, a square complex matrix A has weak group inverse X if it satisfies the equations

$$AX^2 = X, AX = A^\oplus A.$$

(see [28]).

Following Chen and Sheibani, an element $a \in \mathcal{A}$ has generalized group inverse if there exists $x \in \mathcal{A}$ such that

$$x = ax^2, (a^*a^2x)^* = a^*a^2x, \lim_{n \rightarrow \infty} \|a^n - xa^{n+1}\|^{\frac{1}{n}} = 0.$$

Such an x is unique if it exists, and is denoted by $a^\oplus$ (see [5]). Following Chen and Sheibani, an element $a \in \mathcal{A}$ has generalized group inverse if there exists $x \in \mathcal{A}$ such that

$$x = ax^2, (a^*a^2x)^* = a^*a^2x, \lim_{n \rightarrow \infty} \|a^n - xa^{n+1}\|^{\frac{1}{n}} = 0.$$

Such x is unique if exists, denoted by $a^\oplus$ (see [5]). Many properties of weak group inverse are extended to the wider case such as linear operator on an infinitely dimensional Hilbert space.

In [33], Zhou et al. introduced and studied the m -weak group inverse. Recently, m -weak group inverse was extensively studied by many authors (see [13,15,16,22,23,25]). Recently, the authors extended m -weak group inverse and introduced m -generalized group inverse (see [2]).

On the other hand, many authors studied various weighted generalized inverses. Let $a, w \in \mathcal{A}$. An element $a \in \mathcal{A}$ has generalized w -Drazin inverse x if there exists unique $x \in \mathcal{A}$ such that

$$awx = xwa, xwawx = x \text{ and } aw - xw(aw)^2 \in \mathcal{A}^{qnil}.$$

We denote x by $a^{d,w}$ (see [19]). If we replace $\mathcal{A}^{qnil}$ by the set $\mathcal{A}^{nil}$ of all nilpotents in $\mathcal{A}$, $a^{d,w}$ is called the w -Drazin inverse of a , and denote it by $a^{D,w}$. Evidently, $a^{D,w} = x$ if and only if

$$awx = xwa, xwawx = x \text{ and } (aw)^n = xw(aw)^{n+1}$$

for some $n \in \mathbb{N}$. If the weight $w = 1$, we call $a^{d,w}$ ($a^{D,w}$) the g -Drazin (Drazin) inverse of a , and denote it by a^d (a^D).

Definition 1. (see [12]) An element $a \in \mathcal{A}$ has w -group inverse if there exist $x \in \mathcal{A}$ such that

$$awx = xwa, xwawx = x, a = awxwa.$$

The preceding x is unique if it exists, and we denote it by $a_w^\#$. The set of all w -group invertible elements in $\mathcal{A}$ is denoted by $\mathcal{A}_w^\#$.

Evidently, $a \in \mathcal{A}_w^\#$ if and only if $aw, wa \in \mathcal{A}^\#$. In [3], the author introduced and studied generalized w -group inverse in Banach algebras. Many properties of weak group inverse was thereby extended to the generalized inverse with weights.

Definition 2. (see [3]) An element $a \in \mathcal{A}$ has generalized w -group inverse if $a \in \mathcal{A}^{d,w}$ and there exist $x \in \mathcal{A}$ such that

$$x = a(wx)^2, ((aw)^d)^*(aw)^2x = ((aw)^d)^*a, \\ \lim_{n \rightarrow \infty} \|(aw)^n - xw(aw)^{n+1}\|^{\frac{1}{n}} = 0.$$

The preceding x is unique if it exists, and we denote it by $a_w^\oplus$. The set of all generalized w -group invertible elements in $\mathcal{A}$ is denoted by $\mathcal{A}_w^\oplus$.

The objective of this paper is to introduce the (m, w) -generalized group inverse, defined as the sum of a weighted group inverse and weighted quasi-nilpotent elements. Consequently, many properties of weighted and m -generalized group inverses are extended to a broader context.

Definition 3. An element $a \in \mathcal{A}$ has (m, w) -generalized group inverse if $a \in \mathcal{A}^{d,w}$ and there exists $x \in \mathcal{A}$ such that

$$x = a(wx)^2, [(aw)^d]^*(aw)^{m+1}x = [(aw)^d]^*(aw)^{m-1}a, \\ \lim_{n \rightarrow \infty} \|(aw)^n - xw(aw)^{n+1}\|^{\frac{1}{n}} = 0.$$

The preceding x is unique, and is called the (m, w) -generalized group inverse of a . We denote it by $a_w^{\otimes m}$.

In Section 2, we establish the fundamental properties of this new generalized inverse, which in turn reveal several new characteristics of the generalized inverses mentioned above. We prove that $a \in \mathcal{A}_w^{\otimes m}$ if and only if $(aw)^{m-1}a \in \mathcal{A}_w^{\otimes}$. The representations of the (m, w) -generalized group inverse are presented.

Definition 4. An element $a \in \mathcal{A}$ has (m, w) -generalized group decomposition if there exist $x, y \in \mathcal{A}$ such that

$$a = x + y, x^*a^{m-1}y = yx = 0, x \in \mathcal{A}_w^{\#}, y \in \mathcal{A}_w^{qnil}.$$

In Section 3, we prove that $a \in \mathcal{A}$ has (m, w) -generalized group inverse if and only if a has (m, w) -generalized group decomposition.

As is well known, $a \in R$ has generalized Drazin inverse if and only if it has quasi-polar property (see [6, Theorem 6.4.8]). The polar-like property for (m, w) -generalized group inverse is established. Let $a \in \mathcal{A}_w^{\otimes m}$. We prove that there exists an idempotent $p \in \mathcal{A}$ such that

$$aw + p \in \mathcal{A}^{-1}, paw = pawp \in \mathcal{A}^{qnil}, \\ ((aw)^m)^*(aw)^mp)^* = (aw)^m)^*(aw)^mp.$$

We introduce the weighted weak group inverse as an extension of the m -weak group inverse for a complex matrix.

Definition 5. An element $a \in \mathcal{A}$ has (m, w) -weak group inverse provide that there exist $x \in R$ and $n \in \mathbb{N}$ such that

$$x = a(wx)^2, ((aw)^n)^*(aw)^{m+1}x = ((aw)^n)^*a^{m-1}a, (aw)^n = xw(aw)^{n+1}.$$

If x exists, it is unique, and we denote it by $a_w^{\otimes m}$.

Finally, in Section 4, we apply our results to complex matrices and characterize the (m, w) -weak group inverse. We present the formula of weighted weak group inverse by using the related Moore-Penrose inverse, i.e.,

$$A_W^{\otimes m} = (AW)(AW)^D[(AW)^{2m+1}(AW)^D]^{\dagger}(AW)^{m-1}A.$$

Applying the weighted weak group inverse, we prove solvability of the following minimization problem with respect to $X \in \mathbb{C}^{n \times n}$ in the Frobenius norm:

$$\min \|(AW)^{2m}X - (AW)^{m-1}AB\|_F$$

subject to $R(X) \subseteq R(AW)^D$ is $X = A_W^{\otimes m}B$.

2. (m, w) -Generalized Group Inverse

In this section, we explore the fundamental properties of the (m, w) -generalized group inverse of elements in a Banach $*$ -algebra. We start with

Theorem 1. Let $a \in \mathcal{A}$. Then $a \in \mathcal{A}_w^{\otimes m}$ if and only if $(aw)^{m-1}a \in \mathcal{A}_w^{\otimes}$. In this case,

$$a_w^{\otimes m} = (aw)^{m-1}((aw)^{m-1}a)_w^{\otimes}.$$

Proof. $\implies$ Since $a \in \mathcal{A}_w^{\otimes m}$, then $a \in \mathcal{A}^{d,w}$ and there exists $x \in \mathcal{A}$ such that

$$x = a(wx)^2, [(aw)^d]^*(aw)^{m+1}x = [(aw)^d]^*(aw)^{m-1}a, \\ \lim_{n \rightarrow \infty} \|(aw)^n - xw(aw)^{n+1}\|^{\frac{1}{n}} = 0.$$

Let $y = (xw)^{m-1}x$. Then we verify that

$$\begin{aligned} & (aw)^{m-1}a(wy)^2 \\ &= (aw)^{m-1}a(wx)^{2m} = (aw)^{m-1}[a(wx)^2](wx)^{2m-2} \\ &= (aw)^{m-1}x(wx)^{2m-2} = (aw)^{m-2}[a(wx)^2](wx)^{2m-3} = (aw)^{m-2}x(wx)^{2m-3} \\ &= \dots = (aw)^{m-m}x(wx)^{2m-(m+1)} = x(wx)^{m-1} = (xw)^{m-1}x = y, \\ & \quad (((aw)^{m-1}aw)^d)^* ((aw)^{m-1}aw)^2 y \\ &= (((aw)^m)^d)^* (aw)^{2m} (xw)^{m-1}x \\ &= (((aw)^m)^d)^* (aw)^{2m} (xw)^{m-1}x \\ &= (((aw)^m)^d)^* (aw)^{2m-1} [a(wx)^2] w (xw)^{m-3}x \\ &= (((aw)^m)^d)^* (aw)^{2m-1} (xw)^{m-2}x \\ &= \dots = (((aw)^m)^d)^* (aw)^{2m-(m-1)} (xw)^{m-m}x \\ &= (((aw)^m)^d)^* [(aw)^d]^* (aw)^{m+1}x \\ &= (((aw)^{m-1})^d)^* [(aw)^d]^* (aw)^{m-1}a \\ &= (((aw)^m)^d)^* (aw)^{m-1}a, \end{aligned}$$

Moreover, we have

$$\begin{aligned} & ((aw)^{m-1}aw)^n - yw((aw)^{m-1}aw)^{n+1} \\ &= ((aw)^{m-1}aw)^n - (xw)^{m-1}xw((aw)^{m-1}aw)^{n+1} \\ &= ((aw)^m)^n - (xw)^m((aw)^m)^{n+1} \\ &= ((aw)^m)^n - (xw)^{m-1}((aw)^{m-1})^{n+1}((aw)^n) + (xw)^{m-1} \\ & \quad ((aw)^n - (xw)(aw)^{n+1})((aw)^{m-1})^{n+1} \\ &= ((aw)^m)^n - (xw)^{m-2}((xw)(aw)^{n+1})((aw)^{m-2})^{n+1} \\ & \quad ((aw)^n) + (xw)^{m-1}((aw)^n - (xw)(aw)^{n+1})((aw)^{m-1})^{n+1} \\ &= ((aw)^m)^n - (xw)^{m-2}((aw)^{m-2})^{n+1}((aw)^n)^2 + (xw)^{m-2} \\ & \quad ((aw)^n - (xw)(aw)^{n+1})((aw)^{m-2})^{n+1}((aw)^n) \\ &+ (xw)^{m-1}((aw)^n - (xw)(aw)^{n+1})((aw)^{m-1})^{n+1} \\ &= ((aw)^m)^n - (xw)^{m-3}((aw)^{m-3})^{n+1}((aw)^n)^3 + (xw)^{m-3} \\ & \quad ((aw)^n - (xw)(aw)^{n+1})((aw)^{m-3})^{n+1}((aw)^n)^2 \\ &+ (xw)^{m-2}((aw)^n - (xw)(aw)^{n+1})((aw)^{m-2})^{n+1}((aw)^n) \\ &+ (xw)^{m-1}((aw)^n - (xw)(aw)^{n+1})((aw)^{m-1})^{n+1} \\ &= x((aw)^n - (xw)(aw)^{n+1})(aw)^{n+1}((aw)^n)^{m-1} + \dots \\ &+ (xw)^{m-3}((aw)^n - (xw)(aw)^{n+1})((aw)^{m-3})^{n+1}((aw)^n)^2 \\ &+ (xw)^{m-2}((aw)^n - (xw)(aw)^{n+1})((aw)^{m-2})^{n+1}((aw)^n) \\ &+ (xw)^{m-1}((aw)^n - (xw)(aw)^{n+1})((aw)^{m-1})^{n+1}. \end{aligned}$$

Hence, we have

$$\begin{aligned} & \|((aw)^m)^n - yw((aw)^m)^{n+1}\|^{\frac{1}{n}} \\ &\leq \sum_{i=1}^{m-1} \|(xw)^i\|^{\frac{1}{n}} \|(aw)^n - xw(aw)^{n+1}\|^{\frac{1}{n}} \|((aw)^i)^{n+1}((aw)^n)^{m-i-1}\|^{\frac{1}{n}}. \end{aligned}$$

Therefore

$$\lim_{n \rightarrow \infty} \|((aw)^{m-1}a)^n - yw((aw)^{m-1}a)^{n+1}\|^{\frac{1}{n}} = 0.$$

By virtue of [3, Theorem 2.2], $(aw)^{m-1}a \in \mathcal{A}_w^{\otimes}$. Then $((aw)^{m-1}a)_w^{\otimes} = y = (xw)^{m-1}x$, as required.

$\Leftarrow$ Let $y = ((aw)^{m-1}a)_w^{\otimes}$. Then $(aw)^{m-1}a \in \mathcal{A}^{d,w}$, and so $(aw)^m \in \mathcal{A}^d$. In view of [6, Lemma 15.1.13], $a \in \mathcal{A}^{d,w}$. Moreover, we derive

$$\begin{aligned} & [(aw)^{m-1}a](wy)^2 = y, \\ & (((aw)^{m-1}aw)^d)^* ((aw)^{m-1}aw)^2 y = (((aw)^{m-1}aw)^d)^* ((aw)^{m-1}aw), \\ & \lim_{n \rightarrow \infty} \|((aw)^{m-1}aw)^n - yw((aw)^{m-1}aw)^{n+1}\|^{\frac{1}{n}} \\ & = 0. \end{aligned}$$

Hence,

$$\begin{aligned} & [(aw)^{m-1}a](wy)^2 = y, \\ & (((aw)^m)^d)^* (aw)^{2m} y = (((aw)^m)^d)^* (aw)^m, \\ & \lim_{n \rightarrow \infty} \|((aw)^{mn} - yw(aw)^{m(n+1)})\|^{\frac{1}{n}} = 0. \end{aligned}$$

Take $x = (aw)^{m-1}y$. In view of [3, Theorem 3.3], we can find $z \in \mathcal{A}$ such that $y = (aw)(aw)^d z$. Then we verify that

$$\begin{aligned} a(wx)^2 &= a[w(aw)^{m-1}y][w(aw)^{m-1}y] = (aw)^m yw[(aw)^{m-1}y] \\ &= (aw)^m yw(aw)^{m-1}[(aw)(aw)^d z] \\ &= (aw)^m [yw(aw)^m](aw)^d z \\ &= (aw)^m [(yw)((aw)^{m-1}aw)](aw)^d z \\ &= (aw)^m (aw)^d z = (aw)^{m-1}y = x, \\ ((aw)^d)^* (aw)^{m+1}x &= ((aw)^d)^* (aw)^{m+1}(aw)^{m-1}y \\ &= ((aw)^d)^* (aw)^{2m} [((aw)^{m-1}a)_w^{\otimes}] \\ &= ((aw)^m (aw)^d)^* (((aw)^m)^d)^* ((aw)^{m-1}aw)^2 [((aw)^{m-1}a)_w^{\otimes}] \\ &= ((aw)^m (aw)^d)^* (((aw)^m)^d)^* (aw)^{m-1}a \\ &= ((aw)^d)^* (aw)^{m-1}a. \end{aligned}$$

Obviously, we have

$$\begin{aligned} (aw)^{m(n+1)-1} - xw(aw)^{m(n+1)} &= (aw)^{mn+(m-1)} - xw(aw)^{m(n+1)} \\ &= (aw)^{m-1}(aw)^{mn} - ((aw)^{m-1}yw)(aw)^{m(n+1)} \\ &= (aw)^{m-1}[(aw)^m)^n - yw((aw)^m)^{n+1}]. \end{aligned}$$

Hence,

$$\begin{aligned} & \|(aw)^{m(n+1)-1} - xw(aw)^{m(n+1)}\|^{\frac{1}{m(n+1)-1}} \\ & \leq \|(aw)^{m-1}\|^{\frac{1}{m(n+1)-1}} \|((aw)^m)^n - yw((aw)^m)^{n+1}\|^{\frac{1}{m(n+1)-1}} \\ & = \|(aw)^{m-1}\|^{\frac{1}{m(n+1)-1}} (\|((aw)^m)^n - yw((aw)^m)^{n+1}\|^{\frac{1}{n}})^{\frac{n}{mn+m-1}} \\ & \leq \|aw\|^{\frac{m-1}{mn+m-1}} (\|((aw)^m)^n - yw((aw)^m)^{n+1}\|^{\frac{1}{n}})^{\frac{1}{m+\frac{m-1}{n}}} \\ & = \|aw\|^{\frac{m-1}{mn+m-1}} (\|((aw)^{m-1}aw)^n - yw((aw)^{m-1}aw)^{n+1}\|^{\frac{1}{n}})^{\frac{1}{m+\frac{m-1}{n}}}. \end{aligned}$$

Thus, we derive

$$\lim_{n \rightarrow \infty} \|(aw)^{m(n+1)-1} - xw(aw)^{m(n+1)}\|^{\frac{1}{m(n+1)-1}} = 0.$$

Therefore $a_w^{\otimes m} = x = a^{m-1}y = (aw)^{m-1}((aw)^{m-1}a)_w^{\otimes}$, as asserted. $\square$

Corollary 1. Let $a \in \mathcal{A}_w^{\otimes m}$. Then $((aw)^{m-1}a)_w^{\otimes} = (a_w^{\otimes m}w)^{m-1}a_w^{\otimes m}$.

Proof. This is obvious by the proof of Theorem 2.1. $\square$

Set $im(x) = \{xr \mid r \in \mathcal{A}\}$. We are ready to prove:

Theorem 2. Let $a, w \in \mathcal{A}$. Then the following are equivalent:

- (1) $a_w^{\otimes m} = x$.
- (2) $awx = (aw)^m[(aw)^{m-1}a]_w^{\otimes}, a(wx)^2 = x$.
- (3) $awx = (aw)^m[(aw)^{m-1}a]_w^{\otimes}, im(x) \subseteq im(aw)^d$.

Proof. (1) $\Rightarrow$ (2) Obviously, we have $x = a(wx)^2$. By virtue of Theorem 2.1, $x = (aw)^{m-1}[(aw)^{m-1}a]_w^{\otimes}$, and then $awx = (aw)^m[(aw)^{m-1}a]_w^{\otimes}$, as required.

(2) $\Rightarrow$ (3) By hypothesis, we check that

$$\begin{aligned}
 x &= a(wx)^2 = (aw)x(wx) = (aw)x(wx) \\
 &= (aw)^2x(wx)^2 = \dots = (aw)^n x(wx)^n \\
 &= [(aw)^n - (aw)^d(aw)^{n+1}]x(wx)^n + (aw)^d(aw)^{n+1}x(wx)^n \\
 &= [(aw)^n - (aw)^d(aw)^{n+1}]x(wx)^n + (aw)(aw)^d(aw)^n x(wx)^n.
 \end{aligned}$$

Hence,

$$\|x - (aw)(aw)^d x\|^{\frac{1}{n}} \leq \|[(aw)^n - (aw)^d(aw)^{n+1}]\|^{\frac{1}{n}} \|x\|^{\frac{1}{n}} \|wx\|.$$

Since $\lim_{n \rightarrow \infty} \|[(aw)^n - (aw)^d(aw)^{n+1}]\|^{\frac{1}{n}} = 0$, we have $\lim_{n \rightarrow \infty} \|x - (aw)(aw)^d x\|^{\frac{1}{n}} = 0$. Therefore $x = (aw)(aw)^d x$; hence, $im(x) \subseteq im(aw)^d$.

(3) $\Rightarrow$ (1) Since $im(x) \subseteq im(aw)^d$, we see that $x = aw(aw)^d x$. Hence,

$$\begin{aligned}
 x &= a(wx)^2 = (awx)wx \\
 &= ((aw)^m((aw)^{m-1}a)_w^{\otimes})wx \\
 &= ((aw)^{m-1}a)_w^{\otimes} w(aw)^d[awx] \\
 &= ((aw)^m((aw)^{m-1}a)_w^{\otimes})w(aw)^d(aw)^m[(aw)^{m-1}a]_w^{\otimes} \\
 &= (aw)^m((aw)^{m-1}a)_w^{\otimes} w((aw)^{m-1}aw)(aw)^d((aw)^{m-1}a)_w^{\otimes} \\
 &= (aw)^m(aw)^d((aw)^{m-1}a)_w^{\otimes} \\
 &= (aw)^{m-1}((aw)^{m-1}a)_w^{\otimes}.
 \end{aligned}$$

Therefore we complete the proof by Theorem 2.1. $\square$

Corollary 2. Let $a \in \mathcal{A}_w^{\otimes m}$. Then $a_w^{\otimes m} waw a_w^{\otimes m} = a_w^{\otimes m}$.

Proof. By virtue of Theorem 2.3, we verify that

$$\begin{aligned}
 a_w^{\otimes m} waw a_w^{\otimes m} &= a_w^{\otimes m} w[aw a_w^{\otimes m}] \\
 &= a_w^{\otimes m} w(aw)^m[(aw)^{m-1}a]_w^{\otimes} \\
 &= (aw)^{m-1}[(aw)^{m-1}a]_w^{\otimes} \\
 &= a_w^{\otimes m},
 \end{aligned}$$

as asserted. $\square$

We are ready to prove:

Theorem 3. Let $a \in \mathcal{A}$. Then $a \in \mathcal{A}_w^{\otimes m}$ if and only if

- (1) $a \in \mathcal{A}^{d,w}$;
- (2) There exists $x \in \mathcal{A}$ such that

$$(a^{d,w})^* a^{d,w} x = (a^{d,w})^* (aw)^{m-1} a.$$

In this case, $a_w^{\otimes m} = ((aw)^d)^{m+3} ax$.

Proof. $\implies$ Obviously, $a \in \mathcal{A}^{d,w}$. By virtue of Theorem 2.1, $(aw)^{m-1}a \in \mathcal{A}_w^{\otimes}$. Set $z = ((aw)^{m-1}a)_w^{\otimes}$. In light of [3, Theorem 2.2],

$$(((aw)^m)^d)^* ((aw)^m)^2 z = (((aw)^m)^d)^* (aw)^{m-1}a.$$

Write $z = ((aw)^d)^{2m}(aw)^2y$ for some $y \in \mathcal{A}$. Then

$$(a^{d,w})^* ((aw)^m)^2 ((aw)^d)^{2m}(aw)^2y = (a^{d,w})^* (aw)^{m-1}a.$$

Hence,

$$(a^{d,w})^* [(aw)^2a][w((aw)^d)^2a]wy = (a^{d,w})^* (aw)^{m-1}a.$$

Then

$$(a^{d,w})^* a^{d,w}[w((aw)^d)^2a]wy = (a^{d,w})^* (aw)^{m-1}a.$$

Set $x = [w((aw)^d)^2a]wy$. Then $(a^{d,w})^* a^{d,w}x = (a^{d,w})^* (aw)^{m-1}a$, as required.

$\Leftarrow$ By hypothesis,

$$(a^{d,w})^* a^{d,w}x = (a^{d,w})^* (aw)^{m-1}a.$$

for some $x \in \mathcal{A}$. Then

$$(((aw)^d)^2a)^* ((aw)^d)^2ax = (((aw)^d)^2a)^* (aw)^{m-1}a.$$

Hence,

$$(((aw)^d)^2a)^* ((aw)^d)^{m+1}(aw)^{m+1}((aw)^d)^2ax = (((aw)^d)^2a)^* (aw)^{m-1}a.$$

This implies that

$$(((aw)^d)^2a)^* ((aw)^d)^{m+1}(aw)^{m-1}ax = (((aw)^d)^2a)^* (aw)^{m-1}a.$$

Set $z = (wa)^{m-1}x$. As $((aw)^{m-1}a)^{d,w} = ((aw)^d)^{m+1}a$, we see that

$$(((aw)^{m-1}a)^{d,w})^* ((aw)^{m-1}a)^{d,w}z = (((aw)^{m-1}a)^{d,w})^* (aw)^{m-1}a.$$

In view of [3, Theorem 4.4], we have

$$((aw)^{m-1}a)_w^{\otimes} = [((aw)^m)^d]^4(aw)^{m-1}az.$$

According to Theorem 2.1, we have

$$\begin{aligned} a_w^{\otimes m} &= (aw)^{m-1}((aw)^{m-1}a)_w^{\otimes} \\ &= (aw)^{m-1}[((aw)^m)^d]^4(aw)^{m-1}az \\ &= (aw)^{m-1}[((aw)^m)^d]^4(aw)^{m-1}a(wa)^{m-1}x \\ &= ((aw)^d)^{m+3}ax \end{aligned}$$

Therefore we complete the proof. $\square$

Corollary 3. Let $a \in \mathcal{A}_w^{\otimes m}$. Then $a \in \mathcal{A}_w^{\otimes m+1}$. In this case,

$$a_w^{\otimes m+1} = [a_w^{\otimes m}w]^2a.$$

Proof. By virtue of Theorem 2.5, $a \in \mathcal{A}^{d,w}$ and there exists $x \in \mathcal{A}$ such that $(a^{d,w})^* a^{d,w} x = (a^{d,w})^* (aw)^{m-1} a$. Then $(a^{d,w})^* a^{d,w} (xwa) = (a^{d,w})^* (aw)^{m-1} a (wa) = (a^{d,w})^* (aw)^m a$. By using Theorem 2.5 again, $a \in \mathcal{A}_w^{\otimes m+1}$. Moreover, we derive that

$$\begin{aligned} a_w^{\otimes m+1} &= ((aw)^d)^{m+4} a(xwa) \\ &= (aw)^d [((aw)^d)^{m+3} ax] wa \\ &= (aw)^d a_w^{\otimes m} wa \\ &= (aw)^d a [wa_w^{\otimes m}]^2 wa \\ &= [a_w^{\otimes m} w]^2 a \end{aligned}$$

This completes the proof. $\square$

As a consequence, we give an interesting relationship between two projectors formed by the weighted generalized group inverse as follows.

Corollary 4. Let $a \in \mathcal{A}_w^{\otimes m}$. Then $(a_w^{\otimes m} w)(aw) = (aw)(a_w^{\otimes m+1} w)$.

Proof. In view of Corollary 2.6, $a_w^{\otimes m+1} = [a_w^{\otimes m} w]^2 a$. Hence,

$$aw a_w^{\otimes m+1} = aw [a_w^{\otimes m} w]^2 a = a_w^{\otimes m} wa;$$

and so $aw a_w^{\otimes m+1} w = a_w^{\otimes m} waw$, as asserted. $\square$

Recall that $a \in \mathcal{A}$ has w -Drazin inverse if there exists $x \in \mathcal{A}$ such that $a(wx)^2 = x$, $awx = xwa$, $(aw)^n = xw(aw)^{n+1}$ for some $n \in \mathbb{N}$. Such x is unique, if it exists, and we denote it by $a^{D,w}$. We derive

Corollary 5. Let $a \in \mathcal{A}$. Then $a \in \mathcal{A}^{\otimes m}$ if and only if

- (1) $a \in \mathcal{A}^{D,w}$;
- (2) There exists $x \in \mathcal{A}$ such that

$$(a^{D,w})^* a^{D,w} x = (a^{D,w})^* (aw)^{m-1} a.$$

In this case, $a_w^{\otimes m} = ((aw)^D)^{m+3} ax$.

Proof. We easily prove that $a \in \mathcal{A}^{\otimes m}$ if and only if $a \in \mathcal{A}_w^{\otimes m} \cap \mathcal{A}^{D,w}$. Therefore we complete the proof by Theorem 2.5. $\square$

3. Algebraic Properties

In this section we investigate algebraic properties of (m, w) -generalized group inverse. We now establish the relation between (m, w) -generalized group inverse and (m, w) -generalized group decomposition.

Theorem 4. Let $a \in \mathcal{A}$. Then the following are equivalent:

- (1) $a \in \mathcal{A}_w^{\otimes m}$.
- (2) a has (m, w) -generalized group decomposition, i.e., there exist $x, y \in \mathcal{A}$ such that

$$a = x + y, x^* (aw)^{m-1} y = ywx = 0, x \in \mathcal{A}_w^\#, y \in \mathcal{A}_w^{qnil}.$$

Proof. (1) $\Rightarrow$ (2) By hypothesis, a has the (m, w) -generalized group decomposition $a = a_1 + a_2$. Then $(a_1)^*(aw)^{m-1}a_2 = 0$ and $a_2wa_1 = 0$. Let $x = (a_1)_w^\#$. Then we verify that $a(wx)^2 = (a_1 + a_2)(w(a_1)_w^\#)^2 = a_1(w(a_1)_w^\#)^2 = (a_1)_w^\# = x$.

Since $a_2wa_1 = 0$, we have

$$\begin{aligned} & aw - xw(aw)^2 \\ &= (a_1 + a_2)w - ((a_1)_w^\#wa_1 + (a_1)_w^\#wa_2)w(a_1 + a_2)w \\ &= a_1w + a_2w - (a_1)_w^\#wa_1w(a_1 + a_2)w - (a_1)_w^\#wa_2w(a_1 + a_2)w \\ &= a_1w + a_2w - (a_1)_w^\#w(a_1w)^2 - (a_1)_w^\#wa_1wa_2w - (a_1)_w^\#w(a_2w)^2 \\ &= [1 - (a_1)_w^\#wa_1 - (a_1)_w^\#wa_2](wa_2w), \end{aligned}$$

and so

$$\begin{aligned} & ||(aw)^n - (xw)(aw)^{n+1}||^{\frac{1}{n}} \\ &= ||(aw - xw(aw)^2)(aw)^{n-1}||^{\frac{1}{n}} \\ &= ||[1 - (a_1)_w^\#wa_1 - (a_1)_w^\#wa_2](wa_2w)(a_1w + a_2w)^{n-1}||^{\frac{1}{n}} \\ &\leq ||[1 - (a_1)_w^\#wa_1 - (a_1)_w^\#wa_2]w||^{\frac{1}{n}} ||(a_2w)^n||^{\frac{1}{n}}. \end{aligned}$$

Since $a_2 \in \mathcal{A}_w^{qnil}$, we have $\lim_{n \rightarrow \infty} ||(a_2w)^n||^{\frac{1}{n}} = 0$, and then

$$\lim_{n \rightarrow \infty} ||(aw)^n - (xw)(aw)^{n+1}||^{\frac{1}{n}} = 0.$$

Since $a_2wa_1 = 0$, $(a_1w)(a_1w)^\pi = 0$ and $(a_2w)^d = 0$, it follows by [[1] Corollary 3.4] that $aw \in \mathcal{A}^d$ and $(aw)^d = (a_1w)^\# + \sum_{n=1}^{\infty} ((a_1w)^\#)^{n+1}(a_2w)^n$. Then

$$\begin{aligned} & ((aw)^d)^*(aw)^{m-1}a_2 \\ &= ((aw)^\#)^*a_2 + \sum_{n=1}^{\infty} ((a_1w)^\#)^{n+1}(a_2w)^n)^*(aw)^{m-1}a_2 \\ &= (((a_1w)^\#)^2)^*(w^*(a_1)^*(aw)^{m-1}a_2) + \sum_{n=1}^{\infty} (((a_1w)^\#)^{n+2}(a_2w)^n)^*(w^*(a_1)^*a_2) \\ &= 0; \end{aligned}$$

hence, $((aw)^d)^*(a_1w)^{m-1}a_1 = ((aw)^d)^*(aw)^{m-1}a$. Accordingly, we have

$$\begin{aligned} ((aw)^d)^*(aw)^{m+1}x &= ((aw)^d)^*(aw)^m(aw)(a_1)_w^\# \\ &= ((aw)^d)^*(a_1w)^m(a_1 + a_2)w(a_1)_w^\# \\ &= ((aw)^d)^*(a_1w)^{m+1}(a_1)_w^\# \\ &= ((aw)^d)^*(a_1w)^{m-1}a_1 \\ &= ((aw)^d)^*(aw)^{m-1}a. \end{aligned}$$

(2) $\Rightarrow$ (1) By hypothesis, there exists $x \in \mathcal{A}$ such that

$$\begin{aligned} x &= a(wx)^2, [(aw)^d]^*(aw)^{m+1}x = [(aw)^d]^*(aw)^{m-1}a, \\ \lim_{n \rightarrow \infty} ||(aw)^n - xw(aw)^{n+1}||^{\frac{1}{n}} &= 0. \end{aligned}$$

Let $a_1 = (aw)^2x$ and $a_2 = a - (aw)^2x$. Then we check that

$$\begin{aligned} ||a_2wa_1|| &= ||[a - (aw)^2x]w(aw)^2x|| = ||(aw)^3x - (aw)^2x[(wa)^2(wx)]|| \\ &= ||(aw)^3x - (aw)^2x[(wa)^{k+1}(wx)^k]|| \\ &= ||a(wa)^2(wx) - a(wa)(wx)[(wa)^{k+1}(wx)^k]|| \\ &= ||a(wa)^{k+1}(wx)^k - (aw)^2[xw(aw)^k][a(wx)^k]|| \\ &= ||a(wa)^{k+1}(wx)^k - (aw)^{k+1}[a(wx)^k]||. \end{aligned}$$

Then $\lim_{k \rightarrow \infty} \|a_2 w a_1\|^{\frac{1}{k}} = 0$, and then $a_2 w a_1 = 0$.

Moreover, we have

$$\begin{aligned}
 & \|a_1^*(aw)^{m-1}a_2\| \\
 = & \|((aw)^2x)^*(aw)^{m-1}(a - (aw)^2x)\| \\
 = & \|((aw)^2x)^*(aw)^{m-1}a - ((aw)^2x)^*(aw)^{m+1}x\| \\
 = & \|((aw)^2x)^*(aw)^{m-1}a - ((aw)^k(xw)^{k-2}x)^*(aw)^{m+1}x\| \\
 = & \|((aw)^2x)^*(aw)^{m-1}a - [(aw)^k - (aw)^d(aw)^{k+1}](xw)^{k-2}x)^*(aw)^{m+1}x \\
 & - ((aw)^d(aw)^{k+1}(xw)^{k-2}x)^*(aw)^{m+1}x\| \\
 \leq & \|((aw)^k - (aw)^d(aw)^{k+1})^* \|((xw)^{k-2}x)^*\| \| (aw)^{m+1}x \| \\
 + & \|((aw)^2x)^*(aw)^{m-1}a - [(aw)^d(aw)^{k+1}(xw)^{k-2}x]^*(aw)^{m+1}x\| \\
 \leq & \|((aw)^k - (aw)^d(aw)^{k+1})^* \|((xw)^{k-2}x)^*\| \| (aw)^{m+1}x \| \\
 + & \|((aw)^2x)^*(aw)^{m-1}a - [(aw)^{k+1}(xw)^{k-2}x]^*[(aw)^d]^*(aw)^{m+1}x\| \\
 \leq & \|((aw)^k - (aw)^d(aw)^{k+1})^* \|((xw)^{k-2}x)^*\| \| (aw)^{m+1}x \| \\
 + & \|((aw)^2x)^*(aw)^{m-1}a - [(aw)^{k+1}(xw)^{k-2}x]^*[(aw)^d]^*a\| \\
 \leq & \|((aw)^k - (aw)^d(aw)^{k+1})^* \|((xw)^{k-2}x)^*\| \| (aw)^{m+1}x \| \\
 + & \|((aw)^2x)^*(aw)^{m-1}a - [(aw)^{k+1}(xw)^{k-2}x]^*a\| \\
 \leq & \|((aw)^k - (aw)^d(aw)^{k+1})^* \|((xw)^{k-2}x)^*\| \| (aw)^{m+1}x \| \\
 + & \|[(aw)^k - (aw)^d(aw)^{k+1}](xw)^{k-2}x)^*a\| \\
 \leq & \|((aw)^k - (aw)^d(aw)^{k+1})^* \|((xw)^{k-2}x)^*\| \| (aw)^{m+1}x \| \\
 + & \|[(aw)^k - (aw)^d(aw)^{k+1}](xw)^{k-2}x)^*a\| \\
 \leq & \|((aw)^k - (aw)^d(aw)^{k+1})^* \|((xw)^{k-2}x)^*\| \| (aw)^{m+1}x \| \\
 + & \|[(aw)^k - (aw)^d(aw)^{k+1}](xw)^{k-2}x)^*a\|.
 \end{aligned}$$

Then

$$\begin{aligned}
 & \|a_1^*(aw)^{m-1}a_2\|^{\frac{1}{k}} \\
 \leq & \|((aw)^k - (aw)^d(aw)^{k+1})^* \|((xw)^{k-2}x)^*\| \| (aw)^{m+1}x \|^{\frac{1}{k}} \\
 + & \|((aw)^k - (aw)^d(aw)^{k+1})^* \|((xw)^{k-2}x)^*\| \| a \|^{\frac{1}{k}}.
 \end{aligned}$$

By using [5, Lemma 2.1], we have $\lim_{k \rightarrow \infty} \|((aw)^k - (aw)^d(aw)^{k+1})^* \|((xw)^{k-2}x)^*\| \| (aw)^{m+1}x \|^{\frac{1}{k}} = 0$. Therefore $\lim_{k \rightarrow \infty} \|a_1^*(aw)^{m-1}a_2\|^{\frac{1}{k}} = 0$, and then $a_1^*(aw)^{m-1}a_2 = 0$. Moreover, we check that

$$\begin{aligned}
 a_1 w x &= (aw)^2 x w x = aw[a(wx)^2] = awx = xw(aw)^2 x = xwa_1, \\
 a_1 w x w a_1 &= awxw(aw)^2 x = (aw)^2 x = a_1, \\
 x w a_1 w x &= a(wx)^2 = x.
 \end{aligned}$$

Hence $(a_1)_w^\# = x$.

We further verify that

$$\begin{aligned}
 [aw - zw(aw)^2]z &= [aw - zw(aw)^2]a(wz)^2 \\
 &= [(aw)^2 - zw(aw)^3]z(wz) \\
 &= [(aw)^2 - zw(aw)^3][a(wz)^2](wz) \\
 &= [(aw)^3 - zw(aw)^4]z(wz)^2 \\
 &\vdots \\
 &= [(aw)^n - zw(aw)^{n+1}]z(wz)^{n-1}.
 \end{aligned}$$

Therefore

$$\| [aw - zw(aw)^2]z \|^{\frac{1}{n}} \leq \| (aw)^n - zw(aw)^{n+1} \|^{\frac{1}{n}} \| z(wz)^{n-1} \|^{\frac{1}{n}}.$$

Since $\lim_{n \rightarrow \infty} \|a^n - za^{n+1}\|^{\frac{1}{n}} = 0$, we have $\lim_{n \rightarrow \infty} \| [aw - zw(aw)^2]z \|^{\frac{1}{n}} = 0$. This implies that $zw(aw)^2z = awz$.

We check that

$$\begin{aligned}
 & ||(aw - zw(aw)^2)^{n+1}||^{\frac{1}{n+1}} \\
 = & ||[aw - zw(aw)^2]^{n-1}[aw - zw(aw)^2](aw)||^{\frac{1}{n+1}} \\
 = & ||[aw - zw(aw)^2]^{n-1}(aw)^2||^{\frac{1}{n+1}} \\
 & \vdots \\
 = & ||(aw - zw(aw)^2)(aw)^n||^{\frac{1}{n+1}} \\
 \leq & [|| (aw)^n - zw(aw)^{n+1} ||^{\frac{1}{n}}]^{\frac{n}{n+1}} ||(aw)^n||^{\frac{1}{n+1}}.
 \end{aligned}$$

Accordingly, $\lim_{n \rightarrow \infty} ||(aw - zw(aw)^2)^{n+1}||^{\frac{1}{n+1}} = 0$. This implies that $aw - zw(aw)^2 \in \mathcal{A}^{qnil}$. By using Cline's formula (see [17, Theorem 2.1]), $y = a - (aw)^2z \in \mathcal{A}_w^{qnil}$. $\square$

Corollary 6. Let $a \in \mathcal{A}_w^{\otimes m}$. Then $awa_w^{\otimes m} = (aw)^{n-1}a(wa_w^{\otimes m})^n$ for any $n \in \mathbb{N}$.

Proof. These are obvious by the proof of Theorem 3.1. $\square$

We come now to characterize (m, w) -generalized group inverse by the polar-like property.

Theorem 5. Let $a \in \mathcal{A}_w^{\otimes m}$. Then there exists an idempotent $p \in \mathcal{A}$ such that

$$\begin{aligned}
 aw + p & \in \mathcal{A}^{-1}, paw = pawp \in \mathcal{A}^{qnil}, \\
 ((aw)^m)^*(aw)^mp & = (aw)^m)^*(aw)^mp.
 \end{aligned}$$

Proof. By virtue of Theorem 3.1, there exist $z, y \in \mathcal{A}$ such that

$$a = z + y, z^*(aw)^{m-1}y = ywz = 0, z \in \mathcal{A}_w^\#, y \in \mathcal{A}_w^{qnil}.$$

Set $x = z_w^\#$. The we check that

$$\begin{aligned}
 awx & = (z + y)wz_w^\# = zwz_w^\#, \\
 a(wx)^2 & = (awx)wx = z(wz_w^\#)^2 = z_w^\# = x.
 \end{aligned}$$

Since $ywz = 0$, we see that $(aw)^kz = (aw)^{k-1}(y + z)wz = (aw)^{k-1}(zw)z = (aw)^{k-2}(y + z)wz = (aw)^{k-2}(zw)^2z \dots = (zw)^kz$. As $z^*a^{m-1}y = 0$, we derive that

$$\begin{aligned}
 ((aw)^m)^*(zw)^m & = w^*a^*((aw)^{m-1})^*(zw)^m = w^*[z^*(aw)^{m-1}(y + z)]^*(wz)^{m-1}w \\
 & = w^*[z^*(aw)^{m-1}z]^*(wz)^{m-1}w = w^*[z^*(zw)^{m-1}z]^*(wz)^{m-1}w \\
 & = [z^*(zw)^{m-1}zw]^*(wz)^{m-1}w = [(zw)^m]^*z(wz)^{m-1}w \\
 & = [(zw)^m]^*(zw)^m.
 \end{aligned}$$

Hence, we have

$$\begin{aligned}
 ((aw)^m)^*(aw)^{m+1}xw & = ((aw)^m)^*(aw)^m(awxw) = ((aw)^m)^*(aw)^m(zwz_w^\#w) \\
 & = ((aw)^m)^*((aw)^mzw)z_w^\#w = ((aw)^m)^*((zw)^mzw)z_w^\#w \\
 & = [(aw)^m]^*(zw)^mzwz_w^\#w = [(zw)^m]^*(zw)^{m+1}z_w^\#w \\
 & = [(zw)^m]^*(zw)^{m-1}[(zw)^2z_w^\#]w \\
 & = [(zw)^m]^*(zw)^{m-1}zw \\
 & = ((zw)^m)^*(zw)^m.
 \end{aligned}$$

Therefore

$$\begin{aligned}
 [((aw)^m)^*(aw)^{m+1}xw]^* & = [((zw)^m)^*(zw)^m]^* = ((zw)^m)^*(zw)^m \\
 & = ((aw)^m)^*(aw)^{m+1}xw.
 \end{aligned}$$

Let $p = 1 - zwz_w^\#w$. Then $p = 1 - awxw = p^2 \in \mathcal{A}$. Since $aw - awz_w^\#w = aw(1 - zwz_w^\#w) = (z + y)w(1 - zwz_w^\#w) = yw \in \mathcal{A}^{qnil}$, we have $awp = aw(1 - zwz_w^\#w) \in \mathcal{A}^{qnil}$. Therefore we have

$$\begin{aligned} [((aw)^m)^*(aw)^mp]^* &= (((aw)^m)^*(aw)^m - [((aw)^m)^*(aw)^{m+1}xw]^* \\ &= ((aw)^m)^*(aw)^m - ((aw)^m)^*(aw)^{m+1}xw \\ &= (aw)^m)^*(aw)^mp. \end{aligned}$$

Since $paw(1 - p) = (1 - zwz_w^\#w)(z + y)wz_w^\#w = (1 - zwz_w^\#w)(zw)^2z_w^\#w = 0$, we see that $paw = pawp$. Obviously, $zw + 1 - zwz_w^\#w = (z_w^\#w + 1 - zwz_w^\#w)^{-1} \in \mathcal{A}^{-1}$. Since $yw(z_w^\#w + 1 - zwz_w^\#w) = yw \in \mathcal{A}^{qnil}$, it follows by Cline's formula that $(z_w^\#w + 1 - zwz_w^\#w)yw \in \mathcal{A}^{qnil}$. Hence $1 + (z_w^\#w + 1 - zwz_w^\#w)yw \in \mathcal{A}^{-1}$. This implies that

$$\begin{aligned} aw + p &= zw + yw + 1 - zwz_w^\#w \\ &= (zw + 1 - zwz_w^\#w)[1 + (zw + 1 - zwz_w^\#w)^{-1}yw] \\ &\in \mathcal{A}^{-1}, \end{aligned}$$

as desired. $\square$

Corollary 7. Let $a \in \mathcal{A}$ and $w \in \mathcal{A}^{-1}$. Then the following are equivalent:

- (1) $a \in \mathcal{A}_w^{\otimes m}$.
- (2) $a \in \mathcal{A}^{d,w}$ and there exists an idempotent $p \in \mathcal{A}$ such that

$$\begin{aligned} aw + p &\in \mathcal{A}^{-1}, paw = pawp \in \mathcal{A}^{qnil}, \\ (((aw)^m)^*(aw)^mp)^* &= (aw)^m)^*(aw)^mp. \end{aligned}$$

Proof. (1) $\Rightarrow$ (2) This is obvious by Theorem 3.3.

(2) $\Rightarrow$ (1) By hypothesis, there exists an idempotent $p \in \mathcal{A}$ such that

$$\begin{aligned} aw + p &\in \mathcal{A}^{-1}, paw = pawp \in \mathcal{A}^{qnil}, \\ (((aw)^m)^*(aw)^mp)^* &= (aw)^m)^*(aw)^mp. \end{aligned}$$

Set $x = aw(1 - p)w^{-1}$ and $y = awpw^{-1}$. Then $x + y = aw(1 - p)w^{-1} + awpw^{-1} = a$. Since $paw \in \mathcal{A}^{qnil}$, by using Cline's formula, we have $yw = awp \in \mathcal{A}^{qnil}$. Hence, $y \in \mathcal{A}_w^{qnil}$. Since $paw(1 - p) = 0$, we have $ywx = [awpw^{-1}]w[aw(1 - p)w^{-1}] = aw[paw(1 - p)]w^{-1} = 0$.

Set $u = aw + p$. Let $z = (1 - p)u^{-1}w^{-1}$. Then we check that

$$\begin{aligned} z(wx)^2 &= (1 - p)u^{-1}(xw)x = (1 - p)u^{-1}aw(1 - p)x \\ &= (1 - p)u^{-1}[aw + p](1 - p)x = (1 - p)x = x. \end{aligned}$$

Since $a \in \mathcal{A}^{d,w}$, $awp \in \mathcal{A}^d$ and $pwa(1 - p) = 0$, it follows by [29, Lemma 2.2] that $x = aw(1 - p)w^{-1} \in \mathcal{A}^{d,w}$. Hence $xw, wx \in \mathcal{A}^d$. We verify that

$$\begin{aligned} xw &= zw(xw)^2 = (zw)^n(xw)^{n+1} \\ &= (zw)^n((xw)^n - (xw)^d(xw)^{n+1})xw + (zw)^n(xw)^d(xw)^{n+2}, \\ (xw)^2(xw)^d &= (zw(xw)^2)(xw)(xw)^d = (zw)^n(xw)^{n+2}(xw)^d. \end{aligned}$$

Then

$$\begin{aligned} \|(xw) - (xw)^2(xw)^d\|^{\frac{1}{n}} &= \|(zw)^n((xw)^n - (xw)^d(xw)^{n+1})(xw)\|^{\frac{1}{n}} \\ &\leq \|(zw)^n\|^{\frac{1}{n}}\|(xw)^n - (xw)^d(xw)^{n+1}\|^{\frac{1}{n}}\|(xw)\|^{\frac{1}{n}}. \end{aligned}$$

Accordingly,

$$\lim_{n \rightarrow \infty} \|(xw) - (xw)^2(xw)^d\|^{\frac{1}{n}} = 0.$$

Therefore we have $xw = (xw)^2(xw)^d$, and so $xw \in \mathcal{A}^\#$. Likewise, $wx \in \mathcal{A}^\#$. Therefore $x \in \mathcal{A}_w^\#$.

Obviously, $pxw = paw(1 - p) = 0$, and then

$$\begin{aligned} w^*x^*(aw)^{m-1}y &= (xw)^*(aw)^{m-1}y = ((xw)^m((xw)^\#)^{m-1})^*(aw)^{m-1}y \\ &= ((aw)^m((xw)^\#)^{m-1})^*(aw)^{m-1}y \\ &= (((xw)^\#)^{m-1})^*((aw)^m)^*(aw)^{m-1}[awpw^{-1}] \\ &= (((xw)^\#)^{m-1})^*[(aw)^m]^*(aw)^mp[w^{-1}] \\ &= (((xw)^\#)^{m-1})^*((aw)^m)^*(aw)^mp^*w^{-1} \\ &= (p((xw)^\#)^{m-1})^*((aw)^m)^*(aw)^mp^*w^{-1} \\ &= 0. \end{aligned}$$

Hence, $x^*(aw)^{m-1}y = 0$. Therefore $a = x + y$ is the (m, w) -generalized group decomposition of a . By virtue of Theorem 3.1, $a \in \mathcal{A}_w^{\oplus m}$. $\square$

Corollary 8. Let $a \in \mathcal{A}_w^{\oplus m}$ and $w \in \mathcal{A}^{-1}$. Then a is the sum of three invertible elements in $\mathcal{A}$.

Proof. In view of Theorem 3.3, there exists an idempotent $p \in \mathcal{A}$ such that

$$\begin{aligned} u &:= aw + p \in \mathcal{A}^{-1}, paw = pawp \in \mathcal{A}^{qnil}, \\ ((aw)^m)^*(aw)^mp &= (aw)^m)^*(aw)^mp. \end{aligned}$$

Hence,

$$aw = \frac{1}{2} + \frac{1-2p}{2} + u.$$

Obviously, $[\frac{1-2p}{2}]^2 = \frac{1}{4}$ and then $[\frac{1-2p}{2}]^{-1} = 2(1-2p)$. Therefore

$$a = \frac{w^{-1}}{2} + \frac{(1-2p)w^{-1}}{2} + uw^{-1},$$

as desired. $\square$

Let $a, b, c \in \mathcal{A}$. An element a has (b, c) -inverse provide that there exists $x \in \mathcal{A}$ such that

$$xab = b, cax = c \text{ and } x \in bAx \cap xAc.$$

If such x exists, it is unique and denote it by $a^{(b,c)}$ (see [10]). We now establish the relationship among (m, w) -generalized group inverse, m -generalized group inverse and generalized w -group inverse by using the (b, c) -inverse.

Theorem 6. Let $a \in \mathcal{A}_w^{\oplus m}$. Then

$$a_w^{\oplus m} = (waw)^{(aw)^{\oplus m}, ((aw)^{m-1}a)_w^{\oplus}}.$$

Proof. Let $x = a_w^{\oplus m}$. Then there exists $x \in \mathcal{A}$ such that

$$\begin{aligned} x &= a(wx)^2, [(aw)^d]^*(aw)^{m+1}x = [(aw)^d]^*(aw)^{m-1}a, \\ \lim_{n \rightarrow \infty} \|(aw)^n - xw(aw)^{n+1}\|_n^{\frac{1}{n}} &= 0. \end{aligned}$$

Hence,

$$\begin{aligned} xw &= (aw)(xw)^2, [(aw)^d]^*(aw)^{m+1}(xw) = [(aw)^d]^*(aw)^{m-1}(aw), \\ \lim_{n \rightarrow \infty} \|(aw)^n - xw(aw)^{n+1}\|_n^{\frac{1}{n}} &= 0. \end{aligned}$$

By virtue of [2, Lemma 3.1], $xw = (aw)^{\otimes m}$. Thus

$$x = (xw)(aw)x = (aw)^{\otimes m}(aw)x \in (aw)^{\otimes m}\mathcal{A}x.$$

In view of Theorem 2.1, $x = (aw)^{m-1}((aw)^{m-1}a)_w^{\otimes}$. Then

$$x = x(waw)x = x(waw)(aw)^{m-1}((aw)^{m-1}a)_w^{\otimes} \in x\mathcal{A}((aw)^{m-1}a)_w^{\otimes}.$$

Hence, we verify that

$$\begin{aligned} x(waw)[(aw)^{\otimes m}] &= (aw)^{m-1}((aw)^{m-1}a)_w^{\otimes}(waw)[(aw)^{\otimes m}] \\ &= (aw)^{m-1}((aw)^{m-1}a)_w^{\otimes}w(aw)^m[(aw)^{\otimes m}]^m \\ &= (aw)^{m-1}[(aw)^{m-1}a)_w^{\otimes}w][(aw)^{m-1}a]w[(aw)^{\otimes m}]^m \\ &= (aw)^{m-1}[(aw)^{\otimes m}]^m \\ &= (aw)^{\otimes m}, \\ ((aw)^{m-1}a)_w^{\otimes}(waw)x &= ((aw)^{m-1}a)_w^{\otimes}(waw)(aw)^{m-1}((aw)^{m-1}a)_w^{\otimes} \\ &= ((aw)^{m-1}a)_w^{\otimes}w[(aw)^{m-1}a]w((aw)^{m-1}a)_w^{\otimes} \\ &= ((aw)^{m-1}a)_w^{\otimes}. \end{aligned}$$

Therefore

$$a_w^{\otimes m} = (waw)^{(aw)^{\otimes m}, ((aw)^{m-1}a)_w^{\otimes}},$$

as asserted. $\square$

Let $a \in \mathcal{A}$ and $T, S \subseteq \mathcal{A}$. We say that a has $\{2\}$ -inverse x provided that $xax = x, im(a) = T, ker(a) = S$. We denote x by $a_{T,S}^{(2)}$. We next consider the relation between the weighted generalized group inverse and $\{2\}$ -inverse in a Banach algebra.

Theorem 7. Let $a \in \mathcal{A}_w^{\otimes m}$. Then

$$a_w^{\otimes m} = (waw)^{(2)}_{im(aw)^d, ker((aw)^d)^* (aw)^{m-1}a}.$$

Proof. Let $x = a_w^{\otimes m}$. In view of Corollary 2.4, $x = x(waw)x$.

Step 1. $im(x) = im(aw)^d$. In view of Theorem 2.5, we have $im(x) \subseteq im(aw)^d$.

We claim that $xaw(aw)^d = (aw)^d$. One directly checks that

$$\begin{aligned} &|| (aw)^d - a_w^{\otimes m}(aw)(aw)^d || \\ &= || (aw)^n((aw)^d)^{n+1} - a_w^{\otimes m}(aw)^{n+1}((aw)^d)^{n+1} || \\ &= || ((aw)^n - a_w^{\otimes m}(aw)^{n+1})((aw)^d)^{n+1} || \\ &\leq || (aw)^n - a_w^{\otimes m}(aw)^{n+1} || || ((aw)^d)^{n+1} || \end{aligned}$$

Then

$$\begin{aligned} &|| (aw)^d - a_w^{\otimes m}(aw)(aw)^d ||^{\frac{1}{n}} \\ &\leq || (aw)^n - a_w^{\otimes m}(aw)^{n+1} ||^{\frac{1}{n}} || (aw)^d ||^{1+\frac{1}{n}}. \end{aligned}$$

Thus,

$$\lim_{n \rightarrow \infty} || (aw)^d - a_w^{\otimes m}(aw)(aw)^d ||^{\frac{1}{n}} = 0.$$

This implies that $a_w^{\otimes m}(aw)(aw)^d = (aw)^d$. Accordingly, $im(aw)^d \subseteq im(x)$. Therefore $im(x) = im(aa^{\dagger}a^*)$.

Step 2. $ker(x) = ker((aw)^d)^*(aw)^{m-1}a$. If $((aw)^d)^*(aw)^{m-1}ar = 0$ for some $r \in R$, then $((aw)^d)^*(aw)^{m+1}xr = 0$. Write $x = (aw)^dz$ for some $z \in \mathcal{A}$. Then $((aw)^d)^*(aw)^d(aw)^{m+1}zr = ((aw)^d)^*(aw)^{m+1}(aw)^dzr = 0$. Since the involution $*$ is proper, we easily get $(aw)^d(aw)^{m+1}zr = 0$; and then $xr = (aw)^dzr = [(aw)^d]^{m+1}[(aw)^d(aw)^{m+1}zr] = 0$. Hence, $r \in ker(x)$.

If $x(r) = 0$, then $((aw)^d)^*(aw)^{m-1}ar = ((aw)^d)^*(aw)^{m+1}(xr) = 0$. Hence, $r \in \ker((aw)^d)^*(aw)^{m-1}a$, as required.

Therefore the proof is completed. $\square$

Corollary 9. Let $a \in \mathcal{A}^{\otimes m}$. Then

$$a^{\otimes m} = a_{\text{im}(a^d), \ker(a^d)}^{(2)*} a^m.$$

Proof. This is obvious by choosing $w = 1$ in Theorem 3.7. $\square$

4. Applications to Complex Matrices

We now proceed to examine certain applications of the weighted weak group inverse in the context of complex matrices. For further use, we establish the interrelation between the m -generalized group inverse and the (m, w) -weak group inverse.

Theorem 8. Let $a, w \in \mathcal{A}$. Then $a \in \mathcal{A}_w^{\otimes m}$ if and only if $a \in \mathcal{A}^{D,w} \cap R_w^{\otimes m}$. In this case, $a_w^{\otimes m} = a_w^{\otimes m}$.

Proof. $\Rightarrow$ Clearly, $aw \in \mathcal{A}^D$; hence, $a \in \mathcal{A}^{D,w}$. Then $a \in \mathcal{A}_w^{\otimes m}$. According to Theorem 2.1 and [3, Theorem 2.4], we see that $a_w^{\otimes m} = a_w^{\otimes m}$.

$\Leftarrow$ Let $x = a_w^{\otimes m}$. Then there exists $x \in \mathcal{A}$ such that

$$x = a(wx)^2, [(aw)^d]^*(aw)^{m+1}x = [(aw)^d]^*a^{m-1}a, \lim_{n \rightarrow \infty} \|(aw)^n - xw(aw)^{n+1}\|^{\frac{1}{n}} = 0.$$

Since $a \in \mathcal{A}^{D,w}$, we can find some $y \in \mathcal{A}$ such that

$$yw(aw)^{k+1} = (aw)^k, ywaw y = y, awy = ywa$$

for some $k \in \mathbb{N}$. Set $z = (aw)^k y (wx)^k$. Then we verify that

$$\begin{aligned} awz &= ((aw)^{k+1}yw(xw)^{k-1}x = [(yw)(aw)^{k+1}](xw)^{k-1}x \\ &= (aw)^k(xw)^{k-1}xw = (aw)^k(xw)^{k-1}x = (aw)^2xwx. \end{aligned}$$

Claim 1. $z = a(wz)^2$.

$$\begin{aligned} a(wz)^2 &= (awz)wz = [(aw)^2xwx]w(aw)^k y (wx)^k \\ &= [(aw)^2(xw)^2](aw)^k y (wx)^k \\ &= [(aw)(xw)(aw)^k]y(wx)^k \\ &= [(aw)(aw)^{k-1}]y(wx)^k \\ &= (aw)^k y (wx)^k = z. \end{aligned}$$

Claim 2. $[(aw)^d]^*(aw)^{m+1}z = [(aw)^d]^*(aw)^{m-1}a$.

$$\begin{aligned} [(aw)^d]^*(aw)^{m+1}z &= [(aw)^d]^*[(aw)^{k+m+1}y](wx)^k = [(aw)^d]^*[y(wa)^{k+m+1}](wx)^k \\ &= [(aw)^d]^*[(yw)a(wa)^{k+m+1}w]x(wx)^{k-1} \\ &= [(aw)^d]^*[(yw)(aw)^{k+m+1}]awx(wx)^{k-1} \\ &= [(aw)^d]^*(aw)^{k+m+1}x(wx)^{k-1} \\ &= [(aw)^d]^*a(wa)^{k+m-1}(wx)^k \\ &= [(aw)^d]^*a(wa)^m(wx) \\ &= [(aw)^d]^*(aw)^{m+1}x \\ &= [(aw)^d]^*(aw)^{m-1}a. \end{aligned}$$

Claim 3. $(aw)^k = (zw)(aw)^{k+1}$.

Since $(aw)(yw) = (yw)(aw)$, we see that

$$\begin{aligned}
 (aw)^k - (zw)(aw)^{k+1} &= (aw)^k - (aw)^k yw(xw)^k (aw)^{k+1} \\
 &= (aw)^k - (yw)(aw)^k (xw)^k (aw)^{k+1} \\
 &= (aw)^k - (yw)(aw)(xw)(aw)^{k+1} \\
 &= (aw)^k - (yw)[(aw)^{k+1}(xw)^{k+1}](aw)^{k+1} \\
 &= (aw)^k - [(yw)(aw)^{k+1}](xw)^{k+1}(aw)^{k+1} \\
 &= (aw)^k - (aw)^k (xw)^{k+1}(aw)^{k+1} \\
 &= (aw)^k - (aw)(xw)^2(aw)^{k+1} \\
 &= (aw)^k - [a(wx)^2]w(aw)^{k+1} \\
 &= (aw)^k - xw(aw)^{k+1} \\
 &= (aw)^{k+n}(yw)^n - xw(aw)^{k+n+1}(yw)^n.
 \end{aligned}$$

Then

$$\|(aw)^k - (zw)(aw)^{k+1}\| \leq \|(aw)^{k+n} - xw(aw)^{k+n+1}\| \|yw\|^n.$$

Since $\lim_{n \rightarrow \infty} \|(aw)^{k+n} - xw(aw)^{k+n+1}\|^{\frac{1}{n}} = 0$, we see that

$$\lim_{n \rightarrow \infty} \|(aw)^k - (zw)(aw)^{k+1}\|^{\frac{1}{n}} = 0.$$

Hence, $(aw)^k = (zw)(aw)^{k+1}$. Therefore $a \in \mathcal{A}_w^{\otimes m}$. $\square$

The following example illustrates that (m, w) -generalized group inverse is different from the (m, w) -weak group inverse.

Example 1. Let V be a countably generated infinite-dimensional vector space over the complex field $\mathbb{C}$, and let $\{x_1, x_2, x_3, \dots\}$ be a basis of V . Let $\sigma : V \rightarrow V$ be a shift operator given by $\sigma(x_1) = 0$ and $\sigma(x_{i+1}) = x_i$ for all $i \in \mathbb{N}$. Then σ has $(m, 1)$ -generalized group inverse, while it has no any $(m, 1)$ -weak group inverse for any $m \in \mathbb{N}$.

Proof. Define the matrix A given by

$$\begin{aligned}
 \sigma(x_1, x_2, x_3, \dots) &= (0, x_1, x_2, x_3, \dots) \\
 &= (x_1, x_2, x_3, \dots)A.
 \end{aligned}$$

The linear shift operator σ can be regarded as an element in a Banach $*$ -algebra of complex matrix, with conjugate transpose $*$ as the involution. Then σ is quasinilpotent as all its eigenvalues are zero. Hence, it has $(m, 1)$ -generalized group inverse and $\sigma_1^{\otimes m} = 0$. For any $k \in \mathbb{N}$, $\sigma^k(x_1, x_2, x_3, \dots) = (0, 0, \dots, 0, x_1, x_2, \dots)$; whence, σ is not nilpotent. If σ has $(m, 1)$ -weak group inverse; it has Drazin inverse by Theorem 4.1. Then there exists some $n \in \mathbb{N}$ such that $\sigma^n = \sigma^{n+1}\eta$ and $\sigma\eta = \eta\sigma$ for a linear operator η . Hence $\sigma^n(1 - \sigma\eta) = 0$. As σ is quasinilpotent and $\sigma\eta = \eta\sigma$, we have $1 - \sigma\eta$ is invertible, and the $\sigma^n = 0$. This gives a contradiction. Therefore σ has no $(m, 1)$ -weak group inverse, as asserted. $\square$

Theorem 9. Let $A, W, X \in \mathbb{C}^{n \times n}$, with conjugate transpose $*$ as the involution. Then the following are equivalent:

- (1) $A_W^{\otimes m} = X$.
- (2) There exist $Y, Z \in \mathbb{C}^{n \times n}$ such that

$$A = Y + Z, Y^*Z = ZW^{m-1}Y = 0,$$

$Y \in \mathbb{C}^{n \times n}$ has W -group inverse $X, Z \in \mathbb{C}^{n \times n}$ is nilpotent.

(3) There exist $n \in \mathbb{N}$ such that

$$X = A(WX)^2, (AW)^n = XW(AW)^{n+1}, \\ ((AW)^D)^*(AW)^{m+1}X = ((AW)^D)^*(AW)^{m-1}A.$$

(4) There exist $n \in \mathbb{N}$ such that

$$(AW)^n = XW(AW)^{n+1}, \text{Im}(X) = \text{Im}(A^{D,W}), \\ ((AW)^D)^*(AW)^{m+1}X = (AW)^D)^*(AW)^{m-1}A.$$

(5) There exists $S \in \mathcal{A}$ such that $(A^{D,W})^*A^{D,W}S = (A^{D,W})^*(AW)^{m-1}A$ and $X = ((AW)^D)^{m+3}AS$.

Proof. Evidently, $A \in \mathbb{C}^{n \times n}$ has W -Drazin inverse.

(1) $\Leftrightarrow$ (2) For a complex matrix, the quasi-nilpotent and nilpotent properties coincide with each other. This equivalence is proved by Theorem 3.1 and Theorem 4.1.

(1) $\Rightarrow$ (3) In view of Theorem 4.1, we have

$$X = A(WX)^2, ((AW)^D)^*(AW)^{m+1}X = ((AW)^D)^*(AW)^{m-1}A, \\ \lim_{n \rightarrow \infty} \| (AW)^n - XW(AW)^{n+1} \|^{1/n} = 0.$$

This implies that $(AW)^n = XW(AW)^{n+1}$ for some $n \in \mathbb{N}$.

(3) $\Rightarrow$ (1) This is obvious by Theorem 4.1.

(1) $\Leftrightarrow$ (4) This is proved by Theorem 2.1, Theorem 4.1 and [3, Theorem 3.3].

(1) $\Leftrightarrow$ (5) This is similar by using Theorem 2.5 and Theorem 4.1. $\square$

As an immediate consequence of Theorem 4.3, we derive

Corollary 10. Let $A, X \in \mathbb{C}^{n \times n}$, with conjugate transpose $*$ as the involution. Then the following are equivalent:

(1) $A^{\oplus, W} = X$.

(2) There exist $Y, Z \in \mathbb{C}^{n \times n}$ such that

$$A = Y + Z, Y^*Z = ZY = 0,$$

$Y \in \mathbb{C}^{n \times n}$ has W -group inverse $X, Z \in \mathbb{C}^{n \times n}$ is nilpotent.

(3) There exist $n \in \mathbb{N}$ such that

$$X = A(WX)^2, (AW)^n = XW(AW)^{n+1}, \\ ((AW)^D)^*(AW)^2X = ((AW)^D)^*A.$$

(4) There exist $n \in \mathbb{N}$ such that

$$(AW)^n = XW(AW)^{n+1}, \text{Im}(X) = \text{Im}(A^{D,W}), \\ ((AW)^D)^*(AW)^2X = (AW)^D)^*A.$$

(5) There exists $S \in \mathcal{A}$ such that $(A^{D,W})^*A^{D,W}S = (A^{D,W})^*A$ and $X = ((AW)^D)^4AS$.

We are ready to prove:

Theorem 10. Let $A, W \in \mathbb{C}^{n \times n}$. Then

$$A_W^{\oplus m} = (AW)^{4(m-1)}A^{\oplus, W}WA^{\oplus, W}(WA)^m.$$

Proof. In view of Theorem 2.1 and Theorem 4.1, we have

$$A_W^{\oplus m} = (AW)^{m-1}((AW)^{m-1}A)^{\oplus, W}.$$

By virtue of [7, Theorem 3.1], we see that

$$(AW)^{m-1}A = X + Y, X^*Y = 0, YWX = 0, \\ X \in \mathbb{C}^{n \times n} \text{ has } W\text{-weighted core inverse, } Y \in \mathbb{C}^{n \times n} \text{ is nilpotent.}$$

Then

$$[(AW)^{m-1}A]^{\oplus, W} = X^{\oplus, W}.$$

In light of [7, Theorem 2.4], we have

$$X^{\oplus, W} = (XW)^{\#}XW(WXW)^{\dagger}.$$

Then

$$\begin{aligned} ((XW)^{\#})^2 X^{\oplus, W} (WX)^2 &= ((XW)^{\#})^3 X(WXW)(WXW)^{\dagger}(WXW)X \\ &= [(XW)^{\#}]^3 X(WXW)X \\ &= [(XW)^{\#}]^2 X = X_W^{\#}. \end{aligned}$$

Thus, we have

$$\begin{aligned} X_W^{\#} &= ((XW)^{\#})^2 X^{\oplus, W} (WX)^2 \\ &= ((XW)^{\#})^2 XW X^{\oplus, W} W X^{\oplus, W} (WX)^2 \\ &= ((XW)^{\#})^2 XW [XW X^{\oplus, W} W X^{\oplus, W}] W X^{\oplus, W} (WX)^2 \\ &= (XW)^{\#} XW X^{\oplus, W} W X^{\oplus, W} W X^{\oplus, W} W XW X \\ &= [X^{\oplus, W} W]^3 XW X. \end{aligned}$$

Therefore

$$X_W^{\#} = [X^{\oplus, W} W]^3 XW X.$$

By using [7, Theorem 3.1], we have

$$\begin{aligned} X &= (AW)^{m-1} A W X^{\oplus, W} W (AW)^{m-1} A \\ &= (AW)^m X^{\oplus, W} (WA)^m. \end{aligned}$$

Accordingly, we have

$$\begin{aligned} A_W^{\oplus m} &= (AW)^{m-1} X_W^{\#} \\ &= (AW)^{m-1} [X^{\oplus, W} W]^3 XW X \\ &= (AW)^{m-1} [X^{\oplus, W} W]^3 (AW)^m X^{\oplus, W} (WA)^m W (AW)^m X^{\oplus, W} (WA)^m \\ &= (AW)^{m-1} [X^{\oplus, W} W]^3 (AW)^m X^{\oplus, W} (WA)^{2m} W X^{\oplus, W} (WA)^m \\ &= (AW)^{m-1} [A^{\oplus, W} W]^3 (AW)^m A^{\oplus, W} (WA)^{2m} W A^{\oplus, W} (WA)^m. \end{aligned}$$

as asserted. $\square$

Let $A, W \in \mathbb{C}^{n \times n}$ and $k = \max\{\text{ind}(AW), \text{ind}(WA)\}$. The weighted core-EP inverse of A is the unique solution to the system:

$$WAWX = (WA)^k [(WA)^k]^{\dagger}, \mathcal{R}(X) \subseteq \mathcal{R}((AW)^k),$$

and we denote such X by $A^{\oplus, W}$.

Corollary 11. Let $A, W \in \mathbb{C}^{n \times n}$. Then

$$A^{\oplus, W} = (A^{\oplus, W} W)^2 A.$$

Proof. This is obvious by choosing $m = 1$ in Theorem 4.5. $\square$

Lemma 1. Let $A, W \in \mathbb{C}^{n \times n}$. Then the W -weak group inverse of A can be expressed as

$$A^{\oplus, W} = (AW)(AW)^D[(AW)^3(AW)^D]^{\dagger}A.$$

Proof. Set $X = (AW)(AW)^D[(AW)^3(AW)^D]^{\dagger}A$ and $n = \text{ind}(AW)$. Then we verify that

$$\begin{aligned} & A(WX)^2 \\ &= (AW)^2(AW)^D[(AW)^3(AW)^D]^{\dagger}(AW)^2(AW)^D[(AW)^3(AW)^D]^{\dagger}A \\ &= (AW)^D[(AW)^3(AW)^D](AW)^D[(AW)^3(AW)^D]^{\dagger}A \\ &= (AW)(AW)^D[(AW)^3(AW)^D]^{\dagger}A \\ &= X; \\ & ((AW)^D)^*(AW)^2X \\ &= ((AW)^D)^*(AW)^3(AW)^D[(AW)^3(AW)^D]^{\dagger}A \\ &= ((AW)^D)^*((AW)^3(AW)^D[(AW)^3(AW)^D]^{\dagger})^*A \\ &= ((AW)^3(AW)^D[(AW)^3(AW)^D]^{\dagger}(AW)^D)^*A \\ &= [(AW)^3(AW)^D][(AW)^3(AW)^D]^{\dagger}[(AW)^3(AW)^D]((AW)^D)^3)^*A \\ &= [(AW)^3(AW)^D]((AW)^D)^3)^*A \\ &= ((AW)^D)^*A; \\ & XW(AW)^{n+1} \\ &= (AW)(AW)^D[(AW)^3(AW)^D]^{\dagger}AW(AW)^{n+1} \\ &= ((AW)^D)^2[(AW)^3(AW)^D][(AW)^3(AW)^D]^{\dagger}[(AW)^3(AW)^D](AW)^n \\ &= ((AW)^D)^2[(AW)^3(AW)^D](AW)^n \\ &= (AW)^D(AW)^{n+1} = (AW)^n. \end{aligned}$$

Therefore $A^{\oplus, W} = X$, as asserted. $\square$

We now develop a formula for the weighted weak group inverse of a complex matrix.

Theorem 11. Let $A, W \in \mathbb{C}^{n \times n}$. Then the (m, W) -weak group inverse of A can be expressed as

$$A_W^{\otimes m} = (AW)(AW)^D[(AW)^{2m+1}(AW)^D]^{\dagger}(AW)^{m-1}A.$$

Proof. In view of Theorem 2.1 and Theorem 4.1,

$$A_W^{\otimes m} = (AW)^{m-1}[(AW)^{m-1}A]^{\oplus, W}.$$

By virtue of Lemma 4.7, we have

$$\begin{aligned} [(AW)^{m-1}A]^{\oplus, W} &= (AW)^m((AW)^m)^D[(AW)^{3m}((AW)^m)^D]^{\dagger}(AW)^{m-1}A \\ &= (AW)(AW)^D[(AW)^{2m+1}(AW)^D]^{\dagger}(AW)^{m-1}A. \end{aligned}$$

Therefore we conclude that

$$A_W^{\otimes m} = (AW)^m(AW)^D[(AW)^{2m+1}(AW)^D]^{\dagger}(AW)^{m-1}A,$$

as asserted. $\square$

Corollary 12. Let $A \in \mathbb{C}^{n \times n}$. Then the weak group inverse of A can be expressed as

$$A^{\oplus} = AA^D[A^3A^D]^{\dagger}A.$$

Proof. Straightforward by Theorem 4.8. $\square$

Applying the weighted weak group inverse, we prove solvability of the following minimization problem with respect to $X \in \mathbb{C}^{n \times n}$ in the Frobenius norm:

Theorem 12. Let $A \in \mathbb{C}^{n \times n}$. The minimization problem with respect to $X \in \mathbb{C}^{n \times n}$:

$$\min ||(AW)^{2m}X - (AW)^{m-1}AB||_F$$

subject to $R(X) \subseteq R(AW)^D$ is

$$X = A^{\oplus_m} B.$$

Proof. Since $R(X) \subseteq R(AW)^D$, we may write $X = (AW)(AW)^DY$ for some $Y \in \mathbb{C}^{n \times n}$. Then X is the solution of the preceding minimization problem if and only if Y is the solution of the following minimization problem:

$$\min ||(AW)^{2m+1}(AW)^DY - (AW)^{m-1}AB||_F.$$

By the relation between the minimization problem and the Moore-Penrose inverse, we claim that

$$Y = ((AW)^{2m+1}(AW)^D)^{\dagger}(AW)^{m-1}AB.$$

Therefore

$$X = (AW)(AW)^D([(AW)^3(AW)^D])^{\dagger}(AW)^{m-1}AB = A^{\oplus_m} B,$$

as asserted. $\square$

Corollary 13. Let $A \in \mathbb{C}^{n \times n}$. The minimization problem with respect to $X \in \mathbb{C}^{n \times n}$:

$$\min ||A^{2m}X - A^mB||_F$$

subject to $R(X) \subseteq R(A^D)$ is

$$X = A^{\oplus_m} B.$$

Proof. This is immediate by choosing $W = I_n$ in Theorem 4.10. $\square$

Corollary 14. Let $A, W \in \mathbb{C}^{n \times n}$. The minimization problem with respect to $X \in \mathbb{C}^{n \times n}$:

$$\min ||(AW)^2X - AB||_F$$

subject to $R(X) \subseteq R(AW)^D$ is

$$X = A^{\oplus,W} B.$$

Proof. This is obvious by choosing $m = 1$ in Theorem 4.10. $\square$

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