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Binary Sequences with Low Aperiodic Autocorrelations: Small Peak Sidelobe Levels

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Abstract

Binary sequences (binary codes), where the elements are -1 or $+1$, are useful in many fields, including communications, radar, sonar, mathematics, physics, and cryptography. This paper considers binary sequences with low aperiodic autocorrelations and focuses on the small peak sidelobe levels alongside the merit factor. Two families of binary sequences are considered, namely Rudin-Shapiro and Legendre sequences. For both families, we applied a heuristic algorithm to minimize the peak sidelobe levels for sequences of lengths up to 2^{16} and $2^{20}-1$, respectively. The main contribution of the article is two conjectures associated with Legendre sequences: (1) The obtained binary sequences with the best-known peak sidelobe levels have merit factor ≈ 5.0, (2) The number of elements that differ between the resulting binary sequences and the initial Legendre sequences follows a linear dependence on the sequence length (n), namely $\approx 0.01n$. The Rudin-Shapiro sequences do not exhibit these properties, as worse peak sidelobe level and merit factor values were obtained. The number of elements that differ between the resulting binary sequences and the initial Rudin-Shapiro sequences is also much higher compared to that of the Legendre sequences.

Keywords: binary sequence; aperiodic autocorrelation; peak sidelobe level; Legendre sequence; Rudin-Shapiro sequence; merit factor

MSC: 94A55

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1. Introduction

A *binary sequence* of length n is an n-tuple $A = (a_0, a_1, \ldots, a_{n-1})$, where each a_i, for $i \in \{0, 1, \cdots, n-1\}$ takes the value -1 or $+1$. The *aperiodic autocorrelation* function of the binary sequence A at shift k is defined as

$$C_A(k) := \sum_{i=0}^{n-k-1} a_i a_{i+k} \qquad \text{for } k = 0, 1, \ldots, n-1. \tag{1}$$

The second type of autocorrelation function is periodic. The *periodic autocorrelations* are usually much easier to study than their aperiodic counterparts [1]. In this work, we consider aperiodic autocorrelations.

Several authors have studied different measures for the collective smallness of the aperiodic autocorrelations of sequences. Two measures of "smallness" have been used:

1. merit factor, and
2. peak sidelobe level.

The *merit factor* was introduced by Golay [2,3], and it is defined as

$$F(A) := \frac{n^2}{2\sum_{k=1}^{n-1}(C_A(k))^2} \qquad \text{for } n > 1. \tag{2}$$

The *Peak Sidelobe Level* (PSL) is defined as

$$\mathrm{PSL}(A) := \max_{1 \le k < n} |C_A(k)|. \tag{3}$$

The autocorrelation function of A at shift 0, $C_A(0)$, is called the *main-lobe* level, and it is not included in (3). The rest, $C_A(k), k = 1, 2, \ldots, n-1$, are called *side-lobe* levels. The aperiodic autocorrelation function in (1) is even, since $C_A(-k) = C_A(k)$. Therefore, it is sufficient to consider only the shifts $k = 0, 1, \ldots, (n-1)$.

Sequence design methods in the literature can be broadly classified into two categories: algebraic constructions and algorithmic design approaches [4]. This work focuses mainly on algorithmic approaches.

In practice, we prefer binary sequences that have a lower PSL value and a higher F value. Some algorithms minimize a combination of metrics to create binary sequences with low aperiodic autocorrelation. A sequence with the optimal PSL usually has a merit factor which is much lower than the optimal merit factor, and vice versa [5,6]. Various algorithms have been proposed recently to solve this challenging optimization problem for F metric. For example, massively parallelized implementation of the memetic tabu search algorithm [7], the quantum approximate optimization algorithm [8], quantum-enhanced memetic tabu search [9], and a competitive NISQ and qubit-efficient solver for the LABS problem. The reader interested in the F metric is referred to the survey [1](Chapter 3.5) and recent works [10,11].

Let $\mathcal{A}_n$ denote the set of all binary sequences of length n. We would like to understand the behavior of

$$\mu(n) := \min_{A \in \mathcal{A}_n} \mathrm{PSL}(A), \tag{4}$$

as $n \to \infty$. If someone wishes to compute $\mu(n)$ for a given length n, then they need, using the most naive approach, to probe 2^n different binary sequences. The exponential term of the time complexity can be reduced from $O(2^n)$ to approximately $O(1.4^n)$ by more efficient algorithms [12,13]. The considerable numerical computation effort in finding binary sequences with small PSL values has been put in by various authors, and, nowadays, $\mu(n)$ for $n \le 61$ and $n = 64$ are known, while for larger lengths, the best-known PSL values are reported in the literature. For $71 \le n \le 105$, the PSL values are presented in [14]. Currently, results for all lengths $106 \le n \le 300$ are reported in [5,15,16], PSL values for selected lengths for $301 \le n \le 4000$ are published by several authors (see [17] and references therein), for some n up to 8191 in [18,19], and for even longer n (up to 10^6) in [5,17,20–22] and up to $2^{25} - 1$ in [13].

1.1. Contributions of the Paper

In this paper, two families of binary sequences are considered, namely Rudin-Shapiro and Legendre sequences. Experimental works include the minimization of the peak sidelobe levels of Legendre sequences of length up to $2^{20} - 1$.

Based on numerical experimentation, two conjectures associated with Legendre sequences are proposed:

- The obtained resulting binary sequences with the best-known peak sidelobe levels have merit factor ≈ 5.0. It appears to be close to a whole number.
- The number of elements that differ between the resulting binary sequences and the initial Legendre sequences follows a linear dependence on the sequence length (n), namely $\approx 0.01n$.

Additional experimentation indicates that no such results can be observed with the Rudin-Shapiro sequences, where worse peak sidelobe level and merit factor values were obtained.

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1.2. Structure of the Paper

The paper is organized as follows. Section 2 provides the necessary background knowledge to understand the main ideas of the paper. Related work is discussed in Section 3. Section 4 outlines the methodology used for the experimental part. Extensive experiments conducted to find binary sequences with small peak sidelobe levels and experimental results are presented in Section 5. Based on the results obtained, we also proposed two conjectures in this section. Section 6 highlights some additional notes. Finally, Section 7 summarizes the findings and concludes the paper.

2. Preliminary

In this section, we give a brief overview of some families of binary sequences, how we generate them, their properties, and some known results.

The *Rudin-Shapiro* sequence requires a four-letter alphabet with the following substitution rule [23]:

$$\begin{aligned}
&(1) \quad \mathrm{S} \mapsto \mathrm{ST}, \quad \mathrm{T} \mapsto \mathrm{SU}, \quad \mathrm{U} \mapsto \mathrm{VT}, \quad \mathrm{V} \mapsto \mathrm{VU},\\
&(2) \quad \mathrm{S}, \mathrm{T} \mapsto 0, \quad \mathrm{U}, \mathrm{V} \mapsto 1.
\end{aligned}$$

The first symbolic generations are:

$$\begin{aligned}
w^{(0)} &= \mathrm{S},\\
w^{(1)} &= \mathrm{ST},\\
w^{(2)} &= \mathrm{STSU},\\
w^{(3)} &= \mathrm{STSUSTVT},\\
&\cdots
\end{aligned}$$

Applying the binary conversion rules to the Rudin-Shapiro symbolic generations yields:

$$\begin{aligned}
w^{(0)} &= 0,\\
w^{(1)} &= 00,\\
w^{(2)} &= 0001,\\
w^{(3)} &= 00010010,\\
&\cdots
\end{aligned}$$

Changing each symbol 0 to -1 yields the binary sequence in our format. The length of the Rudin-Shapiro sequence is $n = 2^m$, where $m \in \{1, 2, \cdots\}$.

Calculations by Littlewood show that the Rudin–Shapiro sequences have low mean square autocorrelation [24]. We now examine the lower and upper bounds for PSL values of Rudin–Shapiro sequences. Here we use the abbreviated text from [24] [Theorem 3, p. 3457].

Theorem 1 (Katz and van der Linden [24])**.** *If n is a nonnegative integer and x_n and y_n the n-th Rudin-Shapiro sequence and its companion, then $\mathrm{PSL}(x_n) = \mathrm{PSL}(y_n)$, and we have the lower and upper bound*

$$0.382159... \leq \limsup_{n\to\infty} \frac{\mathrm{PSL}(x_n)}{\alpha_0^n} \leq 0.660113...$$

where $\alpha_0 = 1.658967...$ is the real root of $X^3 + X^2 - 2X - 4$.

The *difference* between binary sequences A and B, both of length n, can be written as the Hamming distance:

$$d = d(A, B) = \sum_{i=0}^{n-1} d_i, \tag{5}$$

where

$$d_i = \begin{cases} 1, & A_i \neq B_i, \\ 0, & A_i = B_i. \end{cases} \tag{6}$$

We are interested in how different the two binary sequences are, where the first sequence is an initial sequence and the second sequence has an optimized value of PSL. The second sequence is obtained during the optimization process using the heuristic algorithm, presented in [17].

Ein-Dor et al. [25] proposed an "educated guess" about the growth rate of the function $\mu(n)$.

Conjecture 1 (Ein-Dor et al. [25])**.** *As $n \to \infty$, we have*

$$\frac{\mu(n)}{\sqrt{n}} \to d, \qquad \text{where } d = 0.435\ldots.$$

Theorem 2 (Moon and Moser [26])**.**

(i) *For any fixed $\epsilon > 0$, the proportion of sequences $A \in \mathcal{A}_n$ such that $\mu(n) \leq (2+\epsilon)\sqrt{n \log n}$ approaches 1 as $n \to \infty$.*

(ii) *If $K(n)$ is any function of n such that $K(n) = o(\sqrt{n})$, then the proportion of sequences $A \in \mathcal{A}_n$ for which $\mu(n)$ approaches 1 as $n \to \infty$.*

Theorem 2, the only proven PSL result for general binary sequences, examines the growth rate of the peak sidelobe level of almost all binary sequences and this growth rate lies between order $\sqrt{n}$ and order $\sqrt{n \log n}$.

A *Legendre sequence* A of length p, where p is an odd prime number, is defined as:

$$A(k) = \left(\frac{k}{p}\right), \qquad \text{for } 0 \leq k < p, \tag{7}$$

where $\left(\frac{k}{p}\right)$ is the Legendre symbol [27,28]. The Legendre symbol is defined as:

$$\left(\frac{k}{p}\right) = \begin{cases} +1, & \textbf{if } p \mid k \textbf{ or } k \text{ is a quadratic residue modulo } p, \\ -1, & \text{otherwise.} \end{cases}$$

Note that we use the convention that $\left(\frac{k}{p}\right) = 1$ if $k = 0$.

Dimitrov [21] gave a complete list of all PSL-optimal Legendre sequences, with or without rotations, for lengths up to 432100. The numerical experiments suggest that the PSL values of all PSL-optimal Legendre sequences, with or without rotations, and with lengths n greater than 235723, are strictly greater than $\sqrt{n}$.

Let us consider another family of binary sequences, namely *Galois sequences*, also called m-sequences. Galois sequences can be efficiently generated using linear feedback shift registers [29]. There is a direct correspondence between m-sequences of degree q and primitive polynomials of degree q over $GF(2)$ [30].

Dmitriev and Jedwab [13] performed experiments on m-sequences up to length $2^{25} - 1$ and they showed numerically:

Conjecture 2 (Dmitriev and Jedwab [13])**.** *The PSL of all m-sequences of length n appears to grow like $O(\sqrt{n} \cdot \log \log n)$.*

In the same work, Dmitriev and Jedwab also numerically showed:

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Conjecture 3 (Dmitriev and Jedwab [13])**.** *The PSL of almost all m-sequences of length n appears to grow like* $\Theta(\sqrt{n})$*.*

Numerical evidence in Conjecture 3 relies on an algorithm for calculating the maximum PSL over all cyclic shifts of an m-sequence for $m \leq 25$. The mean (taken over all primitive elements α of GF(n+1) [13,31]) of the maximum PSL over all cyclic shifts appears to be approximately $1.31\sqrt{n}$ for all values of m between 13 and 25.

3. Related Works

In this section, we give an overview of related works. Finding sequences with low autocorrelation sidelobes is a very challenging optimization problem of the digital sequence design [17]. Recently, several heuristic algorithms were proposed to tackle this optimization problem, especially for longer lengths.

Mow et al. [32] performed an experiment for finding long LABS (low autocorrelation binary sequences) with low PSL. Four different fitness functions were used in their evolutionary algorithm for lengths up to 4096. Coxson et al. [33] applied a multi-thread evolutionary algorithm to search for binary sequences (codes) with small PSL values for n up to 3000. Coxson and Russo [12] used exhaustive search for minimum PSL (see Eq. (4)) binary sequences. Lin et al. [20] proposed the 1bCAN algorithm, used for aperiodic binary sequences design. The proposed algorithm is FFT (Fast Fourier Transform) based, and it can be used to design long sequences with lengths up to $n \sim 10^6$. Dimitrov et al. [15] applied an efficient mechanism for single bit flipping calculation. Binary sequences of length up to $m = 17$ ($n = 131071$) were considered. The hybrid strategies for constructing binary sequences with near-optimal PSL values are presented in [16]. Bošković and Brest [34] proposed the two-phase optimization algorithm. It utilized a concurrent computing on the graphic processing units for finding binary sequences on lengths $n = 2^m - 1$ for $14 \leq m \leq 20$. Lin et al. [22] proposed an efficient gradient based algorithm, called 1bG-PSL, to minimize PSL for binary sequence designs with or without low correlation zone requirements. The result for $n = 2^{17} - 1$ is reported, i.e., PSL $= 284$ and $F = 4.59$. It can be observed that the computational methods presented above used a random binary sequence for initialization of the optimization process.

Brest et al. [17] developed a heuristic algorithm that combines the Legendre sequence to seed an initial binary sequence and an efficient stochastic search method with a dynamic fitness function mechanism to generate a final binary sequence up to length $2^{20} - 1$.

Jedwab and Yoshida [35], and Dmitriev and Jedwab [13] studied the growth rate of PSL for binary sequences and their families, numerically. In [13], the authors reported results for $n = 2^m - 1$ for m up to 25.

4. Methodology

We will analyze the PSL values of the binary sequences that were optimized a heuristic algorithm. The initial binary sequences, which served as seeds for the optimization algorithm, were selected from two well-known families of binary sequences, namely the Rudin-Shapiro and Legendre sequences. The applied heuristic algorithm [17] minimizes only the PSL value during the optimization process. We will be interested in the difference between the initial and the optimized binary sequence. For the sequence obtained after optimization, we will also calculate the merit factor and compare it with the merit factor of the initial sequence, whose merit factor is known. The obtained results will also be compared with results from the literature.

The length of Rudin-Shapiro sequence is $n = 2^m$ and our heuristic algorithm performed the optimization process on that length. We denote the initial and optimized Rudin-Shapiro sequences as B^{RS}_{init} and B^{RS}_{opt}, respectively.

A Legendre sequence is defined for length p, where p is an odd prime number. We wish to use a binary sequence of length $n = 2^m - 1$, so we applied the following procedure to ensure the length.

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We used the first prime p that satisfies $n \leq p$. A Legendre binary sequence of length p is generated using (7) and rotated by $1/4$. It is then truncated, if needed, to ensure that the length of the initial binary sequence is equal to n. We denote the initial, $1/4$-rotated, and truncated Legendre sequence as B_{init}^{Leg} and the optimized Legendre sequence B_{opt}^{Leg}.

5. Experiments

In this section, we present the results of finding good PSL values for two families of binary sequences, and then give two conjectures.

5.1. Results on Rudin-Shapiro Sequences

The construction of the Rudin–Shapiro sequences is described in Section 2. For selected lengths, the lower (LB) and upper (UB) bounds alongside the PSL value of the Rudin-Shapiro sequences are shown in Table 1. The bounds are calculated using Theorem 1.

Table 1. The lower and upper bounds, and the PSL value for Rudin-Shapiro sequences.

m	$n = 2^m$	LB	UB	PSL
10	1024	60	104	85
11	2048	100	172	153
12	4096	166	286	217
13	8192	275	475	373
14	16384	457	789	557
15	32768	768	1309	961
16	65536	1257	2172	1717
17	131072	2086	3604	2445
18	262144	3461	5779	4285
19	524288	5743	9920	6257
20	1048576	9527	16457	11153

[1] LB $= 0.382159\alpha_0^m$
[2] UB $= 0.660113\alpha_0^m$

Next, we present experimental results of optimized Rudin-Shapiro sequences. For this purpose, we used the heuristic algorithm [17]. The algorithm started with the initial Rudin-Shapiro sequence (B_{init}^{RS}) with length n, and during the optimization process, the algorithm made improvements to the binary sequence by changing a small number of elements of the sequence, yielding the optimized Rudin-Shapiro sequence (B_{opt}^{RS}).

Table 2 shows PSL of the optimized sequence (B_{opt}^{RS}), and difference between the initial Rudin-Shapiro sequence (B_{init}^{RS}) and the optimized sequence (B_{opt}^{RS}) for $10 \leq m \leq 16$. Comparing the PSL values in Tables 1 and 2, it can be seen that the PSL values of the initial sequences are much higher than those of the optimized sequences. For example, for $m = 16$, the PSL values are 1717 and 209, respectively. Differences between the Rudin-Shapiro sequences (initial) and the optimized sequences are between 30% and 50%. This indicates that the heuristic algorithm had to flip quite a few elements of the binary sequence.

When we compare the corresponding merit factors, we notice that the merit factors in the optimized sequences increased from the initial $F = 3.0$, which is known for Rudin-Shapiro sequences, to values between 3.8 and 4.2. In general, we can see that with the improvement of the PSL value (lower value is better), the merit factor value also managed to improve (higher value is better).

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Table 2. PSL and F of the optimized sequence (B_{opt}^{RS}), and difference between the Rudin-Shapiro sequence (B_{init}^{RS}) and the optimized sequence (B_{opt}^{RS}).

m	$n = 2^m$	PSL	PSL$/\sqrt{n}$	F	d	d/n [%]
10	1024	24	0.7500	3.84815	519	50.68
11	2048	34	0.7513	4.26986	1002	49.93
12	4096	49	0.7656	4.13834	1952	47.66
13	8192	69	0.7623	4.29800	3590	43.82
14	16384	101	0.7891	4.11236	5535	33.78
15	32768	146	0.8065	3.95976	10872	33.18
16	65536	209	0.8164	3.94681	21477	32.77

5.2. Results on Legendre Sequences

Table 3 shows PSL and merit factor values of the optimized sequence starting from the initial Legendre sequence for lengths $n = 2^m - 1, m \in \{10, 20\}$. Values PSL$/\sqrt{n}$ show an increasing trend, but all of them are lower than 0.75. Merit factor values are very close to 5.0, except for the two shortest lengths (1023 and 2047), where F is ≈ 4.8. Note that our heuristic algorithm applied the minimization criteria on the PSL. The difference (d) between the initial Legendre sequence (B_{init}^{Leg}) and the optimized sequence (B_{opt}^{Leg}), and the ratio d/n in percent are presented in the last two columns. One can see that the ratio d/n follows a constant value that is close to 1% for $m \in \{12, 13, \ldots, 20\}$.

Table 3. PSL and F of the optimized sequence (B_{opt}^{Leg}), and difference between the initial Legendre sequence (B_{init}^{Leg}) and the optimized sequence (B_{opt}^{Leg}).

m	$n = 2^m - 1$	PSL	PSL$/\sqrt{n}$	F	d	d/n [%]
10	1023	22	0.6878	4.78203	13	1.27
11	2047	32	0.7073	4.82918	31	1.51
12	4095	45	0.7032	5.09739	40	0.98
13	8191	66	0.7292	5.02281	86	1.05
14	16383	93	0.7266	5.03535	165	1.01
15	32767	133	0.7347	4.99564	353	1.08
16	65535	189	0.7383	5.06485	700	1.07
17	131071	269	0.7430	5.00051	1394	1.06
18	262143	380	0.7422	4.99135	2855	1.09
19	524287	540	0.7457	5.02356	5319	1.01
20	1048575	766	0.7480	4.99350	10873	1.04

Comparing the optimized PSL values of Rudin-Shapiro (Table 2) and the optimized Legendre (Table 3) sequences, we can see that the PSL values of optimized Rudin-Shapiro sequences are much higher. Also, the differences between the initial and optimized binary sequences are much higher for Rudin-Shapiro compared to Legendre ones, i.e., more than 30% for Rudin-Shapiro and only approximately 1% for Legendre.

Results in Table 3 are taken from work [17], except for $m = 13, 18, 19$, and 20, which were obtained using the heuristic algorithm also presented in [17]. PSL values for $m = 18, 19$, and 20 were improved by 1, 1, and 4, respectively, and are currently the best-known values. Note, the best-known PSL is 65 for $m = 13$ [17].

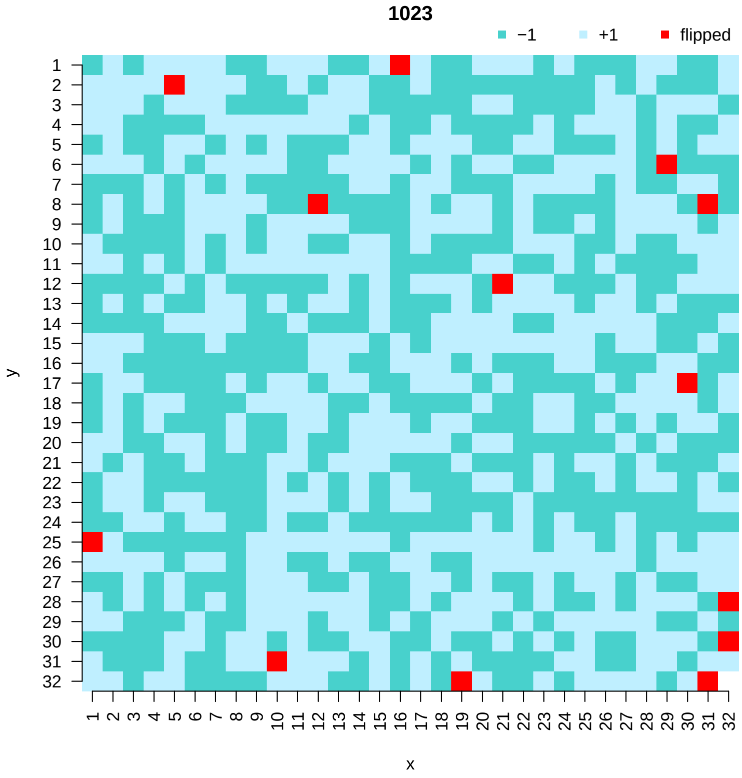


Figure 1. Binary sequence of $L = 1023$ depicted as a square of 32×32. Red color indicates the differences between the rotated Legendre sequence and the optimized sequence with PSL $= 22$. (The figures in this article were created in R [36].)

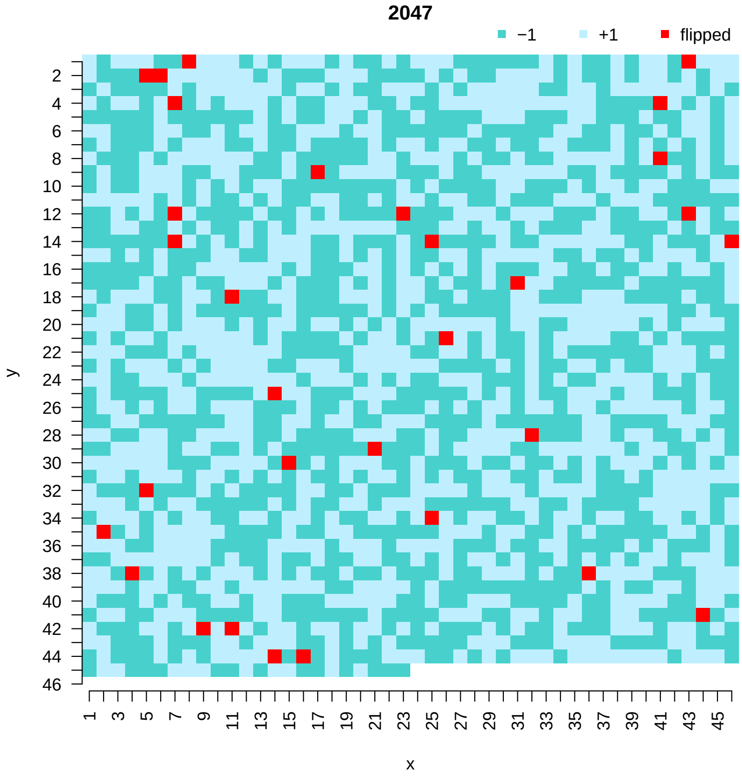


Figure 2. Binary sequence of $n = 2047$ depicted/placed as a square of 46×46. Red color indicates the differences between the rotated Legendre sequence and the optimized sequence with PSL $= 32$.

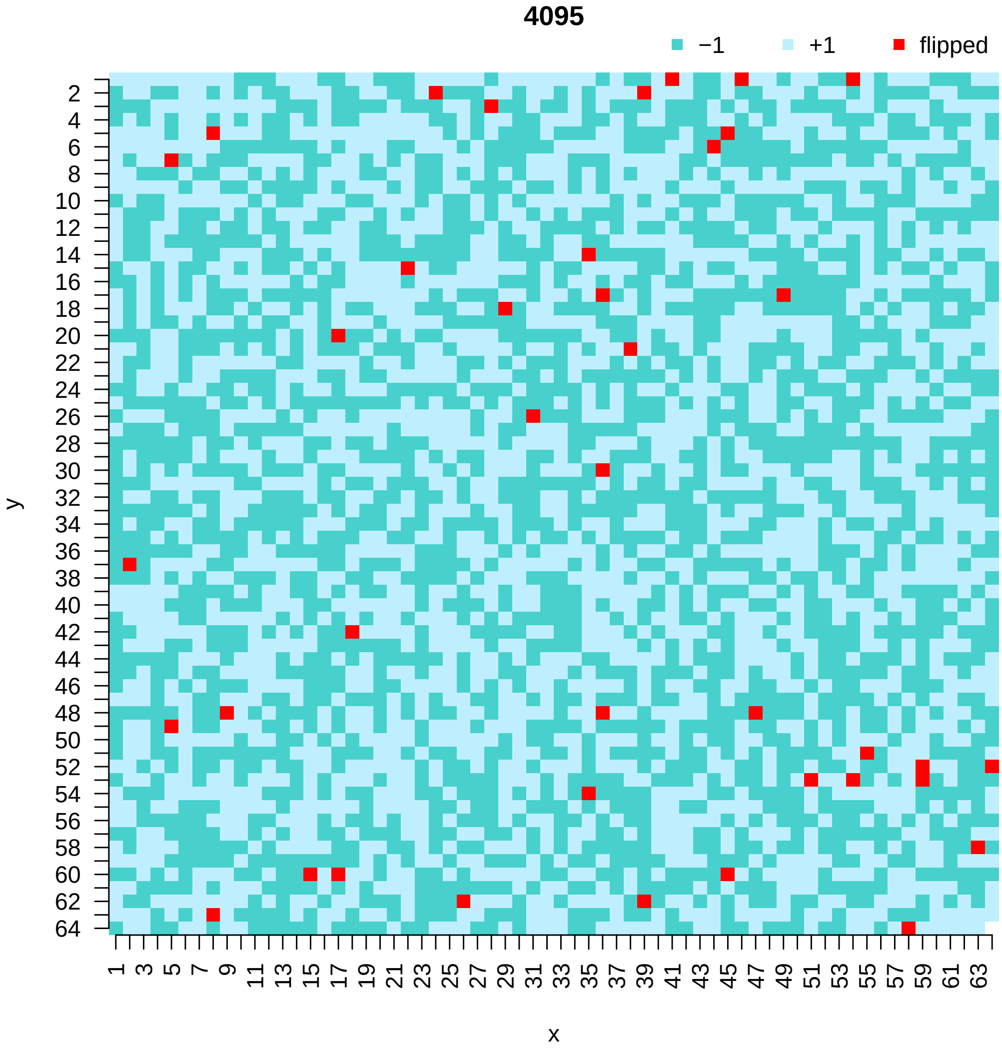
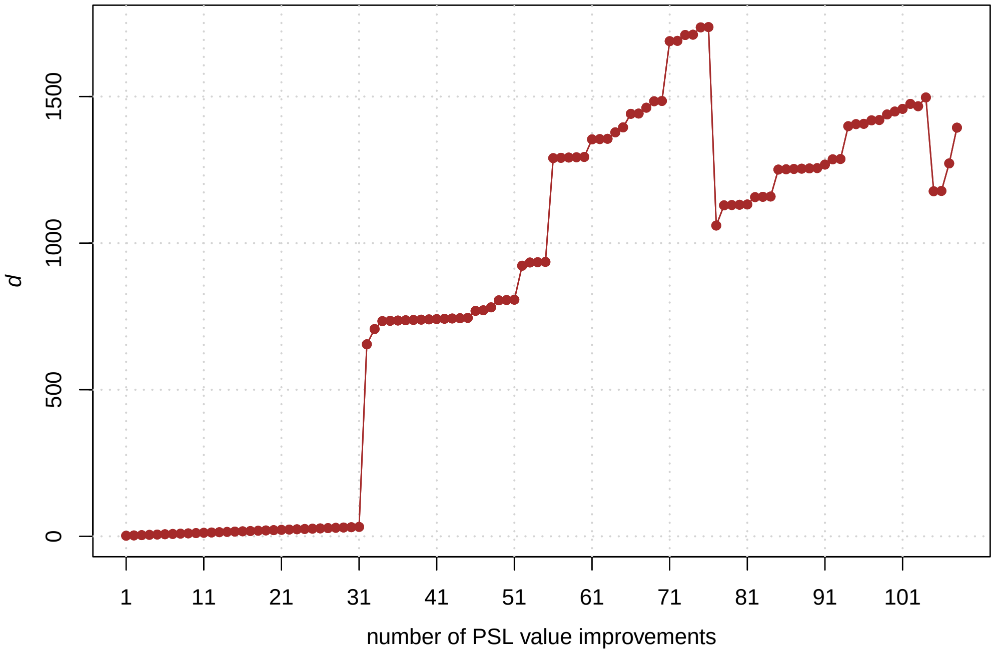


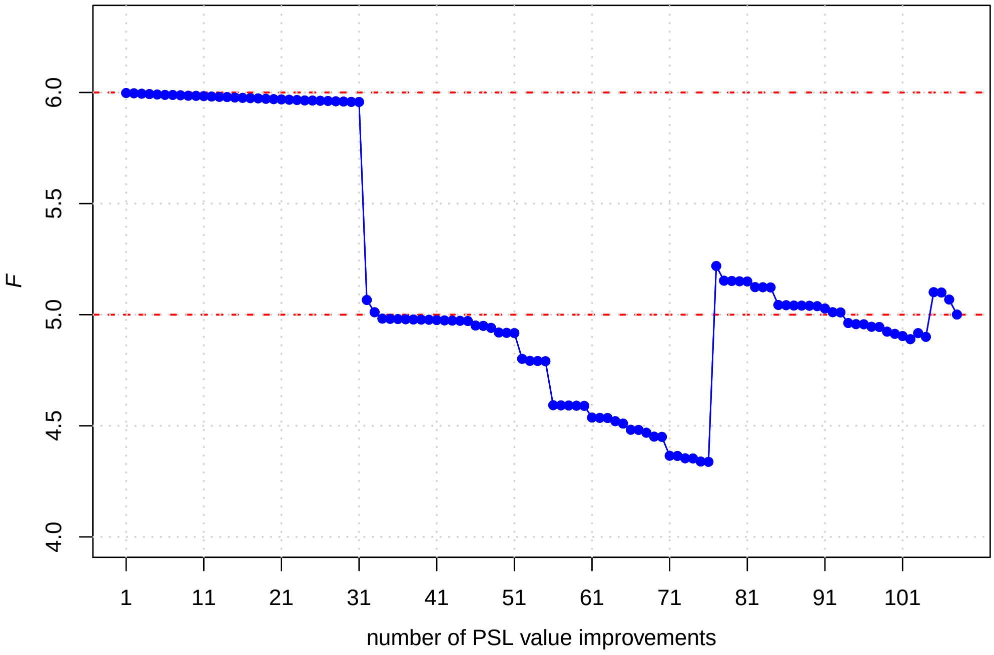
Figure 3. Binary sequence of $n = 4095$ depicted as a square of 64×64. Red color indicates the differences between the rotated Legendre sequence and the optimized sequence with $\text{PSL} = 45$.

Let us briefly compare PSL results of optimized Legendre sequences with selected results from the literature. Mow et al. [32] reported $\text{PSL} = 61$ and $F = 3.4589$ for $n = 4096$, while Zhang et al. [37] obtained $\text{PSL} = 56$ and $F = 4.8197$. Dimitrov et al. [15] gave $\text{PSL} = 77$ for $n = 8191$. $\text{PSL} = 71$ and $F = 4.00397$ for $n = 8191$, and $\text{PSL} = 50$ and $F = 4.0049$ for $n = 4095$ are reported by Bošković and Brest [34]. Brest and Bošković [5] gave $\text{PSL} = 70$ and $\text{PSL} = 48$ for $n = 8191$ and 4095, respectively. In all five works just mentioned, the algorithms were run from a random binary sequence, and from the comparison of their results, we can see slightly higher PSLs (i.e., worse) than those shown in Table 3. Also, the merit factors are usually lower than 5.0.

Figures 1, 2, and 3 illustrate the differences between the initial Legendre sequence (B_{init}^{Leg}) and the resulting optimized sequence (B_{opt}^{Leg}) for length 1023, 2047, and 4095, respectively. Each binary sequence is represented as a square, the sequence starts in the first row in the upper left corner, followed by the second row, etc., the sequence ends in the last row in the lower-right corner. The elements that represent the difference between the initial and resulting sequences are marked in red. For example, there are 40 flipped components, i.e., differences, between the compared sequences (marked with red color) in Figure 3, which is $40/4095 = 0.009768 \approx 1\%$.



(**a**)



(**b**)

Figure 4. (**a**) Difference d and (**b**) merit factor F as a function of the number of PSL value improvements (x-axes) for the sequence of length 131071 ($2^{17} - 1$).

Figure 4 shows how the difference d and merit factor F changed versus the number of PSL value improvements during the optimization process on the sequence of length 131071 ($2^{17} - 1$). The merit

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factor initially starts at 6.0, then decreases to values below 4.4, followed by a more pronounced increase, eventually approaching approximately 5.0 toward the end. One can also observe the increasing and decreasing trends in the number of flipped elements d from the initial sequence. At the end, $d = 1394$ and $d/n = 0.0106 \approx 1\%$.

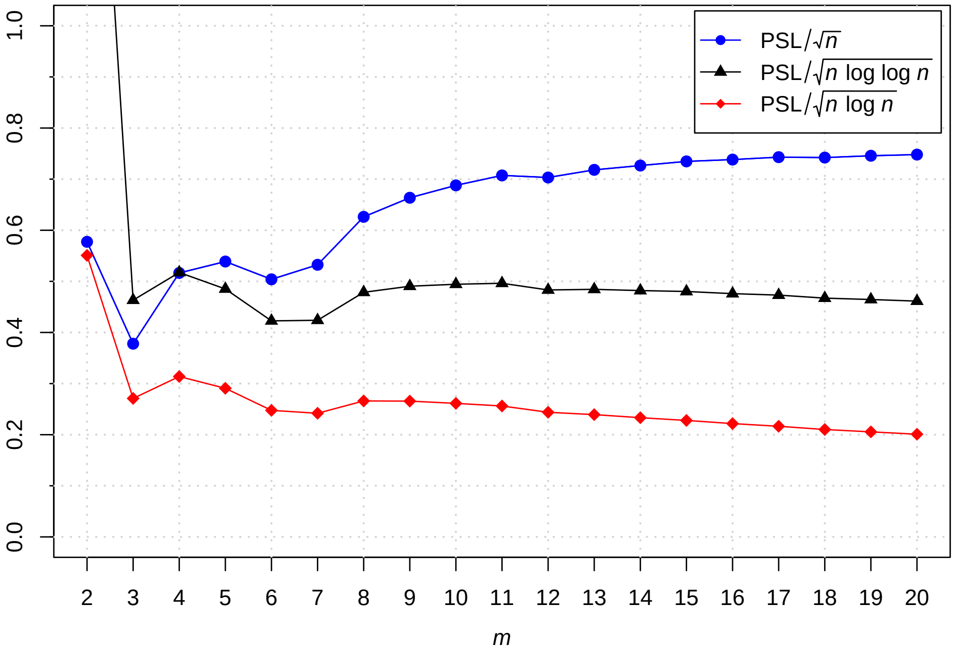

Figure 5. Comparison the growth rate of: $\mathrm{PSL}/\sqrt{n\log n}$, $\mathrm{PSL}/\sqrt{n\log\log n}$, and $\mathrm{PSL}/\sqrt{n}$.

Figure 5 shows the values of $\mathrm{PSL}/\sqrt{n\log n}$, $\mathrm{PSL}/\sqrt{n\log\log n}$, and $\mathrm{PSL}/\sqrt{n}$, respectively. The values of $\mathrm{PSL}/\sqrt{n\log n}$ and $\mathrm{PSL}/\sqrt{n\log\log n}$ show a non-increasing trend for $m \geq 10$, while $\mathrm{PSL}/\sqrt{n}$ shows an increasing trend for $m \geq 10$. This suggests, based on experimental investigation up to $2^{20} - 1$, that currently best-known PSL appears to grow like $O(\sqrt{n\log\log n})$.

5.3. Two Conjectures

Based on the experimental results in this section, we provide the following two conjectures.

Conjecture 4. Let B_{opt}^{Leg} be an optimized Legendre sequence, then $F(B_{opt}^{Leg}) \approx 5.0$.

Conjecture 5. $d(B_{init}^{Leg}, B_{opt}^{Leg}) \approx 0.01n$.

Conjecture 4 is conjectured based on the results of optimized binary sequences (B_{opt}^{Leg}) of length $2^{20} - 1$, where the merit factors are approximately 5.0.

The asymptotic merit factor of the Legendre sequence B_{init}^{Leg} appears to be close to a whole number, although we do not yet have a good explanation as to why. Based on experimental results, a similar observation can be seen for the merit factor of the optimized Legendre sequence B_{opt}^{Leg}, i.e., it is also close to a whole number. Note that the difference between the two merit factors is 1.

Conjecture 5 shows that

$$\frac{d(B_{init}^{Leg}, B_{opt}^{Leg})}{n} \approx 0.01,$$

i.e., ratio d/n appears to be close to a constant value 0.01.

6. Discussion

In the discussion, we highlight three notes:

1. An optimization algorithm can take into account Conjecture 5 as heuristic knowledge. Let (integer) value $q = 0.01n$. Then the number of possible choices of q elements within a binary sequence of length n is equal to $\binom{n}{q}$. Suppose $n = 1023$, then the number of possible choices is approximately 3.3×10^{23}. It becomes a huge value when n is increased.
2. Since the search space for finding (optimal) binary sequences is huge, the obtained PSLs in Table 3 are not necessarily optimal. Binary sequences with even better PSLs may be found in the future, but this will likely require considerable computational effort.
3. Conjectures 2 and 3 proposed by Dmitriev and Jedwab [13] show the excellent grow rate of m-sequences. These sequences may also be good candidates to be used as seed/initial sequences in our heuristic algorithm. This can be a great challenge for further work.

7. Conclusions

In this paper, the experimental work includes the minimization of the peak sidelobe of binary sequences. Two families of binary sequences were considered, namely Rudin-Shapiro and Legendre sequences. The main contributions are as follows: Based on numerical experimentation with Legendre sequences of length up to $2^{20} - 1$ two conjectures associated with Legendre sequences are proposed: (1) The obtained resulting binary sequences with the best-known peak sidelobe levels have merit factor ≈ 5.0, and (2) The number of elements that differ between the resulting binary sequences and the initial Legendre sequences follows a linear dependence on the sequence length (n), namely $\approx 0.01n$. Three new best-known PSL values were obtained for binary sequences of lengths $n = 2^m - 1$ for $m = 18, 19$, and 20.

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