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Probabilistic Modal Logic for Quantum Dynamics

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Abstract: Traditional quantum mechanics provides predictive accuracy but lacks a clear framework for articulating the epistemic status of quantum systems, particularly during measurement. We present Probabilistic Modal Logic for Quantum Dynamics (PML-QD), a formal system that integrates modal logic constraints with probabilistic semantics. Built on the classical modal system K, PML-QD introduces a probabilistic operator that allows reasoning about the likelihood of modal propositions, capturing the transition from possibility before measurement and necessity after measurement. PML-QD supports formal derivations of quantum phenomena like superposition, measurement-induced wavefunction collapse, sequential observations with non-commuting observables, entangled state dependencies and counterfactual reasoning in delayed-choice scenarios. Unlike traditional quantum logics or topos-theoretic approaches, PML-QD preserves classical propositional logic and avoids metaphysical commitments, focusing instead on syntactic clarity and computational feasibility. Operationally, the framework supports experimental design by offering a logical structure for analysing setups involving conditional measurements such as entanglement swapping or quantum erasure. It also helps clarify how changes in experimental context can shape observable outcomes. These capabilities allow researchers to anticipate epistemic transitions, evaluate consistency conditions and refine protocols prior to implementation. As such, PML-QD may serve not only as a conceptual tool for guiding experimental strategy but also as a methodological framework for automated reasoning systems in quantum experiment validation. Overall, PML-QD provides a rigorous means of tracking the epistemic status of quantum systems across pre- and post-measurement states, allowing for precise reasoning about which propositions were possible, probable or necessary at each stage of a quantum process.

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1. Introduction: Logic and Uncertainty in Quantum Theory

The formalism of quantum mechanics, grounded in Hilbert space theory and operator algebra, has achieved extraordinary success in predicting and modeling physical phenomena (Cassinelli and Lahti, 2016; Roy 2023; Svozil, 2024). However, its abstract mathematical structure obscures the conceptual interpretation of key processes such as superposition, entanglement and wavefunction collapse. While probabilistic outcomes are captured by the Born rule and physical dynamics are governed by the Schrödinger equation, the theory provides limited tools for articulating the epistemic and logical structure of quantum transitions (Wieser 2016; Tzemos and Contopoulos, 2021). This has led to persistent foundational debates and ambiguities, particularly around measurement, the role of the observer and the interpretation of quantum states (Sokolovski 2020). Quantum logics, including orthomodular lattices and topos-theoretic approaches, attempt to address these gaps but eschew probabilistic reasoning or lack the power to capture temporal and contextual dependencies in experiments (Gunji et al., 2017; Jorge and Holik, 2020; Gunji and Nakamura, 2022). Likewise, probabilistic and epistemic logics developed in computer science and philosophy provide rich formalisms for uncertainty and belief but are rarely adapted to quantum phenomena (Dalla Chiara et al., 2018; Betz and Richardson, 2023). A conceptual framework that jointly captures the modal and probabilistic dimensions of quantum events, especially in dynamic, measurement-driven contexts, is still underdeveloped.

We introduce a logical system—Probabilistic Modal Logic for Quantum Dynamics (PML-QD)—that unifies modal logic with probabilistic semantics to model quantum events, particularly the transition from indeterminate to determinate states. This system extends standard modal logic with a probability operator defined over accessible worlds, thereby allowing us to represent not only whether a proposition is possible or necessary, but also with what probability it holds. Measurement, in this framework, is formalized as a shift from possibility to necessity, governed by both probabilistic structure and modal constraints. Entangled and sequential measurements are treated via correlated modalities and conditional update rules. We anticipate that this approach will provide a rigorous framework for tracking the epistemic status of quantum systems before and after measurement, enabling precise reasoning about which propositions were possible, probable or necessary at each stage of the quantum process.

Concerning the Formal structure of PML-QD, we adopt the classical modal logic system K, which we extend to incorporate probabilistic semantics (Singleton and Booth, 2023; Quelhas et al., 2024; Litland 2025). Let L denote a standard modal propositional language constructed from a set of propositional variables P, the Boolean connectives ¬ (negation), ∧ (conjunction) and the modal operators □ (necessity) and ◊ (possibility). The semantics are defined using a Kripke frame $F = (W, R)$, where W is a nonempty set of possible worlds and $R \subseteq W \times W$ is an accessibility relation. A Kripke model $M = (W, R, V)$ assigns truth values to each atomic proposition $p \in P$ via a valuation function $V: P \to P(W)$, where $P(W)$ is the powerset of W (Weiss and Birman, 2024). Satisfaction is defined recursively: for any $w \in W, M, w \vDash$p iff $w \in V(p)$; $M, w \vDash \neg\phi$ iff $M, w \nvDash \phi$ *and* $M, w \vDash \phi \wedge \psi$ iff $M, w \vDash \phi$ and $M, w \vDash \psi$; and $M, w \vDash \Box\phi$ iff for all $v \in W$, if wRv, then $M, v \vDash \phi$. The dual operator ◊ϕ is defined as $\neg\Box\neg\phi$. This base system allows us to express propositional necessity and possibility, which we are going to enrich with a quantitative probabilistic extension tailored to quantum systems.

We now extend the Kripke model by introducing a probability measure over the set of possible worlds. A probabilistic Kripke model is defined as $M = (W, R, V, \mu)$, where $\mu: P(W) \to [0,1]$ is a finitely additive probability measure such that $\mu(W) = 1$ and $\mu(\emptyset) = 0$ (Shirazi and Amir, 2007). For each world $w \in W$, we define a conditional probability distribution μw over the accessible worlds $R(w) = \{v \in W \mid wRv\}$. The semantics of the probabilistic modal operator are then defined by introducing a function $P(\phi)$ yielding the probability that ϕ is true in the accessible worlds: formally, $M, w \vDash P(\phi) = p$ iff $\mu w(\{v \in R(w) \mid M, v \vDash \phi\}) = p$. This enables the assessment not only of whether a proposition is necessary or possible, but also of the probability with which it is possibly true. To ensure internal consistency, we assume that μw is defined via restriction and normalization: $\mu w(A) = \mu(A \cap R(w))/\mu(R(w))$ whenever $\mu(R(w)) > 0$. This logic is able to represent graded modal claims such as "it is 70% possible that φ" or more formally, $P(\Diamond\, \phi) = 0.7$, establishing a formal mechanism for assigning probabilities to modal propositions.

The logical language and syntax of PML-QD are built from a base set of propositional variables P, closed under the classical connectives and modal operators □, ◊ and the probability operator P. The syntax includes expressions of the form $P(\phi) = r$, where $r \in [0,1] \cap Q$ and composite formulas such as $P(\Diamond\, \phi) \geq s$ and $P(\Box\phi \to \psi) < t$. Formulas are interpreted over the probabilistic Kripke models defined above. The logic also allows us to define conditional probabilities. For propositions ϕ and ψ, we define $P(\phi \mid \psi) = \mu w(\{v \in R(w) \mid M, v \vDash \phi \wedge \psi\})/\mu w(\{v \in R(w) \mid M, v \vDash \psi\})$, assuming the denominator is nonzero. This allows the formulation of statements like "given ψ, φ is probable with 0.6 likelihood", which corresponds to $P(\phi \mid \psi) = 0.6$.

Overall, this approach yields the formal language of the system—comprising syntax, probability statements, and conditional expressions—necessary for modeling quantum epistemic transitions. Building on this foundation, the next steps involve applying this language to concrete quantum scenarios, analyzing how epistemic statuses evolve across different stages of measurement and inference.

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2. Axiomatic System

The axiomatic core of PML-QD comprises a structured integration of classical propositional logic, normal modal logic (system K) and finite probabilistic logic. The system operates within a Hilbert-style framework with axioms for classical logic, modal logic (K-system) and probability logic, including finite additivity, non-negativity, normalization and probabilistic modus ponens. Well-formed formulas (wffs) are derived from a fixed set of axioms using explicitly defined inference rules. The classical propositional component of the system is based on the axioms of tautology schemas, such as $\phi \to (\psi \to \phi)$, $(\phi \to (\psi \to \chi)) \to ((\phi \to \psi) \to (\phi \to \chi))$ and $\neg\neg\phi \to \phi$. The modal part includes the standard K axiom: $\Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$ and the necessitation rule: if $\vdash \phi$, then $\vdash \Box\phi$. The logic is normal in that it preserves closure under necessitation and respects distribution over implication. The axioms are interpreted over Kripke frames with arbitrary accessibility relations, allowing for flexible modeling of different quantum experimental contexts.

At first, probabilistic axioms are introduced to extend the classical-modal foundation. The core probabilistic principles are drawn from Kolmogorov's axioms, reformulated for integration into logical syntax (Svozil 2022). Let $P(\phi) = r$ be a primitive formula expressing that the probability of ϕ being satisfied in the accessible worlds is $r \in [0,1] \cap Q$. The axioms include non-negativity $P(\phi) \geq 0$, normalization $P(\top) = 1$ and finite additivity: if $\phi \wedge \psi \equiv \bot$, then $P(\phi \vee \psi) = P(\phi) + P(\psi)$. These are supplemented by conditional probability rules: for $\phi, \psi \in L$ with $P(\psi) > 0 P(\psi) > 0, P(\phi \mid \psi) = P(\phi \wedge \psi)/P(\psi)$. The modal-probabilistic interaction is regulated by the schema $P(\Box\phi) \leq P(\phi)$ and similarly $P(\phi) \leq P(\Diamond \phi)$, reflecting that what is necessarily true must also be true and what is true must be possibly true. These constraints are justified by the semantics of modal probability spaces, where $\Box\phi$ implies truth in all accessible worlds, while $\Diamond \phi$ requires only one accessible world where ϕ holds. The derivation rules include probabilistic modus ponens: from $\phi \to \psi$ and $P(\phi) \geq$r, infer $P(\psi) \geq r$, provided all evaluations are over the same accessibility neighborhood. These axioms permit the formal manipulation of graded propositions about truth values in modal contexts.

Therefore, with the K system as a stable scaffolding, the logic acquires the modal structure necessary to represent transitions from epistemic possibility to necessity, forming a base from which the probabilistic dynamics of quantum measurement can be encoded.

Implementation of quantum-specific axioms. To specialize PML-QD to quantum dynamics, we introduce a set of axioms reflecting superposition, measurement and entanglement. Let ϕi denote the proposition "the quantum system is in eigenstate i." For a system in state $\mid \psi\rangle = \sum i\alpha i \mid i\rangle$, we define $P(\Diamond \phi i \mid \alpha i \mid^2$.. This probabilistic modal formulation expresses the pre-measurement epistemic state: the system may be found in state ϕi with probability $\mid \alpha i \mid^2$. Upon measurement and collapse to ϕj, we enforce ϕj and $\Box\neg \Diamond \phi k$ for all $k \neq j$. We also represent entanglement through joint modal constraints. Given an entangled state $\mid \Psi\rangle = \sum i\alpha i \mid ai\rangle \otimes\mid bi\rangle$, let ϕi and ψi be propositions for subsystems A and B, respectively. Then $\Diamond (\phi i \wedge \psi i)$ is true with probability $\mid \alpha i \mid^2$, while $\Diamond \phi i \wedge\Diamond \psi j$ is only allowed if $i = j$ thereby disallowing separable joint modalities. These rules enable the logic to model quantum correlations and non-locality without assuming hidden variables.

Let $\mid \psi\rangle = \sum i\alpha i \mid i\rangle$ be a normalized pure state over an orthonormal basis $\{\mid i\rangle\}$ and let ϕi be the proposition "the system is in state $\mid i\rangle$". We postulate that $P(\Diamond \phi i) =\mid \alpha i \mid^2$, aligning logical possibility with the Born rule. This axiom expresses that, prior to measurement, the system may be found in eigenstate i, with probability equal to the squared modulus of its amplitude. Upon a measurement yielding outcome j, we assert $\Box\phi j$ and $\Box\neg \Diamond \phi k$ for all $k \neq j$, modeling the epistemic update resulting from collapse. This transition encodes the projection postulate as a logical update, i.e., a modal reduction from possibility to necessity and from multiple probabilities to a single certainty. To formalize this shift, we introduce the axiom schema $Measure(\phi j) \to (\Box\phi j \wedge \bigwedge k = j\Box\neg \Diamond \phi k)$. The measurement operator is treated syntactically, marking the transition point in the evaluation of epistemic states. This schema is only applied in cases where the logic designates $P(\Diamond \phi j) > 0$, reflecting the assumption that outcomes with zero probability cannot be observed.

To handle entanglement, we extend the logic with modal rules for joint propositions. Consider a bipartite quantum system with basis $\{\mid ai\rangle \otimes\mid bj\rangle\}$ and an entangled state $\mid \Psi\rangle = \sum i\alpha i \mid ai\rangle \otimes\mid bi\rangle$.

Let ϕi denote "Subsystem A is in state $|ai\rangle$" and ψi denote "Subsystem B is in state $|bi\rangle$." We introduce the entanglement axiom $P(\lozenge(\phi i \wedge \psi i)) = |\alpha i|2$ and the exclusion principle $\neg(\lozenge \phi i \wedge \lozenge \psi j)$ for $i \neq j$. This enforces the non-separability of joint state truth values: subsystems cannot simultaneously assume inconsistent states if derived from an entangled superposition. The logic allows us to evaluate propositions such as $\lozenge \phi i \rightarrow \square\neg\psi j$ for $j \neq$, capturing the non-local correlations implicit in entanglement. These formulations provide a means of expressing Bell-type dependencies as logical constraints within the system, grounded in amplitude-based probability assignments. For measurements, we define the conditional update: observing ϕk implies $\square\psi k$ and $\square\neg\psi j$ for all $j \neq k$ consistent with perfect quantum correlations. These constraints are encoded as derivable rules rather than primitive axioms, preserving flexibility in modeling partial or imperfect entanglement.

Overall, this approach embeds the measurement postulates of quantum mechanics within a modal logical framework, effectively translating quantum dynamics into a structure that supports logical inference and modal reasoning about probability. By doing so, it bridges the gap between quantum formalism and epistemic logic, enabling a systematic analysis of how knowledge and uncertainty evolve through quantum processes. This integration allows for the tracking of necessity, possibility, and likelihood in a manner that aligns with both the probabilistic nature of quantum theory and the inferential tools of modal logic.

Tools and Computational Setup. All logical definitions, axioms and inference rules are formalized using a typed symbolic language implemented in the Lean proof assistant to verify syntactic coherence and logical validity (Löh 2022). For probability measures and accessibility relations, we use custom-built Kripke structures defined programmatically in Python using the networkx library for graph modeling and numpy for probability assignment. Quantum state vectors and projection operations are handled using the qutip library, enabling precise computation of Born-rule probabilities and their mapping to logical probability assignments. Logical expressions are parsed and evaluated using a domain-specific parser that constructs abstract syntax trees, evaluates modal depth and resolves probabilistic truth values based on current world state and transition graphs.

3. Coherence, Consistency and Semantic Soundness of PML-QD

This chapter examines the coherence, consistency and semantic soundness of PML-QD to ensure that its logical structure faithfully represents quantum dynamics and supports reliable epistemic inference. We begin by demonstrating the soundness of PML-QD relative to its probabilistic Kripke semantics. Let $M = (W, R, V, \mu)$ be a model of our system, where R is an arbitrary (possibly non-symmetric, non-transitive) relation to accommodate varied quantum contexts. For each world $w \in W$ and each well-formed formula $\phi \in L$, if $\vdash \phi$ in the PML-QD system, then $M, w \vDash \phi$. Proof proceeds by induction over the structure of derivations. Base cases follow from the validity of classical tautologies. The K modal axiom is validated by the relational condition: $wRv \wedge wRu$ implies that if $M, v \vDash \phi \rightarrow \psi$ and $M, v \vDash \phi$, then $M, v \vDash \psi$. The probabilistic axioms are sound under the standard interpretation of μw as a conditional probability measure on $R(w)$. Additivity, normalization and conditional independence are preserved by construction. Quantum-specific axioms are modeled by assigning $\mu w(\lozenge \phi i) = |\alpha i|2$ for pre-collapse states and using syntactic restrictions post-collapse to enforce that $\square\phi j \rightarrow \mu w(\phi j) = 1$. Thus, the epistemic updates are correctly aligned with quantum measurement rules. This confirms that the axiomatic structure of PML-QD is consistent with its intended semantics, establishing soundness as a necessary condition for subsequent formal evaluation.

Completeness is demonstrated via canonical model construction. We define a canonical model $Mc = (Wc, Rc, Vc, \mu c)$, where Wc is the set of maximally consistent sets (MCSs) of formulas in $LPML - QD$. The accessibility relation c is defined by: $\Gamma Rc \Delta$ iff for every $\square\phi \in \Gamma$, $\phi \in \Delta$. The valuation function $Vc(p) = \{\Gamma \in Wc \mid p \in \Gamma\}$ and the probability function $\mu\Gamma c(\phi) = r$ is defined via the maximal consistent extensions satisfying $P(\phi) = r \in \Gamma$. Completeness then follows: if ϕ is valid (true in every model), then $\phi \in \Gamma$ for all $\Gamma \in Wc$. If $\phi \in/\Gamma$, then there exists a model falsifying ϕ,

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proving that ϕ\phiϕ is not derivable. For quantum-specific modalities, we enrich the canonical model with amplitude maps from abstract syntax to normalized vectors, ensuring that measurement-induced updates correspond to transitions among canonical worlds. The probabilistic collapse axioms are enforced syntactically by excluding extensions that contradict the post-measurement certainty schema.

We thus obtain completeness for the classical, modal and probabilistic layers, with quantum-specific axioms ensured through semantic alignment.

Epistemic closure and logical coherence. An important consideration in any epistemic logic is whether the system permits epistemic closure under valid inference. In PML-QD, closure under modal and probabilistic inference is carefully maintained through explicit syntactic rules. For instance, from $\Box(\phi \rightarrow \psi)$ and $\Box\phi$, one may derive $\Box\psi$, preserving modal consequence. However, in quantum contexts, closure must also handle epistemic transitions: if $\Diamond\, \phi$ and $(\Diamond\, \phi) = 1$, it does not follow that $\Box\phi$, unless a measurement event enforces it. Thus, PML-QD avoids epistemic overreach by enforcing the distinction between high probability and logical certainty. Probabilistic closure is similarly bounded: from $P(\phi \wedge \psi) = 0.9$ and P$P(\psi) = 1$, we may derive $P(\phi \mid \psi) = 0.9$, but not $\Box\phi$ or even $\Diamond\, \phi$ unless additional modal premises are supplied. This disciplined separation of inference domains ensures that conclusions about quantum states are validly drawn only within the scope of their modal and probabilistic constraints. Furthermore, any logical contradiction arising from measurement updates (e.g., asserting $\Box\phi$ and $\Diamond\, \neg\phi$) is syntactically blocked by the collapse axioms.

Counterfactual reasoning and epistemic modality. PML-QD allows for formal engagement with counterfactuals in quantum mechanics, a topic often debated in the context of delayed-choice experiments, weak measurements and quantum nonlocality (Laudisa 2019). The framework supports statements of the form: "If measurement M had been performed, then ϕ would have become necessary," represented formally as $\Diamond\, M \rightarrow (Measure(\phi) \rightarrow \Box\phi)$. Such statements are not metaphysical speculations but logical conditionals grounded in modal accessibility. Suppose a system is in a superposed state $\mid \psi\rangle = \alpha \mid 0\rangle + \beta \mid 1\rangle$ and the measurement is postponed. The logic allows one to say: "Had we measured now, outcome $\phi 0$ would have become necessary with probability $\mid \alpha \mid 2$." This supports rigorous statements about epistemic potential without requiring ontological assertions about unmeasured reality. Importantly, these counterfactuals do not imply retrocausality, but rather preserve the distinction between hypothetical and actualized knowledge states, aligning with quantum experimental setups where delayed configurations define outcome space. The evaluation of counterfactuals is governed by modal consistency: only those conditional claims holding in all accessible paths from a given precondition are allowed.

Model-theoretic consistency with Hilbert Space formalism and temporal evolution. The compatibility and the semantic consistency of PML-QD with the standard Hilbert space formalism can be evaluated. Let H be a finite-dimensional Hilbert space with orthonormal basis $\{\mid i\rangle\}$ and let $\mid \psi\rangle = \sum i\alpha i \mid i\rangle$ be a quantum state. The propositions ϕi correspond to projection operators $P\hat{\ }i = \mid i\rangle\langle i \mid$ and the probability of observing state i is $\langle\psi \mid P\hat{\ }i \mid \psi\rangle = \mid \alpha i \mid 2$. We define a mapping $f: \phi i \mapsto P\hat{\ }i$ and assert that $P(\Diamond\, \phi i) = Tr(\rho P\hat{\ }i)$ where $\rho = \mid \psi\rangle\langle\psi \mid$. The logical model corresponds to a coarse-grained representation of the probabilistic projections across a set of accessible configurations indexed by basis measurements. Post-measurement updates $\Box\phi j$ correspond to Lüders projections: $\rho \mapsto P\hat{\ }j\rho P\hat{\ }j/Tr(\rho P\hat{\ }j)$. For entangled states $\mid \Phi\rangle = \sum i\alpha i \mid ai\rangle \otimes\mid bi\rangle$, joint modal formulas $\Diamond\, (\phi i \wedge \psi i)$ are semantically validated by $Tr(\rho(P\hat{\ }i \otimes Q\hat{\ }i)) = \mid \alpha i \mid 2$. This means that logical constructs are supported by standard operator theory. In this context, the modal structure represents pre-measurement epistemic range, while probability reflects amplitude-squared outcomes.

To evaluate the temporal behavior and dynamical evolution of PML-QD, we consider its capacity to represent quantum evolution between measurement events. Let $U(t) = e - iHt$ be the unitary evolution operator associated with a Hamiltonian H and $\mid \psi(t)\rangle = U(t) \mid \psi(0)\rangle$. In logical terms, we represent a temporal sequence $w0 \rightarrow w1 \rightarrow \cdots \rightarrow wn$, where each wi is a world indexed by the system's state at time ti. If ϕi is the proposition "system is in state $\mid \psi(ti)\rangle$", then temporal modal transitions $\Diamond\, \phi i + 1$ are defined by the Schrödinger evolution. The logic supports this via time-indexed accessibility: $wiRwi$ +1 iff $|\psi(ti+1)\rangle = U(ti+1-ti)|\psi(ti)\rangle$. We extend the model by assigning probability functions $\mu wi(\phi j) = |\langle\phi j|\psi(ti)\rangle|^2$, yielding a dynamic probability assignment compatible with unitary evolution. If a measurement occurs at tj, the update collapses the epistemic structure: all future-accessible paths inconsistent with the observed outcome are pruned from $R(wj)$.

In summary, we establish the dynamic consistency of PML-QD by demonstrating that modal transitions respect unitary evolution up to the point of measurement and that state updates conform to the projection postulate. This validation reinforces the semantic soundness of the framework, confirming its capacity to model quantum epistemic dynamics with logical and physical fidelity.

4. From Theory to Practice: Logical Modeling of Quantum Systems in PML-QD

This chapter presents representative examples illustrating how PML-QD models quantum phenomena through logical structures and modal reasoning.

Derivation of single-measurement collapse. We begin with a formal derivation illustrating how PML-QD syntactically captures the transition from a probabilistic possibility to post-measurement necessity in the single-qubit case. Let $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ be a superposed state, with $|\alpha|2+|\beta|2 = 1$. Let $\phi 0$ denote "the system is in state $|0\rangle$" and $\phi 1$ denote "the system is in state $|1\rangle$." In the logic, prior to measurement we assume $P(\Diamond\phi 0) = |\alpha|2$ and $P(\Diamond\phi 1) = |\beta|2$, encoded via axioms of amplitude-based modal probability. Suppose the system is measured and the outcome is $|0\rangle$. The axiom of modal collapse gives $Measure(\phi 0) \rightarrow \Box\phi 0 \wedge \Box\neg\Diamond\phi 1$. If we assume $Measure(\phi 0)$, then using modus ponens, we derive $0\Box\phi 0$ and $\Box\neg\Diamond\phi 1$. Applying the modal logic equivalence $\Box\neg\Diamond\phi 1 \equiv \neg\Diamond\phi 1$, we deduce that $\phi 1$ is no longer even possible. The derivation shows that the probabilistic possibility $P(\Diamond\phi 1) = |\beta|2$ is syntactically replaced by $\neg\Diamond\phi 1$ and all future logical inferences involving $\phi 1$ become false under modal necessity.

Sequential measurements. Next, we construct a proof sequence for a scenario involving two successive measurements along non-commuting observables. Let the system initially be in state $|\psi\rangle = 21(|0\rangle+|1\rangle)$ and define $\phi 0, \phi 1$ as before. Let $\chi+$ and $\chi-$ denote the propositions corresponding to the diagonal basis $|-\rangle = 21(|0\rangle-|1\rangle)$. Suppose the first measurement is in the $\{|0\rangle,|1\rangle\}$ basis and the outcome is $|0\rangle$. By modal collapse, we derive $\Box\phi 0$ and $\Box\neg\Diamond\phi 1$. Now we consider $P(\Diamond\chi+)$, i.e., the probability of subsequently observing $|+\rangle$. Since the post-measurement state is $|0\rangle$, we compute $P(\Diamond\chi+) = |\langle+|0\rangle|2 = 1/2$. We syntactically derive $\Box\phi 0 \rightarrow P(\Diamond\chi+) = 1/2$ from the substitution of amplitude-based definitions into the probability axioms. Upon performing the second measurement and observing $|+\rangle$, we apply the measurement collapse rule again to derive $\Box\chi+$ and $\Box\neg\Diamond\chi-$. These results can be then used to show that all further inferences about $\phi 1$ or $\chi-$ must fail in all accessible worlds.

We now consider the effect of sequential measurements when the observables do not commute. Let the initial state again be $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ and define a second measurement basis, such as the $\{|+\rangle,|-\rangle\}$ basis, where $|+\rangle = 21(|0\rangle+|1\rangle)$ and $|-\rangle = 21(|0\rangle-|1\rangle)$. Let $\chi+$ and $\chi-$ represent the corresponding modal propositions. If the system is first measured in the computational basis and yields $|0\rangle$, then we assert $\Box\phi 0$ and immediately invalidate $\Diamond\phi 1$. We may now inquire about $P(\Diamond\chi+)$, that is, the possibility of subsequently observing $|+\rangle$. In the post-collapse model, the system state become $|0\rangle$, so we calculate $P(\Diamond\chi+) = |\langle+|0\rangle|2 = 1/2$. Formally, this is encoded as $\Box\phi 0 \rightarrow P(\Diamond\chi+) = 1/2$, reflecting the non-commutativity of the measurement sequence. The logic tracks not only the collapsed state but its implications for future probabilistic possibilities. If we now measure in the diagonal basis and obtain outcome $|+\rangle$, the epistemic update becomes $\Box\chi+ \wedge \Box\neg\Diamond\chi-$, which supersedes the prior assignment. PML-QD may thus support reasoning about nested and sequential updated updates and the logical impact of measurement order.

Single-Qubit superposition and measurement. We begin with a basic scenario: a single qubit in a superposition of eigenstates relative to a particular observable, such as spin along the z-axis. Let the quantum state be $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, where $\alpha, \beta \in C$ and $|\alpha|2+|\beta|2 = 1$. We define the modal propositions $\phi 0$ and $\phi 1$ corresponding respectively to the system being in state $|0\rangle$ and $|$

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1⟩. In the PML-QD framework, the epistemic state prior to measurement is captured by the formulas $P(\lozenge\ 2P(\lozenge\ \phi 1) =\mid \beta \mid 2$. These assertions express the degree of possibility—based on the probabilistic modal semantics—associated with each eigenstate. A measurement in the $\{\mid 0\rangle, \mid 1\rangle\}$ basis is treated as a transition: if outcome $\mid 0\rangle$ is observed, the updated state becomes $\Box\phi 0 \wedge \Box\neg\lozenge\phi 1$, formalizing collapse in logical terms. The inference chain follows directly from the axiom $Measure(\phi j) \rightarrow (\Box\phi j \wedge \wedge k\ = j\Box\neg\lozenge\phi k)$. The model also permits conditional queries such as $P(\lozenge\ \phi 0 \mid \neg\phi 1) = 1$, maintaining coherence with the binary outcome structure of projective measurement. This means that PML-QD may accommodate elementary state transitions and probability-based reasoning using modal assertions grounded in amplitude-based truth.

Collapse simulation and world pruning algorithms. The logic requires that upon measurement possible but unrealized outcomes are no longer epistemically accessible. To simulate this, a world-pruning algorithm could be implemented that dynamically restructures the Kripke model. Upon observing outcome ϕj at world w, this algorithm first may verify $P(\lozenge\ \phi j) > 0$; then, identify the unique subset $R'(w) \subseteq R(w)$ such that $\forall v \in R'(w), M, v \vDash \phi j$. The accessibility relation can be updated by setting $R(w) := R'(w)$ and the probability distribution μw renormalized over $R'(w)$. All ϕk such that $k \neq j$ may be set to evaluate as $\neg\lozenge\phi$k and $\Box\neg\phi k$. The update is conservative and preserves modal truth: previously necessary propositions remain necessary unless invalidated by the measurement result. In sequential measurements, the system may track update sequences using a stack of Kripke structures, allowing rollback and re-evaluation. A collapse consistency check may ensure that $\sum kP(\lozenge\ \phi k) = 1$ prior to collapse and that after measurement such that exactly one ϕj becomes necessary while the others are impossible. In entangled cases, a measurement on one subsystem may automatically trigger pruning on correlated worlds in the partner system, enforcing non-local modal synchrony.

Entanglement constraints via modal dependencies. Let us consider the entangled state $\mid \Phi\rangle = 21(\mid 0\rangle A \mid 1\rangle B - \mid 1\rangle A \mid 0\rangle B)$. Define $\phi 0$, $\phi 1$ for particle A and $\psi 0$, $\psi 1$ for particle B. The axiom of correlated possibility gives $P(\lozenge\ (\phi 0 \wedge \psi 1)) = 1/2$, $P(\lozenge\ (\phi 1 \wedge \psi 0)) = 1/2$ and all other $P(\lozenge\ (\phi i \wedge \psi j)) = 0\ for\ i = ji\ = ji = j$. Suppose a measurement on A yields $\phi 1$. By measurement collapse, we derive $0\Box\phi 1 \wedge \Box\neg\lozenge\phi 0$. From the modal correlation rule $\phi 1 \rightarrow \Box\psi 0$, we deduce $\Box\psi 0$, i.e., the state of B becomes necessarily $\mid 0\rangle$. The derivation holds even if the measurement on B occurs later, as the logical dependencies are enforced modally rather than temporally. Suppose instead that we had observed $\phi 0$; the derivation would then yield $\psi 1$. Importantly, PML-QD prevents derivation of any statement $\lozenge\ (\phi 1 \wedge \psi 1)$, since this is ruled out by the amplitude-based probability axiom. Overall, the logical constraints imposed by PML-QD on entangled propositions may correctly capture the exclusivity and correlation properties inherent in entangled quantum states.

Entanglement Swapping. Entanglement swapping is a protocol in which two initially independent quantum systems become entangled through joint measurement on intermediary particles (Ning et al., 2019; Zangi et al., 2023). Consider two entangled pairs in the states $\mid \psi AB\rangle = 21(\mid 0\rangle A \mid 1\rangle B - \mid 1\rangle A \mid 0\rangle B)$ and $\mid \psi CD\rangle = 21(\mid 0\rangle C \mid 1\rangle D - \mid 1\rangle C \mid 0\rangle D)$. Let a Bell-state measurement be performed on particles B and C. Define modal propositions ϕij the four Bell states between B and C and ψAD for the corresponding entanglement state between A and D. Before the measurement, the logic encodes the system as $\bigvee i, jP(\lozenge\ \phi ij) = 41$, with no assignment of $\Box\psi AD$. Upon observing Bell state $\phi 00$, we assert $\Box\phi 00 \rightarrow \Box\psi AD00$, with $\psi AD00$ denoting a corresponding entangled state between particles A and D. This captures the epistemic update that makes entanglement between distant, non-interacting particles a logical necessity only *after* the intermediate measurement. This avoids invoking retrocausality by localizing modal transitions to the knowledge structure. Conditional probabilities are also updated: $P(\lozenge\ \psi AD00 \mid \phi 00) = 1$, while other $\psi ADij$ are assigned zero. Therefore, PML-QD may capture entanglement swapping via conditional modal updates, supporting rigorous reasoning about delayed entanglement onset.

Delayed-choice interference and quantum eraser. We propose here a derivation involving the delayed-choice quantum eraser (Kim et al., 2000). Define ϕp: "which-path information is accessible," and ϕw: "interference pattern is observed." Before post-selection, we assume $\lozenge\ \phi p \wedge \lozenge\ \phi w$. Let Me

denote the erasure choice and Mp the path-preserving choice. Under the choice Me, the logic enforces $Me \rightarrow \Box\neg\phi p \wedge \Box\phi w$, whereas under Mp, it enforces $Mp \rightarrow \Box\phi p \wedge \Box\neg\phi w$. Assume Me is enacted after the signal photon is measured. Since the epistemic update is contingent, we represent the situation as a conditional modal formula: $\Diamond(\phi p \wedge \phi w) \rightarrow (Me \rightarrow \Box\phi w) \wedge (Mp \rightarrow \Box\phi p)$. Upon performing Me, the system updates to $\Box\phi w$ and $\Diamond\phi p$ is logically rejected. The derivation path shows that the modal state is not solely determined by photon interactions, but by post-measurement contextualization. This is syntactically grounded in modal update axioms, not as an empirical fact but as a derivable transition.

We now examine the delayed-choice quantum eraser, a paradigmatic scenario where information about a quantum system's path is either retained or erased after the system has been measured (Kim and Ham, 2023). Let ϕp denote "which-path information is known" and ϕw denote "interference pattern is visible." The logic must represent conditional dependencies where the post-measurement setup retroactively influences the interpretation of earlier events. We encode the availability of path information as a binary modal variable: if the information is preserved, we assign $\Box\phi p \rightarrow \Box\neg\phi w$ and if it is erased, $\Box\neg\phi p \rightarrow \Box\phi w$. The measurement decision variable Me (erase path info) or Mp (preserve path info) serves as a modal context switch: prior to this, both $\Diamond\phi p$ and $\Diamond\phi w$ are true. After choosing Me, the system updates to $\Box\neg\phi p \wedge \Box\phi w$. PML-QD allows for modal reasoning such as $\Diamond\phi p \wedge \Diamond\phi w \rightarrow (Me \rightarrow \Box\phi w)$, modeling epistemic changes contingent on an action taken after the quantum interaction. This illustrates that PML-QD can represent dynamically dependent modal transitions where final knowledge states depend on future experimental choices.

The examples presented in this chapter demonstrate the capacity of PML-QD to formally capture key aspects of quantum behavior, including measurement collapse, entanglement and contextual inference. These cases highlight the framework's utility in modeling epistemic transitions with logical precision, offering a promising methodology for further theoretical and practical developments.

5. Conclusion

The Probabilistic Modal Logic for Quantum Dynamics (PML-QD) introduced here provides a formal framework that integrates modal logic, probability theory and the epistemic structure of quantum measurement. Its aims to model the dynamic progression from probabilistic possibility to logical necessity that defines quantum behavior under measurement. Built on the classical modal system K, PML-QD introduces a probabilistic valuation mechanism and a compact set of domain-specific axioms governing superposition, collapse and entanglement. Modal propositions are assigned quantitative probability values. Upon measurement, collapse transitions convert possibility into necessity, enforcing expressions like $\Box\varphi$ to indicate that an outcome has become realized. A key non-classical feature of PML-QD is its treatment of epistemic non-monotonicity. In classical modal logic, implications such as $\diamond\varphi \rightarrow \diamond\Box\varphi$ may hold under specific frame conditions. However, this does not persist in PML-QD due to the collapse-induced pruning of modal paths: once a measurement occurs, the model transitions to a substructure where prior possibilities are no longer accessible. This modal reduction is syntactically governed by the axiom $Measure(\varphi j) \rightarrow \Box\varphi j \wedge k \quad = j \wedge \Box\neg\diamond\varphi k$ which ensures logical consistency while rejecting modal monotonicity. In the PML-QD framework, the necessity operator $\Box$ does not collapse into a truth predicate. While $\Box\varphi \rightarrow \varphi$ holds in the sense that a necessary proposition must be true in all accessible worlds, this does not imply absolute truth, but only truth relative to post-measurement substructure. This distinction reinforces the epistemic separation between pre-measurement probability and post-measurement necessity, marking a departure from both classical modal logic and traditional probabilistic logic. Unlike classical knowledge frameworks where a proposition is either known or unknown, PML-QD enables a graded epistemic treatment. It may accommodate statements of the form: possibility without expectation $P(\diamond\varphi) = 0$, expectation without necessity $0 < P(\diamond\varphi) < 1$ and epistemic finality $\Box\varphi$. This enables a more nuanced account of quantum epistemology. For example, the proposition "the particle is spin-up" need not be treated as simply known or unknown; instead, it can be represented as probable but not necessary prior to measurement, with its degree of belief quantified by quantum amplitudes rather than reduced to binary epistemic categories.

A distinctive feature of PML-QD is its logical minimalism, as it extends the basic modal system K only modestly by incorporating probabilistic operators and a small set of collapse-specific axioms. The system deliberately avoids polymodal formulations, higher-order quantification, intensional types, nonclassical connectives, distributed knowledge operators and dynamic logic constructs, focusing instead on a small set of precisely defined epistemic transitions . This design choice emphasizes deductive transparency over maximal expressive power. Each formula expresses a distinct logical claim that can be semantically validated within Kripke models equipped with probability functions. This keeps model-theoretic evaluation computationally feasible while retaining expressiveness sufficient to model quantum experimental structures.

An important advantage of PML-QD is its practical utility in the design and analysis of quantum experiments. By formally modelling the transition from probabilistic possibility to logical necessity, the framework provides a rigorous structure for anticipating and interpreting measurement outcomes. This is especially beneficial in settings involving entanglement swapping, delayed-choice quantum erasers and sequential measurements with non-commuting observables, i.e., scenarios where standard formalisms often obscure epistemic transitions. For example, in designing a delayed-choice interference experiment, PML-QD may provide a framework for anticipating how post-selection contexts modulate which-path information, thereby clarifying the underlying logical dependencies before experimental implementation. In entanglement swapping protocols, PML-QD may aid in tracking modal correlations to ensure consistency across nonlocal updates. Its ability to handle conditional reasoning and epistemic updates supports counterfactual assessments and helps validate whether a given experimental configuration logically aligns with quantum postulates. Overall, PML-QD offers both a conceptual and practical toolkit for optimizing experimental design in foundational quantum research and emerging quantum technologies.

The epistemic distinctions enabled by PML-QD allow a comparison with traditional interpretations of quantum mechanics. In the Copenhagen view, the wavefunction encodes predictive knowledge about measurement outcomes but lacks interpretive content about unmeasured states (Jaeger 2019). PML-QD refines this position by treating superposition as a landscape of modal possibilities rather than ontological ambiguity such that the system can be described as potentially occupying multiple states, each with a distinct logical status and probability. In relation to QBism, which treats quantum states as subjective degrees of belief (Khrennikov 2018; Milgrom 2022), PML-QD provides a formal syntactic structure to those beliefs, grounding them in rules of modal inference and derivability. While QBism invokes a probabilistic agent-centric perspective, PML-QD situates probability within a logical system constrained by axioms, thus offering a structured treatment of epistemic agency. In the many-worlds interpretation, every possible measurement outcome is realized in some branch of reality (Devor 2023; Vaidman 2025). PML-QD models branching through modal accessibility, but does not commit to metaphysical plurality; rather, it treats branching as epistemic openness. The modal relations are not anymore between physically instantiated worlds, but between logically accessible epistemic states.

ML-QD differs from traditional quantum logics, which often replace classical logic with alternative systems to reflect the structural or contextual aspects of quantum theory. For instance, the Birkhoff–von Neumann approach represents quantum propositions as elements of an orthomodular lattice of Hilbert space subspaces, replacing classical Boolean logic with a non-distributive structure that reflects observable incompatibility and quantum geometry (Gunji and Nakamura 2022). In contrast, PML-QD retains classical propositional logic and introduces modal and probabilistic layers to capture quantum uncertainty and measurement dynamics without abandoning logical distributivity. In turn, topos-theoretic quantum logic reformulates quantum theory using intuitionistic logic and category theory, modeling propositions as presheaves over classical contexts and assigning truth values locally without relying on global valuations or classical bivalence (Landsman 2017; Jia et al. 2025). By contrast, PML-QD operates within a bivalent logical system and incorporates probability directly, allowing for global epistemic evaluation and explicit modeling of measurement-induced transitions

PML-QD has certain limitations that warrant acknowledgment. It does not attempt to model open quantum systems or continuous variable states, nor does it incorporate decoherence processes at the level of environment-induced entropy changes. All quantum states are assumed to be finite-dimensional and all measurements are treated as idealized and projective. This simplification enables a syntactically clean and computationally tractable logic but restricts its direct application to more complex or realistic quantum systems. While the system models epistemic transitions triggered by measurements, it does not incorporate temporal indeterminacy or branching-time semantics; instead, it adopts a linear temporal structure via indexed worlds. The exclusion of branching structures limits the logic's ability to model future contingencies or path-dependent quantum evolution. PML-QD also omits any formal treatment of epistemic agents or belief operators. While interpretations such as QBism emphasize agent-centered probabilities and belief updates, PML-QD focuses exclusively on system-level propositions, avoiding subjective or doxastic modalities. Moreover, PML-QD remains neutral regarding the ontological status of the quantum state, avoiding commitment to whether the wavefunction represents physical reality or informational content. Within these limitations, PML-QD provides a coherent logical framework for analyzing the epistemic dynamics of measurement and uncertainty in finite, closed quantum systems. The framework serves not as a replacement but as a complementary language for articulating epistemic features of quantum processes. Within these limitations, PML-QD provides a coherent and complementary logical framework for articulating the epistemic dynamics of measurement and uncertainty in finite, closed quantum systems.

In conclusion, we address whether a unified formal system can capture the modal and probabilistic structure of quantum dynamics, particularly the transition from possibility to necessity induced by measurement. By integrating modal logic with probabilistic semantics, PML-QD offers a framework representing superposition, collapse and entanglement in syntactically precise and semantically consistent terms. The main takeaway is that quantum measurement, often seen as interpretationally opaque, can be rigorously modeled within a logical framework that clearly distinguishes graded possibility from logical necessity across both pre- and post-observation contexts.

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