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Posted Date: 23 March 2026

doi: 10.20944/preprints202603.0786.v2

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Article

A Simpson–type Decomposition of the Euler–Mascheroni Constant (γ) as a Sum of Two Irrational Numbers

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Abstract

An elementary and self-contained approach to the Euler–Mascheroni constant γ is presented, based solely on Simpson’s quadrature rule and the convexity of the function $f(x) = 1/x$. Using Simpson–type weighted harmonic sums, we approximate local logarithmic increments by simple finite linear combinations of reciprocal integers. Exploiting the monotonic and convex nature of $1/x$, sharp two-sided inequalities are established that relate these numerical approximations to the exact logarithmic increments. These inequalities imply that the accumulated quadrature errors form a convergent series, yielding a simple proof of the classical limit defining γ without recourse to the Euler–Maclaurin formula. A central structural observation of the paper is that γ admits a Simpson–type decomposition as a sum of two irrational numbers. More precisely, we show that $\gamma = (\log [2] + 1)/3 + \delta$, where both $(\log [2] + 1)/3$ and δ are irrational. The constant δ arises naturally as the limit of a rational sequence derived from a Simpson–type approximation, and its irrationality is established by an elementary rigidity argument based on prime divisibility in rational approximations.

Keywords: Euler–Mascheroni constant; irrational numbers; harmonic series; numerical quadrature

1. Introduction

The Euler–Mascheroni constant is a classical object of analysis and number theory [1]. Let

$$H(N) = \sum_{k=1}^N \frac{1}{k} \tag{1}$$

denote the N -th harmonic number. Then the Euler–Mascheroni constant can be written succinctly as

$$\gamma = \lim_{N \rightarrow \infty} (H(N) - \log [N]) \approx 0.57721 \dots \tag{2}$$

Figure 1 illustrates the difference between the integral of $1/x$ and its approximation by the rectangular rule. The existence of γ is usually established via asymptotic expansions derived from the Euler–Maclaurin summation formula or related analytic techniques [2]. While this limit is familiar, most standard proofs rely on the Euler–Maclaurin summation formula, and it is natural to ask whether γ can be understood using only elementary numerical ideas.

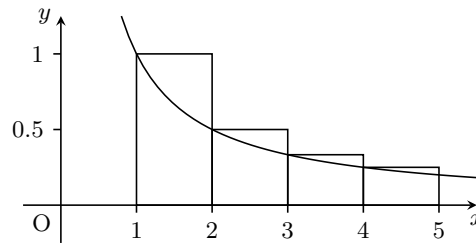


Figure 1. Approximation of the integral of $1/x$ by the left-endpoint rectangular rule. Each rectangle has unit width and height $1/k$, producing the harmonic sum $H(N)$. The difference between this sum and $\log [N]$ converges to the Euler–Mascheroni constant γ .

According to Havil [3], no proof of the irrationality of γ is currently known, and the problem remains open. In this paper, we establish a new structural observation: γ can be represented as the sum of two irrational constants, namely $(\log [2] + 1)/3$ and δ . The constant δ arises naturally as the limit of a rational sequence defined via a Simpson–type approximation.

2. Analytic Approximation of the Logarithmic Integral

2.1. Simpson–Type Approximation of Logarithmic Increments

Definition 1. For integers $n \geq 1$, define

$$f(n) := \int_{2n-1}^{2n+1} \frac{1}{x} dx = \log [2n+1] - \log [2n-1]. \quad (3)$$

This integral is approximated using Simpson’s rule with a step size of 1, as illustrated in Figure 1(b). Using the local Simpson weight

$$g(n) = \frac{1}{3} \left(\frac{1}{2n-1} + \frac{4}{2n} + \frac{1}{2n+1} \right), \quad (4)$$

we now define a global Simpson-regularized approximation of the harmonic sum. Notably, $g(n)$ is a finite linear combination of reciprocal integers and depends solely on the values of $1/x$ at the integer points.

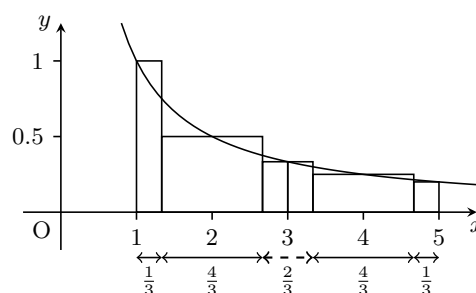


Figure 2. Simpson–type numerical integration of $1/x$, producing the regularized sum $h(N)$. The horizontal width of each rectangle represents the Simpson weight: endpoint contributions carry weight $1/3$, interior subintervals have width $4/3$, and interior boundary nodes contribute a combined weight $2/3$ arising from adjacent subintervals, indicated by a dashed line. The alternating excess and deficit areas explain the higher accuracy compared with the rectangular approximation in Figure 1

2.2. Convexity and Comparison Inequalities

The local quadrature error is defined as follows:

$$d(n) = g(n) - f(n). \quad (5)$$

The function $1/x$ is strictly decreasing and convex on $(0, \infty)$. Let $p_n(x)$ denote a quadratic polynomial interpolating $1/x$ at the three points.

$$x = 2n - 1, 2n, 2n + 1. \quad (6)$$

By construction,

$$g(n) = \int_{2n-1}^{2n+1} p_n(x) dx. \quad (7)$$

As $1/x$ is convex, the interpolation polynomial $p_n(x)$ lies above $1/x$ on $[2n - 1, 2n + 1]$. Thus, $p_n(x) - f(x) > 0$ except at the midpoint, and integration yields $d(n) > 0$. Consequently,

$$d(n) > 0 \quad \text{for } n \geq 1. \quad (8)$$

Moreover, convexity implies a comparison across adjacent intervals; in particular,

$$f(n) > g(n + 1) \quad \text{for } n \geq 2. \quad (9)$$

It follows that

$$0 < d(n) < g(n) - g(n + 1) \quad \text{for } n \geq 2. \quad (10)$$

2.3. Telescoping Bounds and Error Convergence

Summing the aforementioned inequalities for $n = 2, \dots, N$ yields

$$0 < \sum_{n=2}^N d(n) < \sum_{n=2}^N \{g(n) - g(n + 1)\}. \quad (11)$$

Both bounds telescope:

$$0 < \sum_{n=2}^N d(n) < g(2) - g(N + 1) \quad (12)$$

As $g(n) \rightarrow 0$ as $n \rightarrow \infty$ and $d(n) > 0$ for $n \geq 2$, the partial sums

$$\sum_{n=2}^N d(n) \quad (13)$$

form a monotone increasing sequence. Moreover, since

$$\sum_{n=2}^N d(n) < g(2) = \frac{23}{15} \approx 1.5333 \dots, \quad (14)$$

the sequence of partial sums is bounded above and therefore converges.

Definition 2 (The constant δ). *The constant δ is naturally defined by*

$$\delta := \lim_{N \rightarrow \infty} \sum_{n=2}^N d(n) = \lim_{N \rightarrow \infty} \{g(n) - f(n)\}. \quad (15)$$

3. Arithmetic Consequences of Simpson-Regularized Sums

We now assemble the local Simpson approximations introduced above to obtain a global regularized analogue of the harmonic sum.

3.1. Recovery of the Euler–Mascheroni Constant

Definition 3 (Simpson-regularized harmonic sum). *Let N be a positive integer.*

(i) N odd. If N is odd, we partition the interval $[1, N]$ into subintervals of length two,

$$[1, 3], [3, 5], \dots, [N-2, N], \quad (16)$$

and apply Simpson's rule to each subinterval to approximate the integral

$$\int_1^N \frac{1}{x} dx. \quad (17)$$

The resulting approximation is denoted by $h(N)$ and is given by

$$h(N) := \sum_{k=1}^{\frac{N-1}{2}} g(k). \quad (18)$$

(ii) N even. If N is even, then $N-1$ is odd. We apply Simpson's rule on the interval $[1, N-1]$ as above, and approximate the remaining interval $[N-1, N]$ by the trapezoidal rule. Namely, we define

$$h(N) := h(N-1) + \frac{1}{2} \left(\frac{1}{N-1} + \frac{1}{N} \right). \quad (19)$$

Lemma 1 (Representation by integer reciprocals). *The Simpson-regularized harmonic sum $h(N)$ can be written as a finite linear combination of the reciprocals of positive integers.*

(i) N odd. If N is odd, then

$$h(N) = \frac{1}{3} + \frac{1}{3} \sum_{n=2}^{N-1} \frac{c_n}{n} + \frac{1}{3N}, \quad (20)$$

where the coefficients c_n are given by

$$c_n = \begin{cases} 4, & \text{if } n \text{ is even,} \\ 2, & \text{if } n \text{ is odd.} \end{cases} \quad (21)$$

(ii) N even. If N is even, then

$$h(N) = \frac{1}{3} + \frac{1}{3} \sum_{n=2}^{N-1} \frac{c_n}{n} + \frac{1}{6(N-1)} + \frac{1}{2N}. \quad (22)$$

Lemma 2. *The constant δ defined in Definition 2 can be written as*

$$\delta = \lim_{n \rightarrow \infty} \{h(n) - \log[n]\} \quad (23)$$

Proof. By definition, $\delta = \sum_{n=2}^{\infty} d(n)$ with $d(n) = g(n) - f(n)$. Summing the exact logarithmic increments defined in equation (3),

$$\sum_{n=1}^N f(n) = \log[2N+1]. \quad (24)$$

yields a telescoping sum, from which the claim follows. $\square$

Theorem 1 (Limit formula for $H(n) - h(n)$).

$$\lim_{n \rightarrow \infty} \{H(N) - h(N)\} = \frac{\log[2] + 1}{3} = 0.56438 \dots \quad (25)$$

Proof. Let us compare the coefficients of $1/n$ in the representations of $h(N)$ and $H(N)$.

For even n , the coefficient in $h(N)$ is $4/3$, hence

$$\frac{1}{n} - \frac{4}{3n} = -\frac{1}{3n}. \quad (26)$$

For interior odd $n \geq 3$, the coefficient in $h(N)$ is $2/3$, and thus

$$\frac{1}{n} - \frac{2}{3n} = \frac{1}{3n}. \quad (27)$$

Therefore, up to boundary terms, we obtain

$$H(N) - h(N) = \frac{1}{3} \left(\sum_{\substack{3 \leq k \leq N-2 \\ k \text{ odd}}} \frac{1}{k} - \sum_{\substack{2 \leq k \leq N-1 \\ k \text{ even}}} \frac{1}{k} \right) + \frac{2}{3} + O\left(\frac{1}{N}\right). \quad (28)$$

The alternating harmonic series

$$\sum_{k=1}^{\infty} (-1)^{k-1} \frac{1}{k} \quad (29)$$

is known to converge to $\log[2]$. Since the boundary contributions vanish as $N \rightarrow \infty$, we conclude that

$$\lim_{n \rightarrow \infty} \{H(n) - h(n)\} = \frac{\log[2] + 1}{3}. \quad (30)$$

□

Proposition 1 (Recovery of the Euler–Mascheroni constant). *From equations (12) and (28), we obtain*

$$H(N) - \log[N] = \frac{\log[2] + 1}{3} + \delta + O\left(\frac{1}{N}\right). \quad (31)$$

In particular,

$$\gamma = \lim_{N \rightarrow \infty} \{H(N) - \log[N]\} = \frac{\log[2] + 1}{3} + \delta. \quad (32)$$

Equivalently, this yields the decomposition

$$\gamma = \frac{\log[2] + 1}{3} + \delta, \quad (33)$$

which expresses the Euler–Mascheroni constant, as defined in equation (2), in the form of a sum of two constants.

Definition 4 (A rational sequence associated with δ). *Let us define a sequence $a(n)$ by*

$$a(n) := 2h(n) - h(n^2). \quad (34)$$

Since each $h(n)$ is rational, each $a(n)$ is rational.

Lemma 3. *The constant δ satisfies*

$$\delta = \lim_{n \rightarrow \infty} a(n) \approx 0.01283 \dots \quad (35)$$

Proof. From the asymptotic relation established earlier,

$$h(n) - \log n = \delta + O\left(\frac{1}{n}\right), \quad (36)$$

we obtain

$$h(n^2) - \log(n^2) = \delta + O\left(\frac{1}{n^2}\right). \tag{37}$$

Multiplying equation (36) by 2 and subtracting equation (37), we obtain

$$a(n) = 2h(n) - h(n^2) = \delta + O\left(\frac{1}{n}\right). \tag{38}$$

This proves equation (35). $\square$

The numerical values in Table 1 illustrate the rapid convergence of $a(n)$ toward the constant δ . This behavior can be explained by the cancellation of the dominant $O(1/n)$ error terms in the asymptotic expansion of $h(n)$.

Table 1. Numerical convergence of $a(n)$ toward δ .

n	$a(n)$	$\delta - a(n)$
3	0.012169312	6.64×10^{-4}
5	0.012735433	9.78×10^{-5}
7	0.012806754	2.65×10^{-5}
9	0.012823395	9.88×10^{-6}
11	0.012828805	4.47×10^{-6}
13	0.012830969	2.30×10^{-6}

Remark 1. The slow convergence traditionally observed in numerical approximations to the Euler–Mascheroni constant γ [3] can be traced to the $O(1/n)$ contribution associated with the alternating harmonic series introduced in equation (29). In the present framework, this contribution is isolated explicitly, and removing it leads to the constant δ , whose numerical values exhibit markedly faster convergence (see Table 1).

3.2. Proof That δ Is Irrational by an Elementary Divisibility Argument

In this section, we prove that the constant δ cannot be a rational number.

Lemma 4. Let p be an odd prime. Then the rational number $a(p)$ has denominator divisible by p .

In particular, the denominator of $a(p)$ grows with p , which is the key arithmetic input for the irrationality argument. Recall that δ can be expressed in the form

$$\delta = \lim_{n \rightarrow \infty} a(n) = \lim_{n \rightarrow \infty} \{2h(n) - h(n^2)\}. \tag{39}$$

We now exploit the arithmetic structure of $a(n)$ by evaluating it along an increasing sequence of odd primes.

Let p be an odd prime. From the definition of the Simpson-regularized harmonic sum, we have

$$h(p) = \frac{1}{3p} + r_p, \tag{40}$$

where $r_p \in \mathbb{Q}$. The denominator of $h(p)$ contains no factor p , since the term $1/p$ appears only with endpoint weight $1/3$ in the Simpson-type approximation.

Multiplying equation (40) by 2 yields

$$2h(p) = \frac{2}{3p} + 2r_p, \tag{41}$$

where the denominator of $2r_p$ is also coprime to p .

On the other hand, in the Simpson-type approximation of $h(p^2)$, the integer p appears exactly once as an interior boundary point of the partition of $[1, p^2]$ into subintervals of length two, and hence contributes with the same endpoint weight $1/3$. Therefore, we can write

$$h(p^2) = \frac{1}{3p} + s_p, \tag{42}$$

where $s_p \in \mathbb{Q}$ and the denominator of s_p is coprime to p .

Subtracting equation (42) from equation (41), we obtain

$$a(p) = 2h(p) - h(p^2) = \frac{1}{3p} + t_p, \tag{43}$$

where $t_p \in \mathbb{Q}$ and the denominator of t_p is coprime to p .

Since p can be chosen to be an arbitrarily large prime, the sequence appearing in equation (43) consists of rational numbers whose denominators involve arbitrarily large primes.

Lemma 5 (A basic inequality for rational approximations). *Let $\frac{a}{b}$ be a rational number in lowest terms. If $\frac{c}{d} \neq \frac{a}{b}$, then*

$$\left| \frac{a}{b} - \frac{c}{d} \right| \geq \frac{1}{bd}. \tag{44}$$

Proof. Let a/b be a rational number in lowest terms. Consider any rational approximation c/d .

$$\left| \frac{a}{b} - \frac{c}{d} \right| = \frac{|ad - bc|}{bd} \tag{45}$$

Since a, b, c , and d are integers, we have $ad - bc \in \mathbb{Z}$.

Thus, if $\frac{c}{d} \neq \frac{a}{b}$, then $|ad - bc| \geq 1$, and consequently

$$\left| \frac{a}{b} - \frac{c}{d} \right| = \frac{|ad - bc|}{bd} \geq \frac{1}{bd}. \tag{46}$$

□

Corollary 1. *Let $\alpha \in \mathbb{Q}$. For any sequence of rational numbers converging to α , the set of prime divisors appearing in the denominators of sufficiently close approximants is finite.*

Assume, for contradiction, that $\delta = a/b \in \mathbb{Q}$. From Lemma 3 and equation (35), the sequence $a(n)$ converges to δ as $n \rightarrow \infty$. In particular, $a(p) \rightarrow \delta$ along the subsequence of odd primes p .

By Corollary 1, the set of prime divisors appearing in the denominators of $a(p)$ is finite. However, from equations (40), (41), (42), and (43), each $a(p)$ has denominator divisible by the prime p . Since p can be chosen arbitrarily large, this contradicts the finiteness asserted in Corollary 1.

Therefore, δ is irrational.

4. Conclusions

This study demonstrated that the convergence of the harmonic series minus the logarithm can be established using solely Simpson’s rule and the convexity of the function $1/x$. This provides a simple, purely numerical, and analytic alternative to classical proofs based on the Euler–Maclaurin formula.

The method highlights an unexpected connection between elementary quadrature rules and one of the fundamental constants of analysis and suggests natural extensions to higher-order Newton-Cotes formulas [4].

In addition, we introduced the constant δ , defined as the limit of a rational sequence arising from Simpson-type harmonic approximations, and proved that δ is irrational. Combined with the identity

established in equation (33), this shows that the Euler–Mascheroni constant γ can be expressed as the sum of two irrational constants.

Acknowledgments: I thank my wife for her suggestions and encouragement. This work was supported by JSPS KAKENHI Grant Numbers JP25K22410 and Yokohama Foundation for Advancement of Medical Science (Medical Digitalization Grant). I thank Editage (www.editage.com) for providing professional English language editing services. AI-assisted tools (Microsoft Copilot, 2026) were used solely for idea organization and for improving the clarity and fluency of the English text. All mathematical derivations, theoretical arguments, numerical experiments, and final scientific content were entirely produced and verified by the authors.

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