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Article

Efficient Numerical Methods for Time-Fractional Diffusion Equations with Caputo-Type Erdélyi-Kober Operator

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Abstract: This study proposes an L1 discretization scheme to solve time-fractional diffusion equations involving Caputo-type Erdélyi-Kober operator, which are fundamental in modeling anomalous diffusion phenomena. To facilitate analysis, we first reformulate the original problem as an equivalent fractional integral equation. Within this transformed setting, the temporal Caputo-type Erdélyi-Kober operator is discretized via the L1 formula, with rigorous demonstration of its second-order temporal accuracy and detailed analysis of the coefficient properties in the discrete system. The spatial derivative is approximated using the classical second-order centered difference, culminating in a fully discrete scheme that achieves second-order accuracy in both temporal and spatial dimensions. To address computational challenges arising from the nonlocal nature of fractional operators, we further develop a fast implementation algorithm based on sum-of-exponential approximation for the fractional kernel function, significantly reducing memory requirements and computational costs. The numerical experiment validates the theoretical convergence rates and substantiates the computational superiority of the proposed fast algorithm.

Keywords: Caputo-type Erdélyi-Kober operator; time fractional diffusion equation; L1 formula; error estimate; sum-of-exponential approximation

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1. Introduction

In recent decades, fractional integro-differential equations have emerged as critical mathematical tools in interdisciplinary scientific and engineering research [2,6,7,11]. Within this framework, fractional diffusion equations occupy a central position due to their capability to model anomalous transport phenomena. These equations exhibit remarkable versatility in capturing multi-regime diffusion dynamics and characteristic crossover behaviors observed in complex heterogeneous systems [10,12,14]. To address such diverse physical scenarios, multiple fractional operator definitions have been developed, including the Riemann-Liouville calculus, Grünwald-Letnikov integral, Caputo derivative, Hadamard calculus, Erdélyi-Kober calculus and so on. While numerical methods for equations involving the first four operator types have reached considerable maturity, the analysis of Erdélyi-Kober fractional operators remains an open research frontier. This gap in existing literature serves as the primary motivation for our work on developing systematic numerical approaches for this understudied operator class.

The Erdélyi-Kober fractional integral operator, originally introduced by Erdélyi and Hermann K. in 1940 [3] through integral boundary condition formulations, has gained significant traction in modeling multiscale physical systems [13,16,20]. Fundamental properties of these operators, particularly the relationship between Caputo-type Erdélyi-Kober derivatives and their classical counterparts, are systematically detailed in [6,9,21]. Recent advances in Erdélyi-Kober fractional integro-differential equations have yielded several key developments. [22,23] used the fixed point theorem to analyze

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the existence and uniqueness of solutions for Erdélyi-Kober fractional nonlinear integral equations. Płociniczak [17] established a dimension-reduction framework for fractional porous media equations via non-dimensionalization, transforming the original equation into tractable integro-differential forms, thus greatly reducing the complexity of the solution. Moreover, the Erdélyi-Kober operator's equivalence to hyper-Bessel differential operators has enabled novel solution strategies for hyper-Bessel fractional equations [1,24,25].

The inherent non-locality of fractional operators renders analytical solutions computationally prohibitive, necessitating advanced numerical approximations for practical applications. While significant progress has been made in temporal fractional diffusion equations through finite difference [26], finite element [5], and spectral methods [8], numerical treatments for Erdélyi-Kober fractional integro-differential equations remain underdeveloped. Current limitations in this domain include: Płociniczak et al. [19] proposed rectangular, midpoint, and trapezoidal formulations for Erdélyi-Kober fractional integral operator, but provided no error bounds or theoretical justification. Odibat et al. [15] implemented predictor-correction method for Caputo-type Erdélyi-Kober fractional differential equations without convergence analysis. Existing semi-discrete schemes [18] achieve suboptimal first-order accuracy under initial weak regularity assumptions, and analyzed the stability and convergence of the scheme by using the Galerkin-Hermite method. Motivated by these gaps, this work develops novel high-order finite difference schemes for Caputo-type Erdélyi-Kober fractional diffusion equations, specifically targeting second-order temporal accuracy while maintaining rigorous stability guarantees through innovative kernel coefficient analysis.

In what follows, we focus on the following initial-boundary value problem of Caputo-type Erdélyi-Kober time fractional diffusion model

$$\begin{cases} {}_{CEK}\mathrm{D}^{\alpha}_{0,t;\sigma,\eta}u(x,t) = u_{xx}(x,t) + f(x,t), \quad (x,t) \in (0,L) \times (0,T], & (1) \\ u(x,0) = \varphi(x), \quad x \in (0,L), & (2) \\ u(0,t) = 0, \quad u(L,t) = 0, \quad t \in (0,T], & (3) \end{cases}$$

in which f, φ are given suitably smooth functions, Caputo-type Erdélyi-Kober fractional derivative ${}_{CEK}\mathrm{D}^{\alpha}_{0,t;\sigma,\eta}\, p(t)$ is defined by [15]

$${}_{CEK}\mathrm{D}^{\alpha}_{0,t;\sigma,\eta}\, p(t) = \frac{t^{-\sigma\eta}}{\Gamma(1-\alpha)} \int_0^t (t^{\sigma} - s^{\sigma})^{-\alpha} \delta_{\sigma}[s^{\sigma(\alpha+\eta)} p(s)] \mathrm{d}s^{\sigma}, \quad \alpha \in (0,1),\ \sigma > 0,\ \eta \in \mathbb{R}, \tag{4}$$

where $\delta_{\sigma} = \frac{1}{\sigma s^{\sigma-1}} \frac{\mathrm{d}}{\mathrm{d}s}$.

To facilitate subsequent analysis, we apply Erdélyi-Kober fractional integral ${}_{EK}\mathrm{D}^{-\alpha}_{0,t;\sigma,\eta}$ to both sides of (1), yielding an equivalent form

$$u(x,t) = {}_{EK}\,\mathrm{D}^{-\alpha}_{0,t;\sigma,\eta} u_{xx}(x,t) + F(x,t), \quad (x,t) \in (0,L) \times (0,T], \tag{5}$$

where

$${}_{EK}\mathrm{D}^{-\alpha}_{0,t;\sigma,\eta} u_{xx}(x,t) = \frac{t^{-\sigma(\alpha+\eta)}}{\Gamma(\alpha)} \int_0^t (t^{\sigma} - s^{\sigma})^{\alpha-1} s^{\sigma\eta} u_{xx}(x,s) \mathrm{d}s^{\sigma},$$

$$F(x,t) = \frac{t^{-\sigma(\alpha+\eta)}}{\Gamma(\alpha)} \int_0^t (t^{\sigma} - s^{\sigma})^{\alpha-1} s^{\sigma\eta} f(x,s) \mathrm{d}s^{\sigma}.$$

The remainder of this paper is organized as follows. Section 2 develops an L1-type discretization formula for the Erdélyi-Kober fractional integral operator, accompanied by a detailed analysis of truncation errors. Furthermore, we rigorously establish the positive definiteness of the discrete kernel coefficients. Building upon these theoretical foundations, we derive a finite difference scheme to solve (5). To enhance computational efficiency, Section 3 introduces a fast implementation algorithm for the difference scheme. The numerical example presented in Section 4 validate the theoretical predictions

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and demonstrate the effectiveness of our method. Finally, concluding remarks with perspectives are provided in the last section.

2. A Direct L1 Difference Scheme

This section focuses on constructing an L1-type difference scheme for Equation (5) coupled with its governing initial-boundary value conditions (2)-(3). We firstly introduce some spatial notations. Given a positive constant M, denote $h = L/M$. Let $x_i = ih\,(0 \le i \le M)$, $\Omega_h = \{x_i \,|\, 0 \le i \le M\}$. Then the grid function space is defined by

$$\mathcal{V}_h = \{u \,|\, u = (u_0, u_1, \cdots, u_M)\}, \quad \mathring{\mathcal{V}}_h = \{u \,|\, u \in \mathcal{V}_h;\ u_0 = u_M = 0\}.$$

For any $u \in \mathcal{V}_h$, introduce the following notations,

$$\delta_x u_{i-\frac{1}{2}} = \frac{1}{h}(u_i - u_{i-1}), \quad \delta_x^2 u_i = \frac{1}{h^2}(u_{i+1} - 2u_i + u_{i-1}).$$

For any $u, v \in \mathring{\mathcal{V}}_h$, the inner products and norms are defined by

$$(u, v) = h\sum_{i=1}^{M-1} u_i v_i, \quad \|u\| = \sqrt{(u, u)},$$

$$(\delta_x u, \delta_x v) = h\sum_{i=1}^{M} \delta_x u_{i-\frac{1}{2}} \delta_x v_{i-\frac{1}{2}}, \quad \|\delta_x u\| = \sqrt{(\delta_x u, \delta_x u)}.$$

2.1. The Derivation of the Difference Scheme

For any fixed T, take a positive integer N. And denote $\tau = T^\sigma/N$, $t_0 = 0$, $t_k = (t_{k-1}^\sigma + \tau)^{\frac{1}{\sigma}}$, $k = 1, 2, \cdots, N$.

Denote $g(x,t) \triangleq t^{\sigma\eta} u_{xx}(x,t)$. Considering Equation (5) at the point $t = t_n$, we have

$$\begin{aligned} u(x, t_n) =& \frac{t_n^{-\sigma(\alpha+\eta)}}{\Gamma(\alpha)} \int_0^{t_n} (t_n^\sigma - s^\sigma)^{\alpha-1} g(x,s)\mathrm{d}s^\sigma + F(x, t_n) \\ =& \frac{t_n^{-\sigma(\alpha+\eta)}}{\Gamma(\alpha)} \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} (t_n^\sigma - s^\sigma)^{\alpha-1} g(x,s)\mathrm{d}s^\sigma + F(x, t_n), \quad x \in (0, L),\ 1 \le n \le N. \end{aligned} \tag{6}$$

Let $\Pi_{1,k}p(s)$ be linear interpolation polynomial of any function p defined on an interval $[t_k, t_{k+1}]$, $k = 0, 1, \cdots, N-1$. We get

$$\Pi_{1,k}p(s) = \frac{t_{k+1}^\sigma - s^\sigma}{\tau} p(t_k) + \frac{s^\sigma - t_k^\sigma}{\tau} p(t_{k+1}), \tag{7}$$

with the truncation error

$$\theta_p(s) = \Pi_{1,k}p(s) - p(s) = \frac{1}{2}(s^\sigma - t_{k+1}^\sigma)(s^\sigma - t_k^\sigma)\delta_\sigma^2 p(\xi_k(s)), \quad \xi_k(s) \in (t_k, t_{k+1}),\ 0 \le k \le N. \tag{8}$$

We now develop an L1-type discretization formula for the temporal convolution integral in (6). Let we ignore the space variable x for a while. Using (7) we have

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$$
\begin{aligned}
&{}_{EK}D^{-\alpha}_{0,t;\sigma,\eta}g(t)|_{t=t_n} = \frac{t_n^{-\sigma(\alpha+\eta)}}{\Gamma(\alpha)}\sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}}(t_n^\sigma - s^\sigma)^{\alpha-1}g(s)\mathrm{d}s^\sigma \\
&\approx \frac{t_n^{-\sigma(\alpha+\eta)}}{\Gamma(\alpha)}\sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}}(t_n^\sigma - s^\sigma)^{\alpha-1}\Big[\frac{t_{k+1}^\sigma - s^\sigma}{\tau}g(t_k) + \frac{s^\sigma - t_k^\sigma}{\tau}g(t_{k+1})\Big]\mathrm{d}s^\sigma \\
&= \frac{t_n^{-\sigma(\alpha+\eta)}}{\Gamma(\alpha)}\sum_{k=0}^{n-1}\Big[\int_{t_k}^{t_{k+1}}(t_n^\sigma - s^\sigma)^{\alpha-1}(t_{k+1}^\sigma - s^\sigma)\mathrm{d}s^\sigma\frac{g(t_k)}{\tau} + \int_{t_k}^{t_{k+1}}(t_n^\sigma - s^\sigma)^{\alpha-1}(s^\sigma - t_k^\sigma)\mathrm{d}s^\sigma\frac{g(t_{k+1})}{\tau}\Big] \\
&= \frac{t_n^{-\sigma(\alpha+\eta)}}{\Gamma(\alpha)}\Big[\frac{1}{\alpha(\alpha+1)}A^{(\sigma,\alpha)}_{n,n-1}\frac{g(t_n)}{\tau} + \sum_{k=1}^{n-1}\frac{1}{\alpha(\alpha+1)}(A^{(\sigma,\alpha)}_{n,k-1} - A^{(\sigma,\alpha)}_{n,k})\frac{g(t_k)}{\tau} \\
&\quad + (-\frac{A^{(\sigma,\alpha)}_{n,0}}{\alpha(\alpha+1)} + \frac{\tau}{\alpha}t_n^{\sigma\alpha})\frac{g(t_0)}{\tau}\Big] \\
&= \frac{t_n^{-\sigma(\alpha+\eta)}}{\Gamma(\alpha)}\sum_{k=0}^{n} C^{(\sigma,\alpha)}_{n,k}\frac{g(t_k)}{\tau} \triangleq {}_{EK}\mathbb{D}^{-\alpha}_{0,t;\sigma,\eta}g(t_n),
\end{aligned}
\tag{9}
$$

here

$$A^{(\sigma,\alpha)}_{n,k} = (t_n^\sigma - t_k^\sigma)^{\alpha+1} - (t_n^\sigma - t_{k+1}^\sigma)^{\alpha+1}, \quad 0 \le k \le n-1,$$

$$
C^{(\sigma,\alpha)}_{n,k} = \begin{cases}
\frac{\tau}{\alpha}t_n^{\sigma\alpha} - \frac{1}{\alpha(\alpha+1)}\big[t_n^{\sigma(\alpha+1)} - (t_n^\sigma - t_1^\sigma)^{\alpha+1}\big], & k=0, \\
\frac{1}{\alpha(\alpha+1)}\big[(t_n^\sigma - t_{k-1}^\sigma)^{\alpha+1} - 2(t_n^\sigma - t_k^\sigma)^{\alpha+1} + (t_n^\sigma - t_{k+1}^\sigma)^{\alpha+1}\big], & 1 \le k \le n-1, \\
\frac{1}{\alpha(\alpha+1)}\tau^{\alpha+1}, & k=n.
\end{cases}
$$

For the truncation error of the L1 formula ${}_{EK}\mathbb{D}^{-\alpha}_{0,t;\sigma,\eta}g(t_n)$, we have the estimation below.

Theorem 1. *Suppose $g \in C^2[t_0, t_n]$, $\alpha \in (0,1)$. Denote*

$$r_n = {}_{EK}D^{-\alpha}_{0,t;\sigma,\eta}g(t)|_{t=t_n} - {}_{EK}\mathbb{D}^{-\alpha}_{0,t;\sigma,\eta}g(t_n), \quad n = 1, 2, \cdots, N.$$

Then, we have

$$|r_n| \le \frac{t_n^{-\sigma\eta}}{8\Gamma(1+\alpha)}\max_{t_0<\xi_k(s)<t_n}|\delta_\sigma^2 g(\delta_k(s))|\tau^2.$$

Proof. With the help of (8), we get

$$
\begin{aligned}
|r_n| &= \frac{t_n^{-\sigma(\alpha+\eta)}}{\Gamma(\alpha)}\sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}}(t_n^\sigma - s^\sigma)^{\alpha-1}|g(s) - \Pi_{1,k}g(s)|\mathrm{d}s^\sigma \\
&\le \frac{t_n^{-\sigma(\alpha+\eta)}}{2\Gamma(\alpha)}\sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}}(t_n^\sigma - s^\sigma)^{\alpha-1}(t_{k+1}^\sigma - s^\sigma)(s^\sigma - t_k^\sigma)|\delta_\sigma^2 g(\xi_k(s))|\mathrm{d}s^\sigma \\
&\le \frac{t_n^{-\sigma(\alpha+\eta)}}{8\Gamma(\alpha)}\tau^2\max_{t_0<\xi_k(s)<t_n}|\delta_\sigma^2 g(\xi_k(s))|\sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}}(t_n^\sigma - s^\sigma)^{\alpha-1}\mathrm{d}s^\sigma \\
&= \frac{t_n^{-\sigma\eta}}{8\Gamma(1+\alpha)}\max_{t_0<\xi_k(s)<t_n}|\delta_\sigma^2 g(\xi_k(s))|\tau^2.
\end{aligned}
$$

Thus, we complete the proof. □

The coefficients $\{C^{(\sigma,\alpha)}_{n,k}\}$ defined in (9) have the properties below.

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Lemma 1. *For* $\{C_{n,k}^{(\sigma,\alpha)}|_{k=0}^{n}\}$ *defined in* (9), *it holds that*

$$2C_{n,n}^{(\sigma,\alpha)} > C_{n,n-1}^{(\sigma,\alpha)} > \cdots > C_{n,2}^{(\sigma,\alpha)} > C_{n,1}^{(\sigma,\alpha)} > 2C_{n,0}^{(\sigma,\alpha)} > 0; \tag{10}$$

$$C_{n,k}^{(\sigma,\alpha)} - 2C_{n,k-1}^{(\sigma,\alpha)} + C_{n,k-2}^{(\sigma,\alpha)} > 0,\ 3 \le k \le n-1; \tag{11}$$

$$2C_{n,0}^{(\sigma,\alpha)} - 2C_{n,1}^{(\sigma,\alpha)} + C_{n,2}^{(\sigma,\alpha)} > 0,\ C_{n,n-2}^{(\sigma,\alpha)} - 2C_{n,n-1}^{(\sigma,\alpha)} + 2C_{n,n}^{(\sigma,\alpha)} > 0. \tag{12}$$

Proof. (1) Proof of (10). It is easy to know that

$$C_{n,0}^{(\sigma,\alpha)} = \frac{\tau}{\alpha}\big[t_n^{\sigma\alpha} - (t_n^{\sigma} - \xi_1)^{\alpha}\big] > \frac{\tau}{\alpha}\big[t_n^{\sigma\alpha} - t_n^{\sigma\alpha}\big] = 0,\ \xi_1 \in (0, t_1^{\sigma}).$$

For $2 \le k \le n-1$, we have

$$\begin{aligned}
&C_{n,k}^{(\sigma,\alpha)} - C_{n,k-1}^{(\sigma,\alpha)}\\
=&\frac{1}{\alpha(\alpha+1)}\big[-(t_n^{\sigma} - t_{k-2}^{\sigma})^{\alpha+1} + 3(t_n^{\sigma} - t_{k-1}^{\sigma})^{\alpha+1} - 3(t_n^{\sigma} - t_k^{\sigma})^{\alpha+1} + (t_n^{\sigma} - t_{k+1}^{\sigma})^{\alpha+1}\big]\\
=&(1-\alpha)\int_0^{\tau}\mathrm{d}z_1\int_0^{\tau}\mathrm{d}z_2\int_0^{\tau}(t_n^{\sigma} - t_{k+1}^{\sigma} + z_1 + z_2 + z_3)^{\alpha-2}\mathrm{d}z_3 > 0.\\
&2C_{n,n}^{(\sigma,\alpha)} - C_{n,n-1}^{(\sigma,\alpha)}\\
=&\frac{2}{\alpha(\alpha+1)}\tau^{\alpha+1} - \frac{1}{\alpha(\alpha+1)}\big[(t_n^{\sigma} - t_{n-2}^{\alpha})^{\alpha+1} - 2\tau^{\alpha+1}\big]\\
=&\frac{\tau^{\alpha+1}}{\alpha(\alpha+1)}[4 - 2^{\alpha+1}] > 0.\\
&C_{n,1}^{(\sigma,\alpha)} - 2C_{n,0}^{(\sigma,\alpha)}\\
=&\frac{1}{\alpha(\alpha+1)}\big[3t_n^{\sigma(\alpha+1)} - 4(t_n^{\sigma} - t_1^{\sigma})^{\alpha+1} + (t_n^{\sigma} - t_2^{\sigma})^{\alpha+1}\big] - \frac{2\tau}{\alpha}t_n^{\sigma\alpha}\\
=&\frac{\tau^{\alpha+1}}{\alpha}P(n,\alpha),
\end{aligned}$$

where $P(n,\alpha) = \frac{1}{\alpha+1}\big[3n^{\alpha+1} - 4(n-1)^{\alpha+1} + (n-2)^{\alpha+1}\big] - 2n^{\alpha}$. To prove $C_{n,1}^{(\sigma,\alpha)} - 2C_{n,0}^{(\sigma,\alpha)} > 0$, we just need to prove $P(n,\alpha) > 0$. By Taylor's formula, we have

$$\begin{aligned}
P(n,\alpha) =&\frac{1}{\alpha+1}\Big[3n^{\alpha+1} - 4n^{\alpha+1} + 4(\alpha+1)n^{\alpha} - 2\alpha(\alpha+1)n^{\alpha-1} - \frac{2}{3}\alpha(\alpha+1)(1-\alpha)(n-1+\xi_1)^{\alpha-2}\\
&+ n^{\alpha+1} - 2(\alpha+1)n^{\alpha} + 2\alpha(\alpha+1)n^{\alpha-1} + \frac{4}{3}\alpha(\alpha+1)(1-\alpha)(n-2+\xi_2)^{\alpha-2}\Big] - 2n^{\alpha}\\
=&\frac{2}{3}\alpha(1-\alpha)\big[2(n-2+\xi_2)^{\alpha-2} - (n-1+\xi_1)^{\alpha-2}\big],
\end{aligned}$$

in which $\xi_1 \in (0,1)$, $\xi_2 \in (0,2)$.
It is easy to know that

$$P(n,\alpha) > \frac{2}{3}\alpha(1-\alpha)[2n^{\alpha-2} - (n-1)^{\alpha-2}] > 0 \text{ for } n \ge 4.$$

In addition, when $n = 2,3$, $P(n,\alpha) > 0$ can be proved by a simple calculation.

(2) Proof of (11).

$$\begin{aligned}
&C_{n,k}^{(\sigma,\alpha)} - 2C_{n,k-1}^{(\sigma,\alpha)} + C_{n,k-2}^{(\sigma,\alpha)}\\
=&\frac{1}{\alpha(\alpha+1)}\big[(t_n^{\sigma} - t_{k-3}^{\sigma})^{\alpha+1} - 4(t_n^{\sigma} - t_{k-2}^{\sigma})^{\alpha+1} + 6(t_n^{\sigma} - t_{k-1}^{\sigma})^{\alpha+1} - 4(t_n^{\sigma} - t_k^{\sigma})^{\alpha+1} + (t_n^{\sigma} - t_{k+1}^{\sigma})^{\alpha+1}\big]\\
=&(1-\alpha)(2-\alpha)\int_0^{\tau}\mathrm{d}z_1\int_0^{\tau}\mathrm{d}z_2\int_0^{\tau}\mathrm{d}z_3\int_0^{\tau}(t_n^{\sigma} - t_{k+1}^{\sigma} + z_1 + z_2 + z_3 + z_4)^{\alpha-3}\mathrm{d}z_4 > 0.
\end{aligned}$$

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(3) Proof of (12).

$$
\begin{aligned}
&2C_{n,0}^{(\sigma,\alpha)}-2C_{n,1}^{(\sigma,\alpha)}+C_{n,2}^{(\sigma,\alpha)}\\
=&\frac{2\tau}{\alpha}t_n^{\sigma\alpha}+\frac{1}{\alpha(\alpha+1)}\big[-4t_n^{\sigma(\alpha+1)}+7(t_n^\sigma-t_1^\sigma)^{\alpha+1}-4(t_n^\sigma-t_2^\sigma)^{\alpha+1}+(t_n^\sigma-t_3^\sigma)^{\alpha+1}\big]\\
=&\frac{\tau^{\alpha+1}}{\alpha}\Big\{2n^\alpha+\frac{1}{\alpha+1}\big[-4n^{\alpha+1}+7(n-1)^{\alpha+1}-4(n-2)^{\alpha+1}+(n-3)^{\alpha+1}\big]\Big\}.
\end{aligned}
$$

Similar to the treatment of $P(n,\alpha)$, one can also prove that $2C_{n,0}^{(\sigma,\alpha)}-2C_{n,1}^{(\sigma,\alpha)}+C_{n,2}^{(\sigma,\alpha)}>0$.

$$
\begin{aligned}
&C_{n,n-2}^{(\sigma,\alpha)}-2C_{n,n-1}^{(\sigma,\alpha)}+2C_{n,n}^{(\sigma,\alpha)}\\
=&\frac{1}{\alpha(\alpha+1)}\big[(t_n^\sigma-t_{n-3}^\sigma)^{\alpha+1}-4(t_n^\sigma-t_{n-2}^\sigma)^{\alpha+1}+7\tau^{\alpha+1}\big]\\
=&\frac{\tau^{\alpha+1}}{\alpha(\alpha+1)}\big(3^{\alpha+1}-4\cdot 2^{\alpha+3}+7\big)>0.
\end{aligned}
$$

Therefore, the proof is complete. □

Now, we devote to presenting a second-order difference scheme for solving the problem (1)-(3). Suppose $u(x,t)\in C^{(4,2)}([0,L]\times[0,T])$. Denote

$$U_i^n=u(x_i,t_n),\quad F_i^n=F(x_i,t_n),\quad 0\le n\le N;\quad \varphi_i=\varphi(x_i),\quad 0\le i\le M.$$

Considering Equation (5) at the point (x_i,t_n), we get

$$u(x_i,t_n)=\frac{t_n^{-\sigma(\alpha+\eta)}}{\Gamma(\alpha)}\int_0^{t_n}(t_n^\sigma-s^\sigma)^{\alpha-1}s^{\sigma\eta}u_{xx}(x_i,s)\mathrm{d}s^\sigma+F(x_i,t_n),\quad 1\le i\le M-1,\ 1\le n\le N.$$

Applying (9) to approximating the temporal fractional derivative and central difference quotient to the spatial derivative, we can obtain

$$U_i^n=\frac{t_n^{-\sigma(\alpha+\eta)}}{\Gamma(\alpha)}\sum_{k=0}^{n}C_{n,k}^{(\sigma,\alpha)}\frac{t_k^{\sigma\eta}}{\tau}\delta_x^2U_i^k+F_i^n+R_i^n,\quad 1\le i\le M-1,\ 1\le n\le N, \tag{13}$$

where

$$R_i^n=r_i^n+\frac{t_n^{-\sigma(\alpha+\eta)}}{\Gamma(\alpha)}\sum_{k=0}^{n}C_{n,k}^{(\sigma,\alpha)}\frac{t_k^{\sigma\eta}}{\tau}\big[u_{xx}(x_i,t_k)-\delta_x^2U_i^k\big]$$

Noticing Theorem 1, there exists a positive constant c_1 such that

$$
\begin{aligned}
|R_i^k|\le&|r_i^n|+\frac{c_1t_n^{-\sigma\alpha}}{\Gamma(\alpha)\tau}h^2\sum_{k=0}^{n}C_{n,k}^{(\sigma,\alpha)}\\
\le&\frac{t_n^{-\sigma\eta}}{8\Gamma(1+\alpha)}\max_{t_0<\xi_k(s)<t_n}|\delta_\sigma^2g(\delta_k(s))|\tau^2+\frac{c_1t_n^{-\sigma\alpha}}{\Gamma(\alpha)\tau}h^2\cdot\frac{\tau}{\alpha}t_n^{\sigma\alpha}\\
=&\frac{t_n^{-\sigma\eta}}{8\Gamma(1+\alpha)}\max_{t_0<\xi_k(s)<t_n}|\delta_\sigma^2g(\delta_k(s))|\tau^2+\frac{c_1h^2}{\Gamma(\alpha+1)}.
\end{aligned}
\tag{14}
$$

Noticing the initial-boundary value conditions (2) and (3), we have

$$
\begin{cases}
U_i^0=\varphi_i,\quad 1\le i\le M-1, & (15)\\
U_0^n=0,\quad U_M^n=0,\quad 0\le n\le N. & (16)
\end{cases}
$$

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Omitting the small terms in (13) and replacing the grid function U_i^n by its numerical approximation u_i^n, we construct a direct second-order difference scheme for solving the problem (1)-(3) as follows:

$$\begin{cases} u_i^n = \dfrac{t_n^{-\sigma(\alpha+\eta)}}{\Gamma(\alpha)} \displaystyle\sum_{k=0}^{n} C_{n,k}^{(\sigma,\alpha)} \frac{t_k^{\sigma\eta}}{\tau} \delta_x^2 u_i^k + F_i^n, \quad 1 \le i \le M-1,\ 1 \le n \le N, & (17) \\ u_i^0 = \varphi_i, \quad 1 \le i \le M-1, & (18) \\ u_0^n = 0, \quad u_M^n = 0, \quad 0 \le n \le N. & (19) \end{cases}$$

3. A Fast Difference Scheme

The inherent non-locality of fractional operators poses significant implementation challenges in practical computation. To address this fundamental limitation, we implement the fast algorithm developed by Jiang et al. [4] within our methodological framework.

3.1. Fast Approximation of the Erdélyi-Kober Integral

The following lemma is fundamental and needed.

Lemma 2. [4] For the given $\gamma \in (0,1)$ and tolerance error ϵ, cut-off time restriction δ and final time T, there are one positive integer $N_{exp}^{(\gamma)}$, positive points $\{s_l^{(\gamma)} \mid l = 1, 2, \cdots, N_{exp}^{(\gamma)}\}$, and corresponding positive weights $\{w_l^{(\gamma)} \mid l = 1, 2, \cdots, N_{exp}^{(\gamma)}\}$ such that

$$|t^{-\gamma} - \sum_{l=1}^{N_{exp}^{(\gamma)}} w_l^{(\gamma)} e^{-s_l^{(\gamma)} t}| \le \epsilon, \quad \forall\, t \in [\delta, T],$$

where

$$N_{exp}^{(\gamma)} = O\Big((\log\frac{1}{\epsilon})(\log\log\frac{1}{\epsilon} + \log\frac{T}{\delta}) + (\log\frac{1}{\delta})(\log\log\frac{1}{\epsilon} + \log\frac{1}{\delta})\Big).$$

Now, we will derive a fast algorithm for computing the Erdélyi-Kober integral ${}_{EK}D_{0,t;\sigma,\eta}^{-\alpha} g(t)|_{t=t_n}$. Let $\delta = \tau/2$.

$$\begin{aligned} &{}_{EK}D_{0,t;\sigma,\eta}^{-\alpha} g(t)|_{t=t_n} \\ =&\frac{t_n^{-\sigma(\alpha+\eta)}}{\Gamma(\alpha)} \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} (t_n^\sigma - s^\sigma)^{\alpha-1} g(s) \mathrm{d}s^\sigma \\ =&\frac{t_n^{-\sigma(\alpha+\eta)}}{\Gamma(\alpha)} \left[\sum_{k=0}^{n-2} \int_{t_k}^{t_{k+1}} (t_n^\sigma - s^\sigma)^{\alpha-1} g(s) \mathrm{d}s^\sigma + \int_{t_{n-1}}^{t_n} (t_n^\sigma - s^\sigma)^{\alpha-1} g(s) \mathrm{d}s^\sigma \right] \\ \approx&\frac{t_n^{-\sigma(\alpha+\eta)}}{\Gamma(\alpha)} \left[\sum_{l=1}^{N_{exp}^{(1-\alpha)}} w_l^{(1-\alpha)} \sum_{k=0}^{n-2} \int_{t_k}^{t_{k+1}} e^{-s_l^{(1-\alpha)}(t_n^\sigma - s^\sigma)} \Pi_{1,k} g(s) \mathrm{d}s^\sigma + \int_{t_{n-1}}^{t_n} (t_n^\sigma - s^\sigma)^{\alpha-1} \Pi_{1,n-1} g(s) \mathrm{d}s^\sigma \right] \\ =&\frac{t_n^{-\sigma(\alpha+\eta)}}{\Gamma(\alpha)} \left[\sum_{l=1}^{N_{exp}^{(1-\alpha)}} w_l^{(1-\alpha)} F_l^n + \frac{\tau^\alpha}{\alpha+1} g(t_{n-1}) + \frac{\tau^\alpha}{\alpha(\alpha+1)} g(t_n) \right], \end{aligned} \tag{20}$$

where

$$\begin{aligned} F_l^1 &= 0; \\ F_l^n &= \sum_{k=0}^{n-2} \int_{t_k}^{t_{k+1}} e^{-s_l^{(1-\alpha)}(t_n^\sigma - s^\sigma)} \Pi_{1,k} g(s) \mathrm{d}s^\sigma. \end{aligned}$$

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The integral F_l^n can be evaluated by using a recursive algorithm, i.e.

$$\begin{aligned} F_l^n =& e^{-s_l^{(1-\alpha)}\tau} \sum_{k=0}^{n-3} \int_{t_k}^{t_{k+1}} e^{-s_l^{(1-\alpha)}(t_{n-1}^\sigma - s^\sigma)} \Pi_{1,k} g(s) \mathrm{d}s^\sigma + \int_{t_{n-2}}^{t_{n-1}} e^{-s_l^{(1-\alpha)}(t_n^\sigma - s^\sigma)} \Pi_{1,n-2} g(s) \mathrm{d}s^\sigma \\ =& e^{-s_l^{(1-\alpha)}\tau} F_l^{n-1} + A_l \frac{g(t_{n-2})}{\tau} + B_l \frac{g(t_{n-1})}{\tau}, \quad n = 2, 3, \cdots, \end{aligned}$$

where for $1 \le l \le N_{exp}^{(1-\alpha)}$,

$$\begin{aligned} A_l =& \frac{1}{s_l^{(1-\alpha)}} \Big\{ -\tau e^{-2s_l^{(1-\alpha)}\tau} + \frac{1}{s_l^{(1-\alpha)}} \big[e^{-s_l^{(1-\alpha)}\tau} - e^{-2s_l^{(1-\alpha)}\tau} \big] \Big\}, \\ B_l =& \frac{1}{s_l^{(1-\alpha)}} \Big\{ \tau e^{-s_l^{(1-\alpha)}\tau} - \frac{1}{s_l^{(1-\alpha)}} \big[e^{-s_l^{(1-\alpha)}\tau} - e^{-2s_l^{(1-\alpha)}\tau} \big] \Big\}. \end{aligned}$$

Besides, one can show that

$$\begin{aligned} & {}_{EK}D_{0,t;\sigma,\eta}^{-\alpha} g(t)|_{t=t_n} \\ \approx& \frac{t_n^{-\sigma(\alpha+\eta)}}{\Gamma(\alpha)} \Bigg[\sum_{l=1}^{N_{exp}^{(1-\alpha)}} w_l^{(1-\alpha)} \sum_{k=0}^{n-2} \int_{t_k}^{t_{k+1}} e^{-s_l^{(1-\alpha)}(t_n^\sigma - s^\sigma)} \Big(\frac{t_{k+1}^\sigma - s^\sigma}{\tau} g(t_k) + \frac{s^\sigma - t_k^\sigma}{\tau} g(t_{k+1}) \Big) \mathrm{d}s^\sigma \\ &+ \int_{t_{n-1}}^{t_n} (t_n^\sigma - s^\sigma)^{\alpha-1} \Big(\frac{t_n^\sigma - s^\sigma}{\tau} g(t_{n-1}) + \frac{s^\sigma - t_{n-1}^\sigma}{\tau} g(t_n) \Big) \mathrm{d}s^\sigma \Bigg] \\ =& \frac{t_n^{-\sigma(\alpha+\eta)}}{\Gamma(\alpha)} \Bigg[\int_{t_{n-1}}^{t_n} (t_n^\sigma - s^\sigma)^{\alpha-1} (s^\sigma - t_{n-1}^\sigma) \mathrm{d}s^\sigma \cdot \frac{g(t_n)}{\tau} \\ &+ \Big(\sum_{l=1}^{N_{exp}^{(1-\alpha)}} w_l^{(1-\alpha)} \int_{t_{n-2}}^{t_{n-1}} e^{-s_l^{(1-\alpha)}(t_n^\sigma - s^\sigma)} (s^\sigma - t_{n-2}^\sigma) \mathrm{d}s^\sigma + \int_{t_{n-1}}^{t_n} (t_n^\sigma - s^\sigma)^\alpha \mathrm{d}s^\sigma \Big) \cdot \frac{g(t_{n-1})}{\tau} \\ &+ \sum_{l=1}^{N_{exp}^{(1-\alpha)}} w_l^{(1-\alpha)} \sum_{k=1}^{n-2} \Big(\int_{t_k}^{t_{k+1}} e^{-s_l^{(1-\alpha)}(t_n^\sigma - s^\sigma)} (t_{k+1}^\sigma - s^\sigma) \mathrm{d}s^\sigma + \int_{t_{k-1}}^{t_k} e^{-s_l^{(1-\alpha)}(t_n^\sigma - s^\sigma)} (s^\sigma - t_{k-1}^\sigma) \mathrm{d}s^\sigma \Big) \cdot \frac{g(t_k)}{\tau} \\ &+ \sum_{l=1}^{N_{exp}^{(1-\alpha)}} w_l^{(1-\alpha)} \int_0^{t_1} e^{-s_l^{(1-\alpha)}(t_n^\sigma - s^\sigma)} (t_1^\sigma - s^\sigma) \mathrm{d}s^\sigma \cdot \frac{g(t_0)}{\tau} \Bigg] \\ =& \frac{t_n^{-\sigma(\alpha+\eta)}}{\Gamma(\alpha)} \sum_{k=0}^{n} \widehat{C}_{n,k}^{(\sigma,\alpha)} \frac{g(t_k)}{\tau} \triangleq {}_{FEK}\mathbb{D}_{0,t;\sigma,\eta}^{-\alpha} g(t_n), \end{aligned}$$

in which

$$\widehat{C}_{n,k}^{(\sigma,\alpha)} = \begin{cases} \sum_{l=1}^{N_{exp}^{(1-\alpha)}} w_l^{(1-\alpha)} \int_0^{t_1} e^{-s_l^{(1-\alpha)}(t_n^\sigma - s^\sigma)} (t_1^\sigma - s^\sigma) \mathrm{d}s^\sigma, \quad k = 0, \\ \sum_{l=1}^{N_{exp}^{(1-\alpha)}} w_l^{(1-\alpha)} \Big(\int_{t_k}^{t_{k+1}} e^{-s_l^{(1-\alpha)}(t_n^\sigma - s^\sigma)} (t_{k+1}^\sigma - s^\sigma) \mathrm{d}s^\sigma \\ \qquad + \int_{t_{k-1}}^{t_k} e^{-s_l^{(1-\alpha)}(t_n^\sigma - s^\sigma)} (s^\sigma - t_{k-1}^\sigma) \mathrm{d}s^\sigma \Big), \quad 1 \le k \le n-2, \\ \sum_{l=1}^{N_{exp}^{(1-\alpha)}} w_l^{(1-\alpha)} \int_{t_{n-2}}^{t_{n-1}} e^{-s_l^{(1-\alpha)}(t_n^\sigma - s^\sigma)} (s^\sigma - t_{n-2}^\sigma) \mathrm{d}s^\sigma + \frac{1}{\alpha+1} \tau^{\alpha+1}, \quad k = n-1, \\ \frac{1}{\alpha(\alpha+1)} \tau^{\alpha+1}, \quad k = n. \end{cases}$$

By the definition of $C_{n,k}^{(\sigma,\alpha)}$ and $\widehat{C}_{n,k}^{(\sigma,\alpha)}$, it is easy to know that

$$C_{n,n}^{(\sigma,\alpha)} - \widehat{C}_{n,n}^{(\sigma,\alpha)} = 0; \quad |C_{n,k}^{(\sigma,\alpha)} - \widehat{C}_{n,k}^{(\sigma,\alpha)}| \le \epsilon, \quad 0 \le k \le n-1.$$

Lemma 3. *The coefficient* $\{\widehat{C}_{n,k}^{(\sigma,\alpha)}|_{k=0}^{n}\}$ *satisfies*

$$2\widehat{C}_{n,n}^{(\sigma,\alpha)} > \widehat{C}_{n,n-1}^{(\sigma,\alpha)} > \cdots > \widehat{C}_{n,2}^{(\sigma,\alpha)} > \widehat{C}_{n,1}^{(\sigma,\alpha)} > 2\widehat{C}_{n,0}^{(\sigma,\alpha)} > 0; \tag{21}$$

$$\widehat{C}_{n,k}^{(\sigma,\alpha)} - 2\widehat{C}_{n,k-1}^{(\sigma,\alpha)} + \widehat{C}_{n,k-2}^{(\sigma,\alpha)} > 0, \ 3 \le k \le n-1; \tag{22}$$

$$2\widehat{C}_{n,0}^{(\sigma,\alpha)} - 2\widehat{C}_{n,1}^{(\sigma,\alpha)} + \widehat{C}_{n,2}^{(\sigma,\alpha)} > 0, \ \widehat{C}_{n,n-2}^{(\sigma,\alpha)} - 2\widehat{C}_{n,n-1}^{(\sigma,\alpha)} + 2\widehat{C}_{n,n}^{(\sigma,\alpha)} > 0. \tag{23}$$

Proof. It is clear that all $\widehat{C}_{n,k}^{(\sigma,\alpha)}$ are positive. For $1 \le k \le n-2$, the mean value theorem yields

$$\begin{aligned}
&\widehat{C}_{n,k}^{(\sigma,\alpha)} \\
=& \sum_{l=1}^{N_{exp}^{(1-\alpha)}} w_l^{(1-\alpha)} \Big(\frac{1}{s_l^{(1-\alpha)}} \int_{t_k^\sigma}^{t_{k+1}^\sigma} (t_{k+1}^\sigma - s) \mathrm{d}e^{-s_l^{(1-\alpha)}(t_n^\sigma - s)} + \frac{1}{s_l^{(1-\alpha)}} \int_{t_{k-1}^\sigma}^{t_k^\sigma} (s - t_{k-1}^\sigma) \mathrm{d}e^{-s_l^{(1-\alpha)}(t_n^\sigma - s)} \Big) \\
=& \sum_{l=1}^{N_{exp}^{(1-\alpha)}} \frac{w_l^{(1-\alpha)}}{s_l^{(1-\alpha)}} \Big(\int_{t_k^\sigma}^{t_{k+1}^\sigma} e^{-s_l^{(1-\alpha)}(t_n^\sigma - s)} \mathrm{d}s - \int_{t_{k-1}^\sigma}^{t_k^\sigma} e^{-s_l^{(1-\alpha)}(t_n^\sigma - s)} \mathrm{d}s \Big) \\
=& \sum_{l=1}^{N_{exp}^{(1-\alpha)}} \frac{w_l^{(1-\alpha)}}{s_l^{(1-\alpha)}} \int_{t_k^\sigma}^{t_{k+1}^\sigma} e^{-s_l^{(1-\alpha)}(t_n^\sigma - s)} (1 - e^{-s_l^{(1-\alpha)}\tau}) \mathrm{d}s \\
=& \sum_{l=1}^{N_{exp}^{(1-\alpha)}} \frac{w_l^{(1-\alpha)}}{s_l^{(1-\alpha)}} (1 - e^{-s_l^{(1-\alpha)}\tau}) \tau e^{-s_l^{(1-\alpha)}(t_n^\sigma - \xi_k)}, \quad \xi_k \in (t_k^\sigma, t_{k+1}^\sigma),
\end{aligned}$$

which means that $\widehat{C}_{n,k}^{(\sigma,\alpha)}$ increase monotonically with respect to k and is convex.

$$\begin{aligned}
&\widehat{C}_{n,n-1}^{(\sigma,\alpha)} - \widehat{C}_{n,n-2}^{(\sigma,\alpha)} \\
=& \sum_{l=1}^{N_{exp}^{(1-\alpha)}} w_l^{(1-\alpha)} \Big(\int_{t_{n-2}^\sigma}^{t_{n-1}^\sigma} e^{-s_l^{(1-\alpha)}(t_n^\sigma - s)} (s - t_{n-2}^\sigma) \mathrm{d}s - \int_{t_{n-3}^\sigma}^{t_{n-2}^\sigma} e^{-s_l^{(1-\alpha)}(t_n^\sigma - s)} (s - t_{n-3}^\sigma) \mathrm{d}s \Big) \\
&+ \frac{1}{\alpha+1}\tau^{\alpha+1} - \sum_{l=1}^{N_{exp}^{(1-\alpha)}} w_l^{(1-\alpha)} \int_{t_{n-2}^\sigma}^{t_{n-1}^\sigma} e^{-s_l^{(1-\alpha)}(t_n^\sigma - s)} (t_{n-1}^\sigma - s) \mathrm{d}s \\
\ge& \sum_{l=1}^{N_{exp}^{(1-\alpha)}} w_l^{(1-\alpha)} \int_{t_{n-2}^\sigma}^{t_{n-1}^\sigma} e^{-s_l^{(1-\alpha)}(t_n^\sigma - s)} (s - t_{n-2}^\sigma)(1 - e^{-s_l^{(1-\alpha)}\tau}) \mathrm{d}s \\
&+ \frac{1}{\alpha+1}\tau^{\alpha+1} - \int_{t_{n-2}^\sigma}^{t_{n-1}^\sigma} \big[\epsilon + (t_n^\sigma - s)^{\alpha-1}\big](t_{n-1}^\sigma - s) \mathrm{d}s \\
\ge& \frac{\tau^{\alpha+1}}{\alpha(\alpha+1)}(1-\alpha)(2^\alpha - 1) - \frac{\epsilon}{2}\tau^2 > 0 \ \text{ for } \ \epsilon < \frac{2\tau^{\alpha-1}}{\alpha(\alpha+1)}(1-\alpha)(2^\alpha - 1).
\end{aligned}$$

Similarity, we have

$$\begin{aligned}
&2\widehat{C}_{n,n}^{(\sigma,\alpha)} - \widehat{C}_{n,n-1}^{(\sigma,\alpha)} \\
\ge& \frac{2\tau^{\alpha+1}}{\alpha(\alpha+1)}(2 - 2^\alpha) - \frac{\epsilon}{2}\tau^2 > 0 \ \text{ for } \ \epsilon < \frac{4\tau^{\alpha-1}}{\alpha(\alpha+1)}(2 - 2^\alpha).
\end{aligned}$$

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Furthermore, we have

$$
\begin{aligned}
&\widehat{C}^{(\sigma,\alpha)}_{n,1} - 2\widehat{C}^{(\sigma,\alpha)}_{n,0} \\
\geq &\int_{t_1}^{t_2} (t_n^\sigma - s^\sigma)^{\alpha-1}(t_2^\sigma - s^\sigma)\mathrm{d}s^\sigma + \int_0^{t_1} (t_n^\sigma - s^\sigma)^{\alpha-1} s^\sigma \mathrm{d}s^\sigma - 2\int_0^{t_1} (t_n^\sigma - s^\sigma)^{\alpha-1}(t_1^\sigma - s^\sigma)\mathrm{d}s^\sigma \\
&- \epsilon \int_{t_1}^{t_2} (t_2^\sigma - s^\sigma)\mathrm{d}s^\sigma - \epsilon \int_0^{t_1} s^\sigma \mathrm{d}s^\sigma - 2\epsilon \int_0^{t_1} (t_1^\sigma - s^\sigma)\mathrm{d}s^\sigma \\
\geq & C^{(\sigma,\alpha)}_{n,1} - 2C^{(\sigma,\alpha)}_{n,0} - 2\epsilon\tau^2 > 0 \ \text{ for } \ \epsilon < \big(C^{(\sigma,\alpha)}_{n,1} - 2C^{(\sigma,\alpha)}_{n,0}\big)/(2\tau^2),
\end{aligned}
$$

and

$$
\begin{aligned}
&2\widehat{C}^{(\sigma,\alpha)}_{n,0} - 2\widehat{C}^{(\sigma,\alpha)}_{n,1} + \widehat{C}^{(\sigma,\alpha)}_{n,2} \\
\geq & 2C^{(\sigma,\alpha)}_{n,0} - 2C^{(\sigma,\alpha)}_{n,1} + C^{(\sigma,\alpha)}_{n,2} - 4\epsilon\tau^2 > 0 \ \text{ for } \ \epsilon < \big(2C^{(\sigma,\alpha)}_{n,0} - 2C^{(\sigma,\alpha)}_{n,1} + C^{(\sigma,\alpha)}_{n,2}\big)/(4\tau^2).
\end{aligned}
$$

In addition, one has

$$
\begin{aligned}
&\widehat{C}^{(\sigma,\alpha)}_{n,n-2} - 2\widehat{C}^{(\sigma,\alpha)}_{n,n-1} + 2\widehat{C}^{(\sigma,\alpha)}_{n,n} \\
\geq & C^{(\sigma,\alpha)}_{n,n-2} - 2C^{(\sigma,\alpha)}_{n,n-1} + 2C^{(\sigma,\alpha)}_{n,n} - 2\epsilon\tau^2 \\
= & \frac{\tau^{\alpha+1}}{\alpha(\alpha+1)}(3^{\alpha+1} - 2^{\alpha+3} + 7) - 2\epsilon\tau^2 > 0 \ \text{ for } \ \epsilon < \frac{3^{\alpha+1} - 2^{\alpha+3} + 7}{2\alpha(\alpha+2)}\tau^{\alpha-1}.
\end{aligned}
$$

□

Theorem 2. *Suppose the function $g \in C^2[t_0, t_n]$, $\alpha \in (0,1)$. we have*

$$
\begin{aligned}
&|{}_{EK}D^{-\alpha}_{0,t;\sigma,\eta} g(t)|_{t=t_n} - {}_{FEK}\mathbb{D}^{-\alpha}_{0,t;\sigma,\eta} g(t_n)| \\
\leq & \frac{t_n^{-\sigma\eta}}{8\Gamma(1+\alpha)} \max_{t_0<\xi_k(s)<t_n} |\delta_\sigma^2 g(\delta_k(s))|\tau^2 + \frac{\epsilon t_n^{-\sigma(\alpha+\eta-1)}}{\Gamma(\alpha)} \max_{t_0\leq t\leq t_{n-1}} |g(t)|, \quad n = 1, 2, \cdots, N.
\end{aligned}
$$

Proof. Since

$$
\begin{aligned}
&{}_{EK}D^{-\alpha}_{0,t;\sigma,\eta} g(t)|_{t=t_n} - {}_{FEK}\mathbb{D}^{-\alpha}_{0,t;\sigma,\eta} g(t_n) \\
= & \big[{}_{EK}D^{-\alpha}_{0,t;\sigma,\eta} g(t)|_{t=t_n} - {}_{EK}\mathbb{D}^{-\alpha}_{0,t;\sigma,\eta} g(t_n)\big] + \big[{}_{EK}\mathbb{D}^{-\alpha}_{0,t;\sigma,\eta} g(t_n) - {}_{FEK}\mathbb{D}^{-\alpha}_{0,t;\sigma,\eta} g(t_n)\big] \\
= & r_n + \widehat{r}_n.
\end{aligned} \tag{24}
$$

With the help of Theorem 1, we obtain

$$
|r_n| \leq \frac{t_n^{-\sigma\eta}}{8\Gamma(1+\alpha)} \max_{t_0<\xi_k(s)<t_n} |\delta_\sigma^2 g(\delta_k(s))|\tau^2.
$$

On the other hand, Lemma 2 yields

$$
\begin{aligned}
|\widehat{r}_n| = & \frac{t_n^{-\sigma(\alpha+\eta)}}{\Gamma(\alpha)} \sum_{k=0}^{n-2} \int_{t_k}^{t_{k+1}} \Pi_{1,k} g(s) \Big| (t_n^\sigma - s^\sigma)^{\alpha-1} - \sum_{l=1}^{N_{exp}^{(1-\alpha)}} w_l^{(1-\alpha)} e^{-s_l^{(1-\alpha)}(t_n^\sigma - s^\sigma)} \Big| \mathrm{d}s^\sigma \\
\leq & \frac{\epsilon t_n^{-\sigma(\alpha+\eta)}}{\Gamma(\alpha)} \int_{t_0^\sigma}^{t_{n-1}^\sigma} |g(s)| \mathrm{d}s \leq \frac{\epsilon t_n^{-\sigma(\alpha+\eta-1)}}{\Gamma(\alpha)} \max_{t_0\leq t\leq t_{n-1}} |g(t)|.
\end{aligned}
$$

Thus, we complete the proof. □

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3.2. The Fast Difference Scheme

Now, we present a fast difference scheme for the problem (1)-(3). Similar to the treatment of the direct difference scheme, applying (20) in temporal direction, we get

$$\begin{cases} U_i^n = \dfrac{t_n^{-\sigma(\alpha+\eta)}}{\Gamma(\alpha)}\Big[\sum\limits_{l=1}^{N_{exp}^{(1-\alpha)}} w_l^{(1-\alpha)} F_{l,i}^n + \dfrac{\tau^\alpha}{\alpha+1} t_{n-1}^{\sigma\eta}\delta_x^2 U_i^{n-1} + \dfrac{\tau^\alpha}{\alpha(\alpha+1)} t_n^{\sigma\eta}\delta_x^2 U_i^n\Big] \\ \quad +F_i^n + \widehat{R}_i^n, \quad 1\le i\le M-1,\ 1\le n\le N, & (25)\\ F_{l,i}^1 = 0, \quad 1\le l\le N_{exp}^{(1-\alpha)},\ 1\le i\le M-1, & (26)\\ F_{l,i}^n = e^{-s_l^{(1-\alpha)\tau}} F_{l,i}^{n-1} + A_l \dfrac{t_{n-2}^{\sigma\eta}}{\tau}\delta_x^2 U_i^{n-2} + B_l \dfrac{t_{n-1}^{\sigma\eta}}{\tau}\delta_x^2 U_i^{n-1}, \\ \qquad 1\le l\le N_{exp}^{(1-\alpha)},\ 1\le i\le M-1,\ 2\le n\le N, & (27) \end{cases}$$

where there exists a constant c_2 such that

$$|\widehat{R}_i^n| \le c_2(\tau^2 + h^2 + \epsilon), \quad 1\le i\le M-1,\ 1\le n\le N. \tag{28}$$

Omitting the small term $\widehat{R}_i^n$ in (25), replacing the grid functions $\{U_i^n, F_{l,i}^n\}$ by their numerical approximations $\{u_i^n, f_{l,i}^n\}$ and noticing the initial-boundary value conditions (15)-(16), we arrive at following fast difference scheme

$$\begin{cases} u_i^n = \dfrac{t_n^{-\sigma(\alpha+\eta)}}{\Gamma(\alpha)}\Big[\sum\limits_{l=1}^{N_{exp}^{(1-\alpha)}} w_l^{(1-\alpha)} f_{l,i}^n + \dfrac{\tau^\alpha}{\alpha+1} t_{n-1}^{\sigma\eta}\delta_x^2 u_i^{n-1} + \dfrac{\tau^\alpha}{\alpha(\alpha+1)} t_n^{\sigma\eta}\delta_x^2 u_i^n\Big] \\ \quad +F_i^n, \quad 1\le i\le M-1,\ 1\le n\le N, & (29)\\ f_{l,i}^1 = 0, \quad 1\le l\le N_{exp}^{(1-\alpha)},\ 1\le i\le M-1, & (30)\\ f_{l,i}^n = e^{-s_l^{(1-\alpha)\tau}} f_{l,i}^{n-1} + A_l \dfrac{t_{n-2}^{\sigma\eta}}{\tau}\delta_x^2 u_i^{n-2} + B_l \dfrac{t_{n-1}^{\sigma\eta}}{\tau}\delta_x^2 u_i^{n-1}, \\ \qquad 1\le l\le N_{exp}^{(1-\alpha)},\ 1\le i\le M-1,\ 2\le n\le N, & (31)\\ u_i^0 = \varphi_i, \quad 1\le i\le M-1, & (32)\\ u_0^n = 0, \quad u_M^n = 0, \quad 0\le n\le N. & (33) \end{cases}$$

4. Numerical Analysis

This section provides numerical validation demonstrating the effectiveness of our novel discretization framework for Erdélyi-Kober integration operator. Through a detailed comparative analysis of CPU time between the direct difference scheme (17)-(19) and the fast difference scheme (29)-(33), our computational experiments reveal that the accelerated algorithm significantly enhances computational efficiency with complexity reduced from $O(N^2)$ to $O(N\log N)$. The absolute tolerance error $\epsilon = 10^{-12}$ is adopted in our numerical simulations.

Example 1. *We choose the exact solution of the problem* (5) *to be* $u(x,t) = t^{\alpha+2-\sigma\eta}\sin x$. *The right hand side function is determined by the exact solution as follows:*

$$F(x,t) = \left[1 + \frac{\Gamma(1+\frac{\alpha+2}{\sigma})}{\Gamma(\alpha+1+\frac{\alpha+2}{\sigma})}\right] t^{\alpha+2-\sigma\eta}\sin x.$$

Let the finial time $T=1$. The spatial domain $\Omega = (0, 2\pi)$. To examine the accuracy and efficiency of the difference schemes, the L^2-norm error

$$E(h,\tau) = \max_{0\le n\le N} \|U^n - u^n\|$$

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is recorded in each run and the experimental orders are evaluated by

$$R_\tau = \log_2\left(\frac{E(h,2\tau)}{E(h,\tau)}\right), \quad R_h = \log_2\left(\frac{E(2h,\tau)}{E(h,\tau)}\right).$$

The accuracy tests are performed by taking the parameter $\eta = 0.2$, $\sigma = 0.9$ for three fractional orders $\alpha = 0.1, 0.5$ and 0.9. The error analyses for both the direct and accelerated difference schemes establish an asymptotic convergence order of $O(\tau^2 + h^2)$ with the theoretical upper bounds demonstrating remarkable agreement with empirical convergence rates observed in our numerical simulations.

Table 1 shows the numerical accuracy and CPU time of two difference schemes in time. To isolate temporal errors, we employ a refined spatial discretization with $h = \pi/2500$. The numerical verification results confirm second-order temporal accuracy, demonstrating remarkable agreement between empirical convergence rates and theoretical predictions.

Table 1. Truncation errors and temporal convergence orders.

α	N	Direct difference scheme (17)-(19)			Fast difference scheme (29)-(33)		
		$E(h,\tau)$	R_τ	CPU(s)	$E(h,\tau)$	R_τ	CPU(s)
0.1	80	1.02e-5		34.98	1.02e-5		33.46
	160	2.67e-6	1.9323	78.63	2.67e-6	1.9323	75.52
	320	6.60e-7	2.0159	214.52	6.60e-7	2.0159	161.22
	640	1.50e-7	2.1320	325.09	1.50e-7	2.1320	320.56
0.5	80	2.78e-5		34.74	2.78e-5		38.59
	160	6.97e-6	1.9943	89.78	6.97e-6	1.9943	78.55
	320	1.72e-6	2.0195	155.16	1.72e-6	2.0195	157.49
	640	3.97e-7	2.1129	326.34	3.97e-7	2.1129	316.44
0.9	80	3.09e-5		53.27	3.09e-5		33.52
	160	7.70e-6	2.0035	141.53	7.70e-6	2.0039	73.62
	320	1.91e-6	2.0158	278.45	1.91e-6	2.0147	154.67
	640	4.55e-7	2.0661	468.05	4.46e-7	2.0954	314.03

Next, we turn to test the numerical accuracy and corresponding CPU time of two difference schemes in space. Fix a fine temporal step size $\tau = 1/20000$ such that the spatial errors dominate the temporal errors. Table 2 records the numerical results of two difference schemes when solving the example for different α. One can observe that convergence orders of two difference schemes are good agreement with theoretical results. Furthermore, the computational cost of the direct difference scheme is largely higher than that of the fast difference scheme, which also reflects from the side the high efficiency of the fast algorithm.

Table 2. Truncation errors and spatial convergence orders.

α	M	Direct difference scheme (17)-(19)			Fast difference scheme (29)-(33)		
		$E(h,\tau)$	R_h	CPU(s)	$E(h,\tau)$	R_h	CPU(s)
0.1	8	2.44e-2		2315.75	2.44e-2		0.85
	6	6.09e-3	2.0038	5880.66	6.09e-3	2.0038	0.95
	32	1.52e-3	2.0010	3411.01	1.52e-3	2.0010	1.08
	64	3.80e-4	2.0003	2217.43	3.80e-4	2.0003	2.06
0.5	8	1.78e-2		2214.54	1.78e-2		0.88
	16	4.46e-3	1.9968	5235.00	4.46e-3	1.9968	0.99
	32	1.12e-3	1.9992	2617.88	1.12e-3	1.9992	1.14
	64	2.79e-4	1.9998	5548.97	2.79e-4	1.9998	2.02
0.9	8	1.10e-2		2791.54	1.10e-2		1.25
	16	2.78e-3	1.9896	2885.54	2.78e-3	1.9899	1.12
	32	6.95e-4	1.9974	2475.10	6.94e-4	1.9987	1.41
	64	1.74e-4	1.9994	2438.99	1.73e-4	2.0044	2.32

5. Concluding Remarks

This paper presented two temporal second-order difference schemes for solving time-fractional diffusion equations involving the Caputo-type Erdélyi-Kober operator, accompanied by rigorous error estimation. While a complete theoretical framework for the difference schemes remains to be developed, we nevertheless establish critical properties of the temporal L1-type formula's coefficients. Domain knowledge suggests that these coefficient properties constitute fundamental prerequisites for rigorous stability and convergence analyses in such fractional calculus contexts. Consequently, the present work establishes a methodological foundation that may inform subsequent theoretical investigations. The numerical validation section conclusively demonstrates the theoretical convergence rates through systematic implementation.

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