Article

Twisted Wang Transform Distribution

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Abstract. The twisted Wang transform distribution family, defined as the composition of parameter shifted inverse CDF function with an original CDF function, is found to be most suitable for matching low shape factor distributions, characterizing hard to fit or to simulate reinsurance portfolio losses for some perils from our previous study. Among them, the best form for matching a hard-to-fit empirical loss distribution for a specific peril, is the Exponential Fractional Extra Power 0 Distribution in (0,1) with CDF:

$$\mathbf{E}^{-\mathbf{q}\frac{(1-\mathbf{x}^{\mathbf{k}})^{\mathbf{m}}}{\mathbf{x}^{\mathbf{l}}}}.$$

The simplest yet still a good form of this family is the Transformed Hyperbolic Tangent Distribution with CDF:

$$\frac{\mathbf{1}}{\mathbf{1}+\mathbf{q}^{\frac{\mathbf{1}-\mathbf{x}^{\mathbf{k}}}{\mathbf{x}^{\mathbf{k}}}}},$$

which has analytical formulas for the moments. The twisted Wang transform distribution family is compared and confirmed to be superior to all other well-known distribution families through extensive numerical optimization practice, distribution forms guesses, and computer-aided exploration experiments.

Keywords: twisted Wang transform; shape factor; EP curve; reinsurance; numerical optimization

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1. Introduction

1.1 Background

In our previous study [1], it is found that a distribution with high coefficient of variation (CV) is hard to simulate with fixed simulation iterations. And such a distribution in extreme case is represented by a bi-valued distribution. To interpolate a bi-valued distribution cumulative distribution function (CDF) by smooth function, the hyperbolic tangent function $\frac{e^x-e^{-x}}{e^x+e^{-x}}$ is considered. With transformation to re-lay the range to (0, 1), and shift the central point, we get: $\frac{1}{1+qe^{-2x}}$. When compositing a transformation of the domain from (-∞,∞) also to (0,1): Log$\frac{x}{1-x}$, we get $\frac{1}{1+q(\frac{1-x}{x})^2}$. Considering that the power function is (0,1)→(0,1) and (0,∞)→(0,∞), we can add different power to x and etc. to add asymmetry to the distribution (for experiment on power function, see [2][3]). The final transformation is

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$$\frac{1}{1+q(\frac{1-x^k}{x^l})^m}:(0,1)\to(0,1). \quad (1)$$

When $k=l\&\&m=1$, the transformation become $\frac{1}{1+q\frac{1-x^k}{x^k}}$, we name it Transformed Hyperbolic Tangent (THT) distribution, whose moments can be expressed explicitly by hypergeometric function. When $k=1$ and with some reparametrization, it become $\frac{1}{1+q\frac{(1-x)^k}{x^l}}$, we name it Alternative Transformed Hyperbolic Tangent (ATHT) distribution. When $k\neq l\&\&m=1$, it is $\frac{1}{1+q\frac{1-x^k}{x^l}}$, we name it Generalized Transformed Hyperbolic Tangent (GTHT) distribution, and when $k\neq l\&\&m\neq 1$, with reparametrization of Eq. 1 to $\frac{1}{1+q\frac{(1-x^k)^m}{x^l}}$, we name it Generalized Transformed Hyperbolic Tangent Extended (GTHTE) distribution: even though their moments cannot be explicitly expressed by known functions, we can use numerical integration to get them.
In [2][3], a shape factor (SF) is defined using kurtosis divided by squared skewness, which can measure the intensity of the asymmetry and steepness of the distribution probability density function (PDF) when the SF is approaching 1; where it is also observed that the finite interval distribution generally works better than infinite range distribution for fitting small SF distributions, thus point to study for distributions just in the unit interval. Although the so created THT in our first round of study cannot fit such cases, but GTHT and GTHTE tested work better.
Completely independent of the CV and SF reasoning, the construction of GTHTE by compositing CDF with Quantile function also largely fall in the general Wang Transformation schema [4], while the most general Wang Transformation can use different type of distribution for CDF and Quantiles, we just used the CDF and Quantile function of the same distribution family but with different parameters for it. Also note that the positive real numbers domain is scale invariant, and the whole real number domain is scale and shift invariant which is a major twisting in the Wang Transformation. We use the parameter deviation of CDF and Quantile in place of a direct central shift. The power function transformation is also positive real number domain invariant and the unit interval domain invariant, we use this twist too to add asymmetry to the distribution.
Now apply this idea to the most natural form of transformation from $(-\infty,\infty)\to(0,1)$: $e^{-e^{-x}}, \frac{1}{1+e^{-x}}, 1-e^{-e^{x}}$, the first form gives power function, the second gives GTHTE, and the third gives function below:

$$(1-(1-x^k)^q)^l:(0,1)\to(0,1). \quad (2)$$

This is a generalized Kumaraswamy distribution (GK), called EKw in [5].
Apply similar idea to the most natural form of transformation from $(0,\infty)\to(0,1)$: $1-e^{-x}, \frac{x}{1+x}$, the first form still gives generalized Kumaraswamy distribution, the second gives function below:

$$(\frac{qx^k}{1+(q-1)x^l})^m:(0,1)\to(0,1). \quad (3)$$

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We name it the Twisted Wang Transform (TWT) distribution, even though all the GTHTE type and generalized Kumaraswamy distribution belongs to this category, but only Eq. 3 do not have alternative names.
More generally, the twisted Wang transform can be thought of as a way of construct CDF directly from composition of simple monotonic functions, different from approaches which define PDF first. This will give us large set of candidate distributions to be tested. And since the two different lines of reasoning arrive at the same construction, the twisted Wang transform is more than merely an arbitrary algebraic manipulation for generating the distribution forms.

1.2 Research objective

Our research objective is to find probability distribution that can fit distributions with small shape factors. The Quantile function of loss for a given probability in the reinsurance industry, inverse function of the CDF, is called VaR; and the average quantile above a given loss is called TVaR. The TVaR curve for all the probabilities is called the EP curve. The fitness of the distribution is judged by the matching of the EP curves. For risk management purpose, usually only the higher end of the EP curve is of interest. We will use three points of the EP curve, TVaR of probability 0.96, 0.99, and 0.996, as our main matching objective. A loss reference portfolio for the North American Tornado Hail peril (NATH) [3] is used as a test case that can be used to compare the well-known probability distribution families, as well as our tailor made probability distributions, to see what kind of distribution function is the best for describing the small SF distributions. For the top distribution thus found, conduct further research of its EP curve change tendency with respect to its parameters changes (similar to SF tendency study in [2][3]).

1.3 Method

Numerical integration, optimization [6], and root finding are used to calculate the VaR and TVaR for given distribution; the calculated numbers can be regarded as the theoretical limit that simulation can reach when iterations turn to infinite, while the simulated numbers are what are seen in practice. If quantile function $Q(t)$ can be solved in explicit form, we use the following formula for TVaR:

$$\frac{\int_q^1 Q(t)\,dt}{1-q}. \tag{4}$$

When only the PDF is known or the CDF is hard to find inverse, we use FindRoot to calculate VaR for given quantile q, and use the following formula for TVaR (when distribution is on (0,1)):

$$\frac{\int_{\mathrm{VaR}}^1 x\mathrm{PDF}(x)\,dx}{1-q}. \tag{5}$$

Numerical experiments confirm that the skewness and kurtosis are intrinsically correlated with EP curve shape. Optimize by matching skewness and kurtosis yield better TVaR match than doing the TVaR match directly. Combining the two objectives together yield even better results. The appeared redundant or repetitive objectives actually help with finding the optimum. This is also observed in [7][8]. So the combination will be our objective function for optimization.
Dimension reduction method is used to overcome the numerical optimization difficult, starting from distribution with fewer parameters, gradually add more parameters in. We use Mathematica Table function, or one dimensional plot to find the initial value approximate position or region, and

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use Mathematica FoldList function to do one dimensional iterative search when fix a parameter, and change that parameter gradually using previously find solution as consequent starting point. Local minimization FindMinimum or one dimensional line search by FoldList is faster, so it is good for find parameters approximate regions, i.e., should it be large or very small. But the local minimization seems never find the true solution, and at times give the wrong conclusion that fit is impossible, if the true solution is outside of the conceivable and tested ranges. So the slower global minimization function NMinimize and iterative through NestWhile will always be used (details of using these functions and the complete code are in the Appendix A and B).

The main contents of this paper are detailed report of our numerical experiments for finding the best fit of the NATH distribution.

2. Results

2.1 Preliminary results

The NATH has SF 1.83 and {Mean, StandardDeviation, Skewness, Kurtosis} = {7418611.10904006, 9517336.93024634, 5.99378199789956, 65.8901734355745}. Its TVaR for the three probabilities is {41113929.8424838, 68867612.8345741, 8579409.238445}. Even though only the mean, standard deviation and the three TVaRs are directly used in reinsurance pricing and equity allocating, the study in [2] shows that the skewness is the most important factor that affect the EP curve shape, the kurtosis and the shape factor follows. Optimization with the additional constraints of matching these characteristics yield better results than just fit the three TVaRs. Distribution with finite range of values fit better, as well as distribution with infinite range values but are truncated to finite interval. For example, the truncated generalized gamma (TGG) distribution has error of 3.37%, and the largest deviation for the whole spectrum of TVaR practically used is only 12.5% (Table 1 and 2).

Table 1. NATH 0.96 0.99 0.996 TVaR fit error.

Distribution	Fit error
GTHTE	1.52
TGG	3.37
GTHT	3.79
TLG	4.4
NIG	4.6
GB2	5
GB1	5
GH	5
GnH	5.4
DoubleLog	5.59
FD4	5.8
Kumaraswamy	6
EIDL	6.48
GG	6.5
Meixner	7.3
LogGamma	7.6
VG	8
Beta	8.7
EIG	9.4

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Fit error is the maximum of the absolute of the ratio of fitted TVaR to input TVaR minus 1.

In Table 1, the TLG is the truncated Log Gamma distribution, GH is the generalized hyperbolic distribution. DoubleLog distribution has CDF $\frac{\int_0^x (-\text{Log}[1-t])^b (-\text{Log}[t])^a\,dt}{\int_0^1 (-\text{Log}[1-x])^b (-\text{Log}[x])^a\,dx}$. EIDL (abbreviation of extended by power function factor infinite interval double log distribution) is probability distribution in $(1,\infty)$ with PDF proportional to $(\text{Log}[x])^a(-\text{Log}[1-\frac{1}{x}])^b x^{-c}$. These two are some of the results out of the attempt to explore the CDF function space generated from functions of types other than the power function and exponential function. All other distributions are well known and are studied in [2][3].

2.2 Results with the broad twisted Wang transform distribution family

Through extensive numerical experiment, the best fit of the broad twisted Wang transform distribution family is in Table 2.

Table 2. Fit power of TWT families of distributions.

Distribution	Fit error	Maximum fit error
GTHTE	1.52	16.15
PTHT	1.77	5.11
THT	2.09	8.55
GTHT	2.52	11.89
APTHT	2.55	11.29
TGG	3.37	12.54
THT	3.81	5.58
PK	4.95	46.99
GK	5.1	48.04
Kumaraswamy	6.09	41.09
GG	6.52	42.76
Beta	8.61	14.23
TWT	15.05	25.25

Fit error is the maximum of the absolute of the ratio of fitted TVaR to input TVaR minus 1 for probability 0.96 to 0.996.

The maximum fit error in Table 2 is the maximum of the absolute of the ratio of fitted TVaR to input TVaR minus 1 for probability 0.5 to 0.99999; it is usually attained near probability 0.9999 or 0.99999. Now that the best fit for the three selected quantile TVaR is close to within 3%, so the wider quantile range is checked to see whether we find the "real" analytical expression for the empirical distribution.

In Table 2, the Beta, GG, Kumaraswamy, and TGG are for comparison, where Kumaraswamy used 4 parameters, and TGG used 5 parameters. The PTHT is the productive transformed hyperbolic tangent distribution, with CDF $\frac{1}{1+q\frac{(1-x^k)(1-x^m)}{x^l}}$; the APTHT is the alternative productive transformed hyperbolic tangent distribution, with CDF $\frac{1}{1+q\frac{(1-x^k)(1-x)^m}{x^l}}$. They are alteration of the GTHTE distribution by a similar productive factor. An additional extension of the Kumaraswamy distribution (KK), called $KwKw$ in [5], with CDF $1-(1-(1-(1-x^k)^q)^l)^m$, is not included in Table. 2 since we cannot find a better fit than GK from it, perhaps due to numerical optimization difficulties with KK. The

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PK is the productive Kumaraswamy distribution, with CDF $(1-(1-x^k)^q)(1-(1-x^m)^l)$, similar in spirit to PTHT.

The GTHTE and TWT are special case of the twisted Wang transform extended distribution TWTE, with CDF $\frac{1}{(x^n+q\frac{(1-x^k)^m}{x^l})^o}$, since $(\frac{qx^k}{1+(q-1)x^l})^m=\frac{1}{(x^{l-k}+\frac{1}{q}\frac{1-x^l}{x^k})^m}$. The alternative generalization for THT, named ATHT, with CDF $\frac{1}{1+q\frac{(1-x)^k}{x^l}}$ would be another choice in place of GTHT.

While the ATHT do not have special subfamily, the GTHT have the special subfamily of THT that have analytical representation for all moments:

$$M[h]=\frac{q-\text{Hypergeometric2F1}\left[1,\frac{h}{k},\frac{h+k}{k},\frac{-1+q}{q}\right]}{-1+q}$$
$$=1-\frac{h\text{Hypergeometric2F1}\left[1,\frac{h+k}{k},2+\frac{h}{k},\frac{-1+q}{q}\right]}{(h+k)q}$$
$$=\frac{q-\frac{h}{k}\text{HurwitzLerchPhi}\left[1-\frac{1}{q},1,\frac{h}{k}\right]}{q-1}. \quad (6)$$

2.3 Twisted Wang transform distribution second round experiment results

There are three basic types of function forms: power, logarithm, and exponential. The exponential is widely used in distributions, such as the TGG, EIG, NIG, and GH. We also tried the logarithms in DoubleLog and EIDL as described in section 2.1. They seem do not stand out than purely power function embedded in fractional construction. The exponential form will be studied further later, we will focus on power function first, and comparing them afterwards. For the three types of power construction (TWT, GTHTE, GK, in section 1.1), or the TWTE types of distributions, the GTHTE type performs best, followed by Kumaraswamy type. For GTHTE type, the alternative form of factor $(1-x)^m$, such as in ATHT, APTHT, seems do not work as well as the form $(1-x^m)$. To test or confirm this observation, other distributions ATHT(alternative THT), PTHT2(productive THT with 2 additional terms), THTE(THT extended), THTE1 (in Table 3) are formed.

Table 3. Probability distributions to study.

Distribution	CDF
GTHTE	$\frac{1}{1+q\frac{(1-x^k)^m}{x^l}}$
PTHT	$\frac{1}{1+q\frac{(1-x^k)(1-x^m)}{x^l}}$
THT	$\frac{1}{1+q\frac{1-x^k}{x^k}}$
GTHT	$\frac{1}{1+q\frac{1-x^k}{x^l}}$
APTHT	$\frac{1}{1+q\frac{(1-x^k)(1-x)^m}{x^l}}$
KK	$1-(1-(1-(1-x^k)^q)^l)^m$
PK	$(1-(1-x^k)^q)(1-(1-x^m)^l)$
GK	$(1-(1-x^k)^q)^l$

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Kumaraswamy	$1-(1-x^k)^q$
TWT	$(\frac{qx^k}{1+(q-1)x^l})^m$
TWTE	$\frac{1}{(x^n+q\frac{(1-x^k)^m}{x^l})^o}$
TWTE0	$\frac{1}{(1+q\frac{(1-x^k)^m}{x^l})^n}$
TWTE1	$\frac{1}{x^n+q\frac{(1-x^k)^m}{x^l}}$
ATHT	$\frac{1}{1+q\frac{(1-x)^k}{x^l}}$
PTHT2	$\frac{1}{1+q\frac{(1-x^k)(1-x^m)(1-x^n)}{x^l}}$
THTE	$\frac{1}{(1+q(\frac{1-x^k}{x^k})^l)^m}$
THTE1	$\frac{1}{1+q(\frac{1-x^k}{x^k})^l}$
APTHTE0	$\frac{1}{(1+q\frac{(1-x^k)(1-x)^m}{x^l})^n}$
APK	$(1-(1-x^k)^q)x^l$
EF	$E^{-q\frac{1-x^k}{x^l}}$
EFP	$x^m E^{-q\frac{1-x^k}{x^l}}$
ELP	$E^{-q(-\text{Log}[x])^k}$
ERLP	$E^{\frac{q}{\text{Log}[1-x^k]}}$
ERLPE	$E^{-\frac{q}{(-\text{Log}[1-x^k])^l}}$
THTP	$\frac{x^l}{1+q\frac{1-x^k}{x^k}}$
TWTE1*	$\frac{x^n}{1+q\frac{(1-x^k)^m}{x^l}}$
GTHTP	$\frac{x^m}{1+q\frac{1-x^k}{x^l}}$
PTHT2P	$\frac{x^o}{1+q\frac{(1-x^k)(1-x^m)(1-x^n)}{x^l}}$

In Table 3, the APK is the alternative productive Kumaraswamy distribution, to check the effectiveness of productive CDF formed from two CDF with one the power function x^l. The EF is the exponential fractional distribution, EFP is the exponential fractional power distribution, ELP is the exponential logarithmic power distribution, ERLP is the exponential reciprocal logarithmic power distribution, ERLPE is the exponential reciprocal logarithmic power extended distribution: all these distributions are testing generalized form twisted Wang transform involving exponential (and logarithmic) functions, and the GTHT or GTHTP can be regarded as the first order expansion of the EF or EFP distribution. The ALIG is the alternative log InverseGaussian distribution: 1-Exp[-InverseGaussian]; whose PDF has a main factor like the ERLP CDF. The TWTE1* is a reformulation of TWTE1 distribution with opposite signs of allowable region of the parameter n.

We find that even if starting from the best solution that GTHTE get, the TWTE1 cannot find a better solution. If search afresh by its own, TWTE1 will get much worse solution. So either the addition of the x^n post numerical difficult for optimization, or that term is not compatible with small SF distributions. The lesson from this is that a broader distribution family may not be as good as a narrower distribution family for distribution fit in practice. The new round of study starting from best solution from THT, GTHTE and etc. (section 2.2) yield results in Table 4, with the same definitions of the columns as Table 2.

Table 4. Fit power of TWT families of distributions more runs.

Distribution	Fit error	Maximum fit error
EFP	1.412	8.33
TWTE0	1.515	9.51
APTHT	1.519	15.07
GTHTE	1.523	8.59
GTHTP	1.523	13.24
GTHTE	1.523	16.15
TWTE1	1.525	16.1
PTHT2P	1.55	6.49
TWTE1*	1.558	6.12
THTP	1.62	5.16
GTHT	1.644	7.72
PTHT2	1.651	4.61
PTHT	1.767	5.11
EF	1.785	16.65
THT	2.086	8.55
THTP	2.193	6.38
THTE	2.237	9.37
THTE1	2.24	10.31
THTE	2.326	7.95
PTHT2	2.555	10.7
ERLPE	2.653	12.34
GTHTE	3.175	5.35
ATHT	3.271	30.59
TGG	3.375	12.54
APTHTE0	3.525	35.36
APK	3.548	9.47
THTP	3.727	5.92
ELP	3.788	37.42
THT	3.814	5.58

Fit error is the maximum of the absolute of the ratio of fitted TVaR to input TVaR minus 1 for probability 0.96 to 0.996.

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It is a commonly observed phenomenon that a special case of a broader distribution family whose optimum solution is usually a "repeller": most optimization starting from initial value deviated from the optimum will not converge to it but will diverge away from it. The TWTE1 vs GTHTE (in Table 4), GTHT vs THT (in Table 2), and NIG vs GH (in Table 1) are such cases. The THTE, THTE1 have poorer performance than THT. The GTHTE result start from best PTHT2 may be a rare exception, it actually improved than optimization from GTHTE alone. The steps used in the numerical optimizations determined the results to a large extent. But in general, the simpler the CDF form, the easier the optimization process, and the better the results.
Another observation is that a distribution fit starting from another related distribution best fit or starting from a simplified objective best fit will usually give sub-optimum solution, not as good as one starting without using any of such hints. ATHT vs THT is such an example.
The vast numerical experiment with various TWT types of distribution seems suggest that the fit capability limit of them is about 1.5% residual error. It is an indication that types of distribution with function form other than fraction and power may need to be considered. Further attempt of the TWT type's distribution as well as distributions with exponential or logarithmic function factor are made and are inserted in Table 3 and Table 4 for comparison.
In theory, a more general distribution family should fit better than a narrower distribution family. The anomaly observed showed the drawbacks of numerical approach. The ranks or results from numerical optimization may be unreliable or completely wrong. To address the reliability problem of numerical approach, or in alternative terminology, the model risk or numerical risk, we will follow the philosophy represented by the Perl motto: we must try alternative approaches; the more and different approaches tried with matching or compatible results, the less likely that the results are wrong.
Since the fit error is already close to 1.5%, the simplest alternative is to change the objective from fit error to maximum fit error and use different from iterative optimization method. But we will first try more different approaches, analytical and graphical methods, and return to numerical experiment lastly.

2.4 THT Distribution study by plot

The command:

```
Tooltip[ContourPlot[{sktht[q,k]==4.99378199789956`, sktht[q,k]==5.99378199789956`, sktht[q,k]==6.99378199789956`, kttht[q,k]==60.8901734355745`, kttht[q,k]==65.8901734355745`, kttht[q,k]==70.8901734355745`}, {k, .0, .0}, {q, .0001, .0009}, Contours->Automatic, ContourLabels->None, ContourStyle->{Red, Orange, Pink, Blue, Purple, Magenta}, FrameLabel->Automatic, ImageSize->1200, Frame->True, FrameTicks->All, GridLines->Automatic, PlotLabel->{sktht, kttht}, PlotLegends->{4.99, 5.99, 6.99, 60.89, 65.89, 70.89}, PlotRange->Automatic], Dynamic[MousePosition["Graphics"]]]
```

plot the THT distribution skewness and kurtosis, Figure 1. This plot shows the parameters change tendency: to keep the same shape characteristics of skewness and kurtosis, an increasing k need a decreasing q. This observation for THT and TWTE family of distribution in general, is critical for an iterative approach to get better solution, using each solution as the boundary constraints for the next run.

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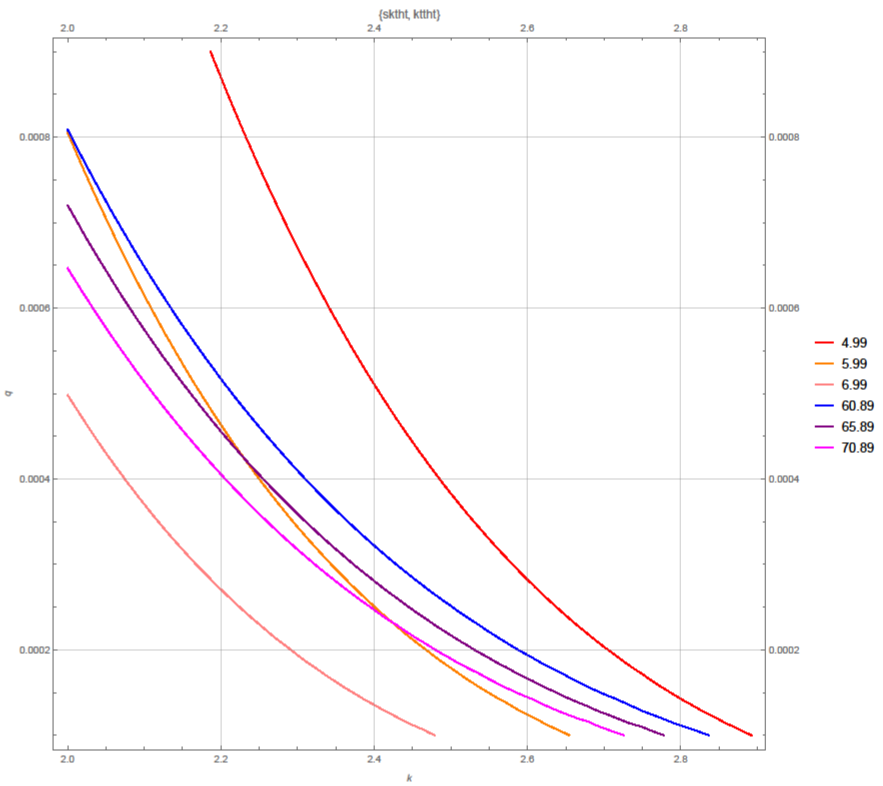

Fig. 1. Contour Plot of THT Skewness and Kurtosis.

2.5 Some notes about THT distribution symbolic analysis

When numerical experiments find some emerging pattern, it is desirable to make further study with symbolic calculation, in case analytical formula is available, which can confirm the numerical results, and is also a better form for representing and sharing the knowledge gained from numerical experiments.
Even though a formula approach may not find better solution than numerical approach, the speed improvement will be significant. The THT distribution is such an example, using numerical integration took 1913.64 seconds in performing the numerical optimization, while using the following formulas for VaR, TVaR and etc. took 3.36 seconds, more than 500 times faster:

$$\text{quantiletht} = (1 + (1/x - 1)/q)^{-\frac{1}{k}};$$
$$\text{tvartht}[\text{q\_?NumericQ}, \text{k\_?NumericQ}, \text{x\_?NumericQ}]:$$
$$= \frac{k(q\text{Hypergeometric2F1}[1,2,2+\frac{1}{k},1-q] - qx^2(\frac{1+(-1+q)x}{qx})^{\frac{-1+k}{k}}\text{Hypergeometric2F1}[1,2,2+\frac{1}{k},x-qx])}{(1+k)(1-x)}. \quad (7)$$

$$\text{meantht} = \frac{q - \text{Hypergeometric2F1}\left[1,\frac{1}{k},1+\frac{1}{k},\frac{-1+q}{q}\right]}{-1+q};$$
$$\text{sdtht} = \sqrt{\frac{-(q-\text{Hypergeometric2F1}[1,\frac{1}{k},1+\frac{1}{k},\frac{-1+q}{q}])^2 + (-1+q)(q-\text{Hypergeometric2F1}[1,\frac{2}{k},\frac{2+k}{k},\frac{-1+q}{q}])}{(-1+q)^2}}. \quad (8)$$

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$$\text{skewtht} = \left(\text{Abs}[-1+q]^3\left(2\left(q-\text{Hypergeometric2F1}\left[1,\frac{1}{k},1+\frac{1}{k},\frac{-1+q}{q}\right]\right)^3 - 3(-1+q)\left(q-\text{Hypergeometric2F1}\left[1,\frac{1}{k},1+\frac{1}{k},\frac{-1+q}{q}\right]\right)\left(q - \text{Hypergeometric2F1}\left[1,\frac{2}{k},\frac{2+k}{k},\frac{-1+q}{q}\right]\right) + (-1+q)^2\left(q-\text{Hypergeometric2F1}\left[1,\frac{3}{k},\frac{3+k}{k},\frac{-1+q}{q}\right]\right)\right)\right) \div \left((-1+q)^3\left(-\left(q-\text{Hypergeometric2F1}\left[1,\frac{1}{k},1+\frac{1}{k},\frac{-1+q}{q}\right]\right)^2 + (-1+q)\left(q-\text{Hypergeometric2F1}\left[1,\frac{2}{k},\frac{2+k}{k},\frac{-1+q}{q}\right]\right)\right)^{3/2}\right). \tag{9}$$

$$\text{kurttht} = (-3\left(q-\text{Hypergeometric2F1}\left[1,\frac{1}{k},1+\frac{1}{k},\frac{-1+q}{q}\right]\right)^4 + 6(-1+q)\left(q-\text{Hypergeometric2F1}\left[1,\frac{1}{k},1+\frac{1}{k},\frac{-1+q}{q}\right]\right)^2 (q - \text{Hypergeometric2F1}[1,\frac{2}{k},\frac{2+k}{k},\frac{-1+q}{q}]) - 4(-1+q)^2(q-\text{Hypergeometric2F1}[1,\frac{1}{k},1+\frac{1}{k},\frac{-1+q}{q}])(q - \text{Hypergeometric2F1}[1,\frac{3}{k},\frac{3+k}{k},\frac{-1+q}{q}]) + (-1+q)^3(q - \text{Hypergeometric2F1}[1,\frac{4}{k},\frac{4+k}{k},\frac{-1+q}{q}])) \div ((q-\text{Hypergeometric2F1}[1,\frac{1}{k},1+\frac{1}{k},\frac{-1+q}{q}])^2 - (-1+q)(q-\text{Hypergeometric2F1}[1,\frac{2}{k},\frac{2+k}{k},\frac{-1+q}{q}]))^2. \tag{10}$$

Should the numerical optimization find the true optimum, the order we see in table 2 would be different. We know GTHT must be better than THT, but the optimization difficult gives the opposite results. Since this paper will focus on numerical study, we will leave an asymptotic analysis of THT SF to another paper (see [9] for shape factor asymptotic analysis for various distributions).

2.6 Twisted Wang transform distribution final experiment results

After eliminated using logarithmic type function as the ingredients (for example check Table. 4 for ERLPE and ELP performance), we focus on using power function and fractional functions, as well as exponential function as factors, utilizing the comparison to GTHT vis-à-vis EF where the first can be considered as a first order expansion of the second, and the second as the exponential counterpart of the first form, we made an extensive search of all such forms of distributions. To overcome the

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limitation of nested interval or region method of the optimization, we use random seed NelderMead simplex method for large number of seeds. The optimization results are still very much dependent on the variable ranges used, too wide or too narrow range will all gives incorrect results. But all the TWTE type distributions seems have similar ranges of the comparable parameters when attend the optimum, so the correct range find of one distribution can be used as a reference for the other distribution. (If a wider distribution cannot find a better solution, one possible reason is incorrect variable range.) The results are presented in Table 5.

Table 5. Final order of TWT families of distributions fit power for NATH.

Distribution Symbol	Distribution Name	CDF Formula	Maximum fit error
EFEP0	Exponential Fractional Extra Power 0 Distribution	$E^{-q\frac{(1-x^k)^m}{x^l}}$	2.65131
EFEP	Exponential Fractional Extra Power Distribution	$x^n E^{-q\frac{(1-x^k)^m}{x^l}}$	2.6618
PTHT	Productive Transformed Hyperbolic Tangent Distribution	$\frac{1}{1+q\frac{(1-x^k)(1-x^m)}{x^l}}$	3.14909
PTHTP	Productive Transformed Hyperbolic Tangent Power Distribution	$\frac{x^n}{1+q\frac{(1-x^k)(1-x^m)}{x^l}}$	3.18394
PTHT2P	Productive Transformed Hyperbolic Tangent 2 Power Distribution	$\frac{x^o}{1+q\frac{(1-x^k)(1-x^m)(1-x^n)}{x^l}}$	3.1874
EFPP0	Exponential Fractional Productive Power 0 Distribution	$E^{-q\frac{(1-x^k)(1-x^m)}{x^l}}$	3.26582
EFPP	Exponential Fractional Productive Power Distribution	$x^n E^{-q\frac{(1-x^k)(1-x^m)}{x^l}}$	3.27251
TWTE1*	Twisted Wang Transform Extended 1 Distribution	$\frac{x^n}{1+q\frac{(1-x^k)^m}{x^l}}$	3.31532
PTHT2	Productive Transformed Hyperbolic Tangent 2 Distribution	$\frac{1}{1+q\frac{(1-x^k)(1-x^m)(1-x^n)}{x^l}}$	3.40758
GTHTE	Generalized Transformed Hyperboloic Tangent Extended Distribution	$\frac{1}{1+q\frac{(1-x^k)^m}{x^l}}$	3.4145
THTE1	Transformed Hyperbolic Tangent Extended 1 Distribution	$\frac{1}{1+q(\frac{1-x^k}{x^k})^l}$	3.80902
THTE	Transformed Hyperbolic Tangent Extended Distribution	$\frac{1}{(1+q(\frac{1-x^k}{x^k})^l)^m}$	3.88511
EFE1	Exponential Fractional Extended 1 Distribution	$E^{-q(\frac{1-x^k}{x^k})^l}$	4.00827
GTHT	Generalized Transformed Hyperboloic Tangent Distribution	$\frac{1}{1+q\frac{1-x^k}{x^l}}$	4.05377
EF	Exponential Fractional Distribution	$E^{-q\frac{1-x^k}{x^l}}$	4.11104
GTHTP	Generalized Transformed Hyperboloic Tangent Power Distribution	$\frac{x^m}{1+q\frac{1-x^k}{x^l}}$	4.11306

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EFP	Exponential Fractional Power Distribution	$x^m E^{-q\frac{1-x^k}{x^l}}$	4.13766
EF1	Exponential Fractional 1 Distribution	$E^{-q\frac{1-x^k}{x^k}}$	4.22206
EF1P	Exponential Fractional 1 Power Distribution	$x^l E^{-q\frac{1-x^k}{x^k}}$	4.22221
THTP	Transformed Hyperbolic Tangent Power Distribution	$\frac{x^l}{1+q\frac{1-x^k}{x^k}}$	4.27351
THT	Transformed Hyperboloic Tangent Distribution	$\frac{1}{1+q\frac{1-x^k}{x^k}}$	4.27352

Maximum fit error is the minimum of the maximum of the absolute of the deviation of fitted TVaR to input TVaR in percentage for probability 0.5 to 0.99999 of the best fit.

From Table. 5 we see the best distribution for NATH is EFEP0 with an error of 2.65%, while the simplest form distribution THT has an error of 4.27%. We also see that change from the fractional form to the exponential form will sometimes improve the fit, such as from THT to EF1, THTP to EF1P, GTHTE to EFEP0, and TWTE1* to EFEP, but in more than half of the other times become worse, such as from GTHTP to EFP, GTHT to EF, THTE1 to EFE1, PTHT to EFPP0, and PTHT2P to EFPP. Similarly, add the extra power term may improve the fit, such as from GTHTE to TWTE1*, and THT to THTP. But more often will make the fit deteriorated, such as from EFEP0 to EFEP, PTHT to PTHTP, EFPP0 to EFPP, GTHT to GTHTP, EF to EFP, and EF1 to EF1P. Here maybe the difficult with the optimization bring about by the additional parameter is a contributing factor.

The best distribution whose quantile function can be expressed through simple function and do not need a numerical solution is THTE1 with an error of 3.81%. These class include THTE, THTE1, EFE1, EF1, and THT. They may be utilized for numerical simulation which are large scale and time sensitive.

The k parameter in THTE, THTE1, EFE1, EF1, EF1P, THTP, and THT, and the l parameter in other TWT distributions are somewhat similar to the alpha parameter for Pareto distribution [10]. In Table 2.3 of [10], we see for NATH that parameter should be in range 1.8-2.2. Our best fit parameters for NATH are in Table 6.

Table 6. Best fit to NATH parameter k or l.

Distribution	Parameter k or l
EFEP	2.648
EFEP0	2.617
PTHT	2.327
PTHTP	2.314
PTHT2P	2.416
EFPP0	2.539
TWTE1*	2.377
EFPP	2.553
PTHT2	2.453
GTHTE	2.414
THTE	2.029
THTE1	1.905
EFE1	2.236
GTHT	2.073
EF	2.343

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GTHTP	2.166
EFP	2.301
EF1	2.459
EF1P	2.459
THTP	2.511
THT	2.326

k for THTE, THTE1, EFE1, EF1, EF1P, THTP, and THT and l for other distributions.

These k or l parameters are clustered in the range 2.0-2.6. It promotes using a universal parameter per each peril in practise to simplify the distribution fitting.

3. Conclusion and discussions

A new family of distribution, the twisted Wang transform distribution, is devised, which is the algebraic construction of CDF function by composite two simper monotonic function, such as the CDF or quantile function. This type of distribution has root in the CV and SF analysis of empirical distribution from reinsurance portfolio losses, and is demonstrated to fit better than other well-known distributions, such as the GB1, GB2, and GH distributions [2][3], the three best distributions found from more than tens of distribution families. For a specific hard-to-fit example losses distribution, the fit error of the twisted Wang transform distribution family can be less than 1.5% for three given quantiles, while the others are about 5%. Even for the whole range of quantiles practically used, the best twisted Wang transform distribution can have fit error in within 2-4%, with the top one EFEP0 having maximum fit error 2.65%, while the others are more than tens of percent apart.

This study is mainly numerical, which has intrinsic uncertainty. An analytical study of the skewness, kurtosis, and shape factor of the twisted Wang transform distribution is desirable and will be done later, especially for the simplest form of TWT distribution, the Transformed Hyperbolic Tangent distribution, which has analytical expression for skewness, kurtosis, and shape factor.

Other kind of distribution construction than composition, such as mixture, is also of interest to check and compare. For example, GH is better than GB1 and GB2 [2][3], and GH is the normal-variance-mean-mixture of $InverseGaussianDistribution$ [11][12], we may naturally guess that when we extend the mixing distribution to extended inverse Gaussian distribution [13][14], we should get better than GH distribution family which may rival TWT distribution family or even better. Experiment along this line did not find promising results better than GH, perhaps because of the not convergence or slow convergence of the infinite interval integration. Progress in numerical integration and numerical optimization may change this scenario in the future.

Appendix A

The Mathematica code of the tool for distribution study is included here. Generic program that can be used for any type of distribution with analytical expression CDF is achieved through variable numbers of arguments function. The difficult with accessing the first argument is solved by a Block indirection. The mixed up of symbolic derivation and numerical calculation and hold attribute is solved by Module structure and a two steps approach:

```
    ClearAll[definedistribution,defineobjectives]
    definedistribution[name_, func_]:=Module[{f=(D[func,x]//Evaluate//Simplify[#, q>0&&k>0&&l>0&&
m>0&& n>0&& o>0]&)},Off[Part::partw];Symbol["cdf"<>name][p__?NumericQ]//Evaluate:=Block[{},func/.{q->{p}[[1]],k->{p}[[2]],l->{p}[[3]],m->{p}[[4]],n->{p}[[5]],o->{p}[[6]],x->{p}[[-1]]}//Evaluate];
    Symbol[ "pdf"<>name][p__?NumericQ]//Evaluate:=Block[{},
```

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```
    f/.{q->{p}[[1]],k->{p}[[2]],l->{p}[[3]],m->{p}[[4]],n->{p}[[5]],o->{p}[[6]],x->{p}[[-1]]}//Evalu-
ate];Symbol["m"<>name][p__?NumericQ,h_?NumericQ]//Evaluate:=NIntegrate[(1-Evaluate[Sym-
bol["cdf"<>name][p,x]] )h x^(h-1), {x,0.,1.}] ;
    {Evaluate[Symbol["sk"<>name]][p__?NumericQ],Evaluate[Symbol["kt"<>name]][p__?Numer-
icQ],Evaluate[Symbol["sf"<>name]][p__?NumericQ],Evaluate[Symbol["sd"<>name]][p__?Numer-
icQ]}=({#[[2]]/#[[1]]^(3/2),#[[3]]/#[[1]]^2,  #[[3]]  #[[1]]/#[[2]]^2,  #[[1]]^(1/2)}&@(Thread[MomentCon-
vert[CentralMoment[#]&/@{2,3,4},"Moment"]]//.Moment[h_]:> Evaluate[Symbol["m"<>name][p,h]]));
    Evaluate[Symbol["mean"<>name]][p__?NumericQ]=Evaluate[Symbol["m"<>name][p,1]]]
    defineobjectives[name_]:=Block[{},
    Symbol[name<>"epcurve"][pp___?Numer-
icQ]:=Block[{init,vars,tvar,stats,M,std,S,K,scal,shif,ep,s=Length[{pp}]},
    init=x/.(FindMinimum[Abs[Symbol["cdf"<>name][pp,x]-0.5],  {x,0.001,0.999},  MaxIterations-
>5000][[2]]);
    vars=FoldList[x/.FindRoot[Symbol["cdf"<>name][pp,x]==#2,  {x,#1},  MaxIterations-
>5000]&,init,{0.5,0.75,0.8,0.9,0.95,0.96,0.98,0.99,0.995,0.996,0.998,0.999,0.9998,0.9999,0.99999}]//Rest;t
var=NIntegrate[Symbol[  "pdf"<>name][pp,x]*x//Evaluate,{x,#[[2]],1.}]/(1.-#[[1]])  &  /@  Trans-
pose[{{0.5,0.75,0.8,0.9,0.95,0.96,0.98,0.99,0.995,0.996,0.998,0.999,0.9998,0.9999,0.99999},vars}];
    stats={Symbol[  "mean"<>name][pp],Symbol[  "sd"<>name][pp],Symbol[  "sk"<>name][pp],Symbol[
"kt"<>name][pp]}//Evaluate;
    {M,std,S,K}={7418611.10904006,9517336.93024634,5.99378199789956,65.8901734355745};
    {scal,shif}={std/stats[[2]], M-stats[[1]] std/stats[[2]]};
    ep=shif+scal  tvar;
    Flatten[{shif+scal  *  stats[[1]],  M,scal  *  stats[[2]],std,stats[[3]],S,stats[[4]],K,  ep,MapThread[#1-
>#2&,{Take[{q,k,l,m,n,o},s],{pp}}],  {scale->scal,  shift->shif}}]];Symbol["objective"<>name][pp___?Nu-
mericQ]:=Block[{init,vars,tvar,stats,M,std,S,K,scal,shif,ep},
    init=x/.(FindMinimum[Abs[Symbol["cdf"<>name][pp,x]-0.96],  {x,0.001,0.999},  MaxIterations-
>5000][[2]]);
    vars=FoldList[x/.FindRoot[Symbol["cdf"<>name][pp,x]==#2,  {x,#1},  MaxIterations-
>5000]&,init,{0.96,0.99,0.996}]//Rest;tvar=NIntegrate[Symbol[  "pdf"<>name][pp,x]*x//Evalu-
ate,{x,#[[2]],1.}]/(1.-#[[1]]) & /@ Transpose[{{0.96,0.99,0.996},vars}];
    stats={Symbol[ "mean"<>name][pp],Symbol[ "sd"<>name][pp]}//Evaluate;
    {M,std,S,K}={7418611.10904006,9517336.93024634,5.99378199789956,65.8901734355745};
    {scal,shif}={std/stats[[2]], M-stats[[1]] std/stats[[2]]};
    ep=shif+scal  tvar;
    Max[Abs[ep/{41113929.8424838,68867612.8345741,98579409.238445}-1.0]]
    ];
    Symbol["objective1"<>name][pp___?NumericQ]:=Block[{init,vars,tvar,stats,M,std,S,K,scal,shif,ep},
    init=x/.(FindMinimum[Abs[Symbol["cdf"<>name][pp,x]-0.96],  {x,0.001,0.999},  MaxIterations-
>5000][[2]]);
    vars=FoldList[x/.FindRoot[Symbol["cdf"<>name][pp,x]==#2,  {x,#1},  MaxIterations-
>5000]&,init,{0.96,0.99,0.996}]//Rest;tvar=NIntegrate[Symbol[  "pdf"<>name][pp,x]*x//Evalu-
ate,{x,#[[2]],1.}]/(1.-#[[1]]) & /@ Transpose[{{0.96,0.99,0.996},vars}];
    stats={Symbol[  "mean"<>name][pp],Symbol[  "sd"<>name][pp],Symbol[  "sk"<>name][pp],Symbol[
"kt"<>name][pp]}//Evaluate;
    {M,std,S,K}={7418611.10904006,9517336.93024634,5.99378199789956,65.8901734355745};
    {scal,shif}={std/stats[[2]], M-stats[[1]] std/stats[[2]]};
    ep=shif+scal  tvar;
    Max[Max[Abs[ep/{41113929.8424838,68867612.8345741,98579409.238445}-
1.0]],Max[Abs[stats[[3;;4]]/{S,K}-1.0]]]
    ];
```

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```
Symbol["objective0"<>name][pp___?NumericQ]:=Evaluate[(Log@Symbol[ "kt"<>name][pp]-Log@65.8901734355745`)^2+(Log@Symbol[ "sk"<>name][pp]-Log@5.99378199789956`)^2];
]

definedistribution["aptht",1/(1+q ((1-x^k)(1-x)^m)/x^l)]

defineobjectives["aptht"]
```

Appendix B

An example code for the random seeds search optimization in the THTE1 case:

```
biSection[func_,init_ : 1.0,eps_ : 10.^-6]/;NumericQ[init]&&Positive[eps]:=Module[{c=0.,d=init},While[func[d]<0.,c=d;d=(d+1.)/2.];While[d-c>eps,With[{e=(c+d)/2.},If[func[e]<0.,c=e,d=e]]];
(c+d)/2.]

definedistribution["thte1",1/(1+q ((1-x^k)/x^k)^l)]
defineobjectives["thte1"]

NMinimize[{objective1thte1[q,k,l], 0.1>q>0,k>1.0,l>0}, {k,l,q},MaxIterations->5000]
NestWhileList[NMinimize[{objective1thte1[q,k,l(*,m*)],(q/.#[[2]])>q>0,k>(k/.#[[2]]),l>(l/.#[[2]])(*,m>(m/.#[[2]])*)}, {k,l,(*m,*)q},MaxIterations->5000]&,{0.0846901323778386`,{k->1.65779949660623`,l->1.1706735527905872`,q->0.0008585977585324955`}},(Print@##;(k/.#2[[2]])-(k/.#1[[2]])>10^-7||(l/.#2[[2]])-(l/.#1[[2]])>10^-7)& ,2,5]

defineobjectives35[name_]:=Block[{},
Symbol["objective3"<>name][pp___?NumericQ]:=Block[{vars,tvar,stats,M,std,S,K,scal,shif,ep},
vars=biSection[Function[x, (Symbol["cdf"<>name][pp,x]//Evaluate)-# ]]&/@{0.9,0.96,0.98,0.99,0.996,0.998,0.9998,0.9999,0.99999};tvar=NIntegrate[Symbol[ "pdf"<>name][pp,x]*x//Evaluate,{x,#[[2]],1.}]/(1.-#[[1]]) & /@ Transpose[{{0.9,0.96,0.98,0.99,0.996,0.998,0.9998,0.9999,0.99999},vars}];
stats={Symbol[ "mean"<>name][pp],Symbol[ "sd"<>name][pp],Symbol[ "sk"<>name][pp],Symbol[ "kt"<>name][pp]}//Evaluate;
{M,std,S,K}={7418611.10904006,9517336.93024634,5.99378199789956,65.8901734355745};
{scal,shif}={std/stats[[2]], M-stats[[1]] std/stats[[2]]};
ep=shif+scal tvar;
Max[Max[Abs[ep/{28445066.802226,41113929.8424838,52880882.2899095,68867612.8345741,98579409.238445,123436136.002865,170402622.81885,177904161.5761,203519335.497}-1.0]],Max[Abs[stats[[3;;4]]/{S,K}-1.0]]]
];
Symbol["objective5"<>name][pp___?NumericQ]:=Block[{init,vars,tvar,stats,M,std,S,K,scal,shif,ep},
init=x/.(FindMinimum[Abs[(Symbol["cdf"<>name][pp,x]//Evaluate)-0.9], {x,0.001,0.999}, MaxIterations->5000][[2]]);
vars=FoldList[x/.FindRoot[(Symbol["cdf"<>name][pp,x]//Evaluate)==#2, {x,#1}, MaxIterations->5000]&,init,{0.9,0.96,0.98,0.99,0.996,0.998,0.9998,0.9999,0.99999}]//Rest;tvar=NIntegrate[Symbol[ "pdf"<>name][pp,x]*x//Evaluate,{x,#[[2]],1.}]/(1.-#[[1]]) & /@ Transpose[{{0.9,0.96,0.98,0.99,0.996,0.998,0.9998,0.9999,0.99999},vars}];
```

```
stats={Symbol[ "mean"<>name][pp],Symbol[ "sd"<>name][pp],Symbol[ "sk"<>name][pp],Symbol[ "kt"<>name][pp]}//Evaluate;
{M,std,S,K}={7418611.10904006,9517336.93024634,5.99378199789956,65.8901734355745};
{scal,shif}={std/stats[[2]], M-stats[[1]] std/stats[[2]]};
ep=shif+scal  tvar;
Max[Max[Abs[ep/{28445066.802226,41113929.8424838,52880882.2899095,68867612.8345741,98579409.238445,123436136.002865,170402622.81885,177904161.5761,203519335.497}-1.0]],Max[Abs[stats[[3;;4]]/{S,K}-1.0]]]
];
]

defineobjectives35["thte1"]

{sol=NMinimize[{objective5thte1[q,k,l],0.0005>q>0.,k>2.,l>1.},{q,k,l},MaxIterations->5000, Method->{"NelderMead", "RandomSeed"->#,"ShrinkRatio"->0.95, "ContractRatio"->0.95,"ReflectRatio"->2.0}], objective1thte1[q,k,l]/.sol[[2]],thte1epcurve[q,k,l]/.sol[[2]]}& /@ Range[100]

SortBy[%,{#[[2]],#[[1,1]],}&]//TableForm[#,TableDepth->1]&

SortBy[%,{#[[1,1]],#[[2]]}&]//TableForm[#,TableDepth->1]&
```

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