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Left- and Right-Chiral Dirac Spinors In a Unified 4D Spinor Space and Their Vector-Sum Decomposition Into Four-Component Weyl Spinors

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Abstract

We introduce a novel four-dimensional spinor representation of the Lorentz group in which Dirac and Weyl spinors are realized as four-component objects living in a common vector space. Within this extended framework there is enough room for both left- and right-chiral Dirac spinors. These two distinct species transform under the left- and right-handed representations, respectively. Furthermore, each Dirac spinor (left- or-right-chiral) can be expressed as a vector sum -rather than a direct sum- of left- and right-chiral four-component Weyl spinors. Crucially, the Dirac spinor and its components now transform under the same spinor space, permitting an unambiguous identification of chiral constituents. This formalism provides a symmetric and geometrically transparent reinterpretation of Dirac spinors and may offer new insights into extended spinor models and relativistic field theories.

Keywords: group theory; Lie algebras; Clifford algebra; Dirac algebra; representation theory; Lorentz group; chirality; Weyl spinors

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1. Introduction

In the chiral (Weyl) basis, a Dirac spinor ψ is conventionally expressed as a direct sum of left- and right-chiral spinors:

$$\psi = \xi_L \oplus \chi_R \tag{1}$$

ξ_L and χ_R are two-component Weyl spinors, each of which transforms under the respective two-dimensional spinor representations of the Lorentz group [1–5]:

$$\xi_L \to \xi_L' = L\xi_L, \qquad \chi_R \to \chi_R' = \dot{L}\chi_R \tag{2}$$

Where $\dot{L} = (L^{-1})^\dagger$. To isolate the left- and right-chiral parts of ψ, one uses the standard projection operators, defined in terms of γ^5:

$$P_L = \frac{1}{2}(I_4 - \gamma^5), \qquad P_R = \frac{1}{2}(I_4 + \gamma^5) \tag{3}$$

When these act on ψ, they yield the four-component Dirac spinors ψ_L and ψ_R

$$\psi_L = \xi_L \oplus 0, \qquad \psi_R = 0 \oplus \chi_R \tag{4}$$

In this framework, the projection operators do not truncate the remaining empty spinor components but instead produce objects with extended structures, which are distinct from the original two-component

Weyl spinors ξ_L and χ_R. Therefore, although a Dirac spinor can be considered as a direct sum of two-component Weyl spinors, once assembled, it cannot be decomposed back into its original components by using the standard projection operators.

In what follows, we present an alternative formulation based on a new representation g^μ, which obeys the Clifford algebra of spacetime, but cannot be obtained from the traditional γ^μ by a change of basis. In the new framework, Dirac spinor Ψ can be expressed as a vector sum of left- and right-chiral four-component Weyl spinors $\mathcal{U}^p_{L,R}$, $\mathcal{D}^p_{L,R}$ and $\mathcal{U}^q_{L,R}$, $\mathcal{D}^q_{L,R}$, where $\mathcal{U}$ and $\mathcal{D}$ denote up and down spins and the superscripts p, q label two distinct spin subspaces. Now, all these spinors -including Ψ- live in the same four-dimensional space, and left- and right-projection operators can be defined in terms of g^5 that corresponds to γ^5, such that applying these new projection operators on Ψ, retrieves the full left- and right-chiral components directly.

2. Four-Dimensional Spinor Representation of the Lorentz Group: Extension of the Algebra of SL(2, $\mathbb{C}$)

Lorentz transformation matrix Λ can be written as a direct product of left-and right-handed spinor representations of the Lorentz group:

$$\Lambda = L \otimes L^* \tag{5}$$

Let us write this equation in a different way:

$$L \otimes L^* = (L \otimes I_2)(I_2 \otimes L^*) \tag{6}$$

and define two new matrices Z and Z^* [7,8]:

$$Z = L \otimes I_2, \qquad Z^* = I_2 \otimes L^* \tag{7}$$

These definitions allow us to express Λ as a matrix product:

$$\Lambda = ZZ^* \tag{8}$$

Since Z and Z^* commute we also have $\Lambda = Z^*Z$.

The significance of these rather trivial re-definitions is that Z and Z^* now can be regarded as new four-dimensional left- and right-handed spinor representations for the Lorentz group acting on left- and right-chiral four-component Weyl spinors, respectively.

Z and Z^* are four-dimensional analogues of L and L^*, and in the new framework the spinor space is also four-dimensional. As an immediate consequence of this extension, there are two eigenvectors for spin-up and two for spin-down and the number of Weyl spinors doubles. That is, now there are four four-component Weyl spinors, and in the new representation, Weyl spinors and Dirac spinors live in the same spinor space. This key feature allows us to express any solution to the Dirac equation as a vector sum of four-component Weyl spinors:

$$\mathcal{U}^p_L \pm \mathcal{U}^q_R, \qquad \mathcal{D}^p_L \pm \mathcal{D}^q_R, \qquad etc. \tag{9}$$

3. Basic Tools

Let L and L^* be elements of the group SL(2, $\mathbb{C}$). If we use the standard representation of the Pauli matrices and parametrize the rotation and boost parameters by θ and ϕ, exponential forms of L and L^* reads:

$$L = e^{-\frac{i}{2}(\theta_i + i\phi_i)\sigma_i}, \qquad L^* = e^{\frac{i}{2}(\theta_i - i\phi_i)\sigma_i^*} \tag{10}$$

These matrices correspond to the two-dimensional left- and right-handed spinor representations of the Lorentz group. But, when we use these forms in the expression $\Lambda = L \otimes L^*$ we cannot yield the familiar real form of the Lorentz transformation matrix. To recover the familiar real form a change of basis is required:

$$\Lambda \to U \Lambda U^{-1} \tag{11}$$

where

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & i & -i & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}, \qquad U^{-1} = U^\dagger \tag{12}$$

Accordingly, we re-define Z and Z^* matrices. In the new basis,

$$Z = U(L \otimes I_2)U^{-1}, \qquad Z^* = U(I_2 \otimes L^*)U^{-1} \tag{13}$$

In explicit calculations it is easier to work with the parametric form of L (see Appendix):

$$L = \begin{pmatrix} \alpha_0 + \alpha_3 & \alpha_1 - i\alpha_2 \\ \alpha_1 + i\alpha_2 & \alpha_0 - \alpha_3 \end{pmatrix} \tag{14}$$

This can be compactly written in terms of the Pauli matrices as:

$$L = \alpha_\mu \sigma^\mu \tag{15}$$

Then, according to Eq.(13) parametric form of Z reads

$$Z = \begin{pmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_1 & \alpha_0 & -i\alpha_3 & i\alpha_2 \\ \alpha_2 & i\alpha_3 & \alpha_0 & -i\alpha_1 \\ \alpha_3 & -i\alpha_2 & i\alpha_1 & \alpha_0 \end{pmatrix} \tag{16}$$

The matrix Z can be written in a compact form using a new set of 4×4 matrices Σ^μ, which are analogues of σ^μ:

$$\Sigma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \Sigma^1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}, \quad \Sigma^2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & i \\ 1 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \quad \Sigma^3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \tag{17}$$

Hence we have

$$Z = \alpha_\mu \Sigma^\mu \tag{18}$$

To simplify the calculations further, we define new parameters:

$$a = \alpha_0 + \alpha_3, \qquad b = \alpha_1 - i\alpha_2, \qquad c = \alpha_1 + i\alpha_2, \qquad d = \alpha_0 - \alpha_3 \tag{19}$$

Then, L and $\dot{L}$ take their simplest forms

$$L = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad \dot{L} = \begin{pmatrix} d^* & -c^* \\ -b^* & a^* \end{pmatrix} \tag{20}$$

4. Two- and Four-Component Weyl Spinors

A left-chiral Weyl spinor ξ_L is a two component object that transforms under the $(\frac{1}{2}, 0)$ representation of the Lorentz group. Let $L \in \mathrm{SL}(2, \mathbb{C})$, and let b_1 and b_2 be a basis for the corresponding two dimensional spinor space. These basis vectors are usually chosen to be the eigenvectors of σ_3, associated with $+1$ and -1 eigenvalues respectively. When L acts on this basis, we obtain general forms for the left-chiral spin-up and spin-down states. We denote these by u_L and d_L to match our overall notation:

$$u_L = Lb_1 = \begin{pmatrix} a \\ c \end{pmatrix}, \qquad d_L = Lb_2 = \begin{pmatrix} b \\ d \end{pmatrix} \tag{21}$$

It is important to note that, in general, u_L and d_L are not orthogonal, unless L is unitary, which corresponds to pure spatial rotations.

Similarly, two-component right-chiral Weyl spinors transform under the $(0, \frac{1}{2})$ representation. These can be constructed analogously:

$$u_R = \dot{L}b_1 = \begin{pmatrix} d^* \\ -b^* \end{pmatrix}, \qquad d_R = \dot{L}b_2 = \begin{pmatrix} -c^* \\ a^* \end{pmatrix} \tag{22}$$

We now extend this structure to define four-component Weyl-spinors. In the new formalism the spinor space is four-dimensional, and we use a basis set consisting of four vectors $e_a \quad (a = 1, 2, 3, 4)$, which are the eigenvectors of the matrix Σ^3:

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \qquad e_2 = \begin{pmatrix} 0 \\ 1 \\ -i \\ 0 \end{pmatrix}, \qquad e_3 = \begin{pmatrix} 0 \\ 1 \\ i \\ 0 \end{pmatrix}, \qquad e_4 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} \tag{23}$$

In this basis, e_1 and e_3 correspond to $+1$ eigenvalue, and thus represent spin-up states; e_2 and e_4 correspond to eigenvalue -1 eigenvalue, and represent spin-down states. As a result, we will have two spin-up left-chiral spinors, $\mathcal{U}_L^p$ and $\mathcal{U}_L^q$, and two spin-down left-chiral spinors $\mathcal{D}_L^p$ and $\mathcal{D}_L^q$. Their explicit forms are obtained by applying the left-handed four dimensional transformation Z to the basis vectors e_a:

$$\mathcal{U}_L^p = Ze_1 = \begin{pmatrix} a \\ c \\ -ic \\ a \end{pmatrix}, \quad \mathcal{D}_L^p = Ze_2 = \begin{pmatrix} b \\ d \\ -id \\ b \end{pmatrix}, \quad \mathcal{U}_L^q = Ze_3 = \begin{pmatrix} c \\ a \\ ia \\ -c \end{pmatrix}, \quad \mathcal{D}_L^q = Ze_4 = \begin{pmatrix} d \\ b \\ ib \\ -d \end{pmatrix} \tag{24}$$

Although the basis vectors e_a are mutually orthogonal; in general, the spinors need not be, unless Z is unitary. However, spinors carrying different superscripts $(p \neq q)$ are always orthogonal.

We similarly define right-chiral four-component spinors by acting on the basis vectors e_a with the right-handed representation $\dot{Z} = (Z^{-1})^\dagger$. The explicit forms are:

$$\mathcal{U}_R^p = \dot{Z}e_1 = \begin{pmatrix} d^* \\ -b^* \\ ib^* \\ d^* \end{pmatrix}, \quad \mathcal{D}_R^p = \dot{Z}e_2 = \begin{pmatrix} -c^* \\ a^* \\ -ia^* \\ -c^* \end{pmatrix}, \quad \mathcal{U}_R^q = \dot{Z}e_3 = \begin{pmatrix} -b^* \\ d^* \\ id^* \\ b^* \end{pmatrix}, \quad \mathcal{D}_R^q = \dot{Z}e_4 = \begin{pmatrix} a^* \\ -c^* \\ -ic^* \\ -a^* \end{pmatrix} \tag{25}$$

These are four-component Weyl spinors. They are solutions to the Weyl equations in the Σ^μ basis:

$$(i\Sigma^\mu \partial_\mu)\mathcal{X} = 0, \qquad (i\bar{\Sigma}^\mu \partial_\mu)\mathcal{X} = 0 \tag{26}$$

where

$$\Sigma^\mu = (\Sigma^0, \vec{\Sigma}), \qquad \bar{\Sigma}^\mu = (\Sigma^0, -\vec{\Sigma}) \tag{27}$$

These forms will play a central role in defining the left- and right-chiral Dirac spinors and their decomposition into four-component Weyl spinors.

5. Dirac Algebra in the New Framework

There is no nontrivial fourth 2×2 matrix that anticommutes with all Pauli matrices. To define Dirac γ^μ matrices an extension of the basis is required:

$$\sigma^{\mu\nu} = \sigma^\mu \otimes \sigma^\nu \tag{28}$$

This is a direct product extension of the spinor space in which four-component Dirac spinors live. A subset of $\sigma^{\mu\nu}$ satisfying the Clifford algebra of spacetime is identified as Dirac γ^μ matrices [6]:

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} I_4 \tag{29}$$

The Dirac equation in γ^μ basis is given by:

$$(i\gamma^\mu \partial_\mu - m)\psi = 0 \tag{30}$$

Here ψ is the traditional four-component Dirac spinor. It transforms under the representation $S[\Lambda]$, for which the boost and rotation generators K_i and J_i are defined by:

$$K_i = \frac{i}{4}[\gamma^0, \gamma^i], \qquad J_i = \frac{i}{4}[\gamma^i, \gamma^j] \tag{31}$$

These generators satisfy the Lorentz algebra:

$$[J_i, J_j] = i\varepsilon_{ijk} J_k, \qquad [J_i, K_j] = i\varepsilon_{ijk} K_k, \qquad [K_i, K_j] = -i\varepsilon_{ijk} J_k \tag{32}$$

In terms of the parameters and generators

$$S[\Lambda] = \exp\left(-\frac{i}{2}\omega_{\mu\nu} S^{\mu\nu}\right) \tag{33}$$

where

$$S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu] \tag{34}$$

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In any representation, $S[\Lambda]$ obeys the Lorentz covariance condition:

$$S[\Lambda]^{-1}\gamma^{\mu}S[\Lambda] = \Lambda^{\mu}{}_{\nu}\gamma^{\nu} \tag{35}$$

In the new extended formalism introduced here, the Σ^i matrices play a role analogous to the Pauli matrices, satisfying similar algebraic relations, and there is no nontrivial 4×4 matrix that anticommutes with all Σ^i. Therefore, by themselves, they cannot generate a full Clifford algebra. But, in contrast to the traditional Dirac algebra, to resolve this issue we do not need to extend our formalism to higher dimensions, we simply employ the complete set of 16 matrices $\Sigma^{\mu}(\Sigma^{\nu})^*$, which span the full four-dimensional matrix space.

Within this new framework, we can identify six distinct subsets that satisfy the Clifford algebra of spacetime. One such set, g^{μ}, is compatible with the Z matrix formalism we introduced earlier:

$$g^0 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{pmatrix}, \quad g^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad g^2 = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}, \quad g^3 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \end{pmatrix} \tag{36}$$

These matrices satisfy the Clifford algebra of spacetime

$$\{g^{\mu}, g^{\nu}\} = 2\eta^{\mu\nu} I_4 \tag{37}$$

Structurally, the set $\Sigma^{\mu}(\Sigma^{\nu})^*$ obtained by a matrix product of Σ^{μ} and $(\Sigma^{\nu})^*$ is distinct from the set $\sigma^{\mu} \otimes \sigma^{\nu}$ which is a direct product of Pauli matrices. Nevertheless, there exists a similarity transformation between the g^{μ} and γ^{μ}. If γ^{μ} is given in the chiral basis, then g^{μ} can be obtained from γ^{μ} by the following transformation:

$$g^{\mu} = A\gamma^{\mu}A^{-1} \tag{38}$$

where

$$A = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ i & 0 & 0 & -i \\ 0 & -1 & 1 & 0 \end{pmatrix}, \qquad A^{-1} = A^{\dagger} \tag{39}$$

Therefore, in one sense, our representation is not new. But, when we apply the similarity transformation given in Eq.(39) on the $S[\Lambda]$:

$$\widetilde{S}[\Lambda] = AS[\Lambda]A^{-1} \tag{40}$$

$\widetilde{S}[\Lambda]$ will not generate Dirac spinors compatible with the four-component Weyl spinors we have defined previously (Eqs.(24)and (25)). For that reason, we will build up our spinor formalism from scratch.

First of all, since g^{μ} constitute a proper representation of the Clifford algebra, an associated Dirac equation for a Dirac spinor Ψ in the new basis can be written in the same form:

$$(ig^{\mu}\partial_{\mu} - m)\Psi = 0 \tag{41}$$

Where we use Ψ to distinguish it from ψ. The projection operators for left- and right-chiral components of Ψ can be defined as

$$g^5 = -ig^0g^1g^2g^3 \tag{42}$$

In this definition, the minus sign is just a convention.

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To proceed further, we first set $\vec{p} = 0$ and apply the usual ansatz to find the plane-wave solutions in the g^μ basis. Apart from the phase factors a set of solutions is:

$$\Psi_0^1 = \begin{pmatrix} 1 \\ 1 \\ i \\ 1 \end{pmatrix}, \qquad \Psi_0^2 = \begin{pmatrix} 1 \\ 1 \\ -i \\ -1 \end{pmatrix}, \qquad \Psi_0^3 = \begin{pmatrix} 1 \\ -1 \\ -i \\ 1 \end{pmatrix}, \qquad \Psi_0^4 = \begin{pmatrix} 1 \\ -1 \\ i \\ -1 \end{pmatrix} \tag{43}$$

In terms of the basis vectors given in Eq.(23), we can also write:

$$\Psi_0^1 = e_1 + e_3, \qquad \Psi_0^2 = e_2 + e_4, \qquad \Psi_0^3 = e_1 - e_3, \qquad \Psi_0^4 = -e_2 + e_4, \tag{44}$$

Then, we apply the Lorentz transformation to obtain general solutions. But, instead of using $\widetilde{S}[\Lambda]$, we construct our own Lorentz transformation matrix $S_g[\Lambda]$ for Dirac spinors, based on the new generators expressed in the g^μ basis:

$$K_i = \frac{i}{4}[g^0, g^i], \qquad J_i = \frac{i}{4}[g^i, g^j] \tag{45}$$

Just like the standard formalism, these generators satisfy the same Lorentz algebra given in Eq(32), and we define the transformation matrix $S_g[\Lambda]$ in the usual form:

$$S_g[\Lambda] = \exp\left(-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}\right) \tag{46}$$

where γ^μ is replaced by g^μ:

$$S^{\mu\nu} = \frac{i}{4}[g^\mu, g^\nu] \tag{47}$$

After exponentiation $S_g[\Lambda]$ reads

$$S_g[\Lambda] = \frac{1}{2}\begin{pmatrix} a+a^* & b-b^* & ib+ib^* & a-a^* \\ c-c^* & d+d^* & id-id^* & c+c^* \\ -ic-ic^* & -id+id^* & d+d^* & -ic+ic^* \\ a-a^* & b+b^* & ib-ib^* & a+a^* \end{pmatrix} \tag{48}$$

$S_g[\Lambda]$ satisfies the required Lorentz covariance condition:

$$S_g[\Lambda]^{-1} g^\mu S_g[\Lambda] = \Lambda^\mu{}_\nu g^\nu \tag{49}$$

Here, $S_g[\Lambda]^{-1}$ can be obtained by simply exchanging $a \leftrightarrow d$, and changing the signs of b and c, i.e., $b \to -b$ and $c \to -c$.

As it will be shown in what follows, in this representation, the Dirac spinor and its left- and right-chiral components now live in the same four-dimensional spin space, enabling a straightforward decomposition and reconstruction. That is, we will get back the original Dirac spinor as a vector sum of the components.

We will identify this representation as the left-handed one, and show that there also exists another representation, a right-handed one, $\dot{S}_g[\Lambda] = (S_g[\Lambda]^{-1})^\dagger$, which acts on right-chiral Dirac spinors.

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6. Decomposition of a Dirac Spinor into Four-Component Weyl Spinors

To demonstrate the decomposition procedure, one can proceed by setting $\alpha_0 = \cosh(\phi/2)$, $\alpha_1 = \alpha_2 = 0$, and $\alpha_3 = -\sinh(\phi/2)$ for a boost along the z-axis to get simple expressions. But we will continue with the general form of $S_g[\Lambda]$ and apply it to the plane wave solutions given in Eq.(43):

$$\Psi^a = S_g[\Lambda]\Psi_0^a, \tag{50}$$

where Ψ^a, $a \in \{1,2,3,4\}$ is a solution to the Dirac equation in the g^μ basis. Using the linear combinations given in Eq.(44):

$$\Psi^1 = S_g[\Lambda](e_1 + e_3), \quad \Psi^2 = S_g[\Lambda](e_2 + e_4), \quad \Psi^3 = S_g[\Lambda](e_1 - e_3), \quad \Psi^4 = S_g[\Lambda](-e_2 + e_4), \tag{51}$$

Due to the linearity of $S_g[\Lambda]$ we can write:

$$\Psi^1 = S_g[\Lambda]e_1 + S_g[\Lambda]e_3, \quad etc. \tag{52}$$

Now, it is crucial to observe that

$$S_g[\Lambda]e_1 = Ze_1 = \mathcal{U}_L^p, \qquad S_g[\Lambda]e_2 = Ze_2 = \mathcal{D}_L^p \tag{53}$$

And

$$S_g[\Lambda]e_3 = \dot{Z}e_3 = \mathcal{U}_R^q, \qquad S_g[\Lambda]e_4 = \dot{Z}e_4 = \mathcal{D}_R^q \tag{54}$$

These can be easily checked by comparing the action of $S_g[\Lambda]$ on the basis e_a with the actions of Z and $\dot{Z}$ on the same basis. Hence, Ψ^a can be written as vector sum of four-component Weyl spinors:

$$\Psi^1 = \mathcal{U}_L^p + \mathcal{U}_R^q, \quad \Psi^2 = \mathcal{D}_L^p + \mathcal{D}_R^q, \quad \Psi^3 = \mathcal{U}_L^p - \mathcal{U}_R^q, \quad \Psi^4 = -\mathcal{D}_L^p + \mathcal{D}_R^q \tag{55}$$

In these expressions all objects live in the same spinor space. These are not direct sums in the traditional sense (as in block forms), but rather sums of vectors within the same four-dimensional spinor space. Hence, both chiral components are embedded in the same representation space and distinguished only by their transformation properties: $\mathcal{U}_L^p$ and $\mathcal{D}_L^p$ transform under the left-handed representation Z, while $\mathcal{U}_R^q$ and $\mathcal{D}_R^q$ transform under the right-handed one, $\dot{Z}$.

The key point is that the action of $S_g[\Lambda]$ on e_1 and e_2 is the same as the action of Z. Similarly, The action of $S_g[\Lambda]$ on e_3 and e_4 is the same as the action of $\dot{Z}$. But, now all objects live in the same four dimensional spinor space, and these relations are distinct from the following transformation of a Dirac spinor expressed in the traditional direct sum (reducible) representation:

$$\psi \to \psi' = L \oplus \dot{L}(\xi_L \oplus \chi_R) \tag{56}$$

We can also decompose Dirac spinors Ψ into left- and right-chiral components by means of the projection operators P_L and P_R defined in the g^μ basis:

$$P_L = \frac{1}{2}(I_4 - g^5) = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & i & 0 \\ 0 & -i & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}, \qquad P_R = \frac{1}{2}(I_4 + g^5) = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -i & 0 \\ 0 & i & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix} \tag{57}$$

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It is enough to examine the action of these projections on the spinors Ψ_0^a:

$$P_L\Psi_0^1 = e_1, \quad P_R\Psi_0^1 = e_3, \quad P_L\Psi_0^2 = e_2, \qquad P_R\Psi_0^2 = e_4 \tag{58}$$

and

$$P_L\Psi_0^3 = e_1, \quad P_R\Psi_0^3 = -e_3, \quad P_L\Psi_0^4 = -e_2, \qquad P_R\Psi_0^4 = e_4 \tag{59}$$

These relations immediately yield the left- and right-chiral components of Ψ^a:

$$\Psi_L^1 = \mathcal{U}_L^p, \qquad \Psi_R^1 = \mathcal{U}_R^q \; ; \; \Psi_L^2 = \mathcal{D}_L^p, \qquad \Psi_R^2 = \mathcal{D}_R^q \tag{60}$$

and

$$\Psi_L^3 = \mathcal{U}_L^p, \qquad \Psi_R^3 = -\mathcal{U}_R^q \; ; \; \Psi_L^4 = -\mathcal{D}_L^p \qquad \Psi_R^4 = \mathcal{D}_R^q \tag{61}$$

As a result,

$$\Psi^a = \Psi_L^a + \Psi_R^a \tag{62}$$

In contrast to the Dirac formalism in γ^μ basis, in the new framework, Ψ_L and Ψ_R can be directly associated with four-component Weyl spinors. The new formalism not only clarifies the internal structure of Dirac spinors but also establishes an explicit and symmetric embedding of left- and right-chiral components in a unified spinor space.

Now we can write the Dirac equation as

$$(ig^\mu\partial_\mu - m)(\Psi_L + \Psi_R) = 0 \tag{63}$$

But this equation cannot be split into two uncoupled equations unless $m = 0$ (or we are working in the ultra-relativistic limit). If $m = 0$ we have two separate equations:

$$(ig^\mu\partial_\mu)\Psi_L = 0, \qquad (ig^\mu\partial_\mu)\Psi_R = 0 \tag{64}$$

In other words, when $m = 0$, Ψ_L and Ψ_R satisfy these equations and the Weyl equations given in Eq.(26) simultaneously.

7. Left-and Right-Handed Representations for Dirac Algebra

Thus far we examined the transformation properties of the Dirac spinors, Ψ, which can be decomposed into the following four-component Weyl spinors:

$$\mathcal{U}_L^p, \qquad \mathcal{D}_L^p, \qquad \mathcal{U}_R^q, \qquad \mathcal{D}_R^q \tag{65}$$

Ψ transform under the representation $S_g[\Lambda]$:

$$\Psi \to \Psi' = S_g[\Lambda]\Psi \tag{66}$$

But there is another set of objects in Eqs.(24) and (25):

$$\mathcal{U}_L^q, \qquad \mathcal{D}_L^q, \qquad \mathcal{U}_R^p, \qquad \mathcal{D}_R^p \tag{67}$$

Accordingly, there exist another kind of Dirac spinors that can be decomposed into these four-component Weyl spinors. Let us denote them by $\dot{\Psi}$. We may regard Ψ and $\dot{\Psi}$ as left- and right-chiral Dirac spinors, respectively.

The dotted version of Dirac spinors (right-chiral ones), transform under the right-handed representation $\dot{S}_g[\Lambda] = (S_g[\Lambda]^{-1})^\dagger$:

$$\dot{\Psi} \to \dot{\Psi}' = \dot{S}_g[\Lambda]\dot{\Psi} \tag{68}$$

Explicit form of $\dot{S}_g[\Lambda]$ reads

$$\dot{S}_g[\Lambda] = (S_g[\Lambda]^{-1})^\dagger = \frac{1}{2}\begin{pmatrix} d+d^* & c-c^* & -ic-ic^* & -d+d^* \\ b-b^* & a+a^* & -ia+ia^* & -b-b^* \\ ib+ib^* & ia-ia^* & a+a^* & -ib+ib^* \\ -d+d^* & -c-c^* & ic-ic^* & d+d^* \end{pmatrix} \tag{69}$$

We repeat the same procedure and let $\dot{S}_g[\Lambda]$ act on basis e_a, and observe the following relations:

$$\dot{S}_g[\Lambda]e_1 = \dot{Z}e_1 = \mathcal{U}_R^p, \qquad \dot{S}_g[\Lambda]e_2 = \dot{Z}e_2 = \mathcal{D}_R^p \tag{70}$$

And

$$\dot{S}_g[\Lambda]e_3 = Ze_3 = \mathcal{U}_L^q, \qquad \dot{S}_g[\Lambda]e_4 = Ze_4 = \mathcal{D}_L^q \tag{71}$$

Then it is straightforward to show that $\dot{\Psi}^a$ can be written as a vector sum of four-component Weyl spinors:

$$\dot{\Psi}^1 = \mathcal{U}_L^q + \mathcal{U}_R^p, \quad \dot{\Psi}^2 = \mathcal{D}_L^q + \mathcal{D}_R^p, \quad \dot{\Psi}^3 = -\mathcal{U}_L^q + \mathcal{U}_R^p, \quad \dot{\Psi}^4 = \mathcal{D}_L^q - \mathcal{D}_R^p \tag{72}$$

It is also possible to decompose $\dot{\Psi}^a$ into its left- and right-chiral components by the dotted version of the projection operators:

$$\dot{P}_L = P_R = \frac{1}{2}(I_4 + g^5), \qquad \dot{P}_R = P_L = \frac{1}{2}(I_4 - g^5) \tag{73}$$

8. Conclusion

In this work, we introduce a novel four-dimensional spinor representation of the Lorentz group, in which both the Dirac spinors and Weyl spinors are treated as four-component objects living in the same vector space. Within this extended framework there is enough room for both left- and right-chiral Dirac spinors. These two distinct species transform under the left- and right-handed representations, respectively.

This framework allows the Dirac spinor to be expressed as a vector sum -not as a direct sum- of left- and right-chiral four-component Weyl spinors. Unlike the standard formalism, where the chiral components are obtained via projections onto distinct subspaces, our formulation permits a direct and symmetric decomposition.

This new representation is constructed via an extension of the SL(2, ℂ) algebra, which naturally leads to the doubling of the number of Weyl spinors and enables a more transparent treatment of spin degrees of freedom, with up and down chiral components appearing in pairs. Thus, the left- and right-chiral parts obtained via projection operators defined in terms of g^5 are directly identifiable as the four-component Weyl spinors used in the construction of the Dirac spinor.

This approach not only deepens our understanding of spinor structure in relativistic quantum mechanics but also offers potential advantages for exploring extended spinor spaces, alternative representations, and geometrical interpretation of spin.

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Appendix A

Appendix A.1. Four-Dimensional Spinor Representation of the Lorentz Group: Extension of the Algebra of SL(2, ℂ)

We detail the algebraic structure underlying the Z matrix formalism introduced in the main text by showing that any Lorentz transformation matrix Λ can be expressed as the product of two complex matrices, $\Lambda = ZZ^*$. This matrix product form should not be confused with the polar decomposition or with the tensor product form often used in standard representations..

We begin with the well-known real form of the Lorentz transformation expressed via exponentiation the rotation and boost generators J_i and K_i:

$$\Lambda = e^{(\theta_i J_i + \phi_i K_i)} \tag{A1}$$

Here θ_i and ϕ_i are rotation and boost parameters, and the generators in a fully real representation are:

$$J_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad J_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{A2}$$

$$K_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad K_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad K_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \tag{A3}$$

These satisfy the Lorentz algebra commutation relations:

$$[J_i, J_j] = \varepsilon_{ijk} J_k, \qquad [K_i, K_j] = -\varepsilon_{ijk} J_k, \qquad [J_i, K_j] = \varepsilon_{ijk} K_k \tag{A4}$$

Now, define new generators Σ_i and Σ_i^*:

$$\Sigma_i = K_i + iJ_i, \qquad \Sigma_i^* = K_i - iJ_i \tag{A5}$$

$$\Sigma_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}, \quad \Sigma_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & i \\ 1 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \quad \Sigma_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \tag{A6}$$

$$\Sigma_1^* = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{pmatrix}, \quad \Sigma_2^* = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -i \\ 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}, \quad \Sigma_3^* = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \tag{A7}$$

The matrices Σ_i and Σ_i^* are traceless, Hermitian, and satisfy:

$$[\Sigma_i, \Sigma_j] = i\varepsilon_{ijk}\Sigma_k, \qquad [\Sigma_i^*, \Sigma_j^*] = -i\varepsilon_{ijk}\Sigma_k^*, \qquad [\Sigma_i, \Sigma_j^*] = 0 \tag{A8}$$

We express new Lorentz generators in terms of these matrices:

$$K_i = \frac{1}{2}(\Sigma_i + \Sigma_i^*), \qquad J_i = -\frac{i}{2}(\Sigma_i - \Sigma_i^*) \tag{A9}$$

Substituting into the exponential form, $\Lambda = e^{(\theta_i J_i + \phi_i K_i)}$, we find:

$$\Lambda = e^{[-\frac{i}{2}\theta_i(\Sigma_i - \Sigma_i^*) + \frac{1}{2}\phi_i(\Sigma_i + \Sigma_i^*)]} = e^{[-\frac{i}{2}(\theta_i + i\phi_i)\Sigma_i + \frac{i}{2}(\theta_i - i\phi_i)\Sigma_i^*)]} \tag{A10}$$

Since Σ_i and Σ_j^* commutes with each other for all i, j, this exponential splits cleanly into a matrix product:

$$\Lambda = e^{-\frac{i}{2}(\theta_i + i\phi_i)\Sigma_i} e^{\frac{i}{2}(\theta_i - i\phi_i)\Sigma_i^*} = ZZ^* \tag{A11}$$

where, by definition,

$$Z = e^{-\frac{i}{2}(\theta_i + i\phi_i)\Sigma_i}, \qquad Z^* = e^{\frac{i}{2}(\theta_i - i\phi_i)\Sigma_i^*} \tag{A12}$$

In this work, Z and Z^* will be regarded as four-dimensional left- and right-chiral representations of the Lorentz group, acting on four-component Weyl spinors.

Appendix A.2. Parametric Representations

The matrices Λ, L and Z can be expressed in terms of four complex parameters α_μ. To show this, we write Z in exponential form, $Z = e^M$, where $M = -\frac{i}{2}\vec{\rho} \cdot \vec{\Sigma}$, $\rho_i = \theta_i + i\phi_i$. Here $\vec{\rho}$ is a complex vector parameter representing both rotations and boosts. Let $\rho = \sqrt{\rho_1^2 + \rho_2^2 + \rho_3^2}$. Then the exponential form becomes:

$$Z = \Sigma_0 \cos(\rho/2) - \frac{i \sin(\rho/2)}{\rho}\vec{\rho} \cdot \vec{\Sigma} \quad \rightarrow \quad Z = \begin{pmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_1 & \alpha_0 & -i\alpha_3 & i\alpha_2 \\ \alpha_2 & i\alpha_3 & \alpha_0 & -i\alpha_1 \\ \alpha_3 & -i\alpha_2 & i\alpha_1 & \alpha_0 \end{pmatrix} \tag{A13}$$

where

$$\alpha_0 = \cos(\rho/2), \qquad \alpha_1 = -\frac{i \sin(\rho/2)}{\rho}\rho_1, \qquad \alpha_2 = -\frac{i \sin(\rho/2)}{\rho}\rho_2, \qquad \alpha_3 = -\frac{i \sin(\rho/2)}{\rho}\rho_3. \tag{A14}$$

Z matrix can be concisely written as:

$$Z = \alpha_\mu \Sigma^\mu \tag{A15}$$

Using this parametrization, we can define various forms of Z

$$Z = \alpha_0\Sigma^0 + \alpha_1\Sigma^1 + \alpha_2\Sigma^2 + \alpha_3\Sigma^3, \quad Z^{-1} = \alpha_0\Sigma^0 - \alpha_1\Sigma^1 - \alpha_2\Sigma^2 - \alpha_3\Sigma^3, \tag{A16}$$

$$Z^* = \alpha_0^*\Sigma^{0*} + \alpha_1^*\Sigma^{1*} + \alpha_2^*\Sigma^{2*} + \alpha_3^*\Sigma^{3*}, \quad Z^\dagger = \alpha_0^*\Sigma^0 + \alpha_1^*\Sigma^1 + \alpha_2^*\Sigma^2 + \alpha_3^*\Sigma^3, \tag{A17}$$

From the product $\Lambda = ZZ^*$, we obtain the real 4×4 Lorentz transformation matrix with entries constructed from bilinear combinations $\alpha_\mu\alpha_\nu^*$. And for L we have the following parametric form:

$$L = \begin{pmatrix} \alpha_0 + \alpha_3 & \alpha_1 - i\alpha_2 \\ \alpha_1 + i\alpha_2 & \alpha_0 - i\alpha_3 \end{pmatrix} \quad \text{or} \quad L = \alpha_\mu\sigma^\mu \tag{A18}$$

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