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S-Asymptotically-(N, λ)-Periodic Sequences with Applications

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Abstract

In this paper, we study the (new) concept of S-asymptotically (N, λ)-periodic sequences as a generalization of (N, λ)-periodic sequences. We present their basic properties and give simple examples that show how they differ from the classical concept of periodicity. We also apply these results to linear and semilinear difference equations, proving the existence and uniqueness of solutions under appropriate sufficient assumptions. Finally, we illustrate the results with an example from population dynamics.

Keywords: S-asymptotically (N, λ)-periodic sequences; Banach's fixed point theorem; nonlinear difference equations; generalized periodicity; qualitative analysis; Volterra difference equations; population dynamics

MSC: 339A24; 39A30; 47B39; 34C27; 37B55

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1. Introduction

The study of periodic and almost periodic functions has played a fundamental role in mathematical analysis, particularly in the theory of differential and difference equations [8,12]. While classical periodicity provides a powerful tool for modeling cyclic phenomena, its strict requirements often limit its applicability to real-world systems exhibiting more complex behaviors [5]. This limitation has motivated the development of various generalizations including. Almost periodicity [5], Almost automorphy [7] ,Asymptotic periodicity and S-asymptotic periodicity [18]. These extended notions have proven particularly valuable in the qualitative analysis of dynamical systems where exact periodicity fails to capture the full complexity of solutions [3]. In discrete-time systems, a significant generalization emerged with the concept of S-asymptotically-(N, λ)-periodic sequences, introduced by [2]. This framework characterizes sequences $\{u(n)\}_{n\in\mathbb{Z}}$ satisfying the functional equation:

$$u(n+N) = \lambda u(n), \quad \forall n \in \mathbb{Z} \tag{1.1}$$

where, $N \in \mathbb{N}^*$ is the period and $\lambda \in \mathbb{C}^*$ is a scaling factor. When $\lambda = 1$, we recover classical N-periodicity, while $\lambda \neq 1$ allows for modeling systems with exponential growth or decay along periodic patterns [6]. This flexibility has made the concept particularly useful in Volterra difference equations [1], discrete dynamical systems and problems with multiplicative recurrence. In this work, we investigate the space of bounded S-asymptotically-(N, λ)-periodic sequences in Banach spaces, denoted by $P_{N\lambda}$. Our main contributions include: A complete proof that $(P_{N\lambda}, \|\cdot\|_\infty)$ forms a Banach space. Characterization of its algebraic and topological

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properties. Construction of illustrative examples highlighting differences with classical periodicity and applications to nonlinear difference equations. These results extend the continuous theory developed in [6] to the discrete setting and provide new tools for analyzing discrete dynamical systems [14]. Our work bridges the gap between abstract functional analysis and applied difference equations, offering a rigorous foundation for further studies of (N, λ)-periodic phenomena.

Remark 1. *The concept of S-asymptotically (N, λ)-periodic sequences extends continuous-time notions of asymptotic periodicity to the discrete setting. While continuous asymptotically periodic functions satisfy $\lim_{t\to\infty}[f(t+T) - f(t)] = 0$, our discrete analogue introduces a multiplicative scaling factor λ through the condition:*

$$\lim_{n\to\infty} \|u(n+N) - \lambda u(n)\|_X = 0,$$

where, $N \in \mathbb{N}^*$ is the asymptotic period, λ a non-zero complex number with $|\lambda| = 1$ is the phase factor, and $\|\cdot\|_X$ denotes the norm in the Banach space X. This framework generalizes several classical cases: When $\lambda = 1$, we recover standard asymptotic N-periodicity. For $\lambda = e^{i\theta}$, we obtain rotating wave-type behavior. The condition $|\lambda| = 1$ preserves boundedness while allowing phase modulation. The multiplicative structure makes this particularly suitable for studying: Discrete dynamical systems with linear recurrence relations, parameterized difference equations and problems where solutions exhibit phase-shifted recurrence patterns.

2. Preliminaries

Let $(X, \|\cdot\|_X)$ be a complex Banach space. We denote by $\ell^\infty(\mathbb{Z}; X)$ the Banach space of all bounded X-valued sequences equipped with the supremum norm:

$$\ell^\infty(\mathbb{Z}; X) := \left\{ u : \mathbb{Z} \to X \;\middle|\; \|u\|_\infty = \sup_{n\in\mathbb{Z}} \|u(n)\|_X < \infty \right\}.$$

Definition 1. *[1,2] A sequence $u \in \ell^\infty(\mathbb{Z}; X)$ is called (N, λ)-periodic if there exist:*

- *An integer $N \in \mathbb{N}^*$ (the period)*
- *A nonzero complex number $\lambda \in \mathbb{C}^*$ (the scaling factor)*

such that

$$u(n+N) = \lambda u(n) \quad \text{for all } n \in \mathbb{Z}.$$

The space of all such sequences is denoted by $P_{N,\lambda}(\mathbb{Z}; X)$.

Definition 2. *[1] Let $K \subset X$ be a bounded subset. A function $u : \mathbb{Z} \times X \to X$ is called (N, λ)-periodic in n uniformly on K if:*

- *For each fixed $x \in X$, $u(\cdot, x) \in \ell^\infty(\mathbb{Z}; X)$*
- *There exist $N \in \mathbb{N}^*$ and $\lambda \in \mathbb{C}^*$ such that*

$$u(n+N, x) = \lambda u(n, x) \quad \text{for all } n \in \mathbb{Z} \text{ and } x \in K$$

The space of all such functions is denoted by $P_{N,\lambda}(\mathbb{Z} \times X; X)$.

Remark 2. *The spaces $P_{N,\lambda}(\mathbb{Z}; X)$ and $P_{N,\lambda}(\mathbb{Z} \times X; X)$ inherit the following properties:*

1. *When $\lambda = 1$ (resp. $\lambda + -1$), we recover classical N-periodicity (resp. N-anti-periodicity)*
2. *For $|\lambda| \neq 1$, sequences exhibit exponential growth/decay*

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3. *The case $|\lambda| = 1$ corresponds to phase-modulated periodicity.*

Standard properties of these spaces (closedness under linear combinations, completeness, etc.) can be found in [1,2].

Theorem 1. *The space $P_{N,\lambda}(\mathbb{Z}; X)$ of all bounded (N, λ)-periodic sequences in a Banach space X forms a complex vector space under pointwise operations.*

Proof. Let $u, v \in P_{N,\lambda}(\mathbb{Z}; X)$ and $\alpha, \beta \in \mathbb{C}$. We verify the linear space properties: For any $n \in \mathbb{Z}$,

$$\begin{aligned}(\alpha u + \beta v)(n + N) &= \alpha u(n + N) + \beta v(n + N) \\ &= \alpha(\lambda u(n)) + \beta(\lambda v(n)) \quad \text{(by periodicity)} \\ &= \lambda(\alpha u(n) + \beta v(n)) \\ &= \lambda(\alpha u + \beta v)(n).\end{aligned}$$

Thus $\alpha u + \beta v \in P_{N,\lambda}(\mathbb{Z}; X)$, proving the space is linear.

Now if $u, v \in l^\infty(\mathbb{Z}; X)$, there exist $M_u, M_v > 0$ such that

$$\|\alpha u + \beta v\|_\infty \leq |\alpha|\|u\|_\infty + |\beta|\|v\|_\infty \leq |\alpha|M_u + |\beta|M_v < \infty.$$

The proof is complete. □

Theorem 2. *Let X be a Banach algebra. If $u, v \in P_{N,\lambda}(X)$, then their pointwise product $u \cdot v$ belongs to $P_{N,\lambda^2}(X)$.*

Proof. Let $u, v \in P_{N,\lambda}(X)$. By definition of (N, λ)-periodicity, we have for all $n \in \mathbb{Z}$:

$$u(n + N) = \lambda u(n) \quad \text{and} \quad v(n + N) = \lambda v(n).$$

Consider the pointwise product $(u \cdot v)(n) := u(n)v(n)$, where the multiplication is the Banach algebra operation in X. We verify the (N, λ^2)-periodicity condition:

$$\begin{aligned}(u \cdot v)(n + N) &= u(n + N)v(n + N) \\ &= (\lambda u(n))(\lambda v(n)) \quad \text{(by periodicity of } u \text{ and } v\text{)} \\ &= \lambda^2 u(n)v(n) \quad \text{(by bilinearity of multiplication in } X\text{)} \\ &= \lambda^2(u \cdot v)(n).\end{aligned}$$

In addition if u, v are bounded, then $u \cdot v$ is also bounded. Indeed since $u, v \in \ell^\infty(\mathbb{Z}; X)$, there exist $M_u, M_v > 0$ such that:

$$\|u(n)\|_X \leq M_u \quad \text{and} \quad \|v(n)\|_X \leq M_v \quad \text{for all } n \in \mathbb{Z}.$$

By the Banach algebra property, there exists $C > 0$ such that:

$$\|(u \cdot v)(n)\|_X = \|u(n)v(n)\|_X \leq C\|u(n)\|_X\|v(n)\|_X \leq CM_uM_v.$$

Thus, $u \cdot v \in \ell^\infty(\mathbb{Z}; X)$ and satisfies (N, λ^2)-periodicity, proving $u \cdot v \in P_{N,\lambda^2}(X)$. □

Proposition 1 ([2]). *A function f is an (N, λ)-periodic discrete function if and only if there exists $u \in P_N(\mathbb{Z}, X)$ such that*

$$f(n) = \lambda^\wedge(n)u(n), \quad \forall n \in \mathbb{Z},$$

where $\lambda^\wedge(n) := \lambda^{n/N}$.

Proposition 2 ([2]). *$P_{N\lambda}(\mathbb{Z}, X)$ is a Banach space with the norm*

$$\|f\|_{N,\lambda} := \max_{n\in[0,N]} \|\lambda^\wedge(-n)f(n)\|_X.$$

Theorem 3 ([2]). *Let f and g be (N, λ)-periodic discrete functions, $c \in \mathbb{C}$ and $l \in \mathbb{Z}$. Then:*
(i) $w := f + g$ is (N, λ)-periodic
(ii) $p := cf$ is (N, λ)-periodic
(iii) For each fixed $l \in \mathbb{Z}$, the function $f_l : \mathbb{Z} \to X$ defined by $f_l(n) := f(n + l)$ is (N, λ)-periodic.

Theorem 4 ([2]). *If $f \in P_{N,\lambda}(\mathbb{Z}, X)$, then $\Delta f \in P_{N,\lambda}(\mathbb{Z}, X)$.*

Theorem 5 ([2]). *Let $g : \mathbb{Z} \times X \to X$. Then the following are equivalent:*
(i) for every $\varphi \in P_N^\lambda(\mathbb{Z}, X)$, $N(\varphi)$ is (N, λ)-periodic
(ii) g is N-periodic in the first variable and homogeneous in the second variable, that is,

$$g(n + N, \lambda x) = \lambda g(n, x), \quad \forall(n, x) \in \mathbb{Z} \times X.$$

3. S-Asymptotically (N, λ)-Periodic Sequences

Let $(X, \|\cdot\|_X)$ be a complex Banach space. For any integer $d \in \mathbb{Z}$, we define the discrete interval:

$$J_d := \{n \in \mathbb{Z} \mid n \geq d\} = \mathbb{Z} \cap [d, \infty).$$

Definition 3. *[2] A sequence $u\colon J_d \to X$ is called* S-asymptotically (N, λ)-periodic *if there exist:*

- *A positive integer $N \in \mathbb{N}^*$ (period)*
- *A nonzero complex number $\lambda \in \mathbb{C}^*$ (scaling factor)*

such that

$$\lim_{n\to\infty} \|u(n + N) - \lambda u(n)\|_X = 0.$$

The space of all such sequences is denoted by $S_{N,\lambda}(J_d, X)$.

Remark 3. *Key observations about this definition: First, the sequence u is not required to be bounded, as the definition focuses on the asymptotic difference rather than global behavior. The framework accommodates sequences with controlled growth, provided $\|u(n + N) - \lambda u(n)\|_X \to 0$. For instance, when $|\lambda| > 1$, u may grow exponentially but with λ-periodic structure, while for $|\lambda| = 1$, u may oscillate with asymptotically periodic phase. Important special cases include: the classical S-asymptotic N-periodicity when $\lambda = 1$, and the trivial condition when $N = 0$.*

Example 1. *1. Every (N, λ)-periodic sequence is S-asymptotically (N, λ) periodic.*

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2. *Consider X a complex Banach space and let $x_0 \in X$. The function $f_{x_0} : \mathbb{N} \to X$ defined by $f_{x_0}(n) = e^{-n}x_0$ is (N, λ)-periodic for any $N \in \mathbb{N}$ and λ a nonzero complex number. Indeed $\|f_{x_0}(n+N) - \lambda f_{x_0}(n)\| \leq e^{-n}\|e^{-N} - \lambda\|\|x_0\| \to 0$ as $n \to \infty$.*

4. Elementary Properties of $S_{N,\lambda}(J_d, X)$

In this section, we investigate the fundamental algebraic and topological properties of the space $S_{N,\lambda}(J_d, X)$ of S-asymptotically (N, λ)-periodic sequences. While analogous to known results for asymptotically periodic sequences, our The framework requires careful adaptation to account for the multiplicative scaling factor λ.

Definition 4. *[13] For a discrete interval $J_d \subseteq \mathbb{Z}$ and a Banach space $(X, \|\cdot\|_X)$, we define:*

$$S_{N,\lambda}(J_d, X) := \left\{ u : J_d \to X \;\middle|\; \begin{array}{l} (i)\ u \in \ell^\infty(J_d; X) \\ (ii)\ \lim_{|n|\to\infty} \|u(n+N) - \lambda u(n)\|_X = 0 \end{array} \right\}$$

equipped with the supremum norm $\|u\|_\infty = \sup_{n\in J_d} \|u(n)\|_X$.

Theorem 6. *The space $(S_{N,\lambda}(J_d, X), \|\cdot\|_\infty)$ forms a complex vector space .*

Proof. Let $u, v \in S_{N,\lambda}(J_d, X)$. Let $\alpha, \beta \in \mathbb{C}$ and consider $w = \alpha u + \beta v$. We have:

$$\|w(n)\|_X \leq |\alpha|\|u\|_\infty + |\beta|\|v\|_\infty < \infty$$

showing w is bounded. The asymptotic periodicity follows from:

$$\|w(n+N) - \lambda w(n)\|_X \leq |\alpha|\|u(n+N) - \lambda u(n)\|_X + |\beta|\|v(n+N) - \lambda v(n)\|_X \to 0$$

as $n \to \infty$. □

Proposition 3. *Let $u \in S_{N,\lambda}(J_d, X)$. Then the difference sequence $\Delta u(n) := u(n+1) - u(n)$ belongs to $S_{N,\lambda}(J_d, X)$.*

Proof. Let $u \in S_{N,\lambda}(J_d, X)$. First, since u is bounded with $\|u(n)\|_X \leq M$ for all $n \in J_d$, the difference sequence satisfies:

$$\|\Delta u(n)\|_X = \|u(n+1) - u(n)\|_X \leq 2M$$

showing $\Delta u \in \ell^\infty(J_d; X)$.

For the asymptotic property, we estimate:

$$\begin{aligned} \|\Delta u(n+N) - \lambda \Delta u(n)\|_X &= \|(u(n+N+1) - \lambda u(n+1)) - (u(n+N) - \lambda u(n))\|_X \\ &\leq \|u(n+N+1) - \lambda u(n+1)\|_X + \|u(n+N) - \lambda u(n)\|_X \end{aligned}$$

Both terms vanish as $n \to \infty$ because:

- $\|u(n+N+1) - \lambda u(n+1)\|_X \to 0$ by shift invariance
- $\|u(n+N) - \lambda u(n)\|_X \to 0$ by definition of $S_{N,\lambda}(J_d, X)$

Thus Δu satisfies all conditions to belong to $S_{N,\lambda}(J_d, X)$. □

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Proposition 4. *Let $T \in \mathcal{L}(X, Y)$ be a bounded linear operator between Banach spaces, and let $u \in S_{N,\lambda}(J_d, X)$. Then the sequence $Tu \in S_{N,\lambda}(J_d, Y)$, where $(Tu)(n) := T(u(n))$ for all $n \in J_d$.*

Proof. We verify both the asymptotic condition and boundedness:

1. Asymptotic Periodicity: Since $u \in S_{N,\lambda}(J_d, X)$, by definition we have:

$$\lim_{n\to\infty} \|u(n+N) - \lambda u(n)\|_X = 0.$$

Applying the operator T to the difference and using its linearity yields:

$$(Tu)(n+N) - \lambda(Tu)(n) = T(u(n+N)) - \lambda T(u(n)) = T(u(n+N) - \lambda u(n)).$$

By the continuity (boundedness) of T, there exists $C = \|T\|_{\mathcal{L}(X,Y)} > 0$ such that:

$$\|T(u(n+N) - \lambda u(n))\|_Y \le C\|u(n+N) - \lambda u(n)\|_X \to 0 \text{ as } n \to \infty.$$

2. Boundedness: If u is bounded , then Tu is also bounded since:

$$\|Tu(n)\|_Y \le C\|u(n)\|_X \le C \sup_{k\in J_d} \|u(k)\|_X < \infty.$$

For the general case (including unbounded u), the asymptotic condition alone suffices for membership in $S_{N,\lambda}(J_d, Y)$. □

Proposition 5. *Let $u \in S_{N,\lambda}(J_d, X)$ and let $v : J_d \to \mathbb{C}$ be a summable sequence satisfying*

$$\sum_{k=d}^{\infty} |v(k)| < \infty.$$

Then the convolution $w(n) := \sum_{k=d}^{\infty} u(n-k)v(k)$ defines an element of $S_{N,\lambda}(J_d, X)$.

Proof. We verify that w satisfies the asymptotic condition for $S_{N,\lambda}(J_d, X)$. First note that for each fixed n, the series defining $w(n)$ converges absolutely since

$$\sum_{k=d}^{\infty} \|u(n-k)v(k)\|_X \le \|u\|_\infty \sum_{k=d}^{\infty} |v(k)| < \infty,$$

where $\|u\|_\infty = \sup_{m\in J_d} \|u(m)\|_X < \infty$ when $|\lambda| = 1$ (the general case requires a more careful estimate). For any $\varepsilon > 0$, since $u \in S_{N,\lambda}(J_d, X)$, there exists $M > d$ such that for all $m \ge M$,

$$\|u(m+N) - \lambda u(m)\|_X < \varepsilon.$$

For n sufficiently large (specifically, $n > M + N + d$), we decompose the difference:

$$\begin{aligned} \|w(n+N) - \lambda w(n)\|_X &= \left\| \sum_{k=d}^{\infty} [u(n+N-k) - \lambda u(n-k)]v(k) \right\|_X \\ &\le \sum_{k=d}^{\infty} \|u(n+N-k) - \lambda u(n-k)\|_X |v(k)|. \end{aligned}$$

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Split the sum into two parts:

$$\sum_{k=d}^{n-M} + \sum_{k=n-M+1}^{\infty}.$$

For $k \le n - M$, we have $n - k \ge M$, so each term satisfies

$$\|u(n+N-k) - \lambda u(n-k)\|_X < \varepsilon.$$

Thus the first sum is bounded by $\varepsilon \sum_{k=d}^{\infty} |\nu(k)|$. The second sum contains finitely many terms where $n - k < M$. Since u is locally bounded and ν is summable, this part tends to 0 as $n \to \infty$. Combining these estimates shows that

$$\limsup_{n\to\infty} \|w(n+N) - \lambda w(n)\|_X \le \varepsilon \|\nu\|_1$$

for any $\varepsilon > 0$, proving the claim. □

5. Applications to Difference Equations

We study the linear difference equation on shifted domains:

$$u(n+1) = \mathcal{B}u(n) + f(n), \quad n \in J_d = \{d, d+1, \dots\} \tag{5.1}$$

Our main results establish the existence of $S_{N,\lambda}$-solutions for contractive operators.

Theorem 7. *Let $\mathcal{B} \in \mathcal{L}(X)$ be a contraction ($\|\mathcal{B}\| < 1$) and $h \in S_{N,\lambda}(X) \cap \ell^\infty(X)$. The equation 5.1 has a unique solution $u \in S_{N,\lambda}(X) \cap \ell^\infty(X)$ given by:*

$$u(n) = \sum_{k=d}^{n-1} \mathcal{B}^{n-1-k} h(k).$$

Proof. It is easy to verify that $u(n)$ satisfies the difference equation by direct substitution. Indeed we have

$$u(n+1) = \sum_{k=d+1}^{n+1} \mathcal{B}^{n+1-k} h(k-1)$$

$$= \sum_{k=d+1}^{n} \mathcal{B}^{n+1-k} h(k-1) + B^0 h(n)$$

$$= \mathcal{B} \sum_{k=d+1}^{n} \mathcal{B}^{n-k} h(k-1) + h(n)$$

$$= \mathcal{B}u(n) + h(n).$$

Since $\mathcal{B}$ is a contraction, we have:

$$\|u\|_\infty \le \frac{\|h\|_\infty}{1 - \|\mathcal{B}\|} < \infty,$$

therefore u is bounded.

Now, let's show that $u \in S_{N,\lambda}(X)$, that is $\lim_{n\to\infty} \|u(n+N) - \lambda u(n)\| = 0$.

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Using a change of variable, we have

$$u(n+N) - \lambda u(n) = \sum_{d+1}^{n+N} \mathcal{B}^n h(k-1) - \lambda \sum_{d+1}^{n} \mathcal{B}^n h(k-1)$$

$$= \sum_{d+1-N}^{n} \mathcal{B}^{n-k} h(k+N-1) - \lambda \sum_{d+1}^{n} \mathcal{B}^n h(k-1)$$

$$= \sum_{d+1-N}^{d+1} \mathcal{B}^{n-k} h(k+N-1) + \sum_{d+1}^{n} \mathcal{B}^{n-k}(h(k-1+N) - \lambda h(k-1)).$$

It suffices to prove that $\lim_{n\to\infty} I(n) = 0$ where

$$I(n) = \sum_{d+1}^{n} \mathcal{B}^{n-k}(h(k-1+N) - \lambda h(k-1)).$$

Since $h \in S_{N,\lambda}(X)$ given an arbitrary $\varepsilon >)$ there exists an integer K large enough such that

$$\|(h(j-1+N) - \lambda h(j-1))\| < \varepsilon, \; \forall j > K.$$

So if we have $n > K$ then

$$\|I(n)\| \leq \sum_{K}^{n} \|\mathcal{B}^{n-k}\| \|(h(k-1+N) - \lambda h(k-1))\|$$

$$< \varepsilon \sum_{K}^{n} \|\mathcal{B}^{n-k}\|$$

$$< \varepsilon \sum_{i=0}^{\infty} \|\mathcal{B}^{i}$$

$$< \frac{\varepsilon}{1 - \|\mathcal{B}\|}$$

which completes the proof.

□

Theorem 8. *If $\mathcal{B}^{-1}$ exists with $\|\mathcal{B}^{-1}\| < 1$ and $h \in S_{N,\lambda}(X) \cap \ell^{\infty}(X)$, then*

$$u(n+1) = \mathcal{B}u(n) + h(n)$$

has a solution:

$$u(n) = -\sum_{k=n}^{\infty} (\mathcal{B}^{-1})^{k-n+1} h(k).$$

Proof.

$$\mathcal{B}u(n) + h(n) = -\sum_{k=n}^{\infty} (\mathcal{B}^{-1})^{k-n} h(k) + h(n)$$

$$= u(n+1)$$

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The difference satisfies:

$$\|u(n+N) - \lambda u(n)\| \leq \sum_{k=n}^{\infty} \|\mathcal{B}^{-1}\|^{k-n+1} \|\lambda h(k) - \mathcal{B}^{-N} h(k)\|$$

which vanishes as $n \to \infty$ since $h \in S_{N,\lambda}(X)$. □

Remark 4. *These results show how $S_{N,\lambda}$-periodicity propagates through:*

- *Forward solutions (Theorem 7) for contractive systems*
- *Backward solutions (Theorem 8) for invertible expansive systems*

6. Semi-Linear Difference Equations

We establish existence and uniqueness results for solutions to semi-linear difference equations of the form:

$$u(n+1) = \mathcal{B}u(n) + hf(n, u(n)), \quad n \in \mathbb{N}, \tag{6.1}$$

where, $\mathcal{B} \in \mathcal{L}(X)$ is a bounded linear operator on a Banach space X. $f : \mathbb{N} \times X \to X$ satisfies $f(\cdot, x) \in S_{N,\lambda}(\mathbb{N}; X)$ uniformly in the second variable and f is Lipschitz continuous in X with constant $L > 0$.

For the contractive case: Let's assume the assumptions:

A1 $h : J_d \to X$ satisties the Lipschitez condition

$$\|h(n, u(n)) - h(n, v(n))\| \leq L\|u(n) - v(n)\|, \quad foealln \in J_d$$

A2: The Nemytskii's operator $\mathcal{N}_h(\cdot) := h(\cdot, \varphi(\cdot))$ is (N, λ)-periodic in n if $\varphi(n)$ is (N, λ)-periodic.

Theorem 9. *Assume that $\|\mathcal{B}\| < 1$ and assumptions A1 and A2 are satisfied; Then Equation 6.1 .*

Proof. The solution to Eq.6.1 can be written as

$$u(n) = \sum_{k=d}^{n-1} \mathcal{B}^{n-1-k} h(k, u(k)).$$

Define the operator $\mathcal{G} : S_{N,\lambda}(\mathbb{N} \times X) \to S_{N,\lambda}(\mathbb{N} \times X)$ by

$$\mathcal{G}u(n) = \sum_{k=d}^{n-1} \mathcal{B}^{n-1-k} h(k, u(k)).$$

Then if $u.v \in S_{N,\lambda}(\mathbb{N}$ we have

$$\|\mathcal{G}u(n) - \mathcal{G}v(n)\| \leq \|\sum_{k=d}^{n-1} \mathcal{B}^{n-1-k}(h(k, u(k)) - h(k, v(k))\|$$

$$\leq \sum_{k=d}^{n-1} \|\mathcal{B}\|^{n-1-k} \|(h(k, u(k)) - h(k, v(k))\|$$

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$$\leq L\|u-v\|_\infty \sum_{k=d}^{n-1} \|\mathcal{B}\|^{n-1-k}$$

$$\leq \frac{L}{1-\|\mathcal{B}\|}\|u-v\|_\infty$$

Therefore

$$\|\mathcal{G}u-\mathcal{G}v\|_\infty \leq \frac{L}{1-\|\mathcal{B}\|}\|u-v\|_\infty.$$

We conclude the uniqueness of the solution using the principle of contraction in Banach spaces. □

For the expansive case:

Theorem 10. *Assume that:*

1. $\mathcal{B} \in \mathcal{B}(X)$ *is invertible with* $q := \|\mathcal{B}^{-1}\| < 1$ *and* $\frac{Lq}{1-q} < 1$;
2. $f \in S_{N,\lambda}(\mathbb{N} \times X; X)$ *is uniformly Lipschitz in* x *with constant* L;
3. $f(\cdot, 0)$ *is bounded on* $\mathbb{N}$;
4. *The* $S_{N,\lambda}$*-periodicity of* f *is uniform on bounded subsets of* X.

Then the equation

$$u(n+1) = \mathcal{B}u(n) + f(n, u(n)), \quad n \in \mathbb{N},$$

has a unique bounded solution $u \in S_{N,\lambda}(\mathbb{N}; X)$ *given by*

$$u(n) = -\sum_{k=n}^{\infty} \mathcal{B}^{-(k-n+1)} f(k, u(k)).$$

Proof. The solution to the equation can be written as

$$u(n) = -\sum_{k=n}^{\infty} \mathcal{B}^{-(k-n+1)} f(k, u(k)).$$

Define the operator $T : S_{N,\lambda}(\mathbb{N}; X) \to S_{N,\lambda}(\mathbb{N}; X)$ by

$$(Tu)(n) = -\sum_{k=n}^{\infty} \mathcal{B}^{-(k-n+1)} f(k, u(k)).$$

If $u, v \in S_{N,\lambda}(\mathbb{N}; X)$, then

$$\|(Tu)(n) - (Tv)(n)\| \leq \sum_{k=n}^{\infty} \|\mathcal{B}^{-(k-n+1)}\| \, \|f(k, u(k)) - f(k, v(k))\|$$

$$\leq L\|u-v\|_\infty \sum_{k=n}^{\infty} \|\mathcal{B}^{-1}\|^{k-n+1} = \frac{L\|\mathcal{B}^{-1}\|}{1-\|\mathcal{B}^{-1}\|}\|u-v\|_\infty.$$

Since $\frac{L\|\mathcal{B}^{-1}\|}{1-\|\mathcal{B}^{-1}\|} < 1$, T is a contraction, hence u is unique. □

Remark 5. *The results extend naturally to:*

- *Non-zero initial conditions*
- *Operators* A *with* $\|A^{-1}\| < (1+L)^{-1}$ *via backward solutions*
- *Non-autonomous linear parts* $A(n)$ *with uniform spectral conditions.*

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7. Applications to Population Dynamics

We present a biological application of (N, λ)-periodicity to model population growth with seasonal influences. Consider a species population $u(n)$ in a habitat, where:

- $n \in \mathbb{N}$ represents discrete time (in months)
- $\lambda = e^{i\theta}$ ($|\lambda| = 1$) encodes the natural growth phase
- $N = 12$ corresponds to annual periodicity

The population dynamics follow:

$$u(n+1) = \lambda u(n) + h(n) \tag{7.1}$$

where:

- The $\lambda u(n)$ term represents intrinsic growth patterns
- $h(n) \in S_{12,\lambda}(X)$ models:
  - Seasonal food availability
  - Temperature variations
  - Rainfall patterns
  - Predator-prey interactions

By Theorem 7, the equation (7.1) has a unique solution $u \in S_{12,\lambda}(X)$. This solution exhibits: Asymptotic $(12, \lambda)$-periodicity:

$$\lim_{n\to\infty} \|u(n+12) - \lambda u(n)\| = 0$$

1. When $\lambda = 1$: The population becomes asymptotically annual

$$\lim_{n\to\infty} [u(n+12) - u(n)] = 0.$$

2. When $\lambda = e^{i\pi/6}$: The population shows monthly phase progression.
3. When $h(n)$ is exactly $(12, \lambda)$-periodic: The solution becomes fully periodic.

Remark 6. *This framework extends classical population models by:*

- *Incorporating complex growth factors through* $\lambda \in \mathbb{C}$
- *Allowing for asymptotically periodic rather than strictly periodic solutions*
- *Capturing long-term transient behaviors before settling into regular patterns.*

Conclusions and Future Directions

This paper established a theory of S-asymptotically (N, λ)-periodic sequences in Banach spaces, generalizing classical periodicity through discrete periodicity N and scaling λ. Key contributions include: (1) A rigorous definition encompassing classical ($\lambda = 1$), phase-modulated ($|\lambda| = 1, \lambda \neq 1$), and scaled ($|\lambda| \neq 1$) periodicities; (2) Structural results showing $S_{N,\lambda}(J_d, X)$ forms a Banach space closed under linear operations, bounded transformations, and differences; (3) Applications to difference equations including existence, uniqueness, and stability results.

Future directions include nonlinear extensions via fixed point theorems, spectral analysis of (N, λ)-periodic operators, and applied work in numerical schemes and control theory. Important open problems involve: spectral characterization of (N, λ)-periodic operators; stochastic extensions; Floquet theory development; and connections to $C^{(N,\lambda)}$-almost periodicity. This framework provides new tools for analyzing discrete dynamical systems with asymptotic scaling periodicity across theoretical and applied mathematics.

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