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Posted Date: 8 April 2026

doi: 10.20944/preprints202604.0469.v1

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Article

The Center Problem for Homogeneous Case of Polynomial Maps

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Abstract

We study the center problem for polynomial maps

$$y = f(x) = - \sum_{n=0}^{\infty} a_n x^{n+1},$$

arising from homogeneous algebraic curves

$$x + y + \sum_{k=0}^n \alpha_{n-k,k} x^{n-k} y^k = 0.$$

While explicit conditions were previously known only for low even degrees $n = 2, 4, 6, 8, 10$, their general structure remained conjectural. In this paper we resolve the case $n = 12$ and prove that the observed algebraic patterns completely characterize the center for all even degrees n . More precisely, we show that the center condition is equivalent to one of two explicit families of algebraic relations. This provides a complete classification of the center problem in the homogeneous case.

Keywords: discrete dynamical systems; center variety; polynomial maps; Poincaré return map

MSC: 2020; 34C05; 37G15; 13P10; 93C55

1. Introduction

The center problem for polynomial maps has been extensively studied in both continuous and discrete dynamical systems. In the homogeneous case, explicit results were previously obtained for low degrees $n = 2, 4, 6, 8, 10$, revealing recurring algebraic structures. Although these results suggested a general pattern, it remained unclear whether these conditions are exhaustive for arbitrary even degree. The main result of this paper is that this is indeed the case: we prove that for all even degrees n , the center condition is completely characterized by two families of algebraic relations. Thus, the homogeneous center problem for polynomial maps is fully resolved. A major step forward was achieved by N. N. Bautin in the late 1940s and early 1950s, who established algebraic criteria for distinguishing centers from foci in quadratic systems via the so-called *focus quantities* [1]. These invariants provide a rigorous framework for addressing the local dynamics around singular points and motivated subsequent developments for higher degrees and for discrete settings.

Over time, the concepts of center and focus have been extended beyond the continuous setting of differential equations to discrete dynamical systems [2,3]. In discrete systems, such as polynomial maps arising from algebraic curves, iterations can be locally conjugate to rotations and thus exhibit center-type behavior without spiraling. This discrete analogue has been extensively studied in connection with classical models, including discrete Lotka–Volterra systems [4,5].

Several methodological approaches have been developed to tackle the center problem. On the analytical side, works by Ilyashenko and Yakovenko clarified the structure of analytic differential equations [6]. On the computational side, algebraic geometry and computer algebra have become indispensable: *Gröbner bases* and *radical membership tests*—implemented in the system SINGULAR and its library `primdec.lib`—enable explicit computation of focus quantities and the decomposition of the associated algebraic varieties [7,8]. In practice, one considers the chain of ideals

$$I_k = \langle p_2, p_3, \dots, p_k \rangle$$

which stabilizes by Hilbert’s Basis Theorem, incrementally reduces new relations modulo a chosen Gröbner basis of the current ideal, and uses a radical test to check whether new information is already contained in $\sqrt{I_k}$. This yields an algorithmic description of the *center variety*, consistent with the algebraic viewpoint of ideals and varieties [9,10].

Within this broad context, the present paper focuses on the homogeneous case of polynomial maps. Previous work has addressed specific values of the degree, such as $n = 2, 4, 6, 8$, and 10 , leading to a conjectural picture for general even n [3,11,12]. In this article, we resolve the open case $n = 12$ and prove that the conjecture holds for arbitrary even degrees. Our results contribute both to the algebraic theory of dynamical systems and to the computational study of discrete maps, and they complement related developments on bifurcation phenomena and planar vector fields [13–17].

We now turn to the specific setting of discrete polynomial maps. In particular, we consider maps of the form

$$f(x) = - \sum_{n=0}^{\infty} a_n x^{n+1}, \tag{1}$$

where $f : \mathbb{R} \rightarrow \mathbb{R}$ denotes an irreducible component of an algebraic curve defined by

$$x + y + \sum_{i+k=2}^n \alpha_{i,k} x^i y^k = 0. \tag{2}$$

Such equations have been studied in [11,12] and provide a natural discrete analogue of planar differential systems. If one solves (2) for y and interprets the result as a map $f(x)$, then the dynamics of its iterations describe a discrete dynamical system with a nontrivial (nonhyperbolic) singular point at $x = 0$. While a number of results are known for low-degree cases, the structure of the center problem for higher degrees is more subtle. Building on earlier studies for $n = 2, 4, 6, 8$, and 10 [3,17], we investigate the existence of a center at the origin for (2) when $n = 12$ and extend the conclusions to all even $n \in \mathbb{N}$.

2. Preliminary Results for Center of Maps (1)

In this section we provide an overview of key definitions and results related to the dynamics of maps defined by equation (1). For more details see [3,11,12].

If one solves equation (2) for y and interprets the solution as a function $f(x)$, then the dynamics of its iterations describe a discrete dynamical system possessing a nontrivial (nonhyperbolic) singular point at $x = 0$. The special situation where equation (2) takes the form

$$x + y + \sum_{k=0}^n \alpha_{n-k,k} x^{n-k} y^k = 0, \tag{3}$$

which is homogeneous of degree n , is referred to as the homogeneous case of degree n .

For the map (1), we denote the p -th iterate of f by $f^p(x)$. The singular point $x = 0$ of the map f is then classified as:

- 1. a *stable focus*, if there exists $\varepsilon > 0$ such that for all $|x| < \varepsilon$, $\lim_{n \rightarrow \infty} f^n(x) = 0$,
- 2. an *unstable focus*, if it is a stable focus for the inverse map f^{-1} ,

3. a *center*, if there exists $\varepsilon > 0$ such that for all $|x| < \varepsilon$, $f^2(x) = x$.

These notions mirror the classification of equilibrium points in continuous dynamical systems, but in the discrete setting they are expressed through the iterates of the map rather than through trajectories of a flow. We emphasize that the center condition $f^2(x) = x$ does not, in general, coincide with the symmetry condition $F(x, y) = F(y, x)$. The latter represents only a special subclass of centers.

2.1. Poincaré Map and Focus Quantities

To make this classification effective, one needs algebraic tools that detect whether the origin is a center or a focus. This is achieved through the Poincaré return map and its associated focus quantities.

Let us examine equation (2), where the coefficients $\alpha_{i,k} \in \mathbb{R}$. This equation admits a unique analytic solution expressible in the form of (1). Based on the analysis of such solutions, the following question arises: how can one characterize, within the space of coefficients $\{\alpha_{i,k}\}$, the manifold on which the associated map f has a center at the origin?

In order to study that, we define the Poincaré return map

$$\mathcal{R}(x) = f^2(x) = x + p_2x^3 + p_3x^4 + \dots \quad (4)$$

and the difference map

$$\mathcal{P}(x) = f^2(x) - x = p_2x^3 + p_3x^4 + \dots \quad (5)$$

By definition, a map f has a center at the origin if $\mathcal{P}(x) = 0$ holds for all $|x| < \varepsilon$, where we call the coefficient p_m the m -th *focus quantity*. All focus quantities of the Poincaré map are polynomials in the coefficients $\{\alpha_{i,k}\}$ of (2), and for a center at the origin all must vanish simultaneously.

In this formulation, the coefficients p_m play the same role as Lyapunov quantities in continuous systems: their vanishing characterizes the presence of a center.

2.2. Algebraic Reformulation via Ideals

Since an infinite number of focus quantities are in principle involved, it is natural to rephrase the problem in terms of commutative algebra, where ideals and varieties encode the common vanishing of polynomial conditions.

To find conditions for the center at the origin we introduce

$$I = \langle p_m : m \in \mathbb{N}, m > 1 \rangle,$$

the ideal generated by all focus quantities, and its variety $\mathbf{V}(I)$, which consists of all common zeros of polynomials in I . Because I is generated by infinitely many focus quantities, it cannot be computed directly. We therefore define truncated ideals

$$I_k = \langle p_2, p_3, \dots, p_k \rangle.$$

If $m \geq k$, then $I_k \subseteq I_m$, and we obtain an ascending chain of ideals. By Hilbert's basis theorem [9], this chain stabilizes, meaning there exists $n \in \mathbb{N}$ such that $I_k = I_n$ for all $k \geq n$.

If $J \subseteq k[x_1, x_2, \dots, x_n]$ is an ideal, the radical of J is defined by

$$\sqrt{J} = \{f \in k[x_1, x_2, \dots, x_n] : \exists p \in \mathbb{N}, f^p \in J\}.$$

The radical ideal $\sqrt{J}$ determines the same variety as J , so $\mathbf{V}(\sqrt{J}) = \mathbf{V}(J)$. This reduction from infinitely many focus quantities to a finite generating set is a key structural feature of the problem and underlies the possibility of obtaining explicit center conditions.

2.3. Constructing the Center Variety

To determine the center variety associated with the focus quantities of the Poincaré map for (2), we proceed iteratively. We compute the first non - zero focus quantity, denoted p_{k_1} , and set $G_1 = \{p_{k_1}\}$. We then compute the next non - zero quantity, p_{k_2} , reduce it modulo $\langle p_{k_1} \rangle$ (see Definition 1.2.15 in [10]), and apply the Radical Membership Test [10] to check whether there exists $k \in \mathbb{N}$ with $\tilde{p}_{k_2} \in \langle p_{k_1} \rangle^k$. Here $\tilde{p}_{k_2}$ denotes the reduced form of p_{k_2} .

If $\tilde{p}_{k_2} \notin \sqrt{\langle p_{k_1} \rangle}$, we add it to the generating set, so $G_2 = \{p_{k_1}, \tilde{p}_{k_2}\}$. We then compute the next non - zero quantity p_{k_3} , reduce it modulo the Gröbner basis of $\langle p_{k_1}, \tilde{p}_{k_2} \rangle$, and repeat the procedure. Continuing this way, we arrive at the smallest index n such that

$$\tilde{p}_{k_{n+1}} \in \sqrt{G_n}, \quad G_n = \{p_{k_1}, \tilde{p}_{k_2}, \dots, \tilde{p}_{k_n}\}.$$

This produces a nested chain of radical ideals

$$\sqrt{G_1} \subset \sqrt{G_2} \subset \dots \subset \sqrt{G_n},$$

from which the center variety can be computed.

Finally, one needs to prove that $\mathbf{V}(I) = \mathbf{V}(G_n)$. It is sufficient to note that $\mathbf{V}(I) \supset \mathbf{V}(G_n)$, since the reverse inclusion is obvious. By the irreducible decomposition theorem [9], the variety $\mathbf{V}(G_n)$ decomposes as $V_1 \cup V_2 \cup \dots \cup V_n$, where each V_i is irreducible. For each component V_i , any map corresponding to its coefficients has a center at the origin.

The outcome of this procedure is a finite set of generating conditions whose common zeros define the center variety. This reduction from infinitely many to finitely many conditions is the key step that makes the problem tractable. These definitions and constructions thus provide the formal setting for the analysis of the center problem in homogeneous polynomial maps. In the following section we build on this framework to derive explicit conditions for the existence of a center, first for the case $n = 12$ and then for arbitrary even degrees.

3. Main Results

Building on the framework introduced in the previous section, we now present explicit conditions for the existence of a center in the homogeneous case. To set the stage, we first recall the known results for small values of n . These cases illustrate the recurring algebraic patterns that will guide our treatment of the next open case $n = 12$ and the eventual extension to arbitrary even n .

While several authors have already characterized the center conditions for low-degree cases, these results reveal a recurring structure but not yet a complete picture.

In particular, two distinct algebraic patterns have consistently appeared: mirror symmetries in the coefficients, denoted by (S_n) , and alternating-sum conditions, denoted by (T_n) . These patterns are visible in the known results for $n = 2, 4, 6, 8$, and 10, which are summarized in Table 1.

Table 1. Conditions for the existence of a center at the origin for equation (3). Results for $n = 2, 4, 6, 8$ and 10 are known from [3,17], the case $n = 12$ is obtained in this work.

n	Conditions
2	$S_2: \alpha_{2,0} - \alpha_{0,2} = 0$ $T_2: \sum_{k=0}^2 (-1)^k \alpha_{2-k,k} = 0$
4	$S_4: \alpha_{4,0} - \alpha_{0,4} = \alpha_{3,1} - \alpha_{1,3} = 0$ $T_4: \sum_{k=0}^4 (-1)^k \alpha_{4-k,k} = 0$
6	$S_6: \alpha_{6,0} - \alpha_{0,6} = \alpha_{5,1} - \alpha_{1,5} = \alpha_{4,2} - \alpha_{2,4} = 0$ $T_6: \sum_{k=0}^6 (-1)^k \alpha_{6-k,k} = 0$
8	$S_8: \alpha_{8,0} - \alpha_{0,8} = \alpha_{7,1} - \alpha_{1,7} = \alpha_{6,2} - \alpha_{2,6} = \alpha_{5,3} - \alpha_{3,5} = 0$ $T_8: \sum_{k=0}^8 (-1)^k \alpha_{8-k,k} = 0$
10	$S_{10}: \alpha_{10,0} - \alpha_{0,10} = \alpha_{9,1} - \alpha_{1,9} = \alpha_{8,2} - \alpha_{2,8} = \alpha_{7,3} - \alpha_{3,7} = \alpha_{6,4} - \alpha_{4,6} = 0$ $T_{10}: \sum_{k=0}^{10} (-1)^k \alpha_{10-k,k} = 0$

The next unresolved case in this sequence is $n = 12$. Its resolution confirms that the two known families of conditions, (S_n) and (T_n) , are indeed exhaustive and, at the same time, provides the final step toward a general characterization valid for all even degrees.

In the sequel we first consider the equation

$$x + y + \sum_{k=0}^{12} \alpha_{12-k,k} x^{12-k} y^k = 0.$$

(6)

Theorem 1

The equation (6) defines a center at the origin if and only if one of the following conditions holds:

1.

$(S_{12}): \alpha_{12,0} - \alpha_{0,12} = \alpha_{11,1} - \alpha_{1,11} = \alpha_{10,2} - \alpha_{2,10} = \alpha_{9,3} - \alpha_{3,9} = \alpha_{8,4} - \alpha_{4,8} = \alpha_{7,5} - \alpha_{5,7} = 0.$
2.

$(T_{12}): \sum_{k=0}^{12} (-1)^k \alpha_{12-k,k} x^{12-k} y^k = 0.$

Proof

Step 1: Known results. Cases for $n = 4, 6, 8$, and 10 have been established in [3,17] by using the Lyapunov function approach (cf. [10,17]). Here we use instead an approach based on the Poincaré map (4).

Step 2: Reduction to parameters. Let f be a function of the form (1). It has been proven that equation (2) has a center at the origin if and only if the solution y is of the form f , which itself has a center at the origin. From this we see that the coefficients a_i of (1) are polynomials in the parameters

$$\{\alpha_{12,0}, \alpha_{11,1}, \alpha_{10,2}, \alpha_{9,3}, \alpha_{8,4}, \alpha_{7,5}, \alpha_{6,6}, \alpha_{5,7}, \alpha_{4,8}, \alpha_{3,9}, \alpha_{2,10}, \alpha_{1,11}, \alpha_{0,12}\}.$$

(7)

This means that the focus quantities of the Poincaré map are also polynomials in the same parameters. We see that every eleventh parameter a_i is non - zero, i.e., $a_{11} \neq 0, a_{22} \neq 0, a_{33} \neq 0, \dots$

Step 3: First coefficients. We obtain

$$\begin{aligned} a_{11} &= \alpha_{12,0} - \alpha_{11,1} + \alpha_{10,2} - \alpha_{9,3} + \alpha_{8,4} - \alpha_{7,5} + \alpha_{6,6} - \alpha_{5,7} + \alpha_{4,8} - \alpha_{3,9} + \alpha_{2,10} - \alpha_{1,11} \\ &\quad + \alpha_{0,12} \\ a_{22} &= (12\alpha_{0,12} - 11\alpha_{1,11} + 10\alpha_{2,10} - 9\alpha_{3,9} + 8\alpha_{4,8} - 7\alpha_{5,7} + 6\alpha_{6,6} - 5\alpha_{7,5} + 4\alpha_{8,4} - 3\alpha_{9,3} \\ &\quad + 2\alpha_{10,2} - \alpha_{11,1})(\alpha_{0,12} - \alpha_{1,11} + \alpha_{2,10} - \alpha_{3,9} + \alpha_{4,8} - \alpha_{5,7} + \alpha_{6,6} - \alpha_{7,5} + \alpha_{8,4} - \alpha_{9,3} + \\ &\quad \alpha_{10,2} - \alpha_{11,1} + \alpha_{12,0}) \\ &\vdots \end{aligned}$$

Step 4: Focus quantities. Then we compute the first few focus quantities of the Poincaré map and it turns out that $p_{22}, p_{44}, \dots$ are non-zero. As described in the previous section, we reduce p_{44} modulo the Gröbner basis of the ideal $\langle p_{22} \rangle$ and denote the remainder by $\tilde{p}_{44}$. So we obtain

$$\begin{aligned} p_{22} &= 2(6\alpha_{0,12} - 5\alpha_{1,11} + 4\alpha_{2,10} - 3\alpha_{3,9} + 2\alpha_{4,8} - \alpha_{5,7} + \alpha_{7,5} - 2\alpha_{8,4} + 3\alpha_{9,3} - 4\alpha_{10,2} + \\ &\quad 5\alpha_{11,1} - 6\alpha_{12,0})(\alpha_{0,12} - \alpha_{1,11} + \alpha_{2,10} - \alpha_{3,9} + \alpha_{4,8} - \alpha_{5,7} + \alpha_{6,6} - \alpha_{7,5} + \alpha_{8,4} - \alpha_{9,3} \\ &\quad + \alpha_{10,2} - \alpha_{11,1} + \alpha_{12,0}) \\ \tilde{p}_{44} &= \frac{1}{108}(55\alpha_{1,11} - 80\alpha_{2,10} + 81\alpha_{3,9} - 64\alpha_{4,8} + 35\alpha_{5,7} - 35\alpha_{7,5} + 64\alpha_{8,4} - 81\alpha_{9,3} + \\ &\quad 80\alpha_{10,2} - 55\alpha_{11,1})(-12\alpha_{0,12} + 11\alpha_{1,11} - 10\alpha_{2,10} + 9\alpha_{3,9} - 8\alpha_{4,8} + 7\alpha_{5,7} - 6\alpha_{6,6} + 5\alpha_{7,5} \\ &\quad - 4\alpha_{8,4} + 3\alpha_{9,3} - 2\alpha_{10,2} + \alpha_{11,1})^2(\alpha_{0,12} - \alpha_{1,11} + \alpha_{2,10} - \alpha_{3,9} + \alpha_{4,8} - \alpha_{5,7} + \alpha_{6,6} - \alpha_{7,5} \\ &\quad + \alpha_{8,4} - \alpha_{9,3} + \alpha_{10,2} - \alpha_{11,1} + \alpha_{12,0}). \end{aligned}$$

If we compute $\tilde{p}_{66}$ and $\tilde{p}_{88}$, we see that $\tilde{p}_{66}, \tilde{p}_{88} \in \sqrt{\langle p_{22}, \tilde{p}_{44} \rangle}$. By computing the decomposition of the variety $V(\langle p_{22}, \tilde{p}_{44} \rangle)$ we obtain both conditions of Theorem 1. Therefore, these two conditions are necessary for the equation (6) to have a center at the origin.

Step 5: Sufficiency. Next we have to prove that the two conditions are also sufficient. (S_{12}) is the mirror symmetry of (6), so it gives the center at the origin.

We consider the equation

$$x + y + \sum_{k=0}^{12} \alpha_{k,12-k} x^k y^{12-k} = (x + y) \left(1 + \sum_{i+j=11} c_{ij} x^i y^j \right).$$

Equating coefficients of the same monomials on the left and right-hand side and eliminating parameters c_{ij} gives condition (T_{12}) , which means symmetry with respect to the line $y = -x$ and is equivalent to $f^2(x) = x$, yielding a center at the origin. $\square$

It is proved in [12] that the polynomial (3) with n odd defines a center in the origin if and only if

$$\sum_{i=0}^n (-1)^i \alpha_{n-i,i} = 0.$$

The case $n = 12$ thus confirms that the two algebraic families (S_n) and (T_n) remain both necessary and sufficient for the existence of a center. This naturally raises the question of whether the same pattern persists for all even degrees. But the passage from the case $n = 12$ to arbitrary even degree is not immediate. While the structure of the conditions suggests a general pattern, one must verify that no additional independent algebraic conditions arise in higher degrees. This requires a structural analysis of the focus quantities and their dependence on the parameters.

Remark 1. All focus quantities for even degree n factor through the expressions defining the families (S_n) and (T_n) . In particular, no additional independent algebraic conditions arise.

The following theorem establishes precisely this generalization.

Theorem 2

For all even degrees n ($n = 2k$), the center condition of the polynomial (3) is completely characterized by the two families:

1. $(S_{2k}) : \alpha_{2k,0} - \alpha_{0,2k} = \alpha_{2k-1,1} - \alpha_{1,2k-1} = \alpha_{2k-2,2} - \alpha_{2,2k-2} = \dots = 0$
2. $(T_{2k}) : \sum_{i=0}^{2k} (-1)^i \alpha_{2k-i,i} = 0.$

Proof

The proof follows the same strategy as for Theorem 1, but in a more general setting. The crucial point is that all higher-order focus quantities factor through these fundamental expressions and no new independent generators occur. We divide the argument into three steps.

Step 1: Relations between parameters. We establish a relation between the parameters in (1) and (3). Direct computation gives

$$\begin{aligned} a_{2k-1} &= \sum_{n=0}^{2k} (-1)^n \alpha_{2k-n,n}, \\ a_{4k-2} &= \left(\sum_{n=0}^{2k} (-1)^n \alpha_{2k-n,n} \right) \left(\sum_{n=0}^{2k} n (-1)^n \alpha_{2k-n,n} \right), \\ a_{6k-3} &= \left(\sum_{n=0}^{2k} (-1)^n \alpha_{2k-n,n} \right) \left(\sum_{n=0}^{2k} n (-1)^n \alpha_{2k-n,n} \right)^2 + \left(\sum_{n=0}^{2k} (-1)^n \alpha_{2k-n,n} \right)^2 \\ &\quad \left(\sum_{n=2}^{2k} n (-1)^n \alpha_{2k-n,n} \right). \end{aligned}$$

This shows that every $(2k-1)$ -th parameter of f is non-zero.

Step 2: Focus quantities. The first non-zero focus quantity of the Poincaré map is

$$p_m = 2 \left(\sum_{n=0}^{2k} (-1)^n \alpha_{2k-n,n} \right) \left(\sum_{n=0}^{2k} n (-1)^n \alpha_{2k-n,n} \right) - 2k \left(\sum_{n=0}^{2k} (-1)^n \alpha_{2k-n,n} \right)^2.$$

Computing further focus quantities, we see that every $(4k-2)$ -th quantity, reduced modulo the Gröbner basis generated by the previous ones, is non-zero. In particular,

$$\begin{aligned} \tilde{p}_{2m} &= \left(\sum_{n=0}^{2k} (-1)^n \alpha_{2k-n,n} \right) \left(\sum_{n=0}^{2k} n (-1)^{n+1} \alpha_{2k-n,n} \right)^2 \left(\sum_{n=1}^{k-1} (-1)^n \beta_n^{(1)} (\alpha_{2k-n,n} - \alpha_{n,2k-n}) \right), \\ \tilde{p}_{3m} &= \left(\sum_{n=0}^{2k} (-1)^n \alpha_{2k-n,n} \right) \left(\sum_{n=0}^{2k} n (-1)^{n+1} \alpha_{2k-n,n} \right)^2 \left(\sum_{n=2}^{k-1} (-1)^n \beta_n^{(2)} (\alpha_{2k-n,n} - \alpha_{n,2k-n}) \right), \end{aligned}$$

where $\beta_i^{(1)}, \beta_i^{(2)} \in \mathbb{R}$ and $z = 4k-2$. We observe that each focus quantity contains either the expression $\sum_{n=0}^{2k} (-1)^n \alpha_{2k-n,n}$, corresponding to condition (T_{2k}) , or the factors $\alpha_{2k-n,n} - \alpha_{n,2k-n}$, corresponding to condition (S_{2k}) .

Therefore, the center variety

$$V(\langle p_{4k-2}, \tilde{p}_{8k-2}, \dots, \tilde{p}_{n(4k-2)} \rangle), \quad n \in \mathbb{N},$$

satisfies both (S_{2k}) and (T_{2k}) . This proves that these conditions are necessary.

Step 3: Sufficiency. To prove sufficiency, we proceed as in Theorem 1. Consider the equation

$$(x+y) \left(1 + \sum_{i+j=2k-1} c_{i,j} x^i y^j \right) = x+y + \sum_{i=0}^{2k} \alpha_{2k-i,i} x^i y^{2k-i}.$$

Equating coefficients of monomials on both sides gives relations such as

$$\alpha_{2k,0} = c_{2k-1,0}, \quad \alpha_{2k-1,1} = c_{2k-1,0} + c_{2k-2,1}, \quad \dots, \quad \alpha_{0,2k} = c_{0,2k-1}.$$

From these identities, one recovers condition (T_{2k}) . Similarly, condition (S_{2k}) expresses the mirror symmetry of the polynomial, which also ensures the existence of a center at the origin. $\square$

the case of odd n was already settled in [12]

4. Conclusions

In this paper we have characterized the necessary and sufficient conditions for the center problem in discrete dynamical systems of the polynomial maps of the form (1). We have shown that the center problem for such polynomial maps, arising from homogeneous algebraic curves is completely characterized in both the even and odd degree cases. This provides a full classification of centers in the homogeneous setting, reducing the problem to explicit algebraic conditions. A natural next step is the study of mixed-degree systems, where new interactions between different homogeneous components may arise.

These interactions may produce new phenomena in the center conditions and in the structure of the associated Bautin ideals. One such mixed case was already considered in [11], where the combination of degree two and three terms was shown to yield a center at the origin.

Looking ahead, applying this framework to mixed-degree systems has the potential to deepen our understanding of limit cycle bifurcations and to reveal the intricate geometry of the corresponding center varieties.

Acknowledgments: This work was supported by the European Union's Horizon Europe research and innovation program under the Marie Skłodowska-Curie Staff Exchanges (MSCA SE) grant agreement No. 101130523 (DSYREKI). The second and third authors also acknowledge financial support from the Slovenian Research and Innovation Agency (ARIS) through research core funding No. P1-0306 and P1-0288.

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