Article

Not peer-reviewed version

On Weakly Tripotent and Locally Invo-Regular Rings

S K Pandey *

Posted Date: 25 October 2023

doi: 10.20944/preprints202310.0968.v2

Keywords: tripotent ring; weakly tripotent ring; locally invo-regular ring; involution t-clean ring

Article

On Weakly Tripotent and Locally Invo-Regular Rings

S K Pandey

Faculty of Science, Technology and Forensic, Sardar Patel University of Police, Security and Criminal Justice, Opp. Vigyan Nagar-342037, Jodhpur, India; skpandey12@gmail.com

Abstract: In this article some important observations have been reported on recent works related to weakly tripotent rings and locally invo-regular rings. Our findings give additional results andcorrect some recent results on weakly tripotent rings and locally invo-regular rings appeared in Rendiconti Sem. Mat. Univ. Pol. Torino (2021) and Azerbaijan Journal of Mathematics (2021) respectively. In addition we exhibit that if the Jacobson radical $J(A)$ of a ring A is strongly involution t-clean then the characteristic of $J(A)$ need not be four. This finding improves an important result appeared in Eur. J. Pure Appl. Math (2022).

Keywords: tripotent ring; weakly tripotent ring; locally invo-regular ring; involution t-clean ring

MSC 2010: 16S34; 20C07; 16U99

---

1. Introduction

In this paper A is a unital and associative ring and $J(A)$ and $U(A)$ stand for the Jacobson radical of A and the set of units in A respectively. We denote the set of all nilpotents and the set of all idempotents in A by $N(A)$ and $Id(A)$ respectively. We recall that a ring A is said to be a weakly tripotent ring if $u^3 = u$ or $(1-u)^3 = 1-u$ for each $u \in A$ [1,2] and a ring A is said to be a locally invo-regular ring if $u = uvu$ or $1-u = (1-u)v(1-u)$ for each $u \in A$ and some $v \in A$ with $v^2 = 1$ [3].

A ring A is called an involution clean ring if every element of A is expressible as $a+b$ for some $a \in Inv(A)$ and some $b \in Id(A)$. If $ab = ba$, then A is called strongly involution clean ring [4,5]. A ring A is called an involution t-clean ring if every element of A is expressible as $u+t$ for some $u \in Inv(A)$ and some $t \in Trip(A)$. If $ut = tu$, then A is called a strongly involution t-clean ring [4]. It directly follows from these definitions that each involution-clean ring is an involution t-clean ring.

It may be worth mentioning that weakly tripotent rings, locally ino-regular rings and associated notions have extensively appeared in mathematical literature [1–10]. Motivated by some of our recent works [11,12], here we take an opportunity to report some significant observations and results on weakly tripotent and locally invo-regular rings. In addition we provide some significant results on involution t-clean rings.

In [2] it has been seen that if A is a weakly tripotent ring having no non-trivial idempotents and 2 is nilpotent in A then $\frac{A}{J(A)} \cong Z_2$ and $u^2 = 2u = 0$ holds for each $u \in J(A)$. Similarly it has been seen in [3] that if A is a locally invo-regular ring having no non-trivial idempotents and 2 is nilpotent in A then $\frac{A}{J(A)} \cong Z_2$ and $u^2 = 2u = 0$ holds for each $u \in J(A)$.

However we observe that if A is a weakly tripotent ring and it does not have non-trivial idempotents and 2 is nilpotent in A then $u^2 = 2u = 0$ is not necessarily true for each $u \in J(A)$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 25 October 2023 doi:10.20944/preprints202310.0968.v2

. Similarly we note that if A is a locally invo-regular ring having no non-trivial idempotents and 2 is nilpotent in A then $u^2 = 2u = 0$ is not necessarily true for each $u \in J(A)$.

Moreover we observe that if A is a weakly tripotent (or locally invo-regular) ring having no non-trivial idempotents such that $u^2 = 2u = 0$ for each $u \in J(A)$ then $u^3 = 4u = 0$ for each $u \in J(A)$ but the converse of this result is not valid. We exhibit that if A is a weakly tripotent (or locally invo-regular) ring having no non-trivial idempotents and 2 is nilpotent in A, then $u^3 = 4u = 0$ for each $u \in J(A)$.

Further as per [4, Proposition 2.10], if R is a ring such that $J(R)$ is strongly involution t-clean, then the characteristic of $J(R)$ is four. However we exhibit that if R is a ring such that $J(R)$ is strongly involution t-clean then the characteristic of $J(R)$ need not be four.

We provide our observations and results in the next section.

2. Some Observations and Results

Theorem 2.1. *Let A is a weakly tripotent ring having no non-trivial idempotents and 2 is nilpotent in A, then $u^3 = 4u = 0$ for each $u \in J(A)$.*

Proof. Let A is a weakly tripotent ring having no non-trivial idempotents and 2 is nilpotent in A. By [1, Corollary 10], we have $u^2 = 1$ for each $u \in U(A)$ and $u^2 = 2u$ for each $u \in J(A)$. It may be noted that if $u \in J(A)$ then $1 + u \in U(A)$. Similarly $1 - u \in U(A)$. We note that $1 + u \in U(A)$ gives that $(1+u)^2 = 1 \Rightarrow u^2 = -2u$ and $1 - u \in U(A)$ gives that $(1-u)^2 = 1 \Rightarrow u^2 = 2u$. Hence $u^2 = -2u$ and $u^2 = 2u$ together give that $u^3 = 4u = 0$ for each $u \in J(RA)$.

Theorem 2.2. *Let A is a locally invo-regular ring having no non-trivial idempotents and 2 is nilpotent in A, then $u^3 = 4u = 0$ for each $u \in J(A)$.*

Proof. The proof of this Theorem follows from the proof of Proposition 2.1 and the fact that each weakly tripotent ring is a locally invo-regular ring [3].

Proposition 2.3. *Let A is a weakly tripotent ring having no non-trivial idempotents and 2 is nilpotent in A then $u^2 = 2u = 0$ is not necessarily true for each $u \in J(A)$.*

Proof. Let $A = Z_4$ and $G = \{1, g : g^2 = 1\}$. Clearly G is an abelian group under multiplication. Now we shall construct the group ring AG. It may be noted that if $a_i \in A, g_i \in G$ then $u \in AG$ is expressible as $(a_1 g_1 + a_2 g_2 + \ldots + a_n g_n) \in AG$ [13]. Thus the group ring AG has the following sixteen elements.

$0, 1, 2, 3, g, 2g, 3g, 1+g, 2+g, 3+g, 1+2g, 2+2g, 3+2g, 1+3g, 2+3g, 3+3g$.

One may easily note that each element $u \in AG$ satisfies $u^3 = u$ or $(1-u)^3 = 1-u$. Hence AG is a weakly tripotent ring. We note that 0 and 1 are idempotent elements of R and R does not have any other idempotent element. Also 2 is nilpotent in R. We have

$U(A) = \{1, 3, g, 2+g, 1+2g, 3+2g, 2+3g\}$ and $J(A) = \{0, 2, 2g, 3+g, 2+2g, 1+3g, 3+3g\}$.

Clearly $3+3g \in J(A)$, but $(3+3g)^2 = 2(3+3g) \neq 0$. Hence the proof is complete.

Proposition 2.4. *Let A is a locally invo-regular ring having no non-trivial idempotents and 2 is nilpotent in A then $u^2 = 2u = 0$ is not necessarily true for each $u \in J(A)$.*

Proof. We prove it as follows. Let us consider the ring A given above (we refer the proof of Proposition 2.3). After some computation one finds that $u = uvu$ or $1-u = (1-u)v(1-v)$ holds for each $u \in A$ and some $v \in A$ with $v^2 = 1$. Therefore A is a locally invo-regular ring.

We have already noted that 2 is nilpotent in A and A is has no non-trivial idempotent elements. Further $1+u \in J(A)$ such that $(1+u)^2 = 2(1+u) \neq 0$. Hence the proof is complete.

Proposition 2.5. *Let A is a weakly tripotent ring having no non-trivial idempotents then $u^2 = 2u = 0 \Rightarrow u^3 = 4u = 0$ for each $u \in J(A)$ but the converse of this result is not valid.*

Proof. Let A is a weakly tripotent ring such that it has no non-trivial idempotents. Let $u^2 = 2u = 0$ for each $u \in J(A)$. This gives that $u^3 = 2u^2 = 0$. This in turn implies that $u^3 = 4u = 0$ for each $u \in J(A)$ The converse is not valid. Let us consider the ring A given in the proof of Proposition 2.3. Clearly $1+u \in J(R)$ such that $(1+u)^3 = 4(1+u) = 0$ but $(1+u)^2 = 2(1+u) \neq 0$.

Proposition 2.6. *Let A is a locally invo-regular ring having no non-trivial idempotents then $u^2 = 2u = 0 \Rightarrow u^3 = 4u = 0$ for each $u \in J(A)$ but the converse of this result is not valid.*

Proof. The proof directly follows from the above.

Proposition 2.7. *Let A is a noncommutative ring such that $J(A)$ is strongly involution t-clean, then the characteristic of $J(A)$ is not necessarily four*

Proof. Let $A = \{a_0, e, a, b, e+a, e+b, a+b, e+a+b\}$. Here we take

$$a_0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, a = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, b = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, e+b = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, e+a = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix},$$
$$e+a+b = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ a+b = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

One may verify that A is a noncommutative ring with identity under addition and multiplication of matrices modulo two. We have

$N(A) = J(A) = \{a_0, a+b\}$.
$Id(A) = \{a_0, e, e+a, e+b, a, b\}$.
$Inv(A) = \{e, e+a+b\} = U(A)$.
$Trip(A) = \{e, e+a+b\} \cup Id\{A\}$.

It is clear that a_0 is a strongly involution t- clean element. It may be noted that $a+b = u+t$ such that $u \in Inv(A)$, $t \in Trip(A)$ and $ut = tu$. Here $u = e$ and $t = (e+a+b)$. Therefore $a+b$ is also strongly involution t-clean element. Thus $J(A)$ is strongly involution t-clean. However the characteristic of $J(A)$ is not four. Hence our claim is verified.

Proposition 2.8. *Let A is a commutative ring such that $J(A)$ is strongly involution t-clean, then the characteristic of $J(A)$ is not necessarily four*

Proof. Let $A=\{a_0,e,a,e+a\}$. Here we take

$$a_0=\begin{pmatrix}0&0\\0&0\end{pmatrix}, e=\begin{pmatrix}1&0\\0&1\end{pmatrix}, a=\begin{pmatrix}0&1\\1&0\end{pmatrix}, e+a=\begin{pmatrix}1&1\\1&1\end{pmatrix}.$$

It is easy to check that A is a commutative ring with identity under the addition and the multiplication of matrices modulo two. Clearly in this case we have

$Inv(A)=\{e,a\}=U(A)$

$Id(A)=\{a_0,e\}$

$Trip(A)=Inv(A)\cup\{a_0\}$ and $N(A)=J(A)=\{a_0,e+a\}$.

One may verify that $J(A)$ is strongly involution t-clean. However the characteristic of $J(A)$ is not four. Hence our claim is justified.

Proposition 2.9. *Let A is a ring such that $J(A)$ is strongly involution t-clean, then $a^2=2a=0$ for each $a\in J(A)$ implies $a^3=4a=0$ for each $a\in J(A)$. The converse is not true.*

Proof. The proof easily follows. The converse is not valid can be seen as follows. Let $A=Z_8$. It is easy to see that A is an involution t-clean ring. It may be noted that $a^3=4a=0$ for each $a\in J(A)$. However $a^2=2a=0$ does not hold for each $a\in J(A)$.

In the following Proposition we prove that if A is any ring such that $J(A)$ is strongly involution t-clean, then the characteristic of $J(A)$ is two provided $a^2=0$ for each $a\in J(A)$.

Proposition 2.10. *If A is a ring such that $J(A)$ is strongly involution t-clean, then the characteristic of $J(A)$ is two provided $a^2=0$ for each $a\in J(A)$.*

Proof. Let A is a ring and $J(A)$ is strongly involution t-clean. Let $a^2=0$ for each $a\in J(A)$. Then by Proposition 7 [4], there exist $u\in Inv(A)$ and $t\in Inv(A)$ such that $a=u+t$. This gives $2+2ut=0\Rightarrow 2t(t+u)=2at=0$. Finally $2at=0$ gives $2a=0$ (since $t^2=1$). Therefore $2a=0$ for each $a\in J(A)$. Hence the characteristic of $J(A)$ is two in this case. Thus proof is complete.

Conflict of Interest: The author declares that there is no conflict of interest.

References

1. Breaz, S., Cimpean, A., Weakly tripotent rings, Bull. Korean Math. Soc., 55 (4) (2018), 1179-1187.
2. Danchev, P., A Characterization of Weakly Tripotent Rings, Rendiconti Sem. Mat. Univ. Pol. Torino, 79 (1) (2021), 21-32.
3. Danchev, P. V. , Locally Invo-Regular Rings, Azerbaijan Journal of Mathematics, 11 (1) (2021).
4. Al.Neima, Mohammed, et al., Involution t-clean rings with applications, Eur. J. Pure Appl. Math, 15 (4) (2022), 1637-1648.
5. Danchev, P. V., Invo-clean unital rings, Commun. Korean Math. Soc., 32 (1) (2017), 19-27.
6. Zhou, Y., Rings in which elements are sums of nilpotents, idempotents and tripotents, J. Algebra Appl., 17 (2018).
7. P. V. Danchev, Quasi invo-clean rings, Bulletin of the Transilvania University of Brasov, Series III: Mathematics, Informatics, Physics, 63(1), 2021.
8. Ying, Zhiling , Kosan, Tamer, Zhou, Yiqiang (2016). Rings in which every element is a sum of two tripotents, *Canad. Math. Bull.*, 59 , 661-672.
9. G. Calugareanu, Tripotents: A Class of Strongly, Clean Elements in Rings, An. St. Univ., Ovidius Constanta, 26, 69-80, 2018.

10. P. Danchev, Commutative Weakly Tripotent Group Rings, Bul. Acad. Stiinte Repub. Mold. Mat., 93 (2), 24-29, 2020.
11. Pandey, S. K., Some counterexamples in ring theory, arXiv:2203.02274 [math.RA], 2022.
12. Pandey, S. K., A note on rings in which each element is a sum of two idempotents, Elem. Math. (2023). DOI 10.4171/EM/507.
13. Passman, D. S., The Algebraic Structure of Group Rings, Dover Publications, 2011.