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Article

On Ricci Flow Equation from Perspective of Deformation Kinematics

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Abstract

In this paper, we investigate the Ricci flow from the fundamental perspective of deformation kinematics of Riemannian manifolds, a research direction that has not been systematically explored in existing literature. By combining the geometric framework of Ricci flow with the kinematic theory of continuous medium deformation, we rigorously prove that the Ricci tensor is equivalent to the velocity gradient (strain rate tensor) of the manifold under the Hamilton Ricci flow equation $\frac{\partial g_{ij}}{\partial t} = -2R_{ij}$. We further extend this core conclusion through detailed tensor analysis, dimensional analysis, and the study of manifold area element evolution, and correct the dimensional imbalance of the original Ricci flow equation by introducing a characteristic kinematic parameter κ. The results of this paper provide a new kinematic perspective for the study of Ricci flow, enrich the geometric connotation of Ricci flow, and build a bridge between differential geometry and continuum mechanics, which is expected to provide new analytical tools for the study of Ricci flow singularities, long-time behavior and Poincaré conjecture related problems.

Keywords: Ricci flow; deformation kinematics; velocity gradient; strain rate tensor; metric tensor; Ricci tensor; dimensional analysis; Riemannian manifold

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1. Introduction

1.1. Background of Ricci Flow

The Ricci flow, first introduced and constructed by Richard S. Hamilton in the early 1980s [1], is a profound geometric evolution equation that describes the continuous deformation of the Riemannian metric on a manifold over time. Its core idea is to let the metric evolve along the direction of the negative Ricci curvature tensor, so that the curvature of the manifold tends to be uniform, which is known as the process of "flowing towards a constant curvature metric". This evolution equation is not only a typical nonlinear parabolic partial differential equation, but also a powerful tool to reveal the deep connection between the geometry and topology of Riemannian manifolds.

In 1982, Hamilton proved the existence of short-time smooth solutions of Ricci flow on closed Riemannian manifolds with positive Ricci curvature, and pointed out that the manifold will converge to a Riemannian manifold with constant positive curvature under the Ricci flow [1]. This result laid the foundation for the subsequent development of Ricci flow theory. In the early 2000s, Grigori Perelman made a groundbreaking contribution to the study of Ricci flow: he introduced the Ricci flow with surgery, proved the finite extinction time of Ricci flow solutions on certain 3-manifolds [2], and completed the proof of the Poincaré conjecture and the geometrization conjecture of 3-manifolds, which is one of the most important achievements in modern mathematics.

Up to now, the research of Ricci flow has been extended to arbitrary dimensional Riemannian manifolds, and its research content involves the existence and regularity of solutions, the formation and classification of singularities, the long-time behavior of flow, and the application in topological geometry [3–6]. However, most of the existing research is carried out within the traditional framework of differential geometry and geometric analysis, and there is a lack of in-depth discussion on the kinematic nature of the manifold deformation process described by the Ricci flow. In essence, the Ricci flow treats the manifold as a topological structure that deforms continuously over time, and its metric evolution is a typical kinematic deformation phenomenon. Therefore, it is necessary to study the Ricci flow from the perspective of deformation kinematics, explore the kinematic characteristics such as deformation velocity, velocity gradient and strain rate in the manifold deformation process, and reveal the intrinsic kinematic mechanism of the Ricci flow.

1.2. *Deformation Kinematics of Continuous Media*

Deformation kinematics is a core branch of continuum mechanics, which studies the geometric characteristics and motion laws of continuous medium deformation over time, including the description of motion, the definition of velocity field, the derivation of strain rate tensor and velocity gradient tensor [7]. The key idea of deformation kinematics is to use the material derivative (Lagrangian derivative) to describe the change of physical quantities with time following the medium particle, and to decompose the velocity gradient tensor into the symmetric strain rate tensor (deformation rate tensor) and the antisymmetric spin tensor (vorticity tensor), which respectively describe the pure deformation and rigid rotation of the medium.

For a continuous medium with position vector $\boldsymbol{r} = \boldsymbol{r}(\xi^i, t)$ (Lagrange coordinate system ξ^i), the velocity field is defined as $\boldsymbol{v} = \frac{d\boldsymbol{r}}{dt}$, and the velocity gradient tensor is $\boldsymbol{\nabla} \otimes \boldsymbol{v} = \frac{\partial v^i}{\partial \xi^j} \boldsymbol{g}^i \otimes \boldsymbol{g}^j$, where $\boldsymbol{g}^i$ is the reciprocal base vector of the manifold. The additive decomposition of the velocity gradient tensor is:

$$\boldsymbol{\nabla} \otimes \boldsymbol{v} = \boldsymbol{D} + \boldsymbol{\Omega}$$

where $\boldsymbol{D} = \frac{1}{2}(\boldsymbol{\nabla} \otimes \boldsymbol{v} + \boldsymbol{v} \otimes \boldsymbol{\nabla})$ is the symmetric strain rate tensor, which describes the pure deformation of the medium (stretching and shearing); $\boldsymbol{\Omega} = \frac{1}{2}(\boldsymbol{\nabla} \otimes \boldsymbol{v} - \boldsymbol{v} \otimes \boldsymbol{\nabla})$ is the antisymmetric spin tensor, which describes the rigid rotation of the medium. This decomposition is unique and is the fundamental basis of deformation kinematics [**?**].

1.3. *Ricci Flow as a Physical Deformation Process*

A central premise of this paper is that the Ricci flow equation is not merely a formal geometric evolution equation but inherently describes a physical deformation process. The justification is straightforward: the equation contains an explicit time derivative $\partial/\partial t$, and the metric $g_{ij}(t)$ — which determines lengths, angles, and volumes — varies with this time parameter. Whenever a spatial configuration changes continuously in time, the concepts of displacement, velocity, and deformation are unavoidable. The time derivative of the metric directly yields the strain rate tensor. Consequently, the Ricci flow equation can be read as a relation between curvature (a geometric quantity) and deformation rate (a kinematic quantity). From this perspective, Ricci flow is a problem of deformation kinematics first, and a geometric PDE second. This physical viewpoint is not an analogy or an interpretation — it is a direct consequence of the equation's structure.

1.4. *Research Motivation and Main Results*

The existing literature has not established a direct connection between the Ricci flow and the deformation kinematics of manifolds. The core motivation of this paper is to fill this gap: by combining the tensor analysis of Riemannian manifolds with the basic theory of deformation kinematics, we study the kinematic characteristics of the manifold deformation process described by the Ricci flow, and reveal the intrinsic relationship between the Ricci tensor and the velocity gradient (strain rate tensor) of the manifold.

The main results of this paper are as follows:

1. We rigorously prove the core theorem: under the Hamilton Ricci flow equation $\frac{\partial g_{ij}}{\partial t} = -2R_{ij}$, the Ricci tensor R_{ij} is equal to the negative of the strain rate tensor (velocity gradient) of the manifold, that is $R_{ij} = -\frac{1}{2}(\nabla_i v_j + \nabla_j v_i)$, and the Ricci tensor field $\boldsymbol{R} = -\boldsymbol{D}$ in coordinate-free representation.

2. We derive the evolution equation of the manifold area element from the perspective of deformation kinematics, establish the relationship between the Ricci tensor, spin tensor and the material derivative of the area element, and obtain $R = \Omega + dA \otimes dA$ under the material conservation condition $\boldsymbol{v} \cdot \boldsymbol{\nabla} = \boldsymbol{\nabla} \cdot \boldsymbol{v} = 0$.

3. We carry out a strict dimensional analysis of the original Ricci flow equation, find the dimensional imbalance problem of the equation, and introduce the kinematic parameter κ (with dimension of area velocity L^2T^{-1}) to correct the equation, so that the dimensional consistency of the Ricci flow equation is satisfied: $\frac{\partial g_{ij}}{\partial t} = -2\kappa R_{ij}$.

1.5. Paper Structure

This paper is organized as follows: In Section 2, we review the foundational preliminaries of Riemannian manifolds and tensor analysis, encompassing Lagrange coordinate systems, metric tensors, covariant derivatives, material derivatives, and the definitions of Ricci and Riemannian curvature tensors, which provides the essential theoretical basis for subsequent derivations and proofs. In Section 3, we present and rigorously prove the core theorem establishing the intrinsic relationship between Ricci flow and the manifolds velocity gradient (strain rate tensor), including a detailed step-by-step tensor derivation, an alternative proof based on the metric tensor field, and an in-depth analysis of the theorems geometric meaning; we further decompose the velocity gradient tensor, derive the material derivative of the manifold area element, establish the connection between the Ricci tensor, spin tensor and area element evolution, and interpret the geometric significance of the spin tensor in Ricci flow under the material conservation condition. In Section 4, we conduct a strict dimensional analysis of the core geometric and kinematic quantities of Riemannian manifolds, identify the dimensional imbalance of the original Hamilton Ricci flow equation, introduce the characteristic kinematic parameter κ (with the dimension of area velocity) to correct the equation for dimensional consistency, and prove the generalized form of the core theorem under the corrected Ricci flow equation while elaborating on the physical and geometric meaning of κ. In Section 5, we discuss the key results of the paper, establish the connections between the proposed kinematic framework of Ricci flow and other mathematical and physical theories (including continuum mechanics, geometric group theory, and nonlinear partial differential equations), and draw the main conclusions of this research; we also prospect promising future research directions at the intersection of differential geometry and continuum mechanics, such as the kinematic study of Ricci flow with surgery and the singularity formation mechanism from the perspective of deformation kinematics.

2. Preliminaries of Riemannian Manifolds and Tensor Analysis

In this section, we review the basic concepts and tensor analysis methods of Riemannian manifolds that are necessary for the study of this paper, including Lagrange coordinate system, base vector and reciprocal base vector, metric tensor, covariant derivative, material derivative and Ricci tensor. All manifolds considered in this paper are smooth, connected, closed Riemannian manifolds (or manifolds with boundary under suitable boundary conditions), and all tensor operations are carried out in the smooth category.

2.1. Lagrange Coordinate System and Base Vectors

For an n-dimensional smooth Riemannian manifold $\mathcal{M}$, we choose the Lagrange coordinate system $\{\xi^k\}_{k=1}^n$, which is a material coordinate system that follows the "particles" of the manifold during deformation. In this coordinate system, the position vector of any point on the manifold is a smooth function of the Lagrange coordinate and time:

$$\boldsymbol{r} = \boldsymbol{r}(\xi^i, t), \quad i = 1, 2, \ldots, n$$

where $t \in \mathbb{R}^+$ is the time parameter, and the smoothness means that $\boldsymbol{r}$ has continuous partial derivatives of all orders with respect to ξ^i and t.

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The tangent base vector (covariant base vector) at a point on the manifold is defined as the partial derivative of the position vector with respect to the Lagrange coordinate:

$$\boldsymbol{g}_i = \frac{\partial \boldsymbol{r}}{\partial \xi^i}, \quad i = 1, 2, \ldots, n$$

The tangent base vectors $\{\boldsymbol{g}_i\}$ form a basis of the tangent space $T_p\mathcal{M}$ at the point $p \in \mathcal{M}$, and any tangent vector at point p can be linearly expressed by $\{\boldsymbol{g}_i\}$.

The reciprocal base vector (contravariant base vector) $\{\boldsymbol{g}^j\}$ is defined by the biorthogonality condition:

$$\boldsymbol{g}_i \cdot \boldsymbol{g}^j = \delta_i^j$$

where δ_i^j is the Kronecker delta, satisfying $\delta_i^j = 1$ when $i = j$, and $\delta_i^j = 0$ when $i \neq j$. The reciprocal base vector $\{\boldsymbol{g}^j\}$ forms a basis of the cotangent space $T_p^*\mathcal{M}$ at point p.

The relationship between the covariant base vector and the contravariant base vector can be derived by the determinant of the metric tensor (see Section 2.2). For an n-dimensional manifold, the volume element is given by:

$$dV = \sqrt{g}\, d\xi^1 \wedge d\xi^2 \wedge \cdots \wedge d\xi^n$$

where $g = \det(g_{ij})$ is the determinant of the covariant metric tensor g_{ij}.

2.2. *Metric Tensor*

The covariant metric tensor g_{ij} of the Riemannian manifold $\mathcal{M}$ is defined as the inner product of the tangent base vectors:

$$g_{ij} = \boldsymbol{g}_i \cdot \boldsymbol{g}_j$$

The metric tensor g_{ij} is a symmetric 2-covariant tensor, that is $g_{ij} = g_{ji}$, which is the fundamental quantity describing the metric structure (distance, angle, volume) of the Riemannian manifold. The line element of the manifold is given by:

$$ds^2 = g_{ij} d\xi^i d\xi^j$$

where the Einstein summation convention is adopted: the repeated upper and lower indices represent the summation over the range of the indices (this convention is adopted throughout the paper unless otherwise specified).

The contravariant metric tensor g^{ij} is defined as the inverse of the covariant metric tensor, that is it satisfies:

$$g^{ik} g_{kj} = \delta_j^i$$

The contravariant metric tensor is also a symmetric 2-contravariant tensor, $g^{ij} = g^{ji}$. The inner product of any two tangent vectors $\boldsymbol{u} = u^i \boldsymbol{g}_i$ and $\boldsymbol{v} = v^j \boldsymbol{g}_j$ on the manifold is:

$$\boldsymbol{u} \cdot \boldsymbol{v} = g_{ij} u^i v^j = u_i v^i = u^i v_i$$

where $u_i = g_{ik} u^k$ and $v_j = g_{jl} v^l$ are the covariant components of the tangent vectors, obtained by the lowering index operation of the metric tensor; similarly, $u^i = g^{ik} u_k$ and $v^j = g^{jl} v_l$ are the contravariant components, obtained by the raising index operation.

The mixed metric tensor is the Kronecker delta δ_j^i, which satisfies the index raising and lowering operation: $\delta_j^i u^j = u^i$, $\delta_j^i u_i = u_j$.

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2.3. *Covariant Derivative and Tensor Derivative*

The Levi-Civita connection (Riemannian connection) $\boldsymbol{\nabla}$ is the unique torsion-free metric connection on the Riemannian manifold $\mathcal{M}$, which is the foundation of the covariant derivative of tensors. The covariant derivative of a scalar function $f \in C^\infty(\mathcal{M})$ is the gradient of f, denoted as $\boldsymbol{\nabla}_i f = \frac{\partial f}{\partial \xi^i}$.

The covariant derivative of a contravariant vector field v^j (tangent vector field) is defined as:

$$\boldsymbol{\nabla}_i v^j = \frac{\partial v^j}{\partial \xi^i} + \Gamma^j_{ik} v^k$$

where Γ^j_{ik} is the Christoffel symbol of the second kind, which is determined by the metric tensor and its partial derivatives:

$$\Gamma^j_{ik} = \frac{1}{2} g^{jl} \left(\frac{\partial g_{il}}{\partial \xi^k} + \frac{\partial g_{kl}}{\partial \xi^i} - \frac{\partial g_{ik}}{\partial \xi^l} \right)$$

The Christoffel symbol is symmetric with respect to the lower indices: $\Gamma^j_{ik} = \Gamma^j_{ki}$, which is the torsion-free condition of the Levi-Civita connection.

The covariant derivative of a covariant vector field v_i is defined as:

$$\boldsymbol{\nabla}_j v_i = \frac{\partial v_i}{\partial \xi^j} - \Gamma^k_{ji} v_k$$

For a 2-covariant tensor field T_{ij}, its covariant derivative is:

$$\boldsymbol{\nabla}_k T_{ij} = \frac{\partial T_{ij}}{\partial \xi^k} - \Gamma^l_{ki} T_{lj} - \Gamma^l_{kj} T_{il}$$

The covariant derivative of the metric tensor is zero, which is the metric compatibility condition of the Levi-Civita connection:

$$\boldsymbol{\nabla}_k g_{ij} = 0, \quad \boldsymbol{\nabla}_k g^{ij} = 0$$

This condition means that the inner product of any two tangent vectors is invariant under the parallel transport defined by the Levi-Civita connection.

2.4. *Material Derivative*

In the study of deformation kinematics, we use the material derivative (Lagrangian derivative) to describe the change of physical quantities with time following the "particles" of the manifold, denoted by $\frac{d}{dt}$ or a dot $(\dot{\cdot})$. The material derivative is different from the partial time derivative $\frac{\partial}{\partial t}$ (Eulerian derivative): the partial time derivative describes the change of physical quantities with time at a fixed spatial point, while the material derivative describes the change with time at a fixed material point (Lagrange coordinate point).

For any scalar function $f(\xi^i, t)$, vector field $\boldsymbol{v}(\xi^i, t)$ or tensor field $T(\xi^i, t)$ on the manifold, the material derivative is defined as:

$$\frac{d}{dt} = \frac{\partial}{\partial t} + v^k \boldsymbol{\nabla}_k$$

where v^k is the contravariant component of the velocity field $\boldsymbol{v} = \frac{d\boldsymbol{r}}{dt}$ (the material derivative of the position vector). This formula is the Reynolds transport theorem in the finite-dimensional Riemannian manifold, which establishes the relationship between the material derivative and the partial time derivative and the covariant derivative.

A key property of the material derivative is that it commutes with the partial derivative with respect to the Lagrange coordinate for the position vector:

$$\frac{d}{dt} \frac{\partial \boldsymbol{r}}{\partial \xi^i} = \frac{\partial}{\partial \xi^i} \frac{d\boldsymbol{r}}{dt}$$

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This property is the foundation of deriving the relationship between the base vector derivative and the velocity field, which will be widely used in the subsequent proof of the core theorem.

2.5. *Ricci Tensor and Riemannian Curvature Tensor*

The Riemannian curvature tensor (Riemann-Christoffel tensor) R^l_{ijk} is the fundamental quantity describing the curvature of the Riemannian manifold, which is defined by the covariant derivative of the Levi-Civita connection:

$$R^l_{ijk} = \frac{\partial \Gamma^l_{ij}}{\partial \xi^k} - \frac{\partial \Gamma^l_{ik}}{\partial \xi^j} + \Gamma^m_{ij}\Gamma^l_{mk} - \Gamma^m_{ik}\Gamma^l_{mj}$$

The Riemannian curvature tensor has rich symmetries and algebraic identities, such as the first Bianchi identity and the second Bianchi identity. The second Bianchi identity is particularly important for the study of Ricci flow:

$$\nabla_l R^m_{ijk} + \nabla_j R^m_{ikl} + \nabla_k R^m_{ilj} = 0$$

The Ricci tensor R_{ij} is obtained by the contraction of the Riemannian curvature tensor (contracting the first contravariant index and the third covariant index):

$$R_{ij} = R^k_{ikj}$$

The Ricci tensor is a symmetric 2-covariant tensor, $R_{ij} = R_{ji}$, which is the core quantity in the Ricci flow equation. The Ricci scalar R is obtained by the further contraction of the Ricci tensor with the contravariant metric tensor:

$$R = g^{ij} R_{ij}$$

The Einstein tensor G_{ij} is defined as:

$$G_{ij} = R_{ij} - \frac{1}{2} R g_{ij}$$

The Einstein tensor is divergence-free, that is $\nabla^j G_{ij} = 0$, which is the foundation of Einstein's general relativity. Although the Einstein tensor is not directly involved in the Ricci flow, its geometric meaning is closely related to the Ricci tensor, and it will be discussed in the generalization of high-dimensional manifolds.

3. The Core Theorem: Ricci Flow and Velocity Gradient of Manifold

In this section, we prove the core theorem of this paper: under the Hamilton Ricci flow equation, the Ricci tensor is equal to the negative of the strain rate tensor (velocity gradient) of the manifold. We first give the formal statement of the theorem, then carry out the detailed tensor derivation and proof, and finally give an alternative proof based on the material derivative of the metric tensor field to verify the correctness of the conclusion.

3.1. *Statement of the Core Theorem*

Theorem 3.1 (Ricci Flow and Manifold Velocity Gradient) Let $\mathcal{M}$ be an n-dimensional smooth closed Riemannian manifold, $g_{ij}(t)$ be a one-parameter family of smooth Riemannian metrics on $\mathcal{M}$ satisfying the Hamilton Ricci flow equation:

$$\frac{\partial g_{ij}}{\partial t} = -2R_{ij}$$

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where $R_{ij}(t)$ is the Ricci tensor of the metric $g_{ij}(t)$. Let $\boldsymbol{v} = \frac{d\boldsymbol{r}}{dt}$ be the velocity field of the manifold deformation in the Lagrange coordinate system $\{\xi^i\}$, $\boldsymbol{\nabla}_i v_j$ be the covariant derivative of the covariant component of the velocity field. Then the Ricci tensor satisfies:

$$R_{ij} = -\frac{1}{2}(\boldsymbol{\nabla}_i v_j + \boldsymbol{\nabla}_j v_i)$$

In the coordinate-free representation, let $\boldsymbol{D} = \frac{1}{2}(\boldsymbol{\nabla} \otimes \boldsymbol{v} + \boldsymbol{v} \otimes \boldsymbol{\nabla})$ be the strain rate tensor (velocity gradient tensor) of the manifold, and $\boldsymbol{R} = g_{ij} R^{ij} \boldsymbol{g}^i \otimes \boldsymbol{g}^j$ be the Ricci tensor field. Then:

$$\boldsymbol{R} = -\boldsymbol{D}$$

Remark 1. *The strain rate tensor $D_{ij} = \frac{1}{2}(\boldsymbol{\nabla}_i v_j + \boldsymbol{\nabla}_j v_i)$ is the symmetric part of the velocity gradient tensor $\boldsymbol{\nabla}_i v_j$, which is the core quantity describing the pure deformation of the manifold in deformation kinematics. The theorem shows that the Ricci curvature of the manifold is exactly the negative of the pure deformation rate of the manifold during the Ricci flow, which establishes a direct and fundamental connection between the curvature of the manifold and its kinematic deformation.*

Remark 2. *The theorem is valid for arbitrary n-dimensional smooth Riemannian manifolds, and there is no restriction on the sign and magnitude of the Ricci curvature. For manifolds with positive Ricci curvature, the strain rate tensor is negative, which means that the manifold is undergoing compressive deformation under the Ricci flow; for manifolds with negative Ricci curvature, the strain rate tensor is positive, which means that the manifold is undergoing tensile deformation; for Einstein manifolds ($R_{ij} = \lambda g_{ij}$, λ is a constant), the strain rate tensor is a constant multiple of the metric tensor, which means that the manifold is undergoing uniform pure deformation (no shearing deformation).*

3.2. Detailed Proof of the Core Theorem

We prove the theorem step by step based on the basic knowledge of tensor analysis and deformation kinematics introduced in Section 2, and strictly follow the Einstein summation convention and the definition of covariant derivative and material derivative.

Step 1: Material Derivative of the Tangent Base Vector

First, we calculate the material derivative of the tangent base vector $\boldsymbol{g}_i = \frac{\partial \boldsymbol{r}}{\partial \xi^i}$. According to the commutativity of the material derivative and the partial derivative of the Lagrange coordinate (Section 2.4):

$$\frac{d\boldsymbol{g}_i}{dt} = \frac{d}{dt}\frac{\partial \boldsymbol{r}}{\partial \xi^i} = \frac{\partial}{\partial \xi^i}\frac{d\boldsymbol{r}}{dt}$$

By the definition of the velocity field $\boldsymbol{v} = \frac{d\boldsymbol{r}}{dt}$, the above formula can be rewritten as:

$$\frac{d\boldsymbol{g}_i}{dt} = \frac{\partial \boldsymbol{v}}{\partial \xi^i}$$

We expand the velocity field $\boldsymbol{v}$ by the tangent base vector: $\boldsymbol{v} = v^k \boldsymbol{g}_k$, where v^k is the contravariant component of the velocity field. Substitute this expansion into the above formula:

$$\frac{d\boldsymbol{g}_i}{dt} = \frac{\partial (v^k \boldsymbol{g}_k)}{\partial \xi^i} = \frac{\partial v^k}{\partial \xi^i}\boldsymbol{g}_k + v^k \frac{\partial \boldsymbol{g}_k}{\partial \xi^i}$$

Recall the definition of the Christoffel symbol of the second kind: $\frac{\partial \boldsymbol{g}_k}{\partial \xi^i} = \Gamma^l_{ik} \boldsymbol{g}_l$ (this is the covariant basis derivative formula, which can be derived by the definition of the Christoffel symbol and the

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metric compatibility condition). Substitute this formula into the above formula, and rename the summation index (replace l with k):

$$\frac{d\boldsymbol{g}_i}{dt} = \left(\frac{\partial v^k}{\partial \xi^i} + \Gamma^l_{ik} v^l\right)\boldsymbol{g}_k$$

By the definition of the covariant derivative of the contravariant vector field (Section 2.3): $\nabla_i v^k = \frac{\partial v^k}{\partial \xi^i} + \Gamma^k_{il} v^l$, the above formula can be simplified to the key identity:

$$\frac{d\boldsymbol{g}_i}{dt} = (\nabla_i v^k)\boldsymbol{g}_k$$

In coordinate-free representation, the above formula can be written as:

$$\frac{d\boldsymbol{g}_i}{dt} = (\boldsymbol{v} \otimes \nabla) \cdot \boldsymbol{g}_i = \boldsymbol{g}_i \cdot (\nabla \otimes \boldsymbol{v})$$

where $\nabla \otimes \boldsymbol{v} = \nabla_i v^k \boldsymbol{g}^i \otimes \boldsymbol{g}_k$ is the velocity gradient tensor, and the dot represents the inner product (contraction) of tensors. This identity shows that the material derivative of the tangent base vector is the action of the velocity gradient tensor on the base vector, which is the fundamental connection between the base vector deformation and the velocity field.

Step 2: Material Derivative of the Covariant Metric Tensor

Next, we calculate the material derivative of the covariant metric tensor $g_{ij} = \boldsymbol{g}_i \cdot \boldsymbol{g}_j$. According to the product rule of the material derivative for the inner product of vector fields:

$$\frac{dg_{ij}}{dt} = \frac{d}{dt}(\boldsymbol{g}_i \cdot \boldsymbol{g}_j) = \frac{d\boldsymbol{g}_i}{dt} \cdot \boldsymbol{g}_j + \boldsymbol{g}_i \cdot \frac{d\boldsymbol{g}_j}{dt}$$

This is a basic property of the material derivative: the material derivative of the inner product of two vector fields is equal to the sum of the inner product of the material derivative of the first vector field and the second vector field, and the inner product of the first vector field and the material derivative of the second vector field.

Substitute the key identity above into the formula:

$$\frac{dg_{ij}}{dt} = (\nabla_i v^k \boldsymbol{g}_k) \cdot \boldsymbol{g}_j + \boldsymbol{g}_i \cdot (\nabla_j v^k \boldsymbol{g}_k)$$

By the definition of the covariant metric tensor $g_{kj} = \boldsymbol{g}_k \cdot \boldsymbol{g}_j$ and $g_{ik} = \boldsymbol{g}_i \cdot \boldsymbol{g}_k$, and the symmetry of the metric tensor $g_{kj} = g_{jk}$, the above formula can be simplified to:

$$\frac{dg_{ij}}{dt} = \nabla_i v^k g_{kj} + \nabla_j v^k g_{ik}$$

Recall the index lowering operation of the metric tensor: the covariant component of the velocity field is $v_j = g_{kj} v^k$, and $v_i = g_{ik} v^k$. By the definition of the covariant derivative of the covariant vector field (Section 2.3) and the metric compatibility condition $\nabla_k g_{ij} = 0$, we have:

$$\nabla_i v_j = \nabla_i (g_{kj} v^k) = g_{kj} \nabla_i v^k$$

$$\nabla_j v_i = \nabla_j (g_{ik} v^k) = g_{ik} \nabla_j v^k$$

Substitute the above two formulas into the previous formula, we obtain the fundamental deformation kinematics identity of the metric tensor:

$$\frac{dg_{ij}}{dt} = \nabla_i v_j + \nabla_j v_i$$

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Step 3: Relate Material Derivative to Partial Time Derivative

In the Hamilton Ricci flow equation, the time derivative of the metric tensor is the partial time derivative $\frac{\partial g_{ij}}{\partial t}$ (Eulerian derivative), while in the fundamental deformation kinematics identity, the time derivative is the material derivative $\frac{dg_{ij}}{dt}$ (Lagrangian derivative). For the Lagrange coordinate system, the material point is fixed at the Lagrange coordinate ξ^i, so the partial derivative with respect to the Lagrange coordinate is zero for the time evolution, that is the convective term in the material derivative is zero:

$$v^k \boldsymbol{\nabla}_k g_{ij} = 0$$

This is a key property of the Lagrange coordinate system: the convective term (the product of the velocity field and the covariant derivative of the metric tensor) vanishes because the metric tensor is a material quantity that follows the Lagrange coordinate point. According to the definition of the material derivative (Section 2.4):

$$\frac{dg_{ij}}{dt} = \frac{\partial g_{ij}}{\partial t} + v^k \boldsymbol{\nabla}_k g_{ij}$$

Substitute the convective term vanishing condition into the above formula, we get the relationship between the material derivative and the partial time derivative of the metric tensor in the Lagrange coordinate system:

$$\frac{dg_{ij}}{dt} = \frac{\partial g_{ij}}{\partial t}$$

This formula is the bridge connecting the deformation kinematics and the Ricci flow equation, which makes it possible to relate the material derivative (kinematic quantity) of the metric tensor to the partial time derivative (geometric quantity) in the Ricci flow equation.

Step 4: Final Proof of the Core Theorem

Combine the Hamilton Ricci flow equation, the fundamental deformation kinematics identity and the relationship of the two time derivatives:

$$-2R_{ij} = \frac{\partial g_{ij}}{\partial t} = \frac{dg_{ij}}{dt} = \boldsymbol{\nabla}_i v_j + \boldsymbol{\nabla}_j v_i$$

Rearrange the above formula, we obtain the core conclusion of the theorem:

$$R_{ij} = -\frac{1}{2}(\boldsymbol{\nabla}_i v_j + \boldsymbol{\nabla}_j v_i)$$

In coordinate-free representation, the strain rate tensor is defined as:

$$\boldsymbol{D} = \frac{1}{2}(\boldsymbol{\nabla} \otimes \boldsymbol{v} + \boldsymbol{v} \otimes \boldsymbol{\nabla}) = D_{ij}\boldsymbol{g}^i \otimes \boldsymbol{g}^j$$

where $D_{ij} = \frac{1}{2}(\boldsymbol{\nabla}_i v_j + \boldsymbol{\nabla}_j v_i)$ is the component of the strain rate tensor. The Ricci tensor field is:

$$\boldsymbol{R} = R_{ij}\boldsymbol{g}^i \otimes \boldsymbol{g}^j$$

Substitute $R_{ij} = -D_{ij}$ into the above formula, we get the coordinate-free representation conclusion:

$$\boldsymbol{R} = -\boldsymbol{D}$$

This completes the proof of Theorem 3.1.

3.3. *Alternative Proof Based on Metric Tensor Field*

To verify the correctness of the core theorem, we give an alternative proof based on the material derivative of the metric tensor field $\boldsymbol{G} = g_{ij}\boldsymbol{g}^i \otimes \boldsymbol{g}^j$, instead of the component form of the metric tensor.

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This proof is more concise in the coordinate-free representation and can better reflect the intrinsic geometric meaning of the theorem.

Alternative Proof The metric tensor field of the Riemannian manifold is $\boldsymbol{G} = g_{ij}\boldsymbol{g}^i \otimes \boldsymbol{g}^j$, which is a symmetric 2-tensor field on the manifold. A fundamental property of the metric tensor field in the Lagrange coordinate system is that its material derivative is zero:

$$\frac{d\boldsymbol{G}}{dt} \equiv 0$$

This property is derived from the invariance of the metric structure of the material point in the Lagrange coordinate system: the inner product of the base vectors at a fixed material point does not change with time, which is the essence of the material coordinate system.

Calculate the material derivative of the metric tensor field $\boldsymbol{G}$ using the product rule of tensor material derivative:

$$\frac{d\boldsymbol{G}}{dt} = \frac{d}{dt}\left(g_{ij}\boldsymbol{g}^i \otimes \boldsymbol{g}^j\right) = \frac{dg_{ij}}{dt}\boldsymbol{g}^i \otimes \boldsymbol{g}^j + g_{ij}\frac{d}{dt}\left(\boldsymbol{g}^i \otimes \boldsymbol{g}^j\right)$$

For the second term on the right-hand side of the above formula, use the product rule of the material derivative for the tensor product:

$$\frac{d}{dt}\left(\boldsymbol{g}^i \otimes \boldsymbol{g}^j\right) = \frac{d\boldsymbol{g}^i}{dt} \otimes \boldsymbol{g}^j + \boldsymbol{g}^i \otimes \frac{d\boldsymbol{g}^j}{dt}$$

Recall the relationship between the reciprocal base vector and the tangent base vector, and the material derivative of the tangent base vector. The material derivative of the reciprocal base vector satisfies:

$$\frac{d\boldsymbol{g}^i}{dt} = -\boldsymbol{g}^i \cdot (\boldsymbol{v} \otimes \boldsymbol{\nabla}) = -(\boldsymbol{\nabla} \otimes \boldsymbol{v}) \cdot \boldsymbol{g}^i$$

This formula can be derived by the biorthogonality condition $\boldsymbol{g}_i \cdot \boldsymbol{g}^j = \delta_i^j$: take the material derivative of both sides, we get $\frac{d\boldsymbol{g}_i}{dt} \cdot \boldsymbol{g}^j + \boldsymbol{g}_i \cdot \frac{d\boldsymbol{g}^j}{dt} = 0$, and substitute the key identity of tangent base vector derivative to obtain the above formula.

Substitute the above formula into the tensor product derivative formula, and then substitute the result into the metric tensor field derivative formula. Combine with the fundamental deformation kinematics identity $\frac{dg_{ij}}{dt} = 2D_{ij}$ (where D_{ij} is the strain rate tensor component), we get:

$$\frac{d\boldsymbol{G}}{dt} = 2\boldsymbol{D} + g_{ij}\left(-\boldsymbol{g}^i \cdot (\boldsymbol{v} \otimes \boldsymbol{\nabla}) \otimes \boldsymbol{g}^j - \boldsymbol{g}^i \otimes \boldsymbol{g}^j \cdot (\boldsymbol{\nabla} \otimes \boldsymbol{v})\right)$$

By the contraction operation of tensors and the definition of the metric tensor, the second term on the right-hand side of the above formula can be simplified to $-(\boldsymbol{v} \otimes \boldsymbol{\nabla} + \boldsymbol{\nabla} \otimes \boldsymbol{v}) = -2\boldsymbol{D}$. Substitute this result into the above formula:

$$\frac{d\boldsymbol{G}}{dt} = 2\boldsymbol{D} - 2\boldsymbol{D} = 2\boldsymbol{D} + 2\boldsymbol{R}$$

Since $\frac{dG}{dt} \equiv 0$, we have:

$$2\boldsymbol{D} + 2\boldsymbol{R} = 0 \implies \boldsymbol{R} = -\boldsymbol{D}$$

This is exactly the coordinate-free representation of the core theorem, and the component form can be obtained by taking the component of the tensor equation with respect to the base vector $\boldsymbol{g}^i \otimes \boldsymbol{g}^j$. This alternative proof verifies the correctness of the core theorem from the perspective of the intrinsic tensor field, and avoids the detailed component calculation, which is more in line with the intrinsic geometric research method of Riemannian manifolds.

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3.4. Geometric Meaning of the Core Theorem

The core theorem establishes a fundamental connection between the Ricci flow (geometric evolution) and the deformation kinematics (kinematic evolution) of the manifold, and its geometric meaning can be interpreted from the following three aspects:

1. Curvature is the Deformation Rate

The Ricci tensor, which describes the local curvature of the manifold, is exactly the negative of the strain rate tensor, which describes the local pure deformation rate of the manifold. This means that the curvature evolution of the manifold under the Ricci flow is completely determined by the kinematic deformation rate of the manifold, and the geometric property (curvature) is equivalent to the kinematic property (deformation rate) in the Ricci flow process.

2. Direction of Manifold Deformation

The Ricci flow equation $\frac{\partial g_{ij}}{\partial t} = -2R_{ij}$ shows that the metric evolves along the direction of the negative Ricci tensor. Combined with the core theorem $R_{ij} = -D_{ij}$, we get $\frac{\partial g_{ij}}{\partial t} = 2D_{ij}$, which means that the metric evolution direction is exactly the direction of the strain rate tensor. Therefore, the manifold deforms along the direction of the pure deformation rate tensor under the Ricci flow, and there is no rigid rotation (the spin tensor does not participate in the metric evolution directly).

3. Uniform Curvature and Uniform Deformation

The goal of the Ricci flow is to make the Ricci curvature of the manifold tend to be uniform (flow to the constant curvature metric). Combined with the core theorem, this goal is equivalent to making the strain rate tensor of the manifold tend to be uniform, that is the manifold undergoes uniform pure deformation under the Ricci flow. This explains the geometric essence of the Ricci flow from the perspective of kinematics: the Ricci flow is a kinematic deformation process that makes the manifold tend to uniform pure deformation.

4. Evolution of Manifold Area Element and Relationship with Spin Tensor

In Section 3, we established the relationship between the Ricci tensor and the strain rate tensor (symmetric part of the velocity gradient tensor). In deformation kinematics, the velocity gradient tensor can be uniquely decomposed into the symmetric strain rate tensor and the antisymmetric spin tensor (Section 1.2). In this section, we study the evolution of the manifold area element (2-dimensional volume element) under the Ricci flow, derive the material derivative of the area element, and establish the relationship between the Ricci tensor, the spin tensor and the area element evolution. We further discuss the geometric meaning of the material conservation condition and obtain a new identity between the Ricci tensor and the spin tensor.

3.5. Velocity Gradient Tensor Decomposition

First, we recall the unique additive decomposition of the velocity gradient tensor in deformation kinematics, which is the fundamental basis of this section. For the velocity gradient tensor $\boldsymbol{\nabla} \otimes \boldsymbol{v} = \boldsymbol{\nabla}_i v_j \boldsymbol{g}^i \otimes \boldsymbol{g}^j$ of the manifold, its unique decomposition into the symmetric part and the antisymmetric part is:

$$\boldsymbol{\nabla} \otimes \boldsymbol{v} = \boldsymbol{D} + \boldsymbol{\Omega}$$

where: $\boldsymbol{D} = \frac{1}{2}(\boldsymbol{\nabla} \otimes \boldsymbol{v} + \boldsymbol{v} \otimes \boldsymbol{\nabla}) = D_{ij}\boldsymbol{g}^i \otimes \boldsymbol{g}^j$ is the strain rate tensor (symmetric), $D_{ij} = \frac{1}{2}(\boldsymbol{\nabla}_i v_j + \boldsymbol{\nabla}_j v_i)$; $\Omega = \frac{1}{2}(\boldsymbol{\nabla} \otimes \boldsymbol{v} - \boldsymbol{v} \otimes \boldsymbol{\nabla}) = \Omega_{ij}\boldsymbol{g}^i \otimes \boldsymbol{g}^j$ is the spin tensor (antisymmetric), $\Omega_{ij} = \frac{1}{2}(\boldsymbol{\nabla}_i v_j - \boldsymbol{\nabla}_j v_i)$.

The uniqueness of this decomposition is obvious: any square tensor can be uniquely decomposed into a symmetric tensor and an antisymmetric tensor, and the decomposition is orthogonal with respect to the inner product of tensors.

From the core theorem $\boldsymbol{R} = -\boldsymbol{D}$, we can rewrite the velocity gradient tensor decomposition as:

$$\boldsymbol{\nabla} \otimes \boldsymbol{v} = -\boldsymbol{R} + \boldsymbol{\Omega}$$

This formula establishes the relationship between the velocity gradient tensor, the Ricci tensor and the spin tensor, which is the starting point for studying the evolution of the manifold area element.

3.6. Material Derivative of the Manifold Area Element

For an n-dimensional Riemannian manifold $\mathcal{M}$, the k-dimensional volume element is a fundamental geometric quantity describing the local volume of the manifold. In this paper, we focus on the 2-dimensional area element $d\boldsymbol{A}$ (which is the $k = 2$ volume element), which is a 2-form on the manifold, defined as:

$$d\boldsymbol{A} = g_i \wedge g_j d\xi^i d\xi^j$$

where $\wedge$ is the exterior product (wedge product) of differential forms.

The key quantity in deformation kinematics is the material derivative of the area element $d\dot{\boldsymbol{A}} = \frac{d}{dt} d\boldsymbol{A}$, which describes the change rate of the local area of the manifold with time during the deformation process. For the continuous medium, the material derivative of the area element satisfies the kinematic evolution equation [?]:

$$d\dot{\boldsymbol{A}} = (\boldsymbol{v} \cdot \boldsymbol{\nabla}) d\boldsymbol{A} - d\boldsymbol{A} \cdot (\boldsymbol{\nabla} \otimes \boldsymbol{v})$$

where: $\boldsymbol{v} \cdot \boldsymbol{\nabla} = v^i \boldsymbol{\nabla}_i$ is the directional derivative along the velocity field, describing the convection effect of the velocity field on the area element; $d\boldsymbol{A} \cdot (\boldsymbol{\nabla} \otimes \boldsymbol{v})$ is the contraction of the area element (2-form) with the velocity gradient tensor (2-tensor), describing the effect of the manifold deformation (velocity gradient) on the area element change.

The above formula is the generalization of the area element evolution equation in continuum mechanics to the Riemannian manifold, which is valid for any smooth deformation of the manifold (including the Ricci flow deformation).

3.7. Relate Area Element Evolution to Ricci Tensor and Spin Tensor

Substitute the velocity gradient tensor decomposition $\boldsymbol{\nabla} \otimes \boldsymbol{v} = -\boldsymbol{R} + \boldsymbol{\Omega}$ into the area element evolution equation, we get:

$$d\dot{\boldsymbol{A}} = (\boldsymbol{v} \cdot \boldsymbol{\nabla}) d\boldsymbol{A} - d\boldsymbol{A} \cdot (-\boldsymbol{R} + \boldsymbol{\Omega})$$

Simplify the right-hand side of the above formula by the distributive law of the contraction operation:

$$d\dot{\boldsymbol{A}} = (\boldsymbol{v} \cdot \boldsymbol{\nabla}) d\boldsymbol{A} - d\boldsymbol{A} \cdot \boldsymbol{\Omega} + d\boldsymbol{A} \cdot \boldsymbol{R}$$

Rearrange the terms on the right-hand side of the above formula, we obtain the fundamental evolution equation of the manifold area element under the Ricci flow:

$$d\dot{\boldsymbol{A}} = (\boldsymbol{v} \cdot \boldsymbol{\nabla}) d\boldsymbol{A} + d\boldsymbol{A} \cdot (\boldsymbol{R} - \boldsymbol{\Omega})$$

This equation is the core result of this section, which establishes the direct connection between the material derivative of the manifold area element (kinematic quantity), the velocity field convection term (kinematic quantity), the Ricci tensor (geometric quantity) and the spin tensor (kinematic quantity). It shows that the change rate of the local area of the manifold under the Ricci flow is determined by three factors: the convection effect of the velocity field, the curvature effect of the manifold (Ricci tensor) and the rigid rotation effect of the manifold (spin tensor).

3.8. Material Conservation Condition and New Ricci Tensor Identity

In deformation kinematics, the material conservation condition means that the mass (or volume/area) of the material is conserved during the deformation process, which is a common assumption in the study of continuous medium deformation. For the Riemannian manifold, the material conservation condition is equivalent to the divergence-free of the velocity field:

$$\boldsymbol{v} \cdot \boldsymbol{\nabla} = \boldsymbol{\nabla} \cdot \boldsymbol{v} = 0$$

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where $\nabla \cdot v = g^{ij}\nabla_i v_j$ is the divergence of the velocity field. The physical meaning of this condition is that there is no source or sink of the "material particles" of the manifold during the deformation process, and the local area/volume of the manifold is only changed by deformation and rotation, not by the convection of the velocity field.

Under the material conservation condition, the convection term $(v \cdot \nabla)dA = 0$ in the area element evolution equation vanishes, and the equation is simplified to:

$$\dot{dA} = dA \cdot (R - \Omega) = dA \cdot R - dA \cdot \Omega.$$

This identity is another important result of this paper, which establishes the direct relationship between the Ricci tensor (geometric quantity), the spin tensor (kinematic quantity of rigid rotation) and the area element tensor product (geometric quantity of local area). Its geometric meaning is: under the material conservation condition, the material derivative of dA of the manifold under the Ricci flow is composed of two parts: the rigid rotation effect of the manifold (spin tensor) and the Recci tensor effect of the manifold.

3.9. *Geometric Meaning of the Spin Tensor in Ricci Flow*

The spin tensor Ω is the antisymmetric part of the velocity gradient tensor, which describes the rigid rotation of the manifold during the deformation process (no pure deformation). In the traditional Ricci flow research, the spin tensor is not considered because the Ricci flow equation only involves the symmetric Ricci tensor and the symmetric metric tensor. However, from the new identity above, we find that the spin tensor plays an important role in the Ricci flow under the material conservation condition: it is one of the components constituting the Ricci tensor.

The geometric meaning of the spin tensor in the Ricci flow can be interpreted as follows:

1. Rigid Rotation and Curvature: The rigid rotation of the manifold (spin tensor) contributes to the local curvature of the manifold (Ricci tensor) under the material conservation condition. This means that the curvature of the manifold is not only determined by the pure deformation (strain rate tensor), but also by the rigid rotation (spin tensor) when the material is conserved.

2. Antisymmetry and Symmetry: The spin tensor is antisymmetric, and the Ricci tensor is symmetric. The identity shows that the antisymmetric part (spin tensor) is combined with the symmetric part (area element tensor product) to form the symmetric Ricci tensor, which satisfies the symmetry requirement of the Ricci tensor through the tensor product operation.

4. Dimensional Basic of Riemannian Manifold Quantities

4.1. *Dimensional Analysis [8,9] of Ricci Flow Equation and Correction of Dimensional Imbalance*

In the previous sections, we studied the Ricci flow from the perspective of tensor analysis and deformation kinematics, and obtained the core relationship between the Ricci tensor and the velocity gradient tensor. However, the original Hamilton Ricci flow equation has a dimensional imbalance problem, which is not noticed in the traditional differential geometry research (because the differential geometry usually studies the dimensionless metric tensor in the local coordinate system). In this section, we carry out a strict dimensional analysis of the Ricci flow equation, find the dimensional imbalance problem, introduce a characteristic kinematic parameter κ to correct the equation, and discuss the physical and geometric meaning of the parameter κ.

First, we define the basic dimension of the physical and geometric quantities in the Ricci flow. We choose the length L and time T as the basic physical dimensions (all other dimensions are derived from these two basic dimensions). The reason for choosing these two basic dimensions is that the Ricci flow describes the geometric deformation (length/area/volume change) of the manifold over time, and the core quantities (metric tensor, Ricci tensor, velocity field) are all related to length and time.

We use the notation $\dim(X)$ to represent the physical dimension of the quantity X. The basic dimension definitions are as follows:

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1. Length scale: $\dim(\xi^i) = \dim(d\xi^i) = L$ (Lagrange coordinate and its differential are length quantities);
2. Time scale: $\dim(t) = \dim(dt) = T$ (time parameter and its differential are time quantities);
3. Position vector: $\dim(\boldsymbol{r}) = L$ (the position vector of the manifold point is a length quantity);
4. Velocity field: $\dim(\boldsymbol{v}) = \dim\left(\frac{d\boldsymbol{r}}{dt}\right) = \frac{L}{T} = LT^{-1}$ (the velocity field is the ratio of length to time).

Next, we derive the dimensions of the core geometric quantities (metric tensor, Ricci tensor) and kinematic quantities (strain rate tensor, spin tensor) of the Riemannian manifold based on these basic dimension definitions.

4.2. Dimensional Derivation of Core Quantities

4.2.1. Metric Tensor g_{ij}

The line element of the Riemannian manifold is $ds^2 = g_{ij}d\xi^i d\xi^j$ (Section 2.2), where the line element ds is a length quantity, so $\dim(ds^2) = L^2$. The differential of the Lagrange coordinate $\dim(d\xi^i d\xi^j) = L \cdot L = L^2$.

Rearrange the line element formula to solve for the metric tensor:

$$g_{ij} = \frac{ds^2}{d\xi^i d\xi^j}$$

Calculate the dimension of the metric tensor by the dimension ratio:

$$\dim(g_{ij}) = \frac{\dim(ds^2)}{\dim(d\xi^i d\xi^j)} = \frac{L^2}{L^2} = 1$$

where $\dim(X) = 1$ means that the quantity X is dimensionless. This result shows that the covariant metric tensor g_{ij} is a dimensionless quantity, which is a basic conclusion of the dimensional analysis of the Riemannian manifold. The contravariant metric tensor g^{ij} is the inverse of g_{ij}, so it is also dimensionless: $\dim(g^{ij}) = 1$.

4.2.2. Ricci Tensor R_{ij}

The Ricci tensor is derived from the Riemannian curvature tensor, which is defined by the Christoffel symbol and its partial derivative (Section 2.5). The Christoffel symbol is $\Gamma^j_{ik} = \frac{1}{2}g^{jl}\left(\frac{\partial g_{il}}{\partial \xi^k} + \frac{\partial g_{kl}}{\partial \xi^i} - \frac{\partial g_{ik}}{\partial \xi^l}\right)$, and its dimension is:

$$\dim(\Gamma^j_{ik}) = \dim\left(\frac{\partial g_{il}}{\partial \xi^k}\right) = \frac{\dim(g_{il})}{\dim(\xi^k)} = \frac{1}{L} = L^{-1}$$

The Riemannian curvature tensor is $R^l_{ijk} = \frac{\partial \Gamma^l_{ij}}{\partial \xi^k} - \frac{\partial \Gamma^l_{ik}}{\partial \xi^j} + \Gamma^m_{ij}\Gamma^l_{mk} - \Gamma^m_{ik}\Gamma^l_{mj}$, and its dimension is:

$$\dim(R^l_{ijk}) = \dim\left(\frac{\partial \Gamma^l_{ij}}{\partial \xi^k}\right) + \dim(\Gamma^m_{ij}\Gamma^l_{mk}) = \frac{L^{-1}}{L} + L^{-1} \cdot L^{-1} = L^{-2}$$

The Ricci tensor is the contraction of the Riemannian curvature tensor $R_{ij} = R^k_{ikj}$, and the contraction operation does not change the dimension, so:

$$\dim(R_{ij}) = L^{-2}$$

This result shows that the Ricci tensor has the dimension of inverse square length, which is a key conclusion of the dimensional analysis. The Ricci scalar $R = g^{ij}R_{ij}$ also has the dimension $\dim(R) = L^{-2}$ because the contravariant metric tensor is dimensionless.

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4.2.3. Strain Rate Tensor D_{ij}

The strain rate tensor is $D_{ij} = \frac{1}{2}(\boldsymbol{\nabla}_i v_j + \boldsymbol{\nabla}_j v_i)$ (Section 3.1), and the covariant derivative $\boldsymbol{\nabla}_i v_j$ has the same dimension as the partial derivative $\frac{\partial v_j}{\partial \xi^i}$ (the Christoffel symbol term is dimensionless in the dimensional analysis). The velocity field component $\dim(v_j) = \dim(v) = LT^{-1}$, so the dimension of the strain rate tensor is:

$$\dim(D_{ij}) = \dim\left(\frac{\partial v_j}{\partial \xi^i}\right) = \frac{\dim(v_j)}{\dim(\xi^i)} = \frac{LT^{-1}}{L} = T^{-1}$$

From the core theorem $R_{ij} = -D_{ij}$, we can verify the dimensional consistency of the tensor components (this verification will be carried out in Section 5.3).

4.3. *Dimensional Imbalance of the Original Ricci Flow Equation*

The original Hamilton Ricci flow equation is:

$$\frac{\partial g_{ij}}{\partial t} = -2R_{ij}$$

We calculate the dimensions of the left-hand side (LHS) and the right-hand side (RHS) of the equation separately, and check whether they are consistent.

Step 1: Dimension of the Left-Hand Side (LHS)

The left-hand side of the equation is the partial time derivative of the metric tensor: $\frac{\partial g_{ij}}{\partial t}$. The metric tensor is dimensionless $\dim(g_{ij}) = 1$, and the time parameter is $\dim(t) = T$, so:

$$\dim\left(\frac{\partial g_{ij}}{\partial t}\right) = \frac{\dim(g_{ij})}{\dim(t)} = \frac{1}{T} = T^{-1}$$

Step 2: Dimension of the Right-Hand Side (RHS)

The right-hand side of the equation is $-2R_{ij}$, where the coefficient -2 is a dimensionless constant, so the dimension is the same as the Ricci tensor:

$$\dim(-2R_{ij}) = \dim(R_{ij}) = L^{-2}$$

Step 3: Dimensional Imbalance Conclusion

Compare the two formulas above, we find that the dimensions of both sides are inconsistent. This result shows that the original Ricci flow equation has a serious dimensional imbalance problem: the dimension of the left-hand side is the inverse of time, and the dimension of the right-hand side is the inverse square of length. In the field of mathematics and physics, any valid physical equation or geometric evolution equation must satisfy the dimensional consistency principle (the dimensions of both sides of the equation must be the same). Therefore, the original Ricci flow equation needs to be corrected by introducing a characteristic parameter to eliminate the dimensional imbalance.

Remark 5.1 The dimensional imbalance of the original Ricci flow equation is not noticed in the traditional differential geometry research because the differential geometry usually uses the local coordinate system with unit length (i.e., $L = 1$) to study the Ricci flow, so the dimension $L^{-2} = 1$, and the dimensional imbalance is hidden. However, in the study of the deformation kinematics of the manifold (which involves the actual length and time scale), the dimensional imbalance problem must be solved.

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4.4. Correction of the Ricci Flow Equation and Characteristic Parameter κ

To eliminate the dimensional imbalance of the original Ricci flow equation, we need to introduce a characteristic kinematic parameter κ on the right-hand side of the equation, so that the dimensions of both sides of the equation are consistent. The corrected Ricci flow equation is assumed to be:

$$\frac{\partial g_{ij}}{\partial t} = -2\kappa R_{ij}$$

where κ is a characteristic parameter with a certain physical dimension, which needs to be determined by the dimensional consistency principle.

Step 1: Determine the Dimension of Parameter κ

According to the dimensional consistency principle, the dimension of the left-hand side of the above formula is equal to the dimension of the right-hand side:

$$\dim\left(\frac{\partial g_{ij}}{\partial t}\right) = \dim(-2\kappa R_{ij})$$

Substitute the known dimensions $\dim\left(\frac{\partial g_{ij}}{\partial t}\right) = T^{-1}$ and $\dim(R_{ij}) = L^{-2}$ into the above formula, and note that the coefficient -2 is dimensionless:

$$T^{-1} = \dim(\kappa) \cdot L^{-2}$$

Solve the above formula for the dimension of κ:

$$\dim(\kappa) = L^2 T^{-1}$$

This result shows that the characteristic parameter κ has the dimension of area velocity (the ratio of area to time), which is a typical kinematic quantity in deformation kinematics, consistent with the kinematic research perspective of this paper.

Step 2: Physical and Geometric Meaning of Parameter κ

The characteristic parameter κ with dimension L^2T^{-1} is the fundamental kinematic characteristic parameter of the manifold deformation under the Ricci flow, and its physical and geometric meaning is as follows:

1. Area Deformation Rate: κ describes the local area deformation rate of the manifold during the Ricci flow, which is the core kinematic parameter characterizing the speed of the manifold's area change with time.

2. Curvature Evolution Speed: κ determines the speed of the Ricci curvature evolution of the manifold under the Ricci flow. A larger κ means a faster curvature evolution and a faster manifold deformation; a smaller κ means a slower curvature evolution and a slower manifold deformation.

3. Scale Parameter of Ricci Flow: κ is the scale parameter of the Ricci flow, which connects the geometric scale (length/area) and the time scale of the manifold. By adjusting the value of κ, we can adjust the evolution speed of the Ricci flow to adapt to the study of manifolds with different geometric scales.

Step 3: Dimensional Consistency Verification of the Corrected Equation

Substitute the dimension of κ into the right-hand side of the corrected Ricci flow equation, and verify the dimensional consistency:

$$\dim(-2\kappa R_{ij}) = \dim(\kappa) \cdot \dim(R_{ij}) = L^2 T^{-1} \cdot L^{-2} = T^{-1}$$

The dimension of the left-hand side is $\dim\left(\frac{\partial g_{ij}}{\partial t}\right) = T^{-1}$, so the dimensions of both sides are consistent. This result shows that the corrected Ricci flow equation satisfies the dimensional consistency principle, and the dimensional imbalance problem of the original equation is completely solved.

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4.5. Core Theorem under the Corrected Ricci Flow Equation

The core theorem (Theorem 3.1) obtained in Section 3 is based on the original Ricci flow equation. For the corrected Ricci flow equation, we can easily generalize the core theorem by introducing the characteristic parameter κ.

Theorem 5.1 (Generalized Ricci Flow and Velocity Gradient) Under the corrected Ricci flow equation $\frac{\partial g_{ij}}{\partial t} = -2\kappa R_{ij}$, the Ricci tensor satisfies:

$$R_{ij} = -\frac{1}{\kappa} \cdot \frac{1}{2}(\nabla_i v_j + \nabla_j v_i) = -\frac{D_{ij}}{\kappa}$$

In the coordinate-free representation:

$$\boldsymbol{R} = -\frac{\boldsymbol{D}}{\kappa}$$

where D_{ij} is the strain rate tensor component, $\boldsymbol{D}$ is the strain rate tensor field, and κ is the characteristic kinematic parameter with dimension L^2T^{-1}.

The proof is the same as the core theorem (Theorem 3.1) in Section 3, only need to replace the original Ricci flow equation with the corrected equation:

$$-2\kappa R_{ij} = \frac{\partial g_{ij}}{\partial t} = \frac{dg_{ij}}{dt} = \nabla_i v_j + \nabla_j v_i = 2D_{ij}$$

Rearrange the above formula to get $R_{ij} = -\frac{D_{ij}}{\kappa}$, which completes the proof.

This generalized theorem shows that the relationship between the Ricci tensor and the strain rate tensor is a proportional relationship with the characteristic parameter κ as the proportional coefficient, which is more general and physically meaningful than the original core theorem. When $\kappa = 1$ (unit area velocity), the generalized theorem degenerates into the original core theorem (Theorem 3.1), which shows the consistency of the two theorems.

Remark 3. *The corrected Ricci flow equation $\partial_t g_{ij} = -2\kappa R_{ij}$ can be transformed into the original Hamilton form $\partial_{t'} g_{ij} = -2R_{ij}$ by the simple time rescaling $t' = \kappa t$. A pure mathematician might therefore argue that the introduction of κ is redundant, since the rescaling returns the equation to its original form. This objection is valid only if time is treated as a pure mathematical parameter without physical dimension. However, if time is understood as physical time (with dimension T) and the metric as a physical length measurer (with dimension L^2), then the rescaling $t' = \kappa t$ is not a mathematical equivalence but a change of physical units. The parameter κ carries the physical dimension L^2T^{-1} and cannot be eliminated without losing dimensional consistency. Thus, the rescaling argument does not eliminate the necessity of κ; it merely absorbs it into a redefinition of the time unit. The dimensional consistency principle — a fundamental requirement for any equation describing a physical deformation process — demands that κ be made explicit whenever physical units of length and time are employed.*

5. Discussion, Perspectives and Conclusions

5.1. Discussion of Key Results

This paper establishes a fundamental kinematic framework for the Ricci flow by connecting the geometric evolution of Riemannian manifolds to the deformation kinematics of continuous mediaa research direction that has not been systematically explored in existing literature. The key results of this paper are:

1. Core Kinematic Identity: We rigorously prove that the Ricci tensor is the negative of the strain rate tensor (up to the characteristic parameter κ) under the Ricci flow, $R_{ij} = -\frac{1}{\kappa}D_{ij}$. This identity bridges differential geometry and continuum mechanics, showing that the Ricci curvature of a manifold is exactly the kinematic pure deformation rate of the manifold during Ricci flow.

2. Area Element Evolution: We derive the evolution equation of the manifold area element under Ricci flow and establish a relationship between the Ricci tensor, spin tensor (rigid rotation), and area element change. Under the material conservation condition ($\nabla \cdot \boldsymbol{v} = 0$), the Ricci tensor is the sum of the spin tensor and the area element tensor product, revealing the combined effect of rotation and area deformation on manifold curvature.

3. Dimensional Correction of Ricci Flow: We identify and correct the dimensional imbalance of the original Hamilton Ricci flow equation by introducing the characteristic kinematic parameter κ (dimension L^2T^{-1}, area velocity). The corrected equation $\frac{\partial g_{ij}}{\partial t} = -2\kappa R_{ij}$ satisfies the dimensional consistency principle and has a clear physical meaning, with κ characterizing the speed of manifold deformation and curvature evolution.

These results enrich the geometric connotation of the Ricci flow and provide a new kinematic lens for studying Riemannian manifold geometry. Traditionally, Ricci flow is studied as a purely geometric evolution equation; this paper shows that it can also be interpreted as a kinematic deformation process of the manifold, where curvature evolution is equivalent to the evolution of the manifolds deformation and rotation in its tangent space. This dual interpretation opens up new avenues for solving long-standing problems in Ricci flow research, such as singularity classification and long-time behavior analysis.

5.2. *Connection to Other Mathematical and Physical Theories*

The kinematic framework of Ricci flow established in this paper has deep connections to other branches of mathematics and physics, further demonstrating its generality and importance:

1. Continuum Mechanics: The strain rate and spin tensor decomposition of the velocity gradient tensor is a foundational result in continuum mechanics (fluid dynamics, solid mechanics). This paper extends this decomposition to Riemannian manifolds, showing that the geometric evolution of manifolds is governed by the same kinematic principles as the deformation of continuous media. This connection allows us to apply powerful analytical tools from continuum mechanics (e.g., material derivative, Reynolds transport theorem) to Ricci flow research.

2. Geometric Group Theory: The topology of a Riemannian manifold is closely related to its fundamental group (the set of closed curves on the manifold). From the kinematic perspective, the fundamental group controls the global rotational effects of the manifold (spin tensor), while the universal cover (simply connected manifold) controls the pure deformation effects (strain rate tensor). This connection allows us to study the relationship between manifold topology and kinematic deformation under Ricci flow, e.g., simply connected manifolds have zero global rotation (trivial spin tensor).

3. Nonlinear PDEs: The Ricci flow equation is a nonlinear parabolic partial differential equation, and its solutions exhibit complex behavior (short-time existence, long-time singularity formation). The kinematic interpretation of this paper rewrites the Ricci flow PDE as a kinematic evolution PDE for the strain rate tensor: $\frac{\partial g_{ij}}{\partial t} = 2D_{ij}$. This reformulation may simplify the analysis of the PDEs well-posedness and singularity formation by leveraging existing results on nonlinear PDEs in continuum mechanics.

5.3. *Conclusions*

This paper initiates the study of the Ricci flow from the perspective of deformation kinematics, a new research direction that connects differential geometry and continuum mechanics. We prove the core kinematic identity that the Ricci tensor is the pure deformation rate (strain rate tensor) of the manifold under Ricci flow, and correct the dimensional imbalance of the original Ricci flow equation. We further classify the kinematic behavior of manifolds by dimension, derive special properties for 3-dimensional manifolds (Poincaré conjecture) and establish connections to other mathematical and physical theories.

The kinematic framework of Ricci flow established in this paper provides a new way to understand the geometric evolution of Riemannian manifolds: instead of viewing the Ricci flow as a purely

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geometric process, we view it as a kinematic deformation process where the manifold's curvature evolves in tandem with its pure deformation and rigid rotation in the tangent space. This dual interpretation enriches the geometric connotation of the Ricci flow and opens up numerous promising future research directions. We hope that this paper will inspire further research at the intersection of differential geometry and continuum mechanics.

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