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A Theta-Kernel Reformulation of a Growth Theorem and the Riemann Hypothesis

Michel Planat

Institut FEMTO-ST CNRS UMR 6174, Université Marie et Louis Pasteur, 15 B Avenue des Montboucons,
F-25044 Besançon, France; michel.planat@femto-st.fr

Abstract

We reformulate the Growth Theorem criterion for the Riemann hypothesis as a global positivity problem for a modular oscillatory kernel arising from the Riemann xi function. Using the fact that the squared modulus $|\xi(\sigma+it)|^2$ is strictly increasing with respect to σ for $\sigma > \frac{1}{2}$, we reformulate this as a positivity condition on a theta-kernel double integral. The half-plane $\sigma > 1$ is closed unconditionally using the symmetric Hadamard product. The remaining obstruction is thereby localized to the critical strip $\frac{1}{2} < \sigma \leq 1$. Introducing diagonal coordinates $(a,b) = (\frac{u+v}{2}, \frac{u-v}{2})$, we decompose the kernel into a positive diagonal sector and an oscillatory off-diagonal sector, and show that the Riemann hypothesis is equivalent to the positivity $I(x,y) = \iint_{a>|b|} \Phi(a+b)\Phi(a-b)\, K_{x,y}(a,b)\, da\, db > 0 \qquad (x \in \mathbb{R},\ y > 0)$. We give an exact characterization of the positive-amplitude structure of this integral and state the resulting theta-kernel positivity problem in its sharpest form.

Keywords: Riemann hypothesis; Riemann xi function; theta kernel; oscillatory integrals; positivity conditions; log-concavity; Hadamard product; functional equation

MSC: 11M26; 11M06; 30D15; 42A38; 26A51

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1. Introduction

1.1. Background and Motivation

The Riemann hypothesis (RH) asserts that all non-trivial zeros of the Riemann zeta function $\zeta(s)$ lie on the critical line $\Re s = \frac{1}{2}$. Among the many equivalent reformulations of RH, monotonicity and positivity criteria for the Riemann xi function have attracted sustained attention, see for instance [1–4]. The historical development of positivity and growth criteria for the Riemann zeta function is surveyed in the classical monograph of Edwards [5]. The present work belongs to the broader tradition initiated by Pólya, where positivity properties of kernels and Fourier transforms are used to study the location of zeros of entire functions [6].

Recent work has emphasized that several positivity-based formulations of the Riemann hypothesis admit a localization of the obstruction to a restricted region of parameter space rather than a genuinely global failure mechanism. In particular, the finite-strip obstruction identified through the onset of hyperbolicity of Jensen polynomials in [7] suggests that the critical difficulty may be concentrated in a narrow transition region. The present work exhibits an analogous localization phenomenon within the theta-kernel framework, where the unresolved obstruction is confined to the strip $1/2 < \sigma \leq 1$.

Several modern approaches study positivity and hyperbolicity phenomena associated with transforms of the Riemann xi function and related entire functions [8].

The *Riemann xi function* is the entire function

$$\xi(s) = \tfrac{1}{2}s(s-1)\pi^{-s/2}\Gamma(\tfrac{s}{2})\zeta(s),$$

satisfying the functional equation $\xi(s) = \xi(1-s)$. Its logarithmic derivative ξ'/ξ carries the full information about the location of zeros.

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A classical approach to RH via *monotone growth* asks whether the modulus $|\xi(\sigma+it)|$ is strictly increasing in σ for $\sigma > \frac{1}{2}$. This is equivalent to requiring

$$\Re\frac{\xi'}{\xi}(\sigma+it) > 0 \qquad (\sigma > \tfrac{1}{2}).$$

This paper develops a reformulation of this condition via the theta-kernel integral representation of $|\xi|^2$, reduces RH to a precise positivity statement, and isolates the analytic obstruction.

1.2. Main Contributions

The contributions of this paper are:

(i) A self-contained derivation of the theta-kernel representation of $\partial_\sigma|\xi(\sigma+it)|^2$ (Section 2).

(ii) An unconditional proof that $\Re(\xi'/\xi)(\sigma+it) > 0$ for $\sigma > 1$ (Section 4).

(iii) The introduction of diagonal coordinates and the decomposition of the kernel into positive and oscillatory contributions (Section 5).

(iv) A precise formulation of the remaining theta-kernel positivity problem (Section 6).

The paper does not claim a proof of RH. Its purpose is to establish the framework within which the positivity problem is naturally posed.

1.3. Structure of the Paper

Section 2 presents the Growth Theorem reformulation. Section 3 develops the theta-kernel representation. Section 4 closes the half-plane $\sigma > 1$ unconditionally and localizes the obstruction. Section 5 introduces diagonal coordinates and separates the oscillatory structure. Section 6 states the final positivity problem. Section 7 discusses subsequent work.

2. Growth Theorem Reformulation

2.1. The Growth Identity and Its Consequences

For $s = \sigma + it$ with $\sigma > \frac{1}{2}$, a direct computation gives the *growth identity*:

$$\partial_\sigma|\xi(\sigma+it)|^2 = 2\,|\xi(\sigma+it)|^2\,\Re\frac{\xi'}{\xi}(\sigma+it).$$

Since $|\xi(s)|^2 > 0$ for s away from zeros, the sign of $\partial_\sigma|\xi|^2$ coincides with the sign of $\Re(\xi'/\xi)$.

Related monotonicity criteria for the Riemann xi function have been investigated in [9]. The positivity of the real part of the logarithmic derivative of the Riemann xi function has recently been investigated in detail by the authors of [10], who obtained explicit lower bounds near the critical line and analyzed hypothetical scenarios involving zeros off the critical line. The present work pursues a different direction by converting the same positivity criterion into a theta-kernel positivity problem.

Theorem 1 (Growth Theorem – RH equivalence)**.** *The following are equivalent:*

(i) *The Riemann hypothesis: all non-trivial zeros of $\zeta(s)$ satisfy $\Re\rho = \frac{1}{2}$.*

(ii) *Monotone growth: $\partial_\sigma|\xi(\sigma+it)|^2 > 0$ for all $t \in \mathbb{R}$ and all $\sigma > \frac{1}{2}$.*

(iii) *Positive logarithmic derivative: $\Re(\xi'/\xi)(\sigma+it) > 0$ for all $t \in \mathbb{R}$ and all $\sigma > \frac{1}{2}$.*

Proof. The equivalence of (i) and (iii) follows from the symmetric Hadamard product representation. Writing

$$\xi(s) = \xi(0)\prod_{\rho}^{\text{sym}}\left(1-\frac{s}{\rho}\right),$$

one obtains

$$\Re\frac{\xi'}{\xi}(s) = \sum_{\rho}\frac{\sigma-\beta}{(\sigma-\beta)^2+(t-\gamma)^2},$$

where $\rho = \beta + i\gamma$ ranges over the non-trivial zeros (in the symmetrized pairing with $1 - \rho$). Each summand is positive if and only if $\sigma > \beta$. Hence $\Re(\xi'/\xi)(\sigma + it) > 0$ for all $\sigma > \frac{1}{2}$ and all t if and only if every zero satisfies $\beta = \frac{1}{2}$. The equivalence with (ii) follows from the growth identity and positivity of $|\xi|^2$. □

Remark 1. *The reformulation in part (iii) identifies the positivity of $\Re(\xi'/\xi)$ as the central analytic object. The theta-kernel representation developed in Section 3 provides an explicit formula for this quantity in terms of a modular double integral.*

3. Theta-Kernel Representation

3.1. The Riemann Theta Kernel

The Riemann xi function admits the classical Fourier representation (see [1], §2.6):

$$\Xi(z) = \int_{-\infty}^{\infty} \Phi(u) e^{izu}\, du,$$

where $\Xi(z) = \xi(\frac{1}{2} + iz)$ and the *Riemann theta kernel* is

$$\Phi(u) = \sum_{n=1}^{\infty} \left(2\pi^2 n^4 e^{\frac{9}{2}u} - 3\pi n^2 e^{\frac{5}{2}u}\right) e^{-\pi n^2 e^{2u}}. \tag{1}$$

The function Φ is smooth, strictly positive on $\mathbb{R}$, even ($\Phi(-u) = \Phi(u)$), and rapidly decaying as $|u| \to \infty$. Evenness follows from the Jacobi theta transformation; see [1,6].

Since Φ is even,

$$\Xi(z) = 2\int_0^{\infty} \Phi(u) \cos(zu)\, du.$$

We work throughout with the normalization

$$D(z) = \int_0^{\infty} \Phi(u) \cos(zu)\, du = \tfrac{1}{2}\Xi(z).$$

Positivity of $\partial_y |D(x+iy)|^2$ is equivalent to positivity of $\partial_y |\Xi(x+iy)|^2$, and has the same sign.

3.2. The Two-Variable Kernel

Set $z = x + iy$ with $x, y \in \mathbb{R}$, $y > 0$. Define

$$D(x,y) = |D(x+iy)|^2 = \left| \int_0^{\infty} \Phi(u) \cos((x+iy)u)\, du \right|^2.$$

An explicit computation gives the double integral representation:

$$D(x,y) = \int_0^{\infty}\int_0^{\infty} \Phi(u)\Phi(v)\, \Re\left[\cos((x+iy)u)\overline{\cos((x+iy)v)}\right] du\, dv. \tag{2}$$

The positivity problem concerns the y-derivative:

$$I(x,y) = \tfrac{1}{2}\partial_y D(x,y).$$

Proposition 1 (Kernel formula)**.** *Under the change of variables $u = a + b$, $v = a - b$ (i.e., $a = \frac{u+v}{2}$, $b = \frac{u-v}{2}$), one has $du\, dv = 2\, da\, db$ and*

$$I(x,y) = \iint_{a>|b|} \Phi(a+b)\Phi(a-b)\, K_{x,y}(a,b)\, da\, db, \tag{3}$$

where the symmetrized kernel is

$$K_{x,y}(a,b) = \frac{a}{2}\cos(2xb)\sinh(2ya) + \frac{b}{2}\cos(2xa)\sinh(2yb). \tag{4}$$

Proof. From (2), differentiate under the integral sign in y and substitute $\cos((x+iy)u) = \cos(xu)\cosh(yu) - i\sin(xu)\sinh(yu)$. After expanding and using the change of variables, the resulting integrand simplifies to (4) by standard trigonometric identities. The domain $a > |b| \geq 0$ corresponds to $u, v > 0$. □

Definition 1 (Two-variable theta kernel)**.** *Set*

$$M(a,b) = \Phi(a+b)\Phi(a-b).$$

The growth integral is

$$I(x,y) = \iint_{a>|b|} M(a,b)\, K_{x,y}(a,b)\, da\, db.$$

The kernel M is symmetric and positive: $M(a,b) = M(a,-b) > 0$ for $a > |b|$, since $\Phi > 0$.

3.3. RH as a Positivity Statement

Theorem 2 (Theta-kernel equivalence)**.** *The Riemann hypothesis is equivalent to*

$$I(x,y) > 0 \qquad \text{for all } x \in \mathbb{R},\ y > 0.$$

Proof. By Theorem 1, RH is equivalent to $\partial_\sigma|\xi(\sigma+it)|^2 > 0$ for all $\sigma > \frac{1}{2}$. Translating to the Ξ-normalization, with $\sigma = \frac{1}{2} + y$ and $t = x$, this becomes $\partial_y|D(x+iy)|^2 > 0$, i.e., $I(x,y) > 0$ for all $x \in \mathbb{R}$ and $y > 0$. The formula (3)–(4) provides the explicit theta-kernel representation of this condition. □

Remark 2 (Pointwise vs. global positivity)**.** *For $x = 0$, the kernel reduces to $K_{0,y}(a,b) = \frac{a}{2}\sinh(2ya)$, which is positive, so $I(0,y) > 0$ follows pointwise from $\Phi > 0$. For $x \neq 0$, the cosine factors in (4) oscillate and $K_{x,y}$ is sign-indefinite. Positivity of $I(x,y)$ is therefore a genuinely global statement for $x \neq 0$.*

4. Localization of the Obstruction

4.1. Unconditional Positivity for $\sigma > 1$

The zero-sum formula established in Theorem 1 gives:

$$\Re\frac{\xi'}{\xi}(\sigma+it) = \sum_{\rho} \frac{\sigma-\beta}{(\sigma-\beta)^2 + (t-\gamma)^2},$$

summed over non-trivial zeros $\rho = \beta + i\gamma$ in the symmetric pairing. All known zeros satisfy $0 < \beta < 1$, so for $\sigma > 1$ every term in the sum satisfies $\sigma - \beta > 0$.

Theorem 3 (Unconditional positivity for $\sigma > 1$)**.** *For all $\sigma > 1$ and all $t \in \mathbb{R}$,*

$$\Re\frac{\xi'}{\xi}(\sigma+it) > 0.$$

Proof. Every non-trivial zero satisfies $\Re\rho \leq 1$ (since ζ has no zeros with $\Re s > 1$, by the Euler product). Hence $\beta \leq 1 < \sigma$ for all ρ, and every summand $(\sigma-\beta)/[(\sigma-\beta)^2 + (t-\gamma)^2]$ is strictly positive. As the sum converges absolutely for $\sigma > 1$ and has at least one positive term, the result follows. □

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4.2. The Critical Strip as the Remaining Obstruction

Corollary 1 (Localization). *The entire unresolved obstruction to the growth inequality is confined to the strip*

$$\tfrac{1}{2} < \sigma \leq 1,$$

or equivalently, to $y \in (0, \frac{1}{2}]$ *in the* (x, y) *parameterization.*

Proof. Theorem 3 closes the half-plane $\sigma > 1$ unconditionally. Since all non-trivial zeros satisfy $\Re\rho \geq \frac{1}{2}$ (by the functional equation and the fact that there are no zeros with $\Re s \leq 0$), a negative contribution to the sum can only arise from a zero with $\beta > \sigma$, which requires $\sigma < \beta \leq 1$. Hence any failure of positivity must occur in the strip $\frac{1}{2} < \sigma \leq 1$. □

Remark 3 (Tractability outside the strip). *For* $\sigma > 1$ *the positivity mechanism is algebraic (each summand is positive). The difficulty arises entirely in the strip* $\frac{1}{2} < \sigma \leq 1$*, where zeros could in principle have* $\beta > \sigma$ *and create negative summands. The theta-kernel formulation makes this global obstruction explicit.*

5. Diagonal Variables and Oscillatory Structure

5.1. Diagonal Coordinates

The diagonal coordinates

$$a = \frac{u+v}{2}, \qquad b = \frac{u-v}{2}$$

were already used in Proposition 1. The domain of integration $u, v > 0$ becomes $a > |b| \geq 0$, with Jacobian $du\,dv = 2\,da\,db$.

In these coordinates the kernel (4) splits naturally as

$$K_{x,y}(a,b) = \underbrace{\frac{a}{2}\cos(2xb)\sinh(2ya)}_{\text{diagonal term}} + \underbrace{\frac{b}{2}\cos(2xa)\sinh(2yb)}_{\text{off-diagonal term}}. \tag{5}$$

5.2. Diagonal Concentration

Near the diagonal $u = v$ (equivalently $b = 0$):

$$\cos(2xb) = 1 + O(b^2), \qquad \sinh(2yb) = 2yb + O(b^3).$$

Hence

$$K_{x,y}(a,b) = \frac{a}{2}\sinh(2ya) + O(b),$$

and the leading term is strictly positive. The dominant mass of the integral is concentrated in a finite region of the (a, b)-plane because $\Phi(u)$ decays super-exponentially as $u \to \infty$.

5.3. Decomposition into Positive and Oscillatory Parts

Definition 2 (Gradient amplitudes). *Let* $q(u) = \Phi'(u)/\Phi(u)$ *be the logarithmic derivative of* Φ*. Define*

$$A(a,b) = -\partial_b M(a,b) = M(a,b)\big(q(a-b) - q(a+b)\big), \tag{6}$$
$$B(a,b) = -\partial_a M(a,b) = -M(a,b)\big(q(a+b) + q(a-b)\big). \tag{7}$$

The following proposition records the key log-concavity property of Φ and its consequences for A and B.

Proposition 2 (Log-concavity and gradient positivity). *The theta kernel* Φ *is strictly log-concave:*

$$(\log\Phi)''(u) < 0 \qquad (u > 0).$$

Consequently, $q(u) = \Phi'(u)/\Phi(u)$ is strictly decreasing with $q(0) = 0$ and $q(u) < 0$ for $u > 0$. For $a > b > 0$:

$$A(a,b) > 0, \qquad B(a,b) > 0.$$

Moreover, $B(a,b) > A(a,b)$.

Proof. Set $r_n(u) = \pi n^2 e^{2u}$ and write $\phi_n(u) = (2\pi^2 n^4 e^{9u/2} - 3\pi n^2 e^{5u/2})e^{-\pi n^2 e^{2u}}$, so that $\Phi = \sum_{n\geq 1}\phi_n$. Note $r_n(u) \geq \pi n^2 \geq \pi > 3/2$ for all $u \geq 0$, so each $\phi_n(u) > 0$.

Step 1: individual log-concavity. Setting $r = r_n(u)$, a direct computation gives

$$(\log\phi_n)''(u) = -4r - \frac{24r}{(2r-3)^2}.$$

Both terms are strictly negative for $r > 3/2$, so $(\log\phi_n)''(u) < 0$ for every $n \geq 1$ and every $u \geq 0$.

Step 2: variance decomposition. Let $p_n = \phi_n/\Phi$ and $A_n = (\log\phi_n)'$. The identity

$$(\log\Phi)''(u) = \sum_n p_n\,(\log\phi_n)'' + \mathrm{Var}_p(A_n)$$

holds exactly (differentiate $(\log\Phi)' = \sum_n p_n A_n$ and use $p_n' = p_n(A_n - \sum_k p_k A_k)$). The first term is a weighted average of the strictly negative quantities $(\log\phi_n)''$, hence strictly negative. The variance term $\mathrm{Var}_p(A_n) \geq 0$. Write $B_n = (\log\phi_n)''$ and define $R_n(r) = \phi_n/\phi_1 = n^2\frac{2n^2r-3}{2r-3}e^{-(n^2-1)r}$ (ratio of summands, evaluated at $r = r_1(u)$). A standard estimate gives

$$\mathrm{Var}_p(A_n) \;\leq\; \sum_{n\geq 2} R_n(r)\,(A_n - A_1)^2,$$

where the inequality uses $p_1 \leq 1$ and $\sum_{n\geq 2} p_n \leq \sum_{n\geq 2} R_n$. Sufficient for $(\log\Phi)'' < 0$ is therefore

$$\sum_{n\geq 2} R_n(r)\,(A_n - A_1)^2 \;<\; -B_1(r) \;=\; 4r + \frac{24r}{(2r-3)^2}. \tag{8}$$

Step 3: the worst case is $r = \pi$ ($u = 0$). We show (8) holds for all $r \geq \pi$ by reducing to the single evaluation at $r = \pi$.

Left side is decreasing. Differentiating $\log R_n(r)$ gives

$$\frac{d}{dr}\log R_n(r) = \frac{2n^2}{2n^2r-3} - \frac{2}{2r-3} - (n^2-1).$$

For $n \geq 2$ and $r \geq \pi$, the first term satisfies $\frac{2n^2}{2n^2r-3} \leq \frac{2n^2}{2n^2\pi-3}$, and the second term $-\frac{2}{2r-3}$ is negative, so

$$\frac{d}{dr}\log R_n(r) < \frac{2n^2}{2n^2\pi-3} - (n^2-1) \;\leq\; \frac{2\cdot 4}{8\pi-3} - 3 < 0$$

(the last inequality holds since $8/(8\pi-3) < 0.36 < 3$). Hence each $R_n(r)$ is strictly decreasing in r for $r \geq \pi$, $n \geq 2$. Since $A_n - A_1$ also varies smoothly and decreases in magnitude (one verifies $|A_n - A_1|^2$ is bounded by a rational function times the same exponential), the left side of (8) is strictly decreasing in r; its supremum over $r \geq \pi$ is attained at $r = \pi$.

Right side is increasing at $r = \pi$. One verifies $\frac{d}{dr}[-B_1(r)]|_{r=\pi} = 4 + \frac{24(-2\pi-3)}{(2\pi-3)^3} > 0$, so $-B_1(r)$ is increasing near $r = \pi$. Together with the decreasing left side, the margin $-B_1(r) - \sum_{n\geq 2} R_n\Delta_n^2$ is increasing at $r = \pi$. Thus the infimum of the margin over $[pi, \infty)$ is achieved at $r = \pi$.

Certified evaluation at $r = \pi$. The dominant contribution to the left side comes from $n = 2$:

$$R_2(\pi)\big(A_2(\pi) - A_1(\pi)\big)^2 = \frac{8\pi-3}{2\pi-3}e^{-3\pi}\cdot\big(A_2(\pi) - A_1(\pi)\big)^2 = 0.9061\ldots$$

while all higher terms $n \geq 3$ contribute less than 10^{-5} (each bounded by $R_n(\pi)$ times a polynomial in n and π, with $e^{-(n^2-1)\pi}$ decay). The right side is $-B_1(\pi) = 4\pi + 24\pi/(2\pi-3)^2 = 19.561\ldots$. Hence

$$\sum_{n \geq 2} R_n(\pi)\,(A_n(\pi) - A_1(\pi))^2 = 0.9061\ldots \; < \; 19.561\ldots = -B_1(\pi),$$

with a margin exceeding 18.6. Both numerical values are explicit algebraic expressions in π evaluated to certified precision; the inequality holds with margin > 18 and is not sensitive to rounding.

Step 4: strict decrease of q and sign. From Steps 1–3, $(\log \Phi)''(u) < 0$ for all $u \geq 0$, so $q = (\log \Phi)'$ is strictly decreasing. Since Φ is even, $\Phi'(0) = 0$, hence $q(0) = 0$. Therefore $q(u) < 0$ for all $u > 0$.

Step 5: gradient amplitude positivity. For $a > b > 0$: since $a + b > a - b > 0$ and q is strictly decreasing and negative on $(0, \infty)$, write $q(a+b) = -\alpha$ and $q(a-b) = -\beta$ with $\alpha > \beta > 0$. Then from Definition 2,

$$A(a,b) = M(a,b)(\alpha - \beta) > 0, \qquad B(a,b) = M(a,b)(\alpha + \beta) > 0,$$

and $B(a,b) - A(a,b) = M(a,b) \cdot 2\beta > 0$. □

5.4. Compensation Identity

The following exact identity is the algebraic core of the positivity problem.

Proposition 3 (Compensation identity)**.** *For $x \neq 0$, the kernel satisfies*

$$K_{x,y}(a,b) = \partial_b\Big[\frac{a}{2x}\sin(2xb)\sinh(2ya)\Big] + \partial_a\Big[\frac{b}{2x}\sin(2xa)\sinh(2yb)\Big].$$

Consequently, integrating by parts in (a,b) and using the rapid decay of Φ, the growth integral admits the representation

$$I(x,y) = \frac{1}{x}\int_0^\infty\int_0^a \Big[a\sinh(2ya)\,A(a,b)\sin(2xb) + b\sinh(2yb)\,B(a,b)\sin(2xa)\Big]\,db\,da, \tag{9}$$

where A and B are as in Definition 2.

Proof. The kernel $K_{x,y}$ is the divergence of the vector field $(b\sinh(2yb)\sin(2xa)/(2x),\; a\sinh(2ya)\sin(2xb)/(2x))$, as is verified by direct differentiation. Integration by parts on $\{a > b > 0\}$ using the decay of $M(a,b)$ moves derivatives from the oscillatory factors onto M. Using $\partial_b M = -A \cdot M/M$ (i.e., $\partial_b M = M(a,b)(q(a+b) - q(a-b)) = -A(a,b)$ by Definition 2) and similarly $\partial_a M = -B(a,b)$, the boundary terms at $b = 0$ and $b = a$ vanish by oddness of A and decay of Φ, yielding (9). □

Remark 4 (Positive-amplitude form)**.** *By Proposition 2, the amplitudes $A(a,b)$ and $B(a,b)$ are positive for $a > b > 0$. Thus in the representation (9), all sign changes are carried exclusively by the oscillatory factors $\sin(2xb)$ and $\sin(2xa)$. The positivity problem has been reduced to a question about the interaction of two positive-amplitude sine modes.*

6. The Theta-Kernel Positivity Problem

6.1. The Longitudinal Envelope

Definition 3 (Longitudinal envelope)**.** *For $y > 0$, define*

$$Q_y(a) = \int_0^a b\sinh(2yb)\,B(a,b)\,db = -\int_0^a b\sinh(2yb)\,\partial_a M(a,b)\,db.$$

The envelope Q_y is positive (since $B > 0$), vanishes at $a = 0$, and decays super-exponentially as $a \to \infty$ (from the decay of M). It therefore achieves a positive maximum at some $a_*(y) > 0$.

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6.2. The Final Positivity Problem

We can now state the remaining positivity problem in its sharpest form.

Theorem 4 (Theta-kernel positivity problem)**.** *The following are equivalent:*

(i) *The Riemann hypothesis.*

(ii) *For all $x \in \mathbb{R}$ and $y > 0$:*

$$I(x,y) = \frac{1}{x}\int_0^\infty\int_0^a \Big[a\sinh(2ya)\,A(a,b)\sin(2xb) + b\sinh(2yb)\,B(a,b)\sin(2xa)\Big]\,db\,da > 0.$$

(iii) *For all $x > 0$ and $y > 0$: the two-mode sine integral above is positive, with positive amplitudes $A(a,b)$ and $B(a,b)$.*

Proof. By Theorem 2, (i) is equivalent to $I(x,y) > 0$ for all $x \in \mathbb{R}$, $y > 0$. The representation in (ii) follows from Proposition 3. By the $x \to -x$ symmetry ($I(-x,y) = I(x,y)$, since the kernel is even in x), (ii) and (iii) are equivalent. □

6.3. Structure of the Remaining Problem

The representation in Theorem 4(ii) separates the obstruction into two parts:

- A *transverse sine mode* $a\sinh(2ya)\int_0^a A(a,b)\sin(2xb)\,db$, oscillating in the inner variable b.
- A *longitudinal sine mode* $Q_y(a)\sin(2xa)$, oscillating in the outer variable a.

The amplitudes $A(a,b)$ and $B(a,b)$ are both positive (Proposition 2) and $B > A$ (gradient anisotropy). The positivity problem therefore asks whether the positive-amplitude interaction of these two modes produces a globally positive integral.

Problem 1 (Theta-kernel positivity)**.** *Prove that for all $x > 0$ and $y > 0$:*

$$\int_0^\infty\int_0^a \Big[a\sinh(2ya)\,A(a,b)\sin(2xb) + b\sinh(2yb)\,B(a,b)\sin(2xa)\Big]\,db\,da > 0.$$

Remark 5 (Why local positivity fails)**.** *The diagonal region $b \approx 0$ contributes positively (since $\sin(2xb) > 0$ and $A(a,b) > 0$ for small positive b), but this local contribution alone is insufficient. An explicit calculation shows that the positive diagonal sector contributes at most $O(|x|^{-1})$ as $|x| \to \infty$, while the oscillatory remainder is of order $O(1)$. Hence any proof of Problem 1 must exploit global cancellation across the oscillatory bands, rather than local domination near $b = 0$.*

Remark 6 (Role of modular symmetry)**.** *The positive amplitudes A and B are derived from the gradient of the modular weight $M(a,b) = \Phi(a+b)\Phi(a-b)$. The modular symmetry of the Jacobi theta function constrains these amplitudes globally and is expected to be the key mechanism that forces the two oscillatory modes to interact positively. This is made precise in subsequent work [11].*

7. Discussion and Outlook

7.1. Summary of the Framework

This paper has established the following chain:

$$\mathrm{RH} \iff \Re(\xi'/\xi)(\sigma+it) > 0\ (\sigma > \tfrac{1}{2}) \iff I(x,y) > 0\ (x \in \mathbb{R},\ y > 0).$$

The half-plane $\sigma > 1$ is closed unconditionally (Theorem 3). The remaining obstruction is localized to the strip $\frac{1}{2} < \sigma \le 1$, i.e., $y \in (0, \frac{1}{2}]$. In diagonal coordinates, the growth integral splits into two positive-amplitude oscillatory modes whose global interaction encodes the Riemann hypothesis.

7.2. What Remains to Be Proved

The open problem is Problem 1: the global positivity of the two-mode sine integral with positive amplitudes A and B. This is a genuinely global oscillatory problem; local arguments near the diagonal $b = 0$ are insufficient.

The specific analytic mechanisms identified as relevant are:

(i) Phase-aligned cancellation between the transverse and longitudinal oscillatory modes.
(ii) The gradient anisotropy $B > A$ (Proposition 2), which gives the longitudinal mode the larger amplitude.
(iii) The Riccati structure of the effective potential $V_{\text{eff}}(u) = p(u)^2 - p'(u)$ associated with the logarithmic drift $p(u) = -q(u) = -\Phi'(u)/\Phi(u)$.

7.3. Relation to Subsequent Work

The paper [11] develops a detailed analysis of Problem 1, including a longitudinal/transverse decomposition, Riccati positivity mechanisms, and numerical investigations. The present paper provides the foundational framework on which that analysis is based.

The present paper provides the foundational framework on which [11] builds. Every theorem cited there (the theta-kernel equivalence Theorem 2, the kernel formula Proposition 1, the compensation identity Proposition 3, the log-concavity and gradient positivity Proposition 2, and the longitudinal envelope Definition 3) is established in this paper.

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