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Article

Equivalence of Common Metrics on Trapezoidal Fuzzy Numbers

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Abstract

In this paper, we show that the four kinds of common metrics: the supremum metric, the L_p-type d_p metrics, the sendograph metric and the endograph metric, are equivalent on the trapezoidal fuzzy numbers. Furthermore, we point out that, on the trapezoidal fuzzy numbers, these four kinds of metric convergences are equivalent to the convergence of the corresponding representation quadruples of the trapezoidal fuzzy numbers in $\mathbb{R}^4$.

Keywords: Supremum metric; d_p metrics; sendograph metric; endograph metric; trapezoidal fuzzy numbers

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1. Introduction

The trapezoidal fuzzy numbers and its special case triangular fuzzy numbers have been widely used and discussed in theory and applications [1–5]. The supremum distance, the L_p-type d_p distances, the sendograph distance and the endograph distance on fuzzy sets have been attracted much attention [3,6–12].

Naturally, these four kinds of metrics can also been used and discussed on the set of trapezoidal fuzzy numbers. So it is important to consider the relationship of these four kinds of metrics on the set of trapezoidal fuzzy numbers.

In this paper, we show all these four kinds of metric convergences on the trapezoidal fuzzy numbers are equivalent to the convergence of the corresponding representation quadruples of the trapezoidal fuzzy numbers in $\mathbb{R}^4$. So they are equivalent metrics on the trapezoidal fuzzy numbers.

The remainder of this paper is organized as follows. Section 2 reviews some basic concepts and fundamental conclusions of fuzzy sets and the extended metrics on them. In Section 3, we recall some basic concepts and properties related to the triangular fuzzy numbers and the trapezoidal fuzzy numbers. Section 4 recalls and gives some properties of the trapezoidal fuzzy numbers and triangular fuzzy numbers. Section 5 presents the main results of this paper. At last, we draw our conclusions in Section 6.

2. Fuzzy Sets and Extended Metrics on Them

In this section, we review some basic concepts and fundamental conclusions of fuzzy sets and the extended metrics on them. For fuzzy theory and applications, we refer the readers to [1–4,6,8,9,13–18].

Let $\mathbb{N}$ be the set of all positive integers and let $\mathbb{R}^m$ be the m-dimensional Euclidean space. $\mathbb{R}^1$ is also written as $\mathbb{R}$.

Let Y be a nonempty set. The symbol $P(Y)$ denotes the power set of Y, which is the set of all subsets of Y. The symbol $F(Y)$ denotes the set of all fuzzy sets in Y, i.e., functions from Y to $[0,1]$. Given $u \in F(Y)$ and $\alpha \in (0,1]$, the α-cut $[u]_\alpha$ of u is defined by $[u]_\alpha := \{x \in Y : u(x) \geq \alpha\}$.

Let Y be a topological space. The symbol $C(Y)$ denotes the set of all nonempty closed subsets of Y. $K(Y)$ denotes the set of all nonempty compact subsets of Y. For $u \in F(Y)$, the 0-cut $[u]_0$ of u is

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defined by $[u]_0 := \overline{\{x \in Y : u(x) > 0\}}$, where $\overline{S}$ denotes the topological closure of S in Y. $[u]_0$ is called the support of u, and is also denoted by $\operatorname{supp} u$.

Let Y be a nonempty set. For $u \in F(Y)$, define

$$\operatorname{end} u := \{(x,t) \in Y \times [0,1] : u(x) \geq t\},$$
$$\operatorname{send} u := \operatorname{end} u \cap ([u]_0 \times [0,1]),$$

where $\operatorname{send} u$ is well-defined if and only if Y is a topological space. $\operatorname{end} u$ and $\operatorname{send} u$ are called the endograph and the sendograph of u, respectively. Clearly $\operatorname{end} u = \operatorname{send} u \cup (X \times \{0\})$.

Throughout this paper, we suppose that X *is a nonempty set and it is equipped with a metric* d. For simplicity, we also use X to *denote the metric space* (X, d).

Let (X, d) be a metric space. We use $\boldsymbol{H}$ to denote the ***Hausdorff extended metric*** on $C(X)$ induced by d, i.e.,

$$\boldsymbol{H(U,V)} = \max\{H^*(U,V),\ H^*(V,U)\}$$

for arbitrary $U, V \in C(X)$, where

$$H^*(U,V) = \sup_{u \in U} d\,(u,V) = \sup_{u \in U} \inf_{v \in V} d\,(u,v).$$

For simplicity, we often refer to both the Hausdorff extended metric and the Hausdorff metric as the Hausdorff metric. See also Remark 2.5 of [10].

Let $[a,b]$ and $[c,d]$ be two intervals. Then

$$H([a,b],[c,d]) = \max\{|a-c|, |b-d|\}. \tag{1}$$

The metric $\overline{d}$ on $X \times [0,1]$ is defined as follows: for $(x,\alpha),(y,\beta) \in X \times [0,1]$, $\overline{d}((x,\alpha),(y,\beta)) = d(x,y) + |\alpha - \beta|$. If there is no confusion, we also use H to denote the Hausdorff extended metric on $C(X \times [0,1])$ induced by $\overline{d}$.

Let $F_{USC}(X)$ denote the set of all upper semi-continuous fuzzy sets in X; that is,

$$F_{USC}(X) := \{u \in F(X) : [u]_\alpha \in C(X) \cup \{\emptyset\} \text{ for all } \alpha \in [0,1]\}.$$

Define $F^1_{USC}(X) := \{u \in F_{USC}(X) : [u]_1 \neq \emptyset\}$. Clearly $F^1_{USC}(X) \subsetneqq F_{USC}(X)$.

The supremum distance d_∞, the sendograph distance H_{send} and the endograph distance H_{end} on $F^1_{USC}(X)$ are defined as follows, respectively. For each $u, v \in F^1_{USC}(X)$,

$$d_\infty(u,v) = \sup_{\alpha \in [0,1]} H([u]_\alpha, [v]_\alpha),$$
$$H_{\text{send}}(u,v) = H(\operatorname{send} u, \operatorname{send} v),$$
$$H_{\text{end}}(u,v) = H(\operatorname{end} u, \operatorname{end} v),$$

where H in the definition of d_∞ denotes the Hausdorff extended metric on $C(X)$ induced by d, and H in the definitions of H_{send} and H_{end} denote the Hausdorff extended metric on $C(X \times [0,1])$ induced by $\overline{d}$.

The sendograph metric H_{send} was introduced by Kloeden [19]. Each one of d_∞ and H_{send} on $F^1_{USC}(X)$ is an extended metric but does not need to be a metric. H_{end} on $F^1_{USC}(X)$ is a metric. See also Remark 2.7 of [10].

The L_p-type d_p distance, $1 \leq p < +\infty$, of each $u, v \in F^1_{USC}(X)$ is defined by

$$d_p(u,v) = \left(\int_0^1 H([u]_\alpha, [v]_\alpha)^p \, d\alpha \right)^{1/p},$$

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where $d_p(u,v)$ is well-defined if and only if $H([u]_\alpha, [v]_\alpha)$ is a measurable function of α on $[0,1]$.

We suppose that, in the sequel, "p" which appears in mathematical expressions such as d_p, etc., is an arbitrary number satisfying $1 \le p < +\infty$.

For some metric spaces Y, d_p distances could be not well-defined on $F^1_{USC}(Y)$ (see Example 3.25 of [10]). So the following d_p^* extended metrics on $F^1_{USC}(X)$ are introduced in [20]. For each $u,v \in F^1_{USC}(X)$,

$$d_p^*(u,v) := \inf\left\{ \left(\int_0^1 f(\alpha)^p \, d\alpha\right)^{1/p} : f \text{ is a measurable function from } [0,1] \text{ to } \mathbb{R}^+ \cup \{+\infty\} \text{ satisfying } f(\alpha) \ge H([u]_\alpha, [v]_\alpha) \text{ for all } \alpha \in [0,1] \right\}.$$

Theorem 2.1. *[10,11,20] Let $u,v \in F^1_{USC}(X)$.*
(i) $d_\infty(u,v) \ge H_{\rm send}(u,v) \ge H_{\rm end}(u,v)$.
(ii) $d_\infty(u,v) \ge d_p^*(u,v)$.
(iii) $d_p^*(u,v) \ge H_{\rm end}(u,v)^{1+1/p}$.
(iv) If $d_p(u,v)$ is well-defined, then $d_p^*(u,v) = d_p(u,v)$.
(v) If $X = \mathbb{R}^m$, then $d_p(u,v)$ is well-defined; so $d_p^*(u,v) = d_p(u,v)$.
(vi) If $X = \mathbb{R}^m$, then d_p^* in (ii) and (iii) can be replaced by d_p.

Proof. Clearly (i) holds. (i) may be a known conclusion. ((i) is (1) of [11]. The inequality at the end of Paragraph 2 in Page 2527 of [21] is this kind of conclusions.). (ii) is (13) of [11]. (iii) is Proposition 4.9(i) of [10]. (iv) is given in Remark 3.2 of [20]. (iv) is obvious. A routine proof of (iv) is given below.

Suppose that $d_p(u,v)$ is well-defined; that is, $H([u]_\alpha, [v]_\alpha)$ is a measurable function of α on $[0,1]$. Put $S := \{f : f$ is a measurable function from $[0,1]$ to $\mathbb{R}^+ \cup \{+\infty\}$ satisfying $f(\alpha) \ge H([u]_\alpha, [v]_\alpha)$ for all $\alpha \in [0,1]\}$. Then the function $H([u]_\alpha, [v]_\alpha)$ of α on $[0,1]$ belongs to S. Hence we have (a) $\inf_{f\in S}\left(\int_0^1 f(\alpha)^p \, d\alpha\right)^{1/p} \le \left(\int_0^1 H([u]_\alpha, [v]_\alpha)^p \, d\alpha\right)^{1/p}$. On the other hand, for each $f \in S$, $\left(\int_0^1 f(\alpha)^p \, d\alpha\right)^{1/p} \ge \left(\int_0^1 (H([u]_\alpha, [v]_\alpha)^p \, d\alpha\right)^{1/p}$ as $f(\alpha) \ge H([u]_\alpha, [v]_\alpha)$ for all $\alpha \in [0,1]$. Thus we obtain (b) $\inf_{f\in S}\left(\int_0^1 f(\alpha)^p \, d\alpha\right)^{1/p} \ge \left(\int_0^1 H([u]_\alpha, [v]_\alpha)^p \, d\alpha\right)^{1/p}$. So $d_p^*(u,v) = \inf_{f\in S}\left(\int_0^1 f(\alpha)^p \, d\alpha\right)^{1/p} =$ (by (a) and (b)) $\left(\int_0^1 H([u]_\alpha, [v]_\alpha)^p \, d\alpha\right)^{1/p} = d_p(u,v)$.

Theorem 3.8 of [10] says that if $X = \mathbb{R}^m$, then $d_p(u,v)$ is well-defined. Obviously, by this and (iv), we obtain that if $X = \mathbb{R}^m$, then $d_p^*(u,v) = d_p(u,v)$. So (v) holds. (vi) follows immediately from (v).
□

Let Y be a nonempty set, Z a subset of Y, and ρ_1 and ρ_2 two extended metrics on Y. We say that ρ_1 is stronger than ρ_2 on Z, denoted by $\rho_1 \succeq \rho_2$ (Z), if for each sequence $\{y_n\}$ in Z and each $y \in Z$, $\lim_{n\to\infty} \rho_1(y_n, y) = 0$ implies that $\lim_{n\to\infty} \rho_2(y_n, y) = 0$.

ρ_1 is stronger than ρ_2 on Z is also known as ρ_2 is weaker than ρ_1 on Z and written as $\rho_2 \preceq \rho_1$ (Z). ρ_1 is said to be equivalent to ρ_2 on Z, denoted by $\rho_1 \approx \rho_2$ (Z), if $\rho_1 \succeq \rho_2$ (Z) and $\rho_2 \succeq \rho_1$ (Z).

The expression $\rho_1 \succeq S \succeq \rho_2$ (Z), where S is a set of extended metrics on Y, means that for each $\rho \in S$, $\rho_1 \succeq \rho \succeq \rho_2$ (Z).

Theorem 2.2. *[11] $d_\infty \succeq \{H_{\rm send}, d_p^*\} \succeq H_{\rm end}$ $(F^1_{USC}(X))$.*

Proof. By Theorem 2.1(i), $d_\infty \succeq H_{\rm send} \succeq H_{\rm end}$ $(F^1_{USC}(X))$. By Theorem 2.1(ii), $d_\infty \succeq d_p^*$ $(F^1_{USC}(X))$. By Theorem 2.1(iii), $d_p^* \succeq H_{\rm end}$ $(F^1_{USC}(X))$. Theorem 6.2 of [11] also says that $d_p^* \succeq H_{\rm end}$ $(F^1_{USC}(X))$. So this theorem is indeed given in [11].
□

By Theorems 2.2 and 2.1(v), we have the following conclusion.

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Corollary 2.3. *[10,11,20]* $d_\infty \succeq \{H_{\text{send}}, d_p\} \succeq H_{\text{end}}\ (F^1_{USC}(\mathbb{R}^m))$.

d_∞, d_p, H_{send} and H_{end} are metrics on $F^1_{USCB}(X)$.
The corresponding author of this paper independently gave Section 2.

3. Triangular Fuzzy Numbers and Trapezoidal Fuzzy Numbers

In this section, we review some basic concepts and properties related to the triangular fuzzy numbers and the trapezoidal fuzzy numbers.

Usually, the symbols (a,b,c,d) with a,b,c,d in $\mathbb{R}$ represent the elements in $\mathbb{R}^4$ and the symbols (a,b,c) with a,b,c in $\mathbb{R}$ represent the elements in $\mathbb{R}^3$. In this paper, for each a,b,c,d in $\mathbb{R}$, we use $[a,b,c,d]$ instead of (a,b,c,d) to represent the corresponding element in $\mathbb{R}^4$, and use $[a,b,c]$ instead of (a,b,c) to represent the corresponding element in $\mathbb{R}^3$.

We use T to denote the set $\{[a,b,c,d] \in \mathbb{R}^4 : a \leq b \leq c \leq d\}$ and T_0 to denote the set $\{[a,b,c,d] \in \mathbb{R}^4 : a < b \leq c < d\}$. Clearly $T_0 \subsetneqq T$.

We use G to denote the set $\{[a,b,c] \in \mathbb{R}^3 : a \leq b \leq c\}$ and G_0 to denote the set $\{[a,b,c] \in \mathbb{R}^3 : a < b < c\}$. Clearly $G_0 \subsetneqq G$.

Definition 3.1. We use **Tag** to denote the set of all regular triangular fuzzy numbers. $Tag := \{(a,b,c) : [a,b,c] \text{ in } G_0\}$, where, for any $[a,b,c]$ in G_0, the regular triangular fuzzy number (a,b,c) is defined to be the fuzzy set u in $F(\mathbb{R})$ given by

$$u(x) = \begin{cases} \dfrac{x-a}{b-a}, \text{ if } a \leq x \leq b, \\ \dfrac{c-x}{c-b}, \text{ if } b \leq x \leq c, \\ \quad 0, \text{ if } x \in \mathbb{R} \setminus [a,c]. \end{cases}$$

Definition 3.2. We use **Tap** to denote the set of all regular trapezoidal fuzzy numbers. $Tap := \{(a,b,c,d) : [a,b,c,d] \text{ in } T_0\}$, where, for any $[a,b,c,d]$ in T_0, the regular trapezoidal fuzzy number (a,b,c,d) is defined to be the fuzzy set u in $F(\mathbb{R})$ given by

$$u(x) = \begin{cases} \dfrac{x-a}{b-a}, \text{ if } a \leq x \leq b, \\ \quad 1, \text{ if } b \leq x \leq c, \\ \dfrac{d-x}{d-c}, \text{ if } \ c \leq x \leq d, \\ \quad 0, \text{ if } x \in \mathbb{R} \setminus [a,d]. \end{cases}$$

Remark 3.3. (i) $u \in$ Tag means that there is an $[a,b,c] \in G_0$ satisfying $u = (a,b,c)$. (ii) $u \in$ Tap means that there is an $[a,b,c,d] \in T_0$ satisfying $u = (a,b,c,d)$. (iii) Each regular triangular fuzzy number (a,b,c) is the regular trapezoidal fuzzy number (a,b,b,c). So Tag $\subseteq$ Tap.

We say that two fuzzy sets are *equal* if they have the same membership function.

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Definition 3.4. *We use **Trag** to denote the set of all triangular fuzzy numbers. Trag* $:= \{(a,b,c) : [a,b,c] \text{ in } G\}$, *where, for any* $[a,b,c]$ *in* G, *the triangular fuzzy number* (a,b,c) *is defined to be the fuzzy set* u *in* $F(\mathbb{R})$ *in the following way:*

$$u \text{ is the regular triangular fuzzy number } (a,b,c) \text{ when } a < b < c;$$

$$u(x) = \begin{cases} \dfrac{c-x}{c-b}, \text{ if } b \le x \le c, \\ 0, \text{ if } x \in \mathbb{R} \setminus [b,c], \end{cases} \quad \text{when } a = b < c;$$

$$u(x) = \begin{cases} \dfrac{x-a}{b-a}, \text{ if } a \le x \le b, \\ 0, \text{ if } x \in \mathbb{R} \setminus [a,b], \end{cases} \quad \text{when } a < b = c;$$

$$u(x) = \begin{cases} 1, \text{ if } x = b, \\ 0, \text{ if } x \in \mathbb{R} \setminus \{b\}, \end{cases} \quad \text{when } a = b = c.$$

Clearly each (a,b,c) in Tag is the (a,b,c) in Trag. This means that the concept of triangular fuzzy numbers is a kind of generalization of the concept of regular triangular fuzzy numbers. Hence Tag $\subseteq$ Trag.

Definition 3.5. *We use **Trap** to denote the set of all trapezoidal fuzzy numbers. Trap* $:= \{(a,b,c,d) : [a,b,c,d] \text{ in } T\}$, *where, for any* $[a,b,c,d]$ *in* T, *the trapezoidal fuzzy number* (a,b,c,d) *is defined to be the fuzzy set* u *in* $F(\mathbb{R})$ *in the following way:*

$$u \text{ is the regular trapezoidal fuzzy number } (a,b,c,d) \text{ when } a < b \le c < d;$$

$$u(x) = \begin{cases} 1, \text{ if } b \le x \le c, \\ \dfrac{d-x}{d-c}, \text{ if } c \le x \le d, \\ 0, \text{ if } x \in \mathbb{R} \setminus [b,d], \end{cases} \quad \text{when } a = b \le c < d;$$

$$u(x) = \begin{cases} \dfrac{x-a}{b-a}, \text{ if } a \le x \le b, \\ 1, \text{ if } b \le x \le c, \\ 0, \text{ if } x \in \mathbb{R} \setminus [a,c], \end{cases} \quad \text{when } a < b \le c = d;$$

$$u(x) = \begin{cases} 1, \text{ if } b \le x \le c, \\ 0, \text{ if } x \in \mathbb{R} \setminus [b,c], \end{cases} \quad \text{when } a = b \le c = d.$$

Clearly each (a,b,c,d) in Tap is the (a,b,c,d) in Trap. This means that the concept of trapezoidal fuzzy numbers is a kind of generalization of the concept of regular trapezoidal fuzzy numbers. Hence Tap $\subseteq$ Trap.

Remark 3.6. (i) $u \in$ Trag means that there is an $[a,b,c] \in G$ satisfying $u = (a,b,c)$. (ii) $u \in$ Trap means that there is an $[a,b,c,d] \in T$ satisfying $u = (a,b,c,d)$.

Remark 3.7. Each triangular fuzzy number (a,b,c) is the trapezoidal fuzzy number (a,b,b,c). So Trag $\subseteq$ Trap.

The readers may also refer to the corresponding contents in [22] for details.

4. Some Properties of Trapezoidal Fuzzy Numbers and Triangular Fuzzy numbers

In this section, we recall and give some properties of the trapezoidal fuzzy numbers and triangular fuzzy numbers. These properties are useful to obtain and understand the main results of this paper.

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The following Propositions 4.1 and 4.2 should be known. See [5] and related works. Clearly, Proposition 4.2 is a corollary of Proposition 4.1.

Proposition 4.1. *Let $u \in F(\mathbb{R})$ and $(a,b,c,d) \in$ Trap. Then $u = (a,b,c,d)$ if and only if*

$$\text{for each } \xi \in [0,1],\ [u]_\xi = [\xi(b-a)+a,\ c+(1-\xi)(d-c)]. \tag{2}$$

Proposition 4.2. *Let $u \in F(\mathbb{R})$ and $(a,b,c) \in$ Trag. Then $u = (a,b,c)$ if and only if for each $\xi \in [0,1]$, $[u]_\xi = [\xi(b-a)+a,\ b+(1-\xi)(c-b)]$.*

For any $[a,b,c,d]$ and $[a_1,b_1,c_1,d_1]$ in $\mathbb{R}^4$, $[a,b,c,d] = [a_1,b_1,c_1,d_1]$ means that $a = a_1$, $b = b_1$, $c = c_1$ and $d = d_1$. For any (a,b,c,d) and (a_1,b_1,c_1,d_1) in Trap, $(a,b,c,d) = (a_1,b_1,c_1,d_1)$ means that (a,b,c,d) and (a_1,b_1,c_1,d_1) are the same fuzzy set.

For any $[a,b,c]$ and $[a_1,b_1,c_1]$ in $\mathbb{R}^3$, $[a,b,c] = [a_1,b_1,c_1]$ means that $a = a_1$, $b = b_1$ and $c = c_1$. For any (a,b,c) and (a_1,b_1,c_1) in Trag, $(a,b,c) = (a_1,b_1,c_1)$ means that (a,b,c) and (a_1,b_1,c_1) are the same fuzzy set.

The following Theorem 4.3(ii) states the representation uniqueness of the trapezoidal fuzzy numbers.

Theorem 4.3. *(i) Let $u = (a,b,c,d)$ be in Trap. Then $[u]_0 = [a,d]$ and $[u]_1 = [b,c]$. (ii) Let (a,b,c,d) and (a_1,b_1,c_1,d_1) be in Trap. Then $(a,b,c,d) = (a_1,b_1,c_1,d_1)$ if and only if $[a,b,c,d] = [a_1,b_1,c_1,d_1]$. (iii) Let $u = (a,b,c,d)$ be in Tap. Then $[u]_0 = [a,d]$ and $[u]_1 = [b,c]$. (iv) Let (a,b,c,d) and (a_1,b_1,c_1,d_1) be in Tap. Then $(a,b,c,d) = (a_1,b_1,c_1,d_1)$ if and only if $[a,b,c,d] = [a_1,b_1,c_1,d_1]$.*

Proof. By Definition 3.5 and easy calculations, we obtain (i). (One way to perform these calculations are to do it based on watching the graphs of the membership functions of (a,b,c,d) in the four cases $a < b \le c < d$, $a = b \le c < d$, $a < b \le c = d$ and $a = b \le c = d$.) (i) follows immediately from Proposition 4.1.

Now we show (ii). If $[a,b,c,d] = [a_1,b_1,c_1,d_1]$, i.e. $a = a_1, b = b_1, c = c_1$ and $d = d_1$, then, by Definition 3.5, $(a,b,c,d) = (a_1,b_1,c_1,d_1)$.

Suppose that $(a,b,c,d) = (a_1,b_1,c_1,d_1)$. Then $[(a,b,c,d)]_1 = [(a_1,b_1,c_1,d_1)]_1$ and $[(a,b,c,d)]_0 = [(a_1,b_1,c_1,d_1)]_0$. By (i), this means that $[b,c] = [b_1,c_1]$ and $[a,d] = [a_1,d_1]$. This is equivalent to $a = a_1$, $b = b_1$, $c = c_1$ and $d = d_1$; that is, $[a,b,c,d] = [a_1,b_1,c_1,d_1]$. So (ii) is proved.

As Tap is a subset of Trap, (iii) follows immediately from (i), and (iv) follows immediately from (ii). (iii) is easy and should be known.

□

Proposition 4.4(ii) gives the representation uniqueness of the triangular fuzzy numbers.

Proposition 4.4. *(i) Let $u = (a,b,c)$ be in Trag. Then $[u]_0 = [a,c]$ and $[u]_1 = \{b\}$. (ii) Let (a,b,c) and (a_1,b_1,c_1) be in Trag. Then $(a,b,c) = (a_1,b_1,c_1)$ if and only if $[a,b,c] = [a_1,b_1,c_1]$. (iii) Let $u = (a,b,c)$ be in Tag. Then $[u]_0 = [a,c]$ and $[u]_1 = \{b\}$. (iv) Let (a,b,c) and (a_1,b_1,c_1) be in Tag. Then $(a,b,c) = (a_1,b_1,c_1)$ if and only if $[a,b,c] = [a_1,b_1,c_1]$.*

Proof. By Definition 3.4 and easy calculations, we obtain (i). (One way to perform these calculations are to do it based on watching the graphs of the membership functions of (a,b,c) in the four cases $a < b < c$, $a = b < c$, $a < b = c$ and $a = b = c$.) (i) follows immediately from Proposition 4.2.

Now we show (ii). If $[a,b,c] = [a_1,b_1,c_1]$, i.e. $a = a_1$, $b = b_1$ and $c = c_1$, then, by Definition 3.4, $(a,b,c) = (a_1,b_1,c_1)$.

Suppose that $(a,b,c) = (a_1,b_1,c_1)$. Then $[(a,b,c)]_1 = [(a_1,b_1,c_1)]_1$ and $[(a,b,c)]_0 = [(a_1,b_1,c_1)]_0$. By (i), this means that $\{b\} = \{b_1\}$ and $[a,c] = [a_1,c_1]$. This is equivalent to $a = a_1$, $b = b_1$ and $c = c_1$; that is, $[a,b,c] = [a_1,b_1,c_1]$. So (ii) is proved.

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As Tag is a subset of Trag, (iii) follows immediately from (i), and (iv) follows immediately from (ii). (iii) is easy and should be known.
□

The above proofs of Theorem 4.3 and Proposition 4.4 are similar. Clearly for k=i, ii, iii, iv, Proposition 4.4(k) is a corollary of Theorem 4.3(k) (see also Remark 4.3 in [22] for details).

We know that Tag $\subseteq$ Trag, Tap $\subseteq$ Trap, Tag $\subseteq$ Tap, and Trag $\subseteq$ Trap (see Section 3). Based on Theorem 4.3(ii) and Proposition 4.4(ii), it is easy to see that Tag $\subsetneqq$ Trag, Tap $\subsetneqq$ Trap, Tag $\subsetneqq$ Tap, and Trag $\subsetneqq$ Trap. (see also Remarks 4.7 and 4.8 in [22] for details.)

Define

$$F_{USCB}(X) := \{u \in F_{USC}(X) : [u]_0 \in K(X) \cup \{\emptyset\}\},$$
$$F^1_{USCB}(X) := \{u \in F_{USCB}(X) : [u]_1 \neq \emptyset\}.$$

Clearly $F^1_{USCB}(X) \subsetneqq F_{USCB}(X)$, $F^1_{USCB}(X) \subseteq F^1_{USC}(X)$ and $F_{USCB}(X) \subseteq F_{USC}(X)$.

For $u \in F(\mathbb{R})$, we call u a 1-dimensional compact fuzzy number if u has the following properties:
(i) $[u]_1 \neq \emptyset$; and
(ii) for each $\alpha \in [0,1]$, $[u]_\alpha$ is a compact interval of $\mathbb{R}$.
The set of all 1-dimensional compact fuzzy numbers is denoted by E. For $u \in E$ and $\alpha \in [0,1]$, $[u]_\alpha$ is denoted by $[u^-(\alpha), u^+(\alpha)]$.

Let $u \in$ Trap. Denote $u = (a,b,c,d)$. By Proposition 4.1, $[u]_1 = [b,c] \neq \emptyset$ and for each $\xi \in [0,1]$, $[u]_\xi = [\xi(b-a)+a,\ c+(1-\xi)(d-c)]$ is a compact interval of $\mathbb{R}$. Also $u \in F(\mathbb{R})$. Thus $u \in E$. So Trap $\subseteq E$. Clearly Trap $\subsetneqq E$ (see also [25]), and $E \subsetneqq F^1_{USCB}(\mathbb{R}) \subsetneqq F^1_{USC}(\mathbb{R})$. So Trap $\subsetneqq F^1_{USC}(\mathbb{R})$, and then, by Corollary 2.3, we have that

Corollary 4.5. *[10,11,20] $d_\infty \succeq \{H_{\text{send}}, d_p\} \succeq H_{\text{end}}$ (Trap).*

As Trap $\subsetneqq F^1_{USC}(\mathbb{R})$ is a quite obvious fact and this fact should be known, we think it is reasonable to cite Corollary 2.3 when we use the fact given in Corollary 4.5.

5. Main Results

In this section, we show the equivalence of the four common types of metrics d_∞, d_p, H_{send} and H_{end} are equivalent on the trapezoidal fuzzy numbers. We do this by verifying that the convergence induced by these metrics is the convergence of the corresponding representation quadruples of the trapezoidal fuzzy numbers in $\mathbb{R}^4$.

First we give a characterization of the supremum metric d_∞ on Trap.

Lemma 5.1. *Let $u = (a_1, b_1, c_1, d_1)$ and $v = (a,b,c,d)$ be two trapezoidal fuzzy numbers.*
(i) $\sup_{\xi\in[0,1]} |u^-(\xi) - v^-(\xi)| = \max\{|a_1 - a|, |b_1 - b|\}$.
(ii) $\sup_{\xi\in[0,1]} |u^+(\xi) - v^+(\xi)| = \max\{|c_1 - c|, |d_1 - d|\}$.
(iii) $d_\infty(u,v) = \max\{|a_1 - a|, |b_1 - b|, |c_1 - c|, |d_1 - d|\}$.

Proof. Set $A := \max\{|a_1 - a|, |b_1 - b|\}$ and $B := \max\{|c_1 - c|, |d_1 - d|\}$. Notice that for each $\xi \in [0,1]$,

$$\begin{aligned}
&|u^-(\xi) - v^-(\xi)| \\
&= |\xi(b_1 - a_1) + a_1 - (\xi(b-a)+a)| \quad \text{(by (2), see (I) below)} \\
&= |\xi(b_1 - b) + (1-\xi)(a_1 - a)| \\
&\leq |\xi A + (1-\xi)A| = |A| = A.
\end{aligned}$$

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Thus $\sup_{\xi\in[0,1]} |u^-(\xi) - v^-(\xi)| \leq A$. On the other hand, since $|u^-(0) - v^-(0)| = |a_1 - a|$ and $|u^-(1) - v^-(1)| = |b_1 - b|$, we have that $\sup_{\xi\in[0,1]} |u^-(\xi) - v^-(\xi)| \geq A$. So $\sup_{\xi\in[0,1]} |u^-(\xi) - v^-(\xi)| = A$. Hence (i) is proved.

The proof of (ii) is similar to that of (i). Notice that for each $\xi \in [0,1]$,

$$\begin{aligned}
&|u^+(\xi) - v^+(\xi)| \\
&= |c_1 + (1-\xi)(d_1 - c_1) - (c + (1-\xi)(d-c))| \\
&= |\xi(c_1 - c) + (1-\xi)(d_1 - d)| \\
&\leq |\xi B + (1-\xi)B| = |B| = B.
\end{aligned}$$

Thus $\sup_{\xi\in[0,1]} |u^+(\xi) - v^+(\xi)| \leq B$. On the other hand, since $|u^+(0) - v^+(0)| = |d_1 - d|$ and $|u^+(1) - v^+(1)| = |c_1 - c|$, we have that $\sup_{\xi\in[0,1]} |u^+(\xi) - v^+(\xi)| \geq B$. So $\sup_{\xi\in[0,1]} |u^+(\xi) - v^+(\xi)| = B$. Hence (ii) is proved.

Note that

$$\begin{aligned}
d_\infty(u,v) &= \sup_{\xi\in[0,1]} H([u]_\xi,[v]_\xi) = \text{(by (1))} \sup_{\xi\in[0,1]} (|u^-(\xi) - v^-(\xi)| \vee |u^+(\xi) - v^+(\xi)|) \\
&= \sup_{\xi\in[0,1]} |u^-(\xi) - v^-(\xi)| \vee \sup_{\xi\in[0,1]} |u^+(\xi) - v^+(\xi)| \\
&= \text{(by (i) and (ii))}\ A \vee B = \max\{|a_1 - a|, |b_1 - b|, |c_1 - c|, |d_1 - d|\}.
\end{aligned}$$

So (iii) is proved.

(I) In this paper, we often use (2) without citing since it is easy to see.

□

Remark 5.2. Let $u = (a,b,c,d) = (a_1,b_1,c_1,d_1) \in$ Trap. Then $0 = d_\infty(u,u) =$ (by Lemma 5.1(iii)) $\max\{|a_1 - a|, |b_1 - b|, |c_1 - c|, |d_1 - d|\}$. Thus $a_1 = a$, $b_1 = b$, $c_1 = c$ and $d_1 = d$. Hence "⇒" of Theorem 4.3(ii) holds. "⇐" of Theorem 4.3(ii) holds obviously. So Lemma 5.1(iii) implies Theorem 4.3(ii). As Proposition 4.4(ii) is a corollary of Theorem 4.3(ii), Lemma 5.1(iii) also implies Proposition 4.4(ii).

By Remark 3.7, Lemma 5.1(iii) implies (a) for each $u = (a,b,c)$ and $v = (a_1,b_1,c_1)$ in Trag, $d_\infty(u,v) = \max\{|a_1 - a|, |b_1 - b|, |c_1 - c|\}$. Clearly, (a) implies Proposition 4.4(ii).

Remark 5.3. The conclusions in this remark are easy to see. The symbols in this remark are consistent with those in the proof of Lemma 5.1.
(i) $\sup_{\xi\in[0,1]} |u^-(\xi) - v^-(\xi)| = \max_{\xi\in[0,1]} |u^-(\xi) - v^-(\xi)|$.
(ii) $\sup_{\xi\in[0,1]} |u^+(\xi) - v^+(\xi)| = \max_{\xi\in[0,1]} |u^+(\xi) - v^+(\xi)|$.
(iii) (iii-1) (a) If $d_\infty(u,v) = |a_1 - a|$, then $d_\infty(u,v) = H([u]_0,[v]_0) = \max_{\xi\in[0,1]} H([u]_\xi,[v]_\xi)$
(b) If $d_\infty(u,v) = |a_1 - a|$, then $d_\infty(u,v) = |u^-(0) - v^-(0)| = \max_{\xi\in[0,1]} |u^-(\xi) - v^-(\xi)|$.
(iii-2) If $d_\infty(u,v) = |b_1 - b|$, then $d_\infty(u,v) = H([u]_1,[v]_1) = \max_{\xi\in[0,1]} H([u]_\xi,[v]_\xi)$ and $d_\infty(u,v) = |u^-(1) - v^-(1)| = \max_{\xi\in[0,1]} |u^-(\xi) - v^-(\xi)|$.
(iii-3) If $d_\infty(u,v) = |c_1 - c|$, then $d_\infty(u,v) = H([u]_1,[v]_1) = \max_{\xi\in[0,1]} H([u]_\xi,[v]_\xi)$ and $d_\infty(u,v) = |u^+(1) - v^+(1)| = \max_{\xi\in[0,1]} |u^+(\xi) - v^+(\xi)|$.
(iii-4) If $d_\infty(u,v) = |d_1 - d|$, then $d_\infty(u,v) = H([u]_0,[v]_0) = \max_{\xi\in[0,1]} H([u]_\xi,[v]_\xi)$ and $d_\infty(u,v) = |u^+(0) - v^+(0)| = \max_{\xi\in[0,1]} |u^+(\xi) - v^+(\xi)|$.
(iv) $d_\infty(u,v) = \max_{\xi\in[0,1]} H([u]_\xi,[v]_\xi)$.

Note that $A = |a_1 - a| = |u^-(0) - v^-(0)|$ or $A = |b_1 - b| = |u^-(1) - v^-(1)|$. Combining this and Lemma 5.1(i) yields that the supremum in (i) is attainable; that is, this supremum can be replaced by maximum. Hence (i) holds.

Note that $B = |d_1 - d| = |u^+(0) - v^+(0))|$ or $B = |c_1 - c| = |u^+(1) - v^+(1))|$. Combining this and Lemma 5.1(ii) yields that the supremum in (ii) is attainable. Hence (ii) holds.

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Note that $d_\infty(u,v) = \sup_{\xi\in[0,1]} H([u]_\xi, [v]_\xi) \geq H([u]_0, [v]_0) \geq |u^-(0) - v^-(0)| = |a_1 - a|$. So if $d_\infty(u,v) = |a_1 - a|$, then $d_\infty(u,v) = \sup_{\xi\in[0,1]} H([u]_\xi, [v]_\xi) = H([u]_0, [v]_0)$, and hence this supremum is attainable. Thus (iii-1)(a) holds.

Note that $d_\infty(u,v) = \sup_{\xi\in[0,1]}(|u^-(\xi) - v^-(\xi)| \vee |u^+(\xi) - v^+(\xi)|) \geq \sup_{\xi\in[0,1]} |u^-(\xi) - v^-(\xi)| \geq |u^-(0) - v^-(0)| = |a_1 - a|$. So if $d_\infty(u,v) = |a_1 - a|$, then $d_\infty(u,v) = \sup_{\xi\in[0,1]} |u^-(\xi) - v^-(\xi)| = |u^-(0) - v^-(0)|$, and hence this supremum is attainable. Thus (iii-1)(b) holds. So (iii-1) is proved. The proof of any of (iii-2), (iii-3) and (iii-4) is similar to that of (iii-1).

By Lemma 5.1(iii), $d_\infty(u,v)$ is equal to some of $|a_1 - a|$, $|b_1 - b|$, $|c_1 - c|$ and $|d_1 - d|$. So, by (iii), (iv) is true.

Lemma 5.4. *Let $\{u_n = (a_n, b_n, c_n, d_n) : n \in \mathbb{N}\}$ be a sequence of trapezoidal fuzzy numbers and $u = (a,b,c,d)$ a trapezoidal fuzzy number. Then the following three statements are equivalent: (i) $\lim_{n\to\infty} d_\infty(u_n, u) = 0$; (ii) $\lim_{n\to\infty} \max\{|a_n - a|, |b_n - b|, |c_n - c|, |d_n - d|\} = 0$; (iii) $\lim_{n\to\infty} a_n = a$, $\lim_{n\to\infty} b_n = b$, $\lim_{n\to\infty} c_n = c$ and $\lim_{n\to\infty} d_n = d$.*

Proof. By Lemma 5.1(iii), (i)⇔(ii). Clearly (ii)⇔(iii). So the statements (i), (ii) and (iii) are equivalent. □

Theorem 5.5. *Let $\{u_n = (a_n, b_n, c_n, d_n) : n \in \mathbb{N}\}$ be a sequence of trapezoidal fuzzy numbers and $u = (a,b,c,d)$ a trapezoidal fuzzy number. Then the following statements are equivalent.*
(i) $\lim_{n\to\infty} a_n = a$, $\lim_{n\to\infty} b_n = b$, $\lim_{n\to\infty} c_n = c$, *and* $\lim_{n\to\infty} d_n = d$.
(ii) $\lim_{n\to\infty} d_\infty(u_n, u) = 0$.
(iii) There exist ξ_1 and ξ_2 in $[0,1]$ with $\xi_1 \neq \xi_2$ satisfying

$$\lim_{n\to\infty} H([u_n]_{\xi_1}, [u]_{\xi_1}) = 0, \tag{3}$$
$$\lim_{n\to\infty} H([u_n]_{\xi_2}, [u]_{\xi_2}) = 0. \tag{4}$$

(iv) There exist ξ_1 and ξ_2 in $[0,1]$ with $\xi_1 \neq \xi_2$ satisfying

(iv-1) $\lim_{n\to\infty} (\xi_1(b_n - a_n) + a_n) = \xi_1(b - a) + a$,

(iv-2) $\lim_{n\to\infty} (\xi_2(b_n - a_n) + a_n) = \xi_2(b - a) + a$,

(iv-3) $\lim_{n\to\infty} (c_n + (1 - \xi_1)(d_n - c_n)) = c + (1 - \xi_1)(d - c)$, *and*

(iv-4) $\lim_{n\to\infty} (c_n + (1 - \xi_2)(d_n - c_n)) = c + (1 - \xi_2)(d - c)$.

Proof. By Lemma 5.4, (i)⇔(ii). Clearly (ii)⇒(iii), as for each $v, w \in \text{Trap}$, $d_\infty(v,w) = \sup_{\alpha\in[0,1]} H([v]_\alpha, [w]_\alpha)$.

By (1) and (2), for each $\xi_1 \in [0,1]$, (3) holds means that both (iv-1) and (iv-3) hold. By (1) and (2), for each $\xi_2 \in [0,1]$, (4) holds means that both (iv-2) and (iv-4) hold. So (iii)⇔(iv).

Now we show that (iv)⇒(i). (Obviously, (i)⇒(iv).) Assume that (iv) is true. Computing (iv-1)−(iv-2), we obtain (a) $\lim_{n\to\infty}(\xi_1 - \xi_2)(b_n - a_n) = (\xi_1 - \xi_2)(b - a)$. As $\xi_1 - \xi_2 \neq 0$, (a) is equivalent to (iv-5) $\lim_{n\to\infty}(b_n - a_n) = (b - a)$. Computing (iv-1)−$\xi_1$·(iv-5), we have (iv-6) $\lim_{n\to\infty} a_n = a$. Computing (iv-5)+(iv-6), we obtain $\lim_{n\to\infty} b_n = b$. Similarly, from (iv-3) and (iv-4), we can deduce that $\lim_{n\to\infty} c_n = c$ and $\lim_{n\to\infty} d_n = d$ (see also (I) below). So (i) is true. Hence (iv)⇒(i) is proved.

Thus (i), (ii), (iii) and (iv) are equivalent. This completes the proof.

(I) Computing (iv-3)−(iv-4), we obtain (b) $\lim_{n\to\infty}(\xi_2 - \xi_1)(d_n - c_n) = (\xi_2 - \xi_1)(d - c)$. As $\xi_2 - \xi_1 \neq 0$, (b) is equivalent to (iv-7) $\lim_{n\to\infty}(d_n - c_n) = (d - c)$. Computing (iv-3)−$(1 - \xi_1)$·(iv-7), we have (iv-8) $\lim_{n\to\infty} c_n = c$. Computing (iv-7)+(iv-8), we obtain $\lim_{n\to\infty} d_n = d$.

□

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Let S be a subset of $\mathbb{R}$, and $P(x)$ a statement about real numbers x. If there exists a set S_1 of measure zero such that $P(x)$ holds for all $x \in S \setminus S_1$, then we say that $P(x)$ holds almost everywhere on $x \in S$. For simplicity, "almost everywhere" is also written as "a.e.".

The result of the following Theorem 5.6 was first given in [24]. As $E \subsetneqq \widetilde{S}^1_{nc}$ (E is also written as E^1. See Page 57 of [23] for the definition of $\widetilde{S}^1_{nc}$ and the relation of E and $\widetilde{S}^1_{nc}$), the result of Theorem 5.6 is part of the result of Theorem 9.4 in [23].

Theorem 5.6. *[23,24] Suppose that u, u_n, $n = 1, 2, \ldots$ are fuzzy sets in E. Then the following statements are equivalent. (i) $\lim_{n\to\infty} H_{\text{end}}(u_n, u) = 0$. (ii) $\lim_{n\to\infty} H([u_n]_\alpha, [u]_\alpha) = 0$ holds a.e. on $\alpha \in (0,1)$. (iii) $u = \lim^{(\Gamma)}_{n\to\infty} u_n$.*

Theorem 5.7. *Let $\{u_n = (a_n, b_n, c_n, d_n) : n \in \mathbb{N}\}$ be a sequence of trapezoidal fuzzy numbers and $u = (a, b, c, d)$ a trapezoidal fuzzy number. Then the following statements are equivalent.*
(i) $\lim_{n\to\infty} d_\infty(u_n, u) = 0$.
(ii) $\lim_{n\to\infty} d_p(u_n, u) = 0$.
(iii) $\lim_{n\to\infty} H_{\text{send}}(u_n, u) = 0$.
(iv) $\lim_{n\to\infty} H_{\text{end}}(u_n, u) = 0$.
(v) $\lim_{n\to\infty} a_n = a$, $\lim_{n\to\infty} b_n = b$, $\lim_{n\to\infty} c_n = c$, *and* $\lim_{n\to\infty} d_n = d$.

Proof. By Theorem 5.5, (v)⇔(i). So to show the desired result, it suffices to show that (i)⇔(ii)⇔(iii)⇔(iv). By Corollary 4.5, to show (i)⇔(ii)⇔(iii)⇔(iv), we only need to show that (iv)⇒(i).

Suppose that (iv) holds. As Trap $\subset E$, by Theorem 5.6, (iv) means that $\lim_{n\to\infty} H([u_n]_\alpha, [u]_\alpha) = 0$ holds a.e. on $\alpha \in (0,1)$. Then there exists two distinct ξ_1 and ξ_2 in $(0,1)$ such that $\lim_{n\to\infty} H([u_n]_{\xi_1}, [u]_{\xi_1}) = 0$ and $\lim_{n\to\infty} H([u_n]_{\xi_2}, [u]_{\xi_2}) = 0$. Hence, by Theorem 5.5, (i) holds. Thus (iv)⇒(i). This completes the proof.
□

Theorem 5.8. $d_\infty \approx d_p \approx H_{\text{send}} \approx H_{\text{end}}$ (Trap).

Proof. The desired result follows immediately from Theorem 5.7.
□

Corollary 5.9. *Let $\{u_n = (a_n, b_n, c_n, d_n) : n \in \mathbb{N}\}$ be a sequence of trapezoidal fuzzy numbers and $u = (a, b, d)$ a triangular fuzzy number. Then the following statements are equivalent.*
(i) $\lim_{n\to\infty} d_\infty(u_n, u) = 0$.
(ii) $\lim_{n\to\infty} d_p(u_n, u) = 0$.
(iii) $\lim_{n\to\infty} H_{\text{send}}(u_n, u) = 0$.
(iv) $\lim_{n\to\infty} H_{\text{end}}(u_n, u) = 0$.
(v) $\lim_{n\to\infty} a_n = a$, $\lim_{n\to\infty} b_n = b$, $\lim_{n\to\infty} c_n = b$, *and* $\lim_{n\to\infty} d_n = d$.

Proof. Note that, by Remark 3.7, u is the trapezoidal fuzzy number (a, b, b, d). Thus the desired result follows immediately from Theorem 5.7.
□

Corollary 5.10. *Let $\{u_n = (a_n, b_n, c_n) : n \in \mathbb{N}\}$ be a sequence of triangular fuzzy numbers and $u = (a, b, c)$ a triangular fuzzy number. Then the following statements are equivalent.*
(i) $\lim_{n\to\infty} d_\infty(u_n, u) = 0$.
(ii) $\lim_{n\to\infty} d_p(u_n, u) = 0$.
(iii) $\lim_{n\to\infty} H_{\text{send}}(u_n, u) = 0$.
(iv) $\lim_{n\to\infty} H_{\text{end}}(u_n, u) = 0$.
(v) $\lim_{n\to\infty} a_n = a$, $\lim_{n\to\infty} b_n = b$, *and* $\lim_{n\to\infty} c_n = c$.

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Proof. Note that, by Remark 3.7, u is the trapezoidal fuzzy number (a, b, b, c) and $\{u_n\}$ is the sequence of trapezoidal fuzzy numbers $\{(a_n, b_n, b_n, c_n) : n \in \mathbb{N}\}$. Thus the desired result follows immediately from Theorem 5.7.

Clearly, the desired result also follows from Corollary 5.9.

□

6. Conclusions

In this paper we find that d_∞, d_p, H_{send} and H_{end} are equivalent metrics on the trapezoidal fuzzy numbers. Furthermore we show that convergence induced by these metrics is the convergence of the corresponding representation quadruples of the trapezoidal fuzzy numbers in $\mathbb{R}^4$.

As d_∞, d_p, H_{send} and H_{end} are commonly used metrics on the trapezoidal fuzzy numbers, the results of this paper have potential effects on the analysis and applications of the triangular fuzzy numbers and the trapezoidal fuzzy numbers.

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