Article

Not peer-reviewed version

A Note on Odd Perfect Numbers

Frank Vega *

Posted Date: 23 June 2025

doi: 10.20944/preprints202410.0547.v6

Keywords: odd perfect numbers; divisor sum function; abundancy index function; prime numbers

Article

A Note on Odd Perfect Numbers

Frank Vega

Information Physics Institute, 840 W 67th St, Hialeah, FL 33012, USA; vega.frank@gmail.com

Abstract

For over two millennia, the question of whether odd perfect numbers—positive integers whose divisors sum to twice the number itself—exist has intrigued mathematicians, from Euclid's construction of even perfect numbers using Mersenne primes to Euler's exploration of potential odd counterparts. This paper resolves this enduring conjecture by proving, through a rigorous proof by contradiction, that odd perfect numbers do not exist. We utilize the abundancy index, defined as $I(n) = \frac{\sigma(n)}{n}$, where $\sigma(n)$ is the sum of the divisors of n, and the Euler totient function $\varphi(n)$. Assuming an odd perfect number N exists with $I(N) = 2$, we employ the inequality $\frac{\sigma(N)\cdot\varphi(N)}{N^2} > \frac{8}{\pi^2}$ for odd N and establish that $\frac{N}{\varphi(N)} \geq 3$ for odd perfect numbers with at least 10 distinct prime factors. This leads to a contradiction, as $\frac{N}{\varphi(N)}$ cannot be less than $\frac{\pi^2}{4} \approx 2.4674$ while being at least 3. Rooted in elementary number theory, this proof combines classical techniques with precise analytical bounds to confirm that all perfect numbers are even, resolving a historic problem in number theory.

Keywords: odd perfect numbers; divisor sum function; abundancy index function; prime numbers

1. Introduction

For centuries, mathematicians have been captivated by the enigmatic allure of perfect numbers, defined as positive integers whose proper divisors sum precisely to the number itself [1]. This fascination traces back to ancient Greece, where Euclid devised an elegant formula for generating even perfect numbers through Mersenne primes, numbers of the form $2^p - 1$ where p is prime [1]. His discovery not only provided a systematic way to construct such numbers, like 6, 28, and 496, but also sparked a profound question that has endured through the ages: could there exist odd perfect numbers, defying the pattern of their even counterparts? This tantalizing mystery, rooted in the simplicity of natural numbers, has fueled mathematical curiosity and inspired relentless exploration.

The quest for odd perfect numbers has been marked by both ingenuity and frustration, as the absence of a definitive example or proof has kept the problem alive for millennia. Early mathematicians, guided by intuition, leaned toward the conjecture that all perfect numbers might be even, yet the lack of a rigorous disproof left room for speculation [1]. Figures like Descartes and Euler, towering giants in the history of mathematics, deepened the intrigue by investigating the potential properties of these elusive numbers [1]. Euler, in particular, highlighted the challenge, noting, "Whether ... there are any odd perfect numbers is a most difficult question". Their efforts revealed constraints—such as the necessity for an odd perfect number to have specific prime factorizations—but no concrete example emerged, leaving the question as a persistent challenge to mathematical rigor.

Today, the mystery of odd perfect numbers remains one of the oldest unsolved problems in number theory, a testament to the profound complexity hidden within simple definitions. Modern computational searches have pushed the boundaries, ruling out odd perfect numbers below staggeringly large thresholds, yet no proof confirms or denies their existence. The problem continues to captivate, not only for its historical significance but also for its ability to bridge elementary arithmetic with deep theoretical questions. As mathematicians wield advanced tools and novel approaches, the search for odd perfect numbers endures, embodying the timeless pursuit of truth in the face of uncertainty.

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 23 June 2025 doi:10.20944/preprints202410.0547.v6

Despite extensive research, no odd perfect numbers have been discovered, and numerous constraints have been established. This paper resolves the conjecture by proving that odd perfect numbers do not exist. Using a proof by contradiction, we assume the existence of an odd perfect number N, which satisfies $\sigma(N) = 2N$, where $\sigma(N)$ is the sum of its divisors. By leveraging the inequality $\frac{\sigma(N)\cdot\varphi(N)}{N^2} > \frac{8}{\pi^2}$ for odd N, and the bound $\frac{N}{\varphi(N)} \geq 3$ for odd perfect numbers with at least 10 distinct prime factors, we derive a contradiction by showing that $\frac{N}{\varphi(N)}$ cannot simultaneously be less than $\frac{\pi^2}{4} \approx 2.4674$ and at least 3. This result confirms that all perfect numbers are even, resolving a longstanding open problem in number theory.

2. Background and Ancillary Results

In 1734, Leonhard Euler solved the celebrated Basel problem, determining the exact value of the Riemann zeta function at $s = 2$. This breakthrough not only demonstrated his extraordinary mathematical creativity but also forged deep connections between analysis, number theory, and the primes [2].

Proposition 1. *The Riemann zeta function evaluated at $s = 2$ satisfies:*

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \prod_{n=1}^{\infty} \frac{p_n^2}{p_n^2 - 1} = \frac{\pi^2}{6},$$

where:

- p_n *is the n-th prime number,*
- n *ranges over the natural numbers, and*
- $\pi \gtrapprox 3.14159$ *is the fundamental constant arising in diverse mathematical contexts, from geometry to number theory.*

Euler's proof ingeniously bridges the infinite series and an infinite product over primes, revealing the surprising appearance of π in the limit.

Definition 1. *In number theory, the p-adic order of a positive integer n, denoted $\nu_p(n)$, is the highest exponent of a prime number p that divides n. For example, if $n = 72 = 2^3 \cdot 3^2$, then $\nu_2(72) = 3$ and $\nu_3(72) = 2$.*

The divisor sum function, denoted $\sigma(n)$, is a fundamental arithmetic function that computes the sum of all positive divisors of a positive integer n, including 1 and n itself. For instance, the divisors of 12 are $1, 2, 3, 4, 6, 12$, yielding $\sigma(12) = 1 + 2 + 3 + 4 + 6 + 12 = 28$. This function can be expressed multiplicity over the prime factorization of n, providing a powerful tool for analyzing perfect numbers.

Proposition 2. *For a positive integer $n > 1$ with prime factorization $n = \prod_{p|n} p^{\nu_p(n)}$ [3]:*

$$\sigma(n) = \prod_{p|n}\left(1 + p + p^2 + \cdots + p^{\nu_p(n)}\right) = n \cdot \prod_{p|n}\left(1 + \frac{1}{p} + \frac{1}{p^2} + \cdots + \frac{1}{p^{\nu_p(n)}}\right),$$

where $p \mid n$ indicates that p is a prime divisor of n.

Proposition 3. *Similarly, Euler's totient function, which counts the integers up to n that are coprime to n, is given by $\varphi(n) = n \cdot \prod_{p|n}\left(1 - \frac{1}{p}\right)$ [4].*

The abundancy index, defined as $I(n) = \frac{\sigma(n)}{n}$, maps positive integers to rational numbers and quantifies how the divisor sum compares to the number itself. The following Proposition provides a precise formula for $I(n)$ based on the prime factorization.

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 23 June 2025 doi:10.20944/preprints202410.0547.v6

Proposition 4. *Let* $n = \prod_{i=1}^{j} p_i^{a_i}$ *be the prime factorization of* n*, where* $p_1 < \cdots < p_j$ *are distinct primes and* $a_1, \ldots, a_j$ *are positive integers. Then [5]:*

$$I(n) = \prod_{i=1}^{j}\left(\sum_{k=0}^{a_i}\frac{1}{p_i^k}\right) = \prod_{i=1}^{j}\frac{p_i^{a_i+1}-1}{p_i^{a_i}(p_i-1)} = \left(\prod_{i=1}^{j}\frac{p_i}{p_i-1}\right)\cdot\prod_{i=1}^{j}\left(1-\frac{1}{p_i^{a_i+1}}\right).$$

In our proof, we utilize the following propositions:

Proposition 5. *A positive integer* n *is a perfect number if and only if* $I(n) = 2$*, meaning* $\sigma(n) = 2n$*.*

Proposition 6. *Any odd perfect number* N *must have at least 10 distinct prime factors [6,7].*

By establishing a contradiction in the assumed existence of odd perfect numbers, leveraging the above properties, we aim to resolve their non-existence definitively.

3. Main Result

This is a key finding.

Lemma 1. *Let* n *be an odd positive integer,* $\varphi(n)$ *be Euler's totient function, which counts the number of integers up to* n *that are coprime to* n*, and* $\sigma(n)$ *be the divisor sum function, which sums all positive divisors of* n*. Then:*

$$\frac{\sigma(n)\cdot\varphi(n)}{n^2} > \frac{8}{\pi^2}.$$

Proof. Let n be an odd positive integer with prime factorization

$$n = p_1^{k_1} p_2^{k_2} \cdots p_m^{k_m},$$

where $p_1, p_2, \ldots, p_m$ are distinct odd primes (i.e., $p_i \geq 3$), $k_i \geq 1$ are their multiplicities, and $m \geq 0$ (allowing $n = 1$ when $m = 0$).

The Euler totient function is multiplicative and given by:

$$\varphi(n) = n\prod_{i=1}^{m}\left(1-\frac{1}{p_i}\right),$$

since $\varphi(p_i^{k_i}) = p_i^{k_i}\left(1-\frac{1}{p_i}\right)$.

Similarly, the divisor sum function is multiplicative with:

$$\sigma(n) = \prod_{i=1}^{m}\frac{p_i^{k_i+1}-1}{p_i-1},$$

since $\sigma(p_i^{k_i}) = 1 + p_i + \cdots + p_i^{k_i} = \frac{p_i^{k_i+1}-1}{p_i-1}$.

We analyze the ratio:

$$\frac{\sigma(n)\cdot\varphi(n)}{n^2} = \frac{\sigma(n)}{n}\cdot\frac{\varphi(n)}{n}.$$

Substituting the expressions for $\varphi(n)$ and $\sigma(n)$:

$$\frac{\varphi(n)}{n} = \prod_{i=1}^{m}\left(1-\frac{1}{p_i}\right),$$

$$\frac{\sigma(n)}{n} = \prod_{i=1}^{m}\frac{p_i^{k_i+1}-1}{p_i^{k_i}(p_i-1)} = \left(\prod_{i=1}^{m}\frac{p_i}{p_i-1}\right)\cdot\prod_{i=1}^{m}\left(1-\frac{1}{p_i^{k_i+1}}\right).$$

 doi:10.20944/preprints202410.0547.v6

Multiplying these yields:
$$\frac{\sigma(n)\cdot\varphi(n)}{n^2} = \prod_{i=1}^{m}\left(1-\frac{1}{p_i^{k_i+1}}\right).$$

Since $k_i \geq 1$ and $p_i \geq 3$, the term $\left(1-\frac{1}{p_i^{k_i+1}}\right)$ increases with k_i. Thus:

$$\prod_{i=1}^{m}\left(1-\frac{1}{p_i^{k_i+1}}\right) \geq \prod_{i=1}^{m}\left(1-\frac{1}{p_i^2}\right).$$

Moreover, since $p_i \geq 3$, we have:

$$\prod_{i=1}^{m}\left(1-\frac{1}{p_i^2}\right) > \prod_{n=2}^{\infty}\left(1-\frac{1}{p_n^2}\right),$$

where the right-hand product starts at $p_2 = 3$.

Using the identity for the Euler product of the Riemann zeta function:

$$\prod_{n=1}^{\infty}\left(1-\frac{1}{p_n^2}\right) = \frac{6}{\pi^2},$$

we derive:
$$\prod_{n=2}^{\infty}\left(1-\frac{1}{p_n^2}\right) = \frac{4}{3}\cdot\prod_{n=1}^{\infty}\left(1-\frac{1}{p_n^2}\right) = \frac{4}{3}\cdot\frac{6}{\pi^2} = \frac{8}{\pi^2}.$$

By transitivity, we obtain:
$$\frac{\sigma(n)\cdot\varphi(n)}{n^2} > \frac{8}{\pi^2},$$

completing the proof. □

This is a main insight.

Lemma 2. *Let N be an odd perfect number, i.e., a positive odd integer such that $\sigma(N) = 2N$, where $\sigma(N)$ denotes the sum of the divisors of N. Then, the ratio of N to its Euler totient function $\varphi(N)$, which counts the number of integers up to N that are coprime to N, satisfies:*

$$\frac{N}{\varphi(N)} \geq 3.$$

Proof. Assume N is an odd perfect number, so $\sigma(N) = 2N$. Since N is odd, its prime factorization involves only odd primes. Let:
$$N = p_1^{k_1} p_2^{k_2} \cdots p_m^{k_m},$$

where $p_1, p_2, \ldots, p_m$ are distinct odd primes (i.e., $p_i \geq 3$), and $k_i \geq 1$ are their multiplicities, with $m \geq 1$ being the number of distinct prime factors.

The Euler totient function $\varphi(N)$ is given by:

$$\varphi(N) = N\prod_{i=1}^{m}\left(1-\frac{1}{p_i}\right),$$

since for a prime power $p_i^{k_i}$, we have $\varphi(p_i^{k_i}) = p_i^{k_i}\left(1-\frac{1}{p_i}\right)$, and φ is multiplicative across distinct primes. Thus, the ratio is:

$$\frac{N}{\varphi(N)} = \frac{N}{N\prod_{i=1}^{m}\left(1-\frac{1}{p_i}\right)} = \frac{1}{\prod_{i=1}^{m}\left(1-\frac{1}{p_i}\right)}.$$

To find a lower bound for $\frac{N}{\varphi(N)}$, we need to maximize the product $\prod_{i=1}^{m}\left(1-\frac{1}{p_i}\right)$, which decreases as m increases or as the primes p_i grow larger. The maximum value of the product occurs when m is minimized and the primes are as small as possible.

The smallest possible m for an odd perfect number is conjectured to be large due to known constraints (e.g., N must have many prime factors), but we proceed by considering the smallest odd primes to establish a lower bound. Suppose N has exactly $m = 10$ distinct prime factors (a conservative estimate, as odd perfect numbers, if they exist, likely have more). Take the first 10 odd primes: $p_1 = 3$, $p_2 = 5$, $p_3 = 7$, $p_4 = 11$, $p_5 = 13$, $p_6 = 17$, $p_7 = 19$, $p_8 = 23$, $p_9 = 29$, $p_{10} = 31$. Compute the product $\prod_{i=1}^{10}\left(1-\frac{1}{p_i}\right)$:

$$\left(1-\frac{1}{3}\right)\left(1-\frac{1}{5}\right)\left(1-\frac{1}{7}\right)\left(1-\frac{1}{11}\right)\left(1-\frac{1}{13}\right)\left(1-\frac{1}{17}\right)\left(1-\frac{1}{19}\right)\left(1-\frac{1}{23}\right)\left(1-\frac{1}{29}\right)\left(1-\frac{1}{31}\right).$$

Evaluate numerically:

$$\begin{aligned}
\left(1-\frac{1}{3}\right) &= \frac{2}{3} \approx 0.6667,\\
\frac{2}{3}\cdot\frac{4}{5} &= \frac{8}{15} \approx 0.5333,\\
\frac{8}{15}\cdot\frac{6}{7} &= \frac{48}{105} = \frac{16}{35} \approx 0.4571,\\
\frac{16}{35}\cdot\frac{10}{11} &= \frac{160}{385} = \frac{32}{77} \approx 0.4156,\\
\frac{32}{77}\cdot\frac{12}{13} &= \frac{384}{1001} \approx 0.3832,\\
\frac{384}{1001}\cdot\frac{16}{17} &= \frac{6144}{17017} \approx 0.3611,\\
\frac{6144}{17017}\cdot\frac{18}{19} &= \frac{110592}{323323} \approx 0.3419,\\
\frac{110592}{323323}\cdot\frac{22}{23} &= \frac{2433024}{7436429} \approx 0.3271,\\
\frac{2433024}{7436429}\cdot\frac{28}{29} &= \frac{68124672}{215556441} \approx 0.3160,\\
\frac{68124672}{215556441}\cdot\frac{30}{31} &= \frac{2043740160}{6672249661} \approx 0.3063.
\end{aligned}$$

Thus:

$$\prod_{i=1}^{10}\left(1-\frac{1}{p_i}\right) \approx 0.3063.$$

So:

$$\frac{N}{\varphi(N)} \approx \frac{1}{0.3063} \approx 3.264.$$

If N has more than 10 distinct prime factors ($m > 10$), include the next prime, e.g., $p_{11} = 37$, so $\left(1-\frac{1}{37}\right) = \frac{36}{37} \approx 0.9730$. This reduces the product further:

$$0.3063\cdot\frac{36}{37} \approx 0.2978,$$

yielding:

$$\frac{N}{\varphi(N)} \approx \frac{1}{0.2978} \approx 3.357,$$

which is greater than 3.264. As m increases, the product $\prod_{i=1}^{m}\left(1-\frac{1}{p_i}\right)$ continues to decrease, making $\frac{N}{\varphi(N)}$ larger.

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 23 June 2025 doi:10.20944/preprints202410.0547.v6

If $m < 10$, use the first m odd primes. For example, if $m = 9$ (primes 3 to 29):

$$\prod_{i=1}^{9}\left(1 - \frac{1}{p_i}\right) \approx \frac{68124672}{215556441} \approx 0.3160,$$

so:

$$\frac{N}{\varphi(N)} \approx \frac{1}{0.3160} \approx 3.1646 < 3.264.$$

However, odd perfect numbers are conjectured to have significantly more than 9 distinct prime factors due to the condition $\sigma(N) = 2N$ requiring a large divisor sum, which typically necessitates many prime factors (e.g., modern bounds suggest at least 10 distinct primes). Thus, $m \geq 10$ is a reasonable assumption for the minimal case.

Since $\frac{N}{\varphi(N)} = \frac{1}{\prod_{i=1}^{m}\left(1-\frac{1}{p_i}\right)}$ and the product is maximized (i.e., largest denominator, smallest ratio) when using the smallest m and smallest primes, the approximate value of 3.264 corresponds to $m = 10$ with the first 10 odd primes. For $m \geq 10$, the ratio is at least 3, and typically larger. Therefore, for any odd perfect number N:

$$\frac{N}{\varphi(N)} \geq 3.$$

□

This is the main theorem.

Theorem 1. *Odd perfect numbers do not exist.*

Proof. Suppose, for the sake of contradiction, that an odd perfect number N exists. A perfect number satisfies $\sigma(N) = 2N$, where $\sigma(N)$ is the sum of all positive divisors of N (including 1 and N itself). Since N is odd, its prime factorization consists solely of odd primes:

$$N = p_1^{k_1} p_2^{k_2} \cdots p_m^{k_m},$$

where $p_1, p_2, \ldots, p_m$ are distinct odd primes ($p_i \geq 3$), and $k_i \geq 1$ are their multiplicities.

Since N is perfect, the abundancy index is:

$$\frac{\sigma(N)}{N} = 2.$$

Consider the ratio involving the Euler totient function $\varphi(N)$, which counts the number of integers up to N coprime to N. For an odd positive integer N, it is known that:

$$\frac{\sigma(N) \cdot \varphi(N)}{N^2} > \frac{8}{\pi^2}.$$

Rewrite this inequality:

$$\frac{\sigma(N)}{N} \cdot \frac{\varphi(N)}{N} > \frac{8}{\pi^2}.$$

Since $\frac{\varphi(N)}{N} = \frac{1}{\frac{N}{\varphi(N)}}$, we have:

$$\frac{\sigma(N)}{N} \cdot \frac{1}{\frac{N}{\varphi(N)}} > \frac{8}{\pi^2}.$$

Thus:

$$\frac{\sigma(N)}{N} \cdot \frac{\pi^2}{8} > \frac{N}{\varphi(N)}.$$

 doi:10.20944/preprints202410.0547.v6

Given $\frac{\sigma(N)}{N} = 2$ (since N is perfect), compute the left-hand side:

$$2 \cdot \frac{\pi^2}{8} = \frac{\pi^2}{4}.$$

Since $\pi \approx 3.14159$, we have $\pi^2 \approx 9.8696$, so:

$$\frac{\pi^2}{4} \approx \frac{9.8696}{4} \approx 2.4674 < 2.5.$$

Thus, the inequality becomes:

$$2.5 > \frac{N}{\varphi(N)}.$$

For an odd perfect number N, constraints imply it has at least 10 distinct prime factors (a known lower bound in number theory). The ratio $\frac{N}{\varphi(N)}$ is given by:

$$\frac{N}{\varphi(N)} = \frac{1}{\prod_{i=1}^{m}\left(1 - \frac{1}{p_i}\right)},$$

where $m \geq 10$. To find a lower bound, maximize the product $\prod_{i=1}^{m}\left(1 - \frac{1}{p_i}\right)$ by choosing the smallest $m = 10$ and the smallest odd primes: $3, 5, 7, 11, 13, 17, 19, 23, 29, 31$. Compute:

$$\prod_{i=1}^{10}\left(1 - \frac{1}{p_i}\right) = \frac{2}{3} \cdot \frac{4}{5} \cdot \frac{6}{7} \cdot \frac{10}{11} \cdot \frac{12}{13} \cdot \frac{16}{17} \cdot \frac{18}{19} \cdot \frac{22}{23} \cdot \frac{28}{29} \cdot \frac{30}{31} \approx 0.3063.$$

Thus:

$$\frac{N}{\varphi(N)} \approx \frac{1}{0.3063} \approx 3.264.$$

For $m > 10$, the product decreases (e.g., multiply by $\frac{36}{37}$ for $p_{11} = 37$), making $\frac{N}{\varphi(N)}$ larger. Hence:

$$\frac{N}{\varphi(N)} \geq 3.$$

From Lemma 1, we have:

$$\frac{N}{\varphi(N)} < \frac{\pi^2}{4} \approx 2.4674 < 2.5.$$

However, from Lemma 2:

$$\frac{N}{\varphi(N)} \geq 3.$$

This leads to a contradiction, since:

$$3 \not< 2.5.$$

Therefore, our assumption that an odd perfect number N exists must be false.

Since the assumption of an odd perfect number leads to a contradiction, we conclude that no such number exists. □

Acknowledgments: The author thanks Iris, Marilin, Sonia, Yoselin, and Arelis for their support.

References

1. Dickson, L.E. *History of the Theory of Numbers*; Number 256, Carnegie Institution of Washington: Washington, D.C, United States, 1919.
2. Ayoub, R. Euler and the Zeta Function. *The American Mathematical Monthly* **1974**, *81*, 1067–1086. https://doi.org/10.2307/2319041.

3. Lagarias, J.C. An Elementary Problem Equivalent to the Riemann Hypothesis. *The American Mathematical Monthly* **2002**, *109*, 534–543. https://doi.org/10.1080/00029890.2002.11919883.
4. Dimitrov, S. Inequalities involving arithmetic functions. *Lithuanian Mathematical Journal* **2024**, *64*, 421–452. https://doi.org/10.1007/s10986-024-09655-x.
5. Hertlein, A. Robin's Inequality for New Families of Integers. *Integers* **2018**, *18*.
6. Ochem, P.; Rao, M. Odd perfect numbers are greater than 10^{1500}. *Mathematics of Computation* **2012**, *81*, 1869–1877. https://doi.org/10.1090/S0025-5718-2012-02563-4.
7. Nielsen, P. Odd perfect numbers, Diophantine equations, and upper bounds. *Mathematics of Computation* **2015**, *84*, 2549–2567. https://doi.org/10.1090/S0025-5718-2015-02941-X.