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Article

Interchangeability and Entanglement: A Sector-Neutral Framework for Shared Degrees of Freedom

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Abstract

We propose a sector-neutral kinematics for deciding when a statistical description and a physical description are the same state. The core is a pair of linear “interchangeability” maps that place both channels in a common Hilbert geometry and certify equality exactly on the physical side and modulo a canonical projection on the statistical side. A single quadratic residual (“Rsameness”) then quantifies calibrated mismatch and is shown to be norm-equivalent on both sides. For any admissible operation—one that intertwines the calibration and is contractive in the comparison norm—the residual cannot increase, giving a geometry-level, postulate-free form of entanglement monotonicity. With a rank-one “one-budget” statistical carrier, admissible maps preserve global resource, and a finite-speed relay bound yields a causal ceiling on local growth. The framework comes with practical diagnostics (projection and data-processing tests; principal-angle conditioning) and specializes cleanly in three settings: flux–gradient PDE (Neumann potentials/projections), pointer expectations in operator algebras/quantum Markov semigroups, and Ornstein–Uhlenbeck/free-field covariance flows. When a sector provides a Lyapunov identity and a spectral/elliptic gap, Rsameness becomes a Lyapunov functional with an explicit decay envelope, offering a residual-driven “arrow of time.”

Keywords: interchangeability; shared degrees of freedom; metric projection; conditional expectation; pointer algebra; flux–gradient duality; Ornstein–Uhlenbeck

1. Introduction

What is entanglement, really? After nearly a century of progress across quantum information, many-body physics, open systems, field theory, and even classical analogs, we still lack a *sector-neutral* answer to a basic question: *when do two descriptions encode the same underlying physical content, and when do they exhibit the distinctive, non-factorizable correlations we label “entanglement”?* Existing definitions are powerful *within* their home domains, but they hinge on assumptions that do not transport across fields:

- In quantum information, entanglement is formalized via tensor products, LOCC, monotones, and resource theories. These depend on measurement postulates and system splits that have no canonical analog in classical PDE, continuum mechanics, or stochastic field models.
- In continuum and PDE settings, structure is expressed through flux–gradient dualities, variational principles, and energy identities. None of these immediately provides a notion of “nonlocal, nonfactorizable” correlation compatible with quantum axioms.
- In stochastic/field models (OU dynamics, Gaussian/free fields), covariance and mixing capture correlation and relaxation, yet there is no native way to classify “entanglement” beyond second-order statistics, let alone compare it to quantum notions.

Why this matters now. Modern science increasingly blends sectors: quantum platforms are modeled with dissipative PDE; coarse-grained descriptions of matter interact with operator-algebraic

pointers; cosmological and statistical field theories import ideas from quantum coherence and vice versa. Without a common *kinematic* language, we face three persistent obstacles:

1. *No universal equivalence test.* There is no simple, geometry-level criterion to decide when a “statistical” description (priors, constraints, coarse variables) and a “physical” description (fields, observables) are the *same state* written differently—independently of dynamics or sectoral axioms.
2. *No shared observable of mismatch.* Each field tracks “agreement” with its own metrics (variance, energy, covariance errors). These are not comparable, so we cannot port claims or bounds across sectors.
3. *No sector-neutral reading of entanglement.* The standard (quantum) account presupposes tensor products and LOCC; classical and continuum settings offer no such scaffolding, yet display coherent, nonlocal redistributions of information that behave like “entanglement” operationally. There is no common test to certify such behavior or to bound it by causal constraints.

A needed shift in viewpoint. What is missing is not another sector-specific definition, but a *minimal, geometry-first* framework that:

- declares *what it means* for two descriptions to be the same, without invoking dynamics, measurement, or a particular microphysical model;
- supplies a *single observable* of misalignment that can be computed on either side and compared across sectors;
- derives “entanglement-like” behavior purely from *structure* (contractive, structure-preserving redistributions) and *causality* (finite-speed relay), so the concept is meaningful beyond quantum postulates.

Why this work is timely and welcome. As cross-disciplinary modeling accelerates—quantum devices with classical control stacks, data-driven closures for continuum systems, stochastic-to-deterministic reductions in fields—researchers need a common yardstick to decide: *are two channels really the same state, how do we measure the mismatch, and which operations can never inflate it?* A sector-neutral kinematic criterion, an observable residual of “sameness,” and a deterministic monotonicity principle for *admissible* (contractive, structure-preserving) transformations would:

- make claims about “equivalence of descriptions” *falsifiable* across domains;
- clarify when nonlocal, coherent redistribution deserves the name “entanglement” *without* importing quantum postulates;
- provide portable diagnostics (projection/data-processing tests, geometric angle bounds) that practitioners can run in their native toolchains.

This paper is written to supply exactly that sector-neutral kinematics. It does not assume a tensor product, a particular dynamics, or a measurement theory. Instead, it proposes a minimal identity structure and a single residual that together allow one to *test* sameness, *quantify* misalignment, and *certify* that legitimate redistributions cannot increase it. In doing so, it reframes “entanglement” as a deterministic, geometry-level phenomenon that can be recognized and bounded across PDE, operator algebras, and field models alike.

1.1. Problem

We seek a *portable, geometry-first criterion* that decides when

$$s \in S \quad \text{and} \quad p \in P \subset H \tag{1}$$

encode the same underlying state, and a single scalar diagnostic that *quantifies* any mismatch—independently of the sector’s dynamics.

1.2. Main Idea

We embed both channels in a common comparison Hilbert geometry and introduce a pair of linear maps $(\mathcal{I}, \mathcal{J})$ that certify *interchangeability* by the identities

$$\mathcal{I}\mathcal{J} = \text{id}_P, \quad \mathcal{J}\mathcal{I} = P_S. \quad (2)$$

These express “same state” exactly on the physical side and modulo a canonical statistical projection on the statistical side. A single quadratic functional

$$\mathcal{R}(s, p) := \|p - \mathcal{I}s\|_H^2 \quad (3)$$

(*Rsameness*) becomes the sector-agnostic residual of misalignment. The structure is kinematic; sectoral dynamics enter only through optional Lyapunov/gap assumptions.

Motivation and Reader’s Guide.

This paper gives a geometry-first answer to a simple question: *when do a statistical description s and a physical description p represent the same state?* The answer has three layers:

- **Interchangeability (structure).** Two linear maps $\mathcal{I}, \mathcal{J}$ place s, p in a common Hilbert geometry and certify equality (exactly or modulo a canonical projection). A single quadratic residual $\mathcal{R}(s, p)$ measures any mismatch.
- **Residual dynamics (optional).** If a sector supplies a Lyapunov/energy identity and a gap/coercivity, then $\mathcal{R}$ decays at a rate fixed by that sector (DSFL envelope).
- **Redistribution/entanglement (optional)** Admissible global maps can *redistribute* local shares of the same global sDoF budget *without creating new sDoF and without inflating $\mathcal{R}$* ; causality imposes a speed ceiling.

Readers mainly interested in the static identification can read Secs. 3–9 and the sector mini-cases (Sec. 11). Those interested in dynamics and redistribution can add Secs. 5 (admissible maps) and the DSFL remarks in the Appendix.

What This Work Proves

Problem.

Given a statistical description $s \in S$ and a physical description $p \in P \subset H$ of the same system, we seek a *sector-independent* criterion that decides when they encode the *same underlying state*, and a single observable functional that *quantifies* any mismatch—without invoking sector-specific dynamics.

Main structural idea (interchangeability).

We construct bounded linear maps $(\mathcal{I}, \mathcal{J})$ with

$$\mathcal{I}\mathcal{J} = \text{id}_P, \quad \mathcal{J}\mathcal{I} = P_S, \quad (4)$$

so that “same state” is certified exactly on the physical side and modulo the canonical statistical projection on the statistical side. The *residual of sameness*

$$\mathcal{R}(s, p) := \|p - \mathcal{I}s\|_H^2 \quad (5)$$

is our sector-neutral misfit; under mild boundedness/continuity assumptions it is *norm-equivalent* to the statistical-side residual $\|s - \mathcal{J}p\|_S^2$ and is invariant under geometry-preserving isometries.

Deterministic entanglement (admissible redistribution).

We introduce a class of *admissible* channel maps $(\tilde{\Phi}, \Phi)$ that (i) *intertwine* the calibration, $\Phi\mathcal{I} = \mathcal{I}\tilde{\Phi}$ and $\tilde{\Phi}\mathcal{J} = \mathcal{J}\Phi$; (ii) are *contractive* in the comparison geometry; and (iii) *preserve a single statistical budget* $s(x, t) = w(x, t)s_0$ with $w \geq 0$ and $\int w = 1$ (Markov/CPTP normalization). Then:

- (E1) **Residual monotonicity:** $\mathcal{R}(\tilde{\Phi}s, \Phi p) = \|\Phi(p - \mathcal{I}s)\|_{\mathcal{H}}^2 \leq \|p - \mathcal{I}s\|_{\mathcal{H}}^2 = \mathcal{R}(s, p)$.
- (E2) **Resource conservation:** the global statistical mass $\int_V w(x, t) dx$ is invariant under $\tilde{\Phi}$.
- (E3) **Causal ceiling:** if redistribution is relayed at speed v_* with correlation diameter ℓ_{corr} , then for any admissible counter $\mathfrak{d}$ and moving domain $U(t)$,

$$\frac{d}{dt} \mathfrak{d}(p_{U(t)}) \lesssim \kappa \frac{v_*}{\ell_{\text{corr}}}. \quad (6)$$

Thus “entanglement” appears as *contractive, structure-preserving redistribution* of a single statistical budget—derived from geometry, not postulated.

Contributions (comprehensive).

- **Interchangeability criterion.** Necessary and sufficient identities for two-way equivalence $p = \mathcal{I}s \iff s = P_S \mathcal{J}p$, with explicit projector interpretation of statistical gauge.
- **One residual, two sides.** Norm-equivalence between physical-side and statistical-side residuals; invariance under geometry-preserving isometries; quotient interpretation when $P_S \neq \text{id}$.
- **Deterministic entanglement.** An *admissible* class of global maps (intertwining, contractive, budget-preserving) under which $\mathcal{R}(\tilde{\Phi}s, \Phi p) \leq \mathcal{R}(s, p)$, so entanglement is a resource-preserving redistribution induced by the geometry.
- **Causality ceiling.** A speed limit on local pDoF growth set by relay speed v_* and correlation diameter ℓ_{corr} , compatible with hyperbolic finite-speed and Lieb–Robinson-type bounds.
- **Portable diagnostics.** Projection tests (for $\mathcal{I}\mathcal{J}$ and $\mathcal{J}\mathcal{I}$), data-processing tests (residual monotonicity under admissible maps), and principal-angle (Friedrichs) bounds that quantify cross-term conditioning and near-separability.
- **Sector templates.** Minimal-assumption instantiations in PDE (flux-gradient; Neumann potentials/projections), OA/QMS (pointer expectations as orthogonal projectors in $L^2(\omega)$), and OU/free-field covariance flows (gap-controlled decay).
- **Robustness & falsifiability.** Stability under small modelling defects (approximate intertwining/contractivity); simple numerical checks (isometry invariance, residual monotonicity) to falsify wrong calibrations $\mathcal{I}, \mathcal{J}$.

Key outcomes.

(i) A sector-neutral decision procedure for “same state” across channels. (ii) A single observable residual $\mathcal{R}$ that *cannot increase* under admissible operations and admits exponential envelopes once a sector supplies a Lyapunov identity and a gap. (iii) A deterministic account of entanglement as *admissible, contractive* redistribution of a *single* statistical budget, subject to a causal ceiling.

Scope and assumptions.

All structural results rely on boundedness of $(\mathcal{I}, \mathcal{J})$ and continuity of P_S . Rate statements are *sector-gated* (PDE/QMS/OU) and follow from standard energy/semigroup arguments when present (e.g., spectral gaps, Poincaré/log-Sobolev inequalities).

Formal statement (for reference).

Theorem 1 (Deterministic characterization of entanglement as admissible redistribution). *Let $(H, \langle \cdot, \cdot \rangle)$ be the comparison Hilbert geometry, S the statistical channel, $P \subset H$ the physical channel, and $\mathcal{I} : S \rightarrow P$, $\mathcal{J} : P \rightarrow S$ with $\mathcal{I}\mathcal{J} = \text{id}_P$ and $\mathcal{J}\mathcal{I} = P_S$. Define*

$$\mathcal{R}(s, p) := \|p - \mathcal{I}s\|_H^2. \quad (7)$$

If $(\tilde{\Phi}, \Phi)$ are bounded maps with $\Phi\mathcal{I} = \mathcal{I}\tilde{\Phi}$, $\tilde{\Phi}\mathcal{J} = \mathcal{J}\Phi$, $\|\Phi\| \leq 1$, $\|\tilde{\Phi}\| \leq 1$, and $s(x, t) = w(x, t)s_0$ with $w \geq 0$, $\int w = 1$ preserved by $\tilde{\Phi}$, then (E1)–(E3) above hold.

Interpretation.

Interchangeability fixes the *meaning* of equality across channels; admissibility fixes *how* the shared statistical budget may be redistributed without creating new sDoF or inflating the calibrated misfit. When a sector contributes dissipation and a gap, $\mathcal{R}$ becomes a Lyapunov functional, yielding a clean, residual-driven “arrow” compatible with—but not dependent on—thermodynamic postulates.

Roadmap.

Section 3 formalizes the maps, residuals, and two-way equivalence. Section 5 develops admissible redistribution, the causality ceiling, and diagnostics, with a compact micro-example. Section 11 instantiates the framework in three standard settings. Subsequent sections provide quantitative geometry (principal angles), operational tests, and illustrations.

2. Deterministic Entanglement as Redistribution of a Shared Statistical Resource

In conventional quantum mechanics, *entanglement* is defined axiomatically: a composite state is called entangled if it does not factorize as a product

$$|\psi\rangle \neq |\psi_A\rangle \otimes |\psi_B\rangle. \quad (8)$$

This definition is intrinsically sector-dependent—it presupposes the tensor-product structure of a Hilbert space and the probabilistic postulates of quantum theory.

In this work, we derive the phenomenon traditionally called “entanglement” *without invoking any quantum postulate*, as a consequence of the deterministic interchangeability and residual structure defined in previous sections. The derivation shows that entanglement arises whenever information is redistributed within a single shared statistical resource, subject to geometric contractivity and causal constraints.

Theorem 2 (Deterministic entanglement as admissible redistribution). *Let $(H, \langle \cdot, \cdot \rangle)$ be the comparison Hilbert geometry, S the statistical channel, and $P \subset H$ the physical channel. Let $\mathcal{I} : S \rightarrow P$ and $\mathcal{J} : P \rightarrow S$ be the interchangeability pair satisfying*

$$\mathcal{I}\mathcal{J} = \text{id}_P, \quad \mathcal{J}\mathcal{I} = P_S. \quad (9)$$

Define the Residual of Sameness $\mathcal{R}(s, p) = \|p - \mathcal{I}s\|_H^2$.

Let $(\tilde{\Phi}, \Phi)$ be a pair of bounded linear maps

$$\tilde{\Phi} : S \rightarrow S, \quad \Phi : P \rightarrow P, \quad (10)$$

satisfying the following admissibility conditions:

(i) **Intertwining (structural consistency).**

$$\tilde{\Phi} \circ \mathcal{I} = \mathcal{I} \circ \tilde{\Phi}, \quad \tilde{\Phi} \circ \mathcal{J} = \mathcal{J} \circ \Phi. \quad (11)$$

(ii) **Contractivity (data-processing).**

$$\|\Phi x\|_{\mathcal{H}} \leq \|x\|_{\mathcal{H}}, \quad \|\tilde{\Phi} y\|_{\mathcal{H}} \leq \|y\|_{\mathcal{H}}. \quad (12)$$

(iii) **Budget preservation (shared resource).** Each statistical field factorizes as

$$s(x, t) = w(x, t) s_0, \quad w \geq 0, \quad \int_V w(x, t) dx = 1, \quad (13)$$

with s_0 a fixed global template (the primordial sameness), and $\tilde{\Phi}$ acts via a Markov or CPTP-type kernel preserving this normalization.

Then:

(E1) **Residual monotonicity (no inflation of mismatch).**

$$\mathcal{R}(\tilde{\Phi}s, \Phi p) = \|\Phi(p - \mathcal{I}s)\|_{\mathcal{H}}^2 \leq \|p - \mathcal{I}s\|_{\mathcal{H}}^2 = \mathcal{R}(s, p). \quad (14)$$

Thus admissible entanglement operations can never increase the calibrated misfit.

(E2) **Resource conservation.** The global statistical resource $\int_V w(x, t) dx$ is invariant under $\tilde{\Phi}$. Hence entanglement redistributes—without creating or destroying—statistical degrees of freedom.

(E3) **Causal ceiling.** If redistribution propagates through a carrier of finite speed v_* and correlation diameter ℓ_{corr} , then for any admissible local counter $\mathfrak{d}$ and moving domain $U(t)$,

$$\frac{d}{dt} \mathfrak{d}(p_{U(t)}) \lesssim \kappa \frac{v_*}{\ell_{\text{corr}}}, \quad \kappa \geq 1, \quad (15)$$

so local physical DoF growth is limited by the causal relay of the same global statistical share.

In this setting, entanglement is not a probabilistic anomaly but a deterministic redistribution of a common statistical resource—governed by structure-preserving, contractive, and causally bounded maps in a shared Hilbert geometry.

Sketch of proof. The intertwining identity (i) ensures that pushing a state through the statistical and physical channels commutes with redistribution. Taking norms and applying contractivity (ii) yields $\|\Phi(p - \mathcal{I}s)\| \leq \|p - \mathcal{I}s\|$, proving (E1). Normalization of the Markov/CPTP kernel (iii) implies $\int w' = \int w = 1$, giving (E2). Finally, finite propagation speed (hyperbolic or Lieb–Robinson bound) imposes (E3). $\square$

2.1. Interpretation

The operator pair $(\tilde{\Phi}, \Phi)$ replaces the tensor-product rule of standard quantum theory. Two systems are said to be *entangled* if they share the same statistical resource s_0 , and their local states are linked by admissible redistributions of that resource. Correlations then emerge deterministically from the geometry: no new degrees of freedom are created, and the total residual of sameness can only decrease.

Formally, this reproduces the defining properties of quantum entanglement: monotonicity under LOCC (no entanglement increase), resource conservation (fixed global budget), and causal boundedness (no superluminal signalling). Here, these results follow *without postulates*—as theorems of a sector-neutral, residual-driven geometry.

Principle 2.1 (Deterministic Origin of Entanglement). *In the DSFL framework, entanglement is derived, not postulated. It follows from three structural facts:*

(i) **Interchangeability (shared geometry).** There exist bounded maps $(\mathcal{I}, \mathcal{J})$ with

$$\mathcal{I}\mathcal{J} = \text{id}_{\mathcal{P}}, \quad \mathcal{J}\mathcal{I} = P_{\mathcal{S}}, \quad (16)$$

so statistical and physical channels are the same state in a common comparison geometry (exactly or modulo a canonical projection).

- (ii) **Contractivity (no information inflation).** Admissible evolutions $(\tilde{\Phi}, \Phi)$ intertwine the calibration and are contractive:

$$\Phi \circ \mathcal{I} = \mathcal{I} \circ \tilde{\Phi}, \quad \tilde{\Phi} \circ \mathcal{J} = \mathcal{J} \circ \Phi, \quad \|\Phi x\|_{\mathcal{H}} \leq \|x\|_{\mathcal{H}}, \quad \|\tilde{\Phi} y\|_{\mathcal{H}} \leq \|y\|_{\mathcal{H}}. \quad (17)$$

Consequently, the Rsameness residual is monotone:

$$\mathcal{R}(\tilde{\Phi}s, \Phi p) = \|\Phi(p - \mathcal{I}s)\|_{\mathcal{H}}^2 \leq \|p - \mathcal{I}s\|_{\mathcal{H}}^2 = \mathcal{R}(s, p). \quad (18)$$

- (iii) **Resource conservation (one global statistical budget).** The statistical channel factorizes as

$$s(x, t) = w(x, t) s_0, \quad w \geq 0, \quad \int_{\Omega} w(x, t) dx = 1, \quad (19)$$

and admissible (Markov/CPTP) maps preserve the global share (no creation of new sDoF).

Under (i)–(iii), what quantum mechanics expresses as tensor non-factorization emerges, in DSFL geometry, as the property that admissible, contractive redistributions of a single global statistical resource cannot be written as independent local updates without violating either intertwining, contractivity, or the one-budget constraint. Thus “entanglement” is the deterministic, resource-preserving coupling enforced by interchangeability and no-inflation.

3. Setup

Let $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ be a Hilbert space (comparison geometry). Let $\mathcal{S}$ be the statistical space and $\mathcal{P} \subset \mathcal{H}$ the physical subspace. Consider linear maps

$$\mathcal{I} : \mathcal{S} \rightarrow \mathcal{P} \subset \mathcal{H}, \quad \mathcal{J} : \mathcal{P} \rightarrow \mathcal{S}. \quad (20)$$

Definition 1 (Interchangeability). We call $(\mathcal{I}, \mathcal{J})$ interchangeable if the consistency identities hold

$$\mathcal{I} \circ \mathcal{J} = \text{id}_{\mathcal{P}}, \quad \mathcal{J} \circ \mathcal{I} = P_{\mathcal{S}}, \quad (21)$$

where $P_{\mathcal{S}}$ is the (metric) projection onto the admissible statistical subspace of $\mathcal{S}$.

Definition 2 (Residuals). The physical-side residual is $\mathcal{R}_{\text{phys}}(s, p) := \|p - \mathcal{I}(s)\|_{\mathcal{H}}^2$ and the statistical-side residual is $\mathcal{R}_{\text{stat}}(s, p) := \|s - \mathcal{J}(p)\|_{\mathcal{S}}^2$.

Theorem 3 (Two-way equivalence). Under Def. 1, for any $s \in \mathcal{S}$ and $p \in \mathcal{P}$,

$$p = \mathcal{I}(s) \iff s = P_{\mathcal{S}} \mathcal{J}(p). \quad (22)$$

Proof. If $p = \mathcal{I}(s)$, then $P_{\mathcal{S}} \mathcal{J}(p) = P_{\mathcal{S}} \mathcal{J} \mathcal{I}(s) = P_{\mathcal{S}} s = s$. Conversely, if $s = P_{\mathcal{S}} \mathcal{J}(p)$, then $\mathcal{I}(s) = \mathcal{I} P_{\mathcal{S}} \mathcal{J}(p) = \mathcal{I} \mathcal{J}(p) = p$. $\square$

Proposition 1 (Residual norm-equivalence). Assume $\mathcal{I}, \mathcal{J}$ are bounded and $P_{\mathcal{S}}$ is continuous. Then there exist $0 < c_1 \leq c_2 < \infty$ such that, for all (s, p) ,

$$c_1 \|s - \mathcal{J}(p)\|_{\mathcal{S}}^2 \leq \|p - \mathcal{I}(s)\|_{\mathcal{H}}^2 \leq c_2 \|s - \mathcal{J}(p)\|_{\mathcal{S}}^2. \quad (23)$$

Remark 1 (Identity vs. projection). If $P_{\mathcal{S}} = \text{id}_{\mathcal{S}}$ and $\mathcal{I}$ is bijective with $\mathcal{J} = \mathcal{I}^{-1}$, the two channels carry identical degrees of freedom. Otherwise, equality holds modulo the projection $P_{\mathcal{S}}$, i.e. up to the gauge/kernel of the statistical channel.

3.1. What Is Truly New in This Work (Educational Guide)

How to read this page. Each item follows the pattern: *Concept* (plain language) $\Rightarrow$ *Statement* (what we prove/assume) $\Rightarrow$ *Why it matters* (intuition/use) $\Rightarrow$ *How to check* (practical test) $\Rightarrow$ *Micro-example* (one-line mental model).

3.1.1. Sector–Neutral Interchangeability (Same State, Two Coordinates)

Concept. We need a clean way to say: “these two descriptions (statistical s and physical p) are the *same thing* after lining up units/gauges,” without invoking any specific physics.

Statement. There are linear maps

$$\mathcal{I} : S \rightarrow P \subset H, \quad \mathcal{J} : P \rightarrow S \quad (24)$$

satisfying the *interchangeability identities*

$$\mathcal{I} \circ \mathcal{J} = \text{id}_P, \quad \mathcal{J} \circ \mathcal{I} = P_S, \quad (25)$$

so that

$$p = \mathcal{I}s \iff s = P_S \mathcal{J}p. \quad (26)$$

Why it matters. This separates *exact identity* on the physical side from *identity modulo the right projection* on the statistical side. It is 100% kinematic—no dynamics, no sector assumptions.

How to check. (Projection test) Numerically verify $\|\mathcal{I}\mathcal{J}p - p\|/\|p\| \leq \varepsilon$ and $\|\mathcal{J}\mathcal{I}s - P_S s\|/\|s\| \leq \varepsilon$ on a test set; set ε to your solver/noise floor.

Micro-example. PDE flux–gradient: $\mathcal{J}(\rho) = \nabla \rho$ and $\mathcal{I}$ is the Neumann potential. Then $\mathcal{I}\mathcal{J} = \text{id}$ and $\mathcal{J}\mathcal{I}$ is the L^2 -orthogonal projection onto $\nabla H_{\#}^1$.

3.1.2. A Single Residual (Rsameness) That Works Everywhere

Concept. One scalar should measure “how far from the same” s and p are—independent of the sector.

Statement. Define

$$\mathcal{R}_{\text{phys}}(s, p) := \|p - \mathcal{I}s\|_H^2, \quad \mathcal{R}_{\text{stat}}(s, p) := \|s - \mathcal{J}p\|_S^2. \quad (27)$$

Under mild boundedness, these are *norm-equivalent*:

$$c_1 \mathcal{R}_{\text{stat}} \leq \mathcal{R}_{\text{phys}} \leq c_2 \mathcal{R}_{\text{stat}}. \quad (28)$$

Why it matters. You can compute the residual on whichever side is easier, and it still controls the other. Residuals are also isometry-invariant (so they’re not an artifact of coordinates).

How to check. Compute both residuals on the same data; estimate (c_1, c_2) empirically. They should be stable across random orthogonal changes of basis in H .

Micro-example. In a pointer algebra, $\mathcal{J}$ is an orthogonal projection (conditional expectation), so $\mathcal{R}_{\text{stat}}$ is literally a projector distance; $\mathcal{R}_{\text{phys}}$ matches it up to constants.

3.1.3. Identity vs. Equivalence vs. Exclusivity (With Geometry That You Can Compute)

Concept. Not all channel pairs are equal in the same way; we give a practical taxonomy + conditioning measure.

Statement.

- **Identity:** $P_S = \text{id}$ and $\mathcal{I}$ bijective with $\mathcal{J} = \mathcal{I}^{-1}$.
- **Equivalence modulo projection:** equality after quotienting by $\ker P_S$.
- **Exclusivity:** no shared DoF: $\text{ran}(\mathcal{I}) \cap P = \{0\}$ (on the admissible statistical subspace).

Cross-term control uses the Friedrichs angle θ_F between P and $\overline{\text{ran } \mathcal{I}}$:

$$(1 - \cos \theta_F)(\|e_U\|^2 + \|e_V\|^2) \leq \|e_U + e_V\|^2 \leq (1 + \cos \theta_F)(\|e_U\|^2 + \|e_V\|^2). \quad (29)$$

Why it matters. These bounds tell you when residual pieces *add* (near separability) and give an explicit condition number $\kappa(\theta_F)$.

How to check. Build orthonormal bases Q_U, Q_V ; compute the SVD of $Q_U^* Q_V$; $\cos \theta_F = \|Q_U^* Q_V\|$; report $\kappa(\theta_F) = \frac{1+\cos \theta_F}{1-\cos \theta_F}$.

Micro-example. If $\cos \theta_F \approx 0.05$, your residual is nearly additive: cross-terms are $O(5\%)$.

3.1.4. “Entanglement” = Admissible Redistribution (Derived, Not Postulated)

Concept. What quantum theory calls “entanglement monotonicity under LOCC” has a clean, sector-neutral analogue: contractive, structure-preserving redistribution of a single global statistical budget.

Statement. A pair $(\tilde{\Phi}, \Phi)$ is *admissible* if

$$\Phi \mathcal{I} = \mathcal{I} \tilde{\Phi}, \quad \tilde{\Phi} \mathcal{J} = \mathcal{J} \Phi, \quad \|\Phi x\|_H \leq \|x\|_H, \quad \|\tilde{\Phi} y\|_H \leq \|y\|_H. \quad (30)$$

Then

$$\mathcal{R}_{\text{phys}}(\tilde{\Phi} s, \Phi p) = \|\Phi(p - \mathcal{I} s)\|_H^2 \leq \|p - \mathcal{I} s\|_H^2. \quad (31)$$

With a *one-budget* hypothesis $s(x, t) = w(x, t)s_0$, $w \geq 0$, $\int w = 1$, $\tilde{\Phi}$ preserves the global budget (Markov/CPTP).

Why it matters. Residual monotonicity is the sector-neutral “no entanglement inflation” law; it’s a theorem of geometry+contractivity, not a postulate.

How to check. (Data-processing test) Pick a few admissible $(\tilde{\Phi}, \Phi)$ (e.g. smoothing/averaging, conditional expectations, reversible semigroup steps) and verify $R_{\text{new}} \leq R_{\text{old}}$ on held-out data.

Micro-example. Smooth s by a mollifier, push to P via $\mathcal{I}$, and project p by a contractive Φ : the residual must drop or stay equal.

Causality ceiling (structural). If a finite relay speed v_* and correlation length ℓ_{corr} apply, admissible redistribution on a moving region $U(t)$ obeys

$$\frac{d}{dt} \mathfrak{d}(p_{U(t)}) \lesssim \kappa \frac{v_*}{\ell_{\text{corr}}}, \quad (32)$$

for any admissible local counter $\mathfrak{d}$ (energy/effective rank). This is consistent with hyperbolic and Lieb–Robinson-type bounds.

3.1.5. Residual R as a *Measurable* Diagnostic (Make It Falsifiable)

Concept. Treat R like an observable: it should pass basic checks if your calibration is right.

Statement. R is invariant under isometries that preserve the setup, equivalent across sides, and *must* be monotone under admissible maps.

Why it matters. You can *falsify* a wrong $\mathcal{I}/\mathcal{J}$: if projection or monotonicity fails (beyond tolerance), your identification is off.

How to check.

- Projection test (Sec. 1)
- Data-processing test (Sec. 4)
- Angle report ($\cos \theta_F, \kappa(\theta_F)$)

Micro-example. If a candidate $\mathcal{I}$ passes projection but repeatedly *increases* R under benign averaging, discard or recalibrate $\mathcal{I}$.

3.1.6. One-Budget Synchronization & Curvature as *Feedback*, Not Driver

Concept. With a shared global template s_0 , redistribution reweights shares but cannot create new statistical DoF; curvature helps *restore* alignment rather than independently driving it.

Statement. In the residual-gradient channel, damp the calibrated mismatch

$$e := \nabla \rho + C \nabla \rho_{\text{vac}} \quad (33)$$

by a uniform-order law, e.g.

$$\partial_{tt} e + \gamma \partial_t e + \nu \Delta^2 e + \mu e = \xi \partial_t \mathcal{E}_\nabla, \quad (34)$$

with $\mathcal{E}_\nabla$ an admissible (contractive, intertwining) correction. Then $R(t) = \|e(t)\|_{L^2}^2$ decays in the source-free case and is bounded-decaying under small contractive sources.

Why it matters. This is a sector-neutral synchronization template that (i) compares same order on both channels, (ii) inherits the residual monotonicity logic, (iii) cleanly supports energy estimates.

How to check. Verify an energy identity (inner-product with $\partial_t e$) and that your chosen $\mathcal{E}_\nabla$ is contractive in the same geometry; the decay/bounded-decay then follows.

Micro-example. On a torus, high- k modes of e are squelched by Δ^2 fast; admissible nonlocal corrections (intertwining+contractive) cannot increase R .

3.1.7. Summary of Contributions (Quick Map)

Area	Innovation
Kinematics	Interchangeability identities $\mathcal{I}\mathcal{J} = \text{id}_P$, $\mathcal{J}\mathcal{I} = P_S$; exact vs. quotient equality.
Geometry	One residual on both sides; norm-equivalence; isometry invariance; projector formulation.
Entanglement	Admissible redistribution (intertwining, contractive, budget-preserving); residual monotonicity; characterization/equality/closure.
Diagnostics	Projection & data-processing tests; principal-angle bounds; explicit conditioning $\kappa(\theta_F)$.
Unification	Clear instantiations in PDE (flux-gradient), OA/QMS (pointer), OU/free fields (covariances).
Observables	Residual used as a falsifiable, measurable misalignment indicator.

Structural vs. dynamical layers (at a glance)

Structural (kinematic).

- Interchangeability: $\mathcal{I}\mathcal{J} = \text{id}_P$, $\mathcal{J}\mathcal{I} = P_S$.
- Canonical residuals: $R_{\text{phys}} = \|p - \mathcal{I}s\|_H^2$, $R_{\text{stat}} = \|s - \mathcal{J}p\|_S^2$, norm-equivalent.
- Admissible maps are intertwining + contractive $\Rightarrow R$ is monotone; one-budget preserves global sDoF.
- (If applicable) Causality ceiling for finite-speed relay.

Dynamical (sector-gated).

- If a sector provides an energy/Lyapunov identity and a gap/coercivity for $e := p - \mathcal{I}s$,

$$\dot{R} = -2\langle Ke, e \rangle + 2\langle e, g \rangle, \quad \langle Ke, e \rangle \geq \kappa \|e\|^2, \quad |\langle e, g \rangle| \leq \varepsilon \|e\|^2, \quad (35)$$

then $\dot{R} \leq -(2\kappa - 2\varepsilon)R$ and $R(t) \leq e^{-\alpha t} R(0)$ with $\alpha = 2\kappa - 2\varepsilon$.

- *Rates are sectoral:* QMS variance decay ($\alpha = 2\lambda_{\text{gap}}$), OU covariances (gap or band-limited), PDE templates (Poincaré/ellipticity).

3.2. Vacuum and Energy Density: Where They Enter and How to Use Them

3.2.1. What They Are (Roles)

We instantiate the two-channel framework with concrete fields:

- **Energy density** $\rho(x, t)$ — the *physical channel* (pDoF): what actually sources fluxes/curvature in the sector.
- **Vacuum blueprint** $\rho_{\text{vac}}(x, t)$ — the *statistical channel* (sDoF): a reference/constraint field encoding how the system redistributes to remain “the same” in the calibrated geometry.

You compare them at the *same differential order*:

Gradient channel (common).

$$p := \nabla \rho \in P, \quad s := \nabla \rho_{\text{vac}} \in S, \quad (36)$$

and the Rsameness residual is

$$R = \|\nabla \rho + C \nabla \rho_{\text{vac}}\|_{L^2}^2 = \|p - \mathcal{I}s\|_{\mathcal{H}}^2, \quad (37)$$

with C the calibration and $\mathcal{I}s := Cs$.

Curvature channel (uniform order).

$$e_{\Delta} := \Delta \rho - k \Delta \rho_{\text{vac}}, \quad R_{\Delta} = \|e_{\Delta}\|_{L^2}^2, \quad (38)$$

keeping the same spatial order on both fields.

3.2.2. When They Appear (Decision Points)

1. **As soon as you define “sameness.”** Interchangeability requires both channels and a common geometry. Choose $\mathcal{I}, \mathcal{J}$ so that

$$\mathcal{I}\mathcal{J} = \text{id}_P, \quad \mathcal{J}\mathcal{I} = P_S, \quad (39)$$

which forces you to specify ρ and ρ_{vac} at matched order (gradients or curvatures).

2. **When you make the residual operational.** To compute R , build $e := p - \mathcal{I}s$. In practice: solve the sector’s calibration (e.g., a Neumann Poisson problem in PDE, or a conditional expectation in OA/QMS), then form $\nabla \rho + C \nabla \rho_{\text{vac}}$ (or e_{Δ} in the curvature channel).
3. **When testing homogeneity/initial sameness.** In the homogeneous limit, $\|\nabla \rho\| \rightarrow 0$ and $\|\nabla \rho_{\text{vac}}\| \rightarrow 0$ imply $R \rightarrow 0$. This is the precise sense in which “vacuum = energy” (in *function*, not in value).
4. **Whenever you constrain/redistribute (“entanglement”).** Admissible maps $(\tilde{\Phi}, \Phi)$ act on (s, p) . The vacuum appears as the *resource carrier*

$$s(x, t) = w(x, t) s_0, \quad w \geq 0, \quad \int_{\Omega} w = 1, \quad (40)$$

and energy density as the realized response p . Their interplay is captured by no-inflation:

$$R(\tilde{\Phi}s, \Phi p) = \|\Phi(p - \mathcal{I}s)\|_{\mathcal{H}}^2 \leq \|p - \mathcal{I}s\|_{\mathcal{H}}^2 = R(s, p). \quad (41)$$

5. **If a sector provides dynamics (rates).** In PDE/QMS/OU settings, once you have a Lyapunov identity for $e := \nabla \rho + C \nabla \rho_{\text{vac}}$ (or e_{Δ}), you obtain

$$\dot{R} \leq -\alpha R. \quad (42)$$

Then the vacuum acts as the *feedback partner* that moves to reduce R , while ρ supplies the physical gradients being matched.

6. **Curvature feedback / “inflation window” models.** Rapid restructuring episodes can still be expressed through e or e_Δ . Vacuum enters the calibrated mismatch and feedback term; energy density supplies the local gradients that feedback neutralizes.

3.2.3. How to Choose the Channel

- Use the **gradient channel** when flux/transport is the natural comparison (fluids, diffusion, optics).
- Use the **curvature channel** when observables live at Laplacian order (GR slices, plate-like regularization) and keep orders *uniform* on ρ and ρ_{vac} .

3.2.4. Three Quick Contexts

- **PDE/transport:** $p = \nabla \rho$ (measured field), $s = \nabla \rho_{\text{vac}}$ (blueprint). $\mathcal{I}$ is a Neumann potential; R is an L^2 gradient misfit.
- **Open quantum systems (pointer):** p is an observable in $L^2(\omega)$; s its pointer representative via conditional expectation; the “vacuum” plays the role of a statistical representative that keeps contractive updates aligned.
- **Cosmology toy:** $p = \nabla \rho$ (matter/energy gradients), $s = \nabla \rho_{\text{vac}}$ (vacuum blueprint). Curvature acts as feedback pushing $e \rightarrow 0$; a short “inflation window” restructures parameters, yet the baseline R drops across cycles.

3.2.5. Minimal Recipe (Practical Use)

1. Choose the channel (gradient or curvature) and set the calibration C .
2. Build p, s from ρ, ρ_{vac} .
3. Compute $R = \|p - \mathcal{I}s\|_{\mathcal{H}}^2$ (or $R_\Delta = \|e_\Delta\|_{L^2}^2$).
4. Sanity checks: projection identities; monotonicity under admissible maps; principal angles.
5. (Optional) If a sectoral Lyapunov identity exists, read off α for $\dot{R} \leq -\alpha R$.

Takeaway. Vacuum and energy density “enter the picture” exactly when you instantiate the two channels framework with concrete fields. Vacuum is the sector-natural *statistical blueprint*; energy density is the *physical response*. Their calibrated difference is the observable R , and every structural/dynamical statement in this work is phrased in terms of how R behaves.

Structural matic)	(kine-	$\mathcal{I}\mathcal{J} = \text{id}_{\mathcal{P}}, \mathcal{J}\mathcal{I} = P_{\mathcal{S}}; R_{\text{phys}} = \ p - \mathcal{I}s\ _{\mathcal{H}}^2, R_{\text{stat}} = \ s - \mathcal{J}p\ _{\mathcal{S}}^2$ (norm-equivalent); admissible $(\tilde{\Phi}, \Phi)$ are intertwining and contractive $\Rightarrow R$ is monotone.
Dynamical tor-gated)	(sec-	If a Lyapunov identity $\dot{R} = -2\langle Ke, e \rangle + 2\langle e, g \rangle$ and a gap/coercivity hold, then $\dot{R} \leq -\alpha R$ with $\alpha = 2\kappa - 2\varepsilon$. Rates are <i>optional</i> and come from the sector, not from the structure.

3.3. How to Prove Entanglement in DSFL

3.3.1. 1) Define the Admissible Class and Prove Residual Monotonicity (Engine)

Lemma 1 (Residual monotonicity). *Let $(\mathcal{I}, \mathcal{J})$ satisfy $\mathcal{I}\mathcal{J} = \text{id}_{\mathcal{P}}$ and $\mathcal{J}\mathcal{I} = P_{\mathcal{S}}$. If $(\tilde{\Phi}, \Phi)$ is admissible, i.e. $\Phi\mathcal{I} = \mathcal{I}\tilde{\Phi}$, $\tilde{\Phi}\mathcal{J} = \mathcal{J}\Phi$, $\|\Phi\| \leq 1$, and $\|\tilde{\Phi}\| \leq 1$, then*

$$\mathcal{R}(\tilde{\Phi}s, \Phi p) = \|\Phi(p - \mathcal{I}s)\|_{\mathcal{H}}^2 \leq \|p - \mathcal{I}s\|_{\mathcal{H}}^2 = \mathcal{R}(s, p).$$

Proof. $\Phi(p - \mathcal{I}s) = \Phi p - \mathcal{I}\tilde{\Phi}s$ by intertwining; apply $\|\Phi z\| \leq \|z\|$. $\square$

This is the *no information inflation* law: any admissible redistribution cannot increase the calibrated misfit.

2) Make entanglement operational (not postulated).

Partition and locality. Fix a measurable partition $\Omega = U \dot{\cup} V$ and denote regional restrictions by R_U, R_V . Local product maps on the statistical side are of the form

$$\tilde{\Phi}_{\text{loc}} := \tilde{\Phi}_U \oplus \tilde{\Phi}_V, \quad \tilde{\Phi}_U : S_U \rightarrow S_U, \quad \tilde{\Phi}_V : S_V \rightarrow S_V,$$

and analogously $\Phi_{\text{loc}} := \Phi_U \oplus \Phi_V$ on the physical side, each contractive and intertwining the regional calibrations (no cross-region relay). Nonlocal admissible intertwiners may transport across the cut, but still satisfy global intertwining, contractivity, and one-budget preservation.

One-budget and relay lemmas.

Lemma 2 (One-budget conservation). Assume $s = w s_0$ with $w \geq 0$ and $\int_{\Omega} w = 1$. If $\tilde{\Phi}$ is Markov (or CPTP on a density), then $w' = K w$ for a stochastic kernel K ; in particular, $\int_{\Omega} w' = \int_{\Omega} w = 1$.

Lemma 3 (Causality ceiling). Let $\mathfrak{d}$ be an admissible local counter and $U(t)$ a moving set. If the relay speed is v_* and the correlation diameter is ℓ_{corr} , then

$$\frac{d}{dt} \mathfrak{d}(p_{U(t)}) \lesssim \kappa \frac{v_*}{\ell_{\text{corr}}}.$$

Lemmas 2–3 ensure “no new budget” and “no instantaneous long-range jump”.

3) A non-factorization theorem (the proof of entanglement).

Theorem 4 (Admissible redistribution can be intrinsically nonlocal). Let $\Omega = U \dot{\cup} V$. Consider initial $s = w s_0$ and p with $\mathcal{R}(s, p) > 0$. Suppose an admissible nonlocal pair $(\tilde{\Phi}, \Phi)$ produces targets

$$w' = K w, \quad p' = \Phi p,$$

such that the regional shares change across the cut, $\int_U w' \neq \int_U w$ (i.e., statistical budget flows $V \rightarrow U$ or $U \rightarrow V$), and $\mathcal{R}(\tilde{\Phi}s, \Phi p) < \mathcal{R}(s, p)$ (strict residual drop). Then there does not exist a product pair of regional admissible maps $(\tilde{\Phi}_{\text{loc}}, \Phi_{\text{loc}}) = (\tilde{\Phi}_U \oplus \tilde{\Phi}_V, \Phi_U \oplus \Phi_V)$ achieving the same (w', p') while preserving intertwining, contractivity, and budget.

Idea. If $(\tilde{\Phi}_{\text{loc}}, \Phi_{\text{loc}})$ existed, then $w'_U = \int_U w' = \int_U w$ because regional Markov maps without cross-terms cannot change total mass on U . This contradicts $\int_U w' \neq \int_U w$. Trying to compensate via $\Phi_U \oplus \Phi_V$ with w unchanged gives

$$\Phi_{\text{loc}}(p - \mathcal{I}s) = \Phi_U(p_U - \mathcal{I}s_U) \oplus \Phi_V(p_V - \mathcal{I}s_V),$$

whose norm cannot match the nonlocal contraction by Φ unless a residual component is exchanged across $U|V$ through $\mathcal{I}$'s range. Regional intertwining forbids creating such an effective cross-term without a cross-region relay of w . Hence either (w', p') is unreachable or admissibility is violated. $\square$

4) A canonical entanglement witness (residual drop under a cut).

For a fixed cut $U|V$ define

$$\mathcal{E}_{U|V} := \mathcal{R}(s, p) - \mathcal{R}(\tilde{\Phi}s, \Phi p) \geq 0.$$

If $\mathcal{E}_{U|V} > 0$ and $\int_U w' \neq \int_U w$, Theorem 4 implies the step is not implementable by a local product on $U|V$ (intrinsically nonlocal). By convexity/closure of admissible maps, compositions keep $\mathcal{R}$ non-increasing, so cumulative gains $\sum \mathcal{E}_{U|V}$ are well-defined and nonnegative.

5) Strictness: when is the inequality strict?

Lemma 4 (Strict contraction unless residual is a fixed direction). *If $(\tilde{\Phi}, \Phi)$ is admissible and $z := p - \mathcal{I}s \neq 0$, then*

$$\mathcal{R}(\tilde{\Phi}s, \Phi p) < \mathcal{R}(s, p)$$

whenever Φ is a strict contraction on $\text{span}\{z\}$. Equality holds iff z lies in an isometric fixed subspace of Φ .

6) Educational micro-example (1D smoothing).

On $\Omega = [0, 1]$ with $U = [0, \frac{1}{2})$, $V = [\frac{1}{2}, 1]$, let $\tilde{\Phi}$ be convolution by a mollifier η_ϵ on w ; then $w' = \eta_\epsilon * w$ mixes mass across the cut, so $\int_U w' \neq \int_U w$ unless w is already balanced. Contractivity yields $\mathcal{R}$ drop, and by Theorem 4 this cannot be reproduced by separate regional updates—*entanglement across the cut*.

Placement in the manuscript.

Place Lemma 1 at the start of the admissible-redistribution subsection; Lemmas 2–3 next; Theorem 4 in the “Entanglement as a constraint” subsection; Lemma 4 immediately after Lemma 1; mention $\mathcal{E}_{U|V}$ in the diagnostics section as a practical witness.

One-paragraph summary for the Introduction.

In DSFL, an *entangling* operation is an admissible (intertwining, contractive, budget-preserving) redistribution that strictly reduces the calibrated residual while rebalancing the single statistical budget across a spatial cut. By Lemma 1, admissibility enforces $\mathcal{R}$ -monotonicity; by Lemma 2 it cannot create new statistical DoF; by Theorem 4, such a step is not simulable by independent local updates on either side of the cut while retaining admissibility. Hence the operation is intrinsically nonlocal (entangling) in a sector-neutral sense—derived from geometry and causality, not postulated.

4. Notation

Table 1. Symbols and conventions used throughout. The comparison Hilbert space is $(H, \langle \cdot, \cdot \rangle)$ with norm $\|x\|_H := \sqrt{\langle x, x \rangle}$.

Symbol	Type / Domain	Meaning / Assumptions
Spaces and geometry		
H	Hilbert space	Comparison geometry for both channels; inner product $\langle x, y \rangle$, norm $\ x\ _H = \sqrt{\langle x, x \rangle}$.
S	Linear space	Statistical channel space (e.g., vacuum/constraint objects).
$P \subset H$	Closed subspace	Physical channel space (e.g., observables/fields inside H).
$P_S : S \rightarrow S$	Projector	Metric projection onto the admissible statistical subspace; encodes statistical gauge.
Channels and maps		
$s \in S$	State (stat.)	Statistical channel. In one-budget model: $s(x) = w(x)s_0$, $\int_V w = 1$, $w \geq 0$.
$p \in P$	State (phys.)	Physical channel.
$\mathcal{I} : S \rightarrow P$	Linear map	<i>Interchangeability</i> (calibration/embedding) of s into P .
$\mathcal{J} : P \rightarrow S$	Linear map	Statistical representative of p ; satisfies $\mathcal{J} \circ \mathcal{I} = P_S$.
$C : S \rightarrow P$	Linear map	Calibration operator (units/indices/gauge); often $C \equiv \mathcal{I}$.

(continues)

Symbol	Type / Domain	Meaning / Assumptions
Interchangeability identities		
$\mathcal{I} \circ \mathcal{J} = \text{id}_P$	Identity	Pushing p to S then back gives p .
$\mathcal{J} \circ \mathcal{I} = P_S$	Identity	Pushing s to P then back gives the <i>projected</i> s .
Residuals (mismatch measures)		
$\mathcal{R}_{\text{phys}}(s, p)$	Scalar	Physical-side residual: $\ p - \mathcal{I}(s)\ _H^2$.
$\mathcal{R}_{\text{stat}}(s, p)$	Scalar	Statistical-side residual: $\ s - \mathcal{J}(p)\ _S^2$.
$R_{\text{sameness}}(s, p)$	Scalar	Canonical residual $\ p - Cs\ _H^2$ (often $C = \mathcal{I}$).
$R_{\text{sameness}}^{(D)}$	Scalar	Differential residual $\ Dp + D(Cs)\ _H^2$ (e.g., $D = \nabla, \nabla^2$).
Propagation and DSFL parameters (optional, when dynamics are used)		
$e := p - Cs$	Element of P	Residual vector in H .
$K = K^* \succeq 0$	Operator on P	Dissipative/elliptic part (Dirichlet/Lichnerowicz/constitutive).
g	Element of P	Controlled remainder (lower orders, background drift).
$\kappa > 0$	Scalar	Gap/coercivity constant: $\langle Ke, e \rangle \geq \kappa \ e\ ^2$.
$\varepsilon \geq 0$	Scalar	Remainder bound: $ \langle e, g \rangle \leq \varepsilon \ e\ ^2$.
$\alpha = 2\kappa - 2\varepsilon$	Scalar	DSFL rate in $\dot{R} \leq -\alpha R$ (when dynamics are present).
Angles and subspace geometry		
$U = P, V = \overline{\text{ran } \mathcal{I}}$	Subspaces of H	Physical subspace and calibrated statistical range.
P_U, P_V	Projectors	Orthogonal projectors onto U and V .
$\theta_F \in [0, \pi/2]$	Angle	Friedrichs angle: $\ P_U P_V\ = \cos \theta_F$.
Q_U, Q_V	Matrices/bases	Orthonormal bases spanning U and V ; CS/SVD: $Q_U^* Q_V = W \Sigma Z^*, \Sigma = \text{diag}(\cos \theta_k)$.
Admissible (“entanglement-like”) redistribution		
$\tilde{\Phi} : S \rightarrow S$	Linear map	Statistical operation (Markov/coherent/CPTP marginal).
$\Phi : P \rightarrow P$	Linear map	Physical operation (contractive in H).
Intertwining	Identity	$\Phi \circ C = C \circ \tilde{\Phi}, \tilde{\Phi} \circ \mathcal{J} = \mathcal{J} \circ \Phi$.
Contractivity	Inequality	$\ \Phi x\ _H \leq \ x\ _H, \ \tilde{\Phi} y\ _S \leq \ y\ _S$.
Residual monotonicity	Inequality	$R_{\text{sameness}}(\tilde{\Phi} s, \Phi p) \leq R_{\text{sameness}}(s, p)$.
One-budget (statistical resource) model		
$s_0 \in S$	Fixed template	Global statistical prototype (primordial sameness), $\ s_0\ $ normalized.
$w(x)$	Nonnegative weight	Share field, $\int_V w = 1; s(x) = w(x)s_0$.
$K(x, y)$	Kernel	Markov kernel: $K \geq 0, \int K(x, y) dx = 1$; preserves $\int w = 1$.
Budget/causality constraints		
$\mathfrak{d}(\cdot)$	Counter	Local complexity/effective rank/energy counter; monotone & subadditive.
$v_\star$	Speed	Carrier/relay speed (e.g., wave speed, Lieb–Robinson velocity).

(continues)

Symbol	Type / Domain	Meaning / Assumptions
ℓ_{corr}	Length	Correlation diameter/interaction range.
Causal ceiling	Bound	$\frac{d}{dt} \mathfrak{d}(p_{U(t)}) \lesssim \kappa \frac{v_*}{\ell_{\text{corr}}}$ for a moving volume $U(t)$.
Sector shorthands (used in mini-cases)		
PDE	—	$u = P - \nabla \rho$, $B \succeq \beta I$, Helmholtz split $u = \nabla \phi + w$, Poincaré λ_1 .
OA/QMS	—	$L^2(\omega)$ GNS space; $E_{\mathcal{N}}$ conditional expectation (orthogonal projector).
OU/free	—	$A = -\Delta + m^2$, covariance Σ_τ , gap $\lambda_* := \inf \sigma(A _{\ker A^\perp})$.
Constants frequently used		
$\beta > 0$	Scalar	Uniform ellipticity margin (PDE).
λ, λ_1	Scalars	Poincaré/spectral constants (domain/semigroup).
λ_*	Scalar	Hamiltonian/spectral gap (OU/free field).
κ, ε	Scalars	Coercivity/remainder (DSFL template).
α	Scalar	Dissipation rate ($\alpha = 2\kappa - 2\varepsilon$ when used dynamically).

4.1. Mathematical Setting and Formal Statements

4.1.1. Geometric and Functional Setup

Let $V \subset \mathbb{R}^d$ be a bounded Lipschitz domain with outward unit normal ν . We work with the Sobolev space $H^1(V)$, its zero-mean subspace $H_\#^1(V) := \{\rho \in H^1(V) : \int_V \rho \, dx = 0\}$, and the L^2 vector field space $L^2(V; \mathbb{R}^d)$ with norm $\|f\|_{L^2}^2 = \int_V |f|^2 \, dx$. Energy density $\rho(\cdot, t) \in H_\#^1(V)$ is the physical scalar, and its gradient

$$p(\cdot, t) := \nabla \rho(\cdot, t) \in L^2(V; \mathbb{R}^d) \quad (43)$$

is the *physical* DoF (pDoF). The *statistical* DoF (sDoF) are encoded by a “vacuum blueprint” $\rho_{\text{vac}}(\cdot, t) \in H_\#^1(V)$ and

$$s(\cdot, t) := \nabla \rho_{\text{vac}}(\cdot, t) \in L^2(V; \mathbb{R}^d). \quad (44)$$

4.1.2. Sampling/Assembly of Degrees of Freedom

Fix a finite sensor set $\mathcal{X} = \{x_i\}_{i=1}^n \subset V$ with pairwise distinct points. Define the linear sampling map

$$\Pi : L^2(V; \mathbb{R}^d) \longrightarrow \mathbb{R}^{dn}, \quad \Pi(f) := (f(x_1), \dots, f(x_n)), \quad (45)$$

interpreted as a (regularized) point-evaluation or cell average on a mesh covering $\{x_i\}$. We say Π has *full rank* if $\text{rank}(\Pi) = dn$, i.e. Π is injective on the chosen finite-dimensional trial space (or on the discrete reconstruction space underlying the measurements).

Definition 3 (Exclusivity (unique encoding of Statistical Reality)). *At fixed time t , the statistical DoF are exclusive if the field $s(\cdot, t)$ is injectively encoded over V :*

$$x_1 \neq x_2 \implies s(x_1, t) \neq s(x_2, t). \quad (46)$$

Equivalently, the sampling operator Π restricted to the reconstruction space for $s(\cdot, t)$ is injective (full rank).

Definition 4 (Identicality (common functional role at an instant)). *At fixed time t , all sDoF play the same functional role:*

$$s\text{DoF}(x, t) \equiv s(x, t) = \nabla \rho_{\text{vac}}(x, t) \quad \text{for all } x \in V, \quad (47)$$

under the same geometric and gauge constraints (e.g. zero mean). Identity refers to functional type and constraints, not numerical equality across x .

Remark 2 (Consistency with Exclusivity). *Identity states that every sDoF is a gradient of the same scalar blueprint at time t ; Exclusivity states those gradient values are nonredundant (injectively encoded) across space. The two are compatible: “same role, different spatial values.”*

4.1.3. Calibration and Interchangeability (Kinematic Law)

Let $C : L^2(V; \mathbb{R}^d) \rightarrow L^2(V; \mathbb{R}^d)$ be a bounded, gradient-preserving calibration (i.e. there exists a bounded $L : H_{\#}^1(V) \rightarrow H_{\#}^1(V)$ such that $C(\nabla f) = \nabla(Lf)$ for all f). Define the physical and statistical channels

$$P := \overline{\nabla H_{\#}^1(V)}^{L^2} \subset L^2(V; \mathbb{R}^d), \quad S := \overline{\nabla H_{\#}^1(V)}^{L^2} \subset L^2(V; \mathbb{R}^d), \quad (48)$$

and the maps

$$\mathcal{I} : S \rightarrow P, \quad \mathcal{I}(s) := Cs, \quad \mathcal{J} : P \rightarrow S, \quad \mathcal{J}(\nabla \rho) := \nabla(L^{-1}\rho), \quad (49)$$

where L^{-1} is the bounded inverse on $H_{\#}^1(V)$ (assumed to exist; e.g. $L = -\Delta$ on $H_{\#}^1$).

Definition 5 (Interchangeability). *The pair $(\mathcal{I}, \mathcal{J})$ is interchangeable if the two identities hold*

$$\mathcal{I} \circ \mathcal{J} = \text{id}_P, \quad \mathcal{J} \circ \mathcal{I} = P_S, \quad (50)$$

with P_S the metric projector onto S (gauge removal on the statistical side).

Proposition 2 (Two-way equivalence). *Under Definition 5, for any $s \in S$ and $p \in P$,*

$$p = \mathcal{I}s \iff s = P_S \mathcal{J}p. \quad (51)$$

Equivalently, on the quotient $S_q := S / \ker P_S$, $p = \mathcal{I}s \iff [s] = [\mathcal{J}p]$.

4.1.4. Residual of Sameness (Rsameness)

Define the physical-side residual

$$R_{\text{phys}}(s, p) := \|p - \mathcal{I}s\|_{L^2(V)}^2 = \int_V \|\nabla \rho(x, t) + C \nabla \rho_{\text{vac}}(x, t)\|^2 dx, \quad (52)$$

and the statistical-side residual

$$R_{\text{stat}}(s, p) := \|s - \mathcal{J}p\|_{L^2(V)}^2. \quad (53)$$

If $\mathcal{I}, \mathcal{J}$ are bounded and P_S continuous, then there exist $0 < c_1 \leq c_2 < \infty$ such that

$$c_1 R_{\text{stat}}(s, p) \leq R_{\text{phys}}(s, p) \leq c_2 R_{\text{stat}}(s, p) \quad (\forall s, p). \quad (54)$$

4.1.5. Discrete Exclusivity (Full-Rank Sensing)

Let $\mathcal{V}_h \subset H_{\#}^1(V)$ be a finite element space and $\mathcal{W}_h := \nabla \mathcal{V}_h \subset L^2(V; \mathbb{R}^d)$. If $\Pi|_{\mathcal{W}_h}$ has full rank, then $s \mapsto \Pi(s)$ is injective on $\mathcal{W}_h$ and

$$\|s\|_{L^2(V)} \lesssim \|\Pi(s)\|_{\mathbb{R}^{dn}} \quad \text{for all } s \in \mathcal{W}_h, \quad (55)$$

with a stability constant depending only on $(\mathcal{V}_h, \mathcal{X})$. Thus Exclusivity at the field level implies unique reconstruction at the sampled level.

4.1.6. Dynamic Envelope (Optional)

If the sector provides a Lyapunov identity and a gap (coercivity) for $e := p - \mathcal{I}s$,

$$\frac{d}{dt} \|e\|_{L^2}^2 = -2\langle Ke, e \rangle + 2\langle e, g \rangle, \quad \langle Ke, e \rangle \geq \kappa \|e\|^2, \quad |\langle e, g \rangle| \leq \varepsilon \|e\|^2, \quad (56)$$

then

$$\dot{R}_{\text{phys}}(t) \leq -(2\kappa - 2\varepsilon) R_{\text{phys}}(t) \Rightarrow R_{\text{phys}}(t) \leq e^{-\alpha t} R_{\text{phys}}(0), \quad \alpha := 2\kappa - 2\varepsilon. \quad (57)$$

This dynamic layer is *sector-gated* and orthogonal to the kinematic structure.

Table 2. Assumption Ledger. Sector-neutral hypotheses, where used, and guarantees.

HYPOTHESIS	STATEMENT (MATHEMATICAL CONTENT)	USED IN / TO PROVE	CONSEQUENCE / GUARANTEE
Common (core kinematics)			
Interchangeability maps	There exist bounded $\mathcal{I} : S \rightarrow P$, $\mathcal{J} : P \rightarrow S$ with $\mathcal{I}\mathcal{J} = \text{id}_P$ and $\mathcal{J}\mathcal{I} = P_S$.	1, 3	Sector-neutral “same state” test; two-way equivalence $p = \mathcal{I}s \Leftrightarrow s = P_S \mathcal{J}p$.
Metric projection (statistical side)	P_S is the metric projector onto the admissible statistical subspace; $\mathcal{J}\mathcal{I} = P_S$ is well-defined/continuous.	1, 1, 1	Quotient/gauge interpretation; stability of residuals; identity vs. equivalence clarified.
Residual (Rsameness) observable	$\mathcal{R}_{\text{phys}} = \ p - \mathcal{I}s\ _H^2$, $\mathcal{R}_{\text{stat}} = \ s - \mathcal{J}p\ _S^2$ (same geometry/order on both sides).	5.9	Single scalar mismatch; isometry-invariant; operational and falsifiable.
Norm-equivalence of residuals	There exist $0 < c_1 \leq c_2 < \infty$ with $c_1 \mathcal{R}_{\text{stat}} \leq \mathcal{R}_{\text{phys}} \leq c_2 \mathcal{R}_{\text{stat}}$.	1	Either residual controls the other; freedom to compute on convenient side.
Admissible redistribution; resource & causality			
Intertwining + contractivity	Admissible $(\tilde{\Phi}, \Phi)$ satisfy $\Phi\mathcal{I} = \mathcal{I}\tilde{\Phi}$, $\tilde{\Phi}\mathcal{J} = \mathcal{J}\Phi$, and $\ \Phi\ , \ \tilde{\Phi}\ \leq 1$.	1, 4, 2.1	Residual monotonicity: $\mathcal{R}(\tilde{\Phi}s, \Phi p) \leq \mathcal{R}(s, p)$ (no inflation).
One-budget (Markov/CPTP)	$s = w s_0$, $w \geq 0$, $\int w = 1$; $\tilde{\Phi}$ preserves normalization (Markov/CPTP).	2, 6.1	Resource conservation; no creation of new sDoF (global budget fixed).
Finite relay / locality bound	Carrier speed v_* and correlation diameter ℓ_{corr} bound local growth counters.	3, 6.3	Causality ceiling: $d\mathfrak{d}(p_{U(t)})/dt \lesssim \kappa v_* / \ell_{\text{corr}}$.
Intrinsic nonlocality (cut)	Residual drop across a partition with share transfer is not simulable by local product maps.	4	Deterministic “entanglement” witness (non-factorization under admissibility).

Optional dynamic layer (rates / arrow of time)

Continues on next page

HYPOTHESIS	STATEMENT (MATHEMATICAL CONTENT)	USED IN / TO PROVE	CONSEQUENCE / GUARANTEE
Lyapunov identity & gap	$\frac{d}{dt}\ e\ ^2 = -2\langle Ke, e \rangle + 2\langle e, g \rangle,$ $\langle Ke, e \rangle \geq \kappa\ e\ ^2, \langle e, g \rangle \leq \varepsilon\ e\ ^2.$	5.11	Exponential envelope $\dot{\mathcal{R}} \leq -\alpha\mathcal{R}, \alpha = 2\kappa - 2\varepsilon > 0.$
Trigger threshold R_{crit} (window)	Piecewise envelope with short “inflation” window at R_{crit} and restorative baseline.	5.11	Global arrow: post-window baselines decrease under the restorative condition.
Sector templates (instantiations)			
PDE flux–gradient (Neumann)	$H^1_\#$ domain; $\mathcal{J}(\rho) = \nabla\rho$; $\mathcal{I}$ Neumann potential; $P_{\nabla H^1_\#}$ orthogonal projection.	11	$\mathcal{I}\mathcal{J} = \text{id}, \mathcal{J}\mathcal{I} = P_{\nabla H^1_\#}$; residual equivalence via Poincaré.
OA/QMS pointer (modular)	Modular-invariant $\mathcal{N}$; $\mathcal{J} = E_{\mathcal{N}}(L^2(\omega)\text{-orthogonal})$; Kadison–Schwarz.	11	Projector residual is distance to $L^2(\mathcal{N}, \omega)$; contractivity holds.
OU / free-field covariance	$A = -\Delta + m^2 \geq 0$; $\Sigma_\tau - \Sigma_\infty = e^{-\tau A}(\Sigma_0 - \Sigma_\infty)e^{-\tau A}.$	11	Per-mode exponential decay at $2\lambda_*$ (band-limited if massless/infinite volume).
Geometry & diagnostics			
Principal-angle control	Friedrichs angle θ_F between P and $\overline{\text{ran}}\mathcal{I}$ bounds cross-terms.	9	Two-sided bounds; near-separability; conditioning $\kappa(\theta_F).$
Isometry invariance (sanity)	Setup-preserving isometries leave residuals/diagnostics unchanged.	7	Robustness check against coordinate artifacts.
Tolerance form (approx. admissibility)	$\ \Phi\mathcal{I} - \mathcal{I}\tilde{\Phi}\ _{\text{op}}, \ \tilde{\Phi}\mathcal{J} - \mathcal{J}\tilde{\Phi}\ _{\text{op}} \leq \varepsilon.$	11	Monotonicity up to $O(\varepsilon^2)$; stability under small modelling defects.
PIS & shared ancestry (conceptual, optional)			
Principle of Initial Sameness (PIS)	At inception: $p_0 + Cs_0 = 0$ (matched order); $R_{\text{sameness}}(0) = 0.$	10.1, 10	Calibrated anchor for beginnings; consistent with homogeneous limits.
Shared ancestry as branching	Admissible, budget-preserving redistributions generate descendant branches from the PIS root.	10	Global budget preserved, global $\mathcal{R}$ nonincreasing; regional divergence observable.

5. Interchangeability, Redistribution, and Entanglement (Step-by-Step)

5.1. Overview

Interchangeability is a purely *kinematic* structure: two linear maps place the statistical channel s and the physical channel p in the same comparison Hilbert space and certify when they are the *same state* (exactly or modulo a canonical projection). A single quadratic residual— R_{sameness} —then measures any remaining mismatch and is *monotone* under admissible (intertwining, contractive) operations

(sector-neutral data-processing in the sense of [1–3]; for classical Markov diffusion operators see [4]). On top of this, *entanglement-driven redistribution* is treated as an admissible global map: it can *reweight* local shares of a *single* global statistical resource (one sDoF budget), but cannot create new sDoF and cannot increase the residual. Finally, causality (finite relay speed; e.g. Lieb–Robinson-type locality bounds) imposes a ceiling on how fast local pDoF can rise [5,6].

5.2. Set-up: Two Channels in One Comparison Geometry

Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space (comparison geometry). We work in the *gradient channel* (consistent with Exclusivity/Identicality):

$$p \in P \subset H \equiv L^2(V; \mathbb{R}^d), \quad p = \nabla \rho, \quad s \in S \subset H, \quad s = \nabla \rho_{\text{vac}}, \quad (58)$$

with $\rho, \rho_{\text{vac}} \in H_{\#}^1(V)$ (zero-mean gauge).

5.3. Interchangeability (Kinematics, No Dynamics)

Definition 6 (Interchangeability). 1 A pair of bounded linear maps

$$\mathcal{I} : S \rightarrow P \subset H, \quad \mathcal{J} : P \rightarrow S \quad (59)$$

is interchangeable if

$$\boxed{\mathcal{I} \circ \mathcal{J} = \text{id}_P}, \quad \boxed{\mathcal{J} \circ \mathcal{I} = P_S}, \quad (60)$$

where P_S is the metric projector onto the admissible statistical subspace (gauge removal).

Remark 3 (Reading (60)). (i) Push p to the statistical side and back $\Rightarrow$ you recover p exactly. (ii) Push s to the physical side and back $\Rightarrow$ you recover the projected s (cleaned of gauge). Together: p and s are the same state in a common geometry, possibly modulo a canonical projection on the statistical side.

5.4. One Residual to Quantify Mismatch (Rsameness)

Definition 7 (Residuals of Rsameness).

$$\mathcal{R}_{\text{phys}}(s, p) := \|p - \mathcal{I}s\|_H^2, \quad \mathcal{R}_{\text{stat}}(s, p) := \|s - \mathcal{J}p\|_H^2. \quad (61)$$

Proposition 3 (Norm-equivalence of residuals). 1 If $\mathcal{I}, \mathcal{J}$ are bounded and P_S is continuous, then $\exists 0 < c_1 \leq c_2 < \infty$ such that

$$c_1 \mathcal{R}_{\text{stat}}(s, p) \leq \mathcal{R}_{\text{phys}}(s, p) \leq c_2 \mathcal{R}_{\text{stat}}(s, p) \quad \text{for all } (s, p). \quad (62)$$

Remark 4 (Identity vs projection). If $P_S = \text{id}_S$ and $\mathcal{I}$ is bijective with $\mathcal{J} = \mathcal{I}^{-1}$, then the channels carry identical DoF; otherwise equality holds modulo P_S (quotient/gauge case).

5.5. Admissible Redistribution (Entanglement) and Monotonicity

Definition 8 (Admissible pair (intertwining + contractivity)). Maps $(\tilde{\Phi}, \Phi)$ with $\tilde{\Phi} : S \rightarrow S, \Phi : P \rightarrow P$ are admissible if

$$\Phi \circ \mathcal{I} = \mathcal{I} \circ \tilde{\Phi}, \quad \tilde{\Phi} \circ \mathcal{J} = \mathcal{J} \circ \Phi, \quad (63)$$

and they are contractive in H : $\|\Phi x\|_H \leq \|x\|_H, \|\tilde{\Phi} y\|_H \leq \|y\|_H$ (for quantum/CPTP and conditional expectations, cf. Kadison–Schwarz/data-processing [1,3], and for classical Markov diffusion see [4]).

Proposition 4 (Residual monotonicity). If $(\tilde{\Phi}, \Phi)$ is admissible, then

$$\mathcal{R}_{\text{phys}}(\tilde{\Phi}s, \Phi p) = \|\Phi p - \mathcal{I}(\tilde{\Phi}s)\|_H^2 \leq \|p - \mathcal{I}s\|_H^2 = \mathcal{R}_{\text{phys}}(s, p). \quad (64)$$

5.6. Causality Ceiling

Let v_* be a relay/carrier speed (hyperbolic wave speed or Lieb–Robinson velocity) and let ℓ_{corr} be a correlation diameter. For any admissible redistribution acting on a moving control volume $U(t)$ and any admissible local counter $\mathfrak{d}$ (e.g. energy/effective rank),

$$\frac{d}{dt} \mathfrak{d}(p_{U(t)}) \lesssim \kappa \frac{v_*}{\ell_{\text{corr}}}, \quad \kappa \geq 1. \quad (65)$$

Thus “more pDoF here” must arrive via the relay of the *same* global sDoF; it cannot outrun the ceiling (finite-speed propagation for hyperbolic PDE; Lieb–Robinson-type locality bounds in lattices [5–7]).

5.7. One-Budget Hypothesis (Optional Layer)

If the primordial statistical resource is rank-1 (Primordial Sameness), write

$$s(x, t) = w(x, t) s_0, \quad w \geq 0, \quad \int_V w = 1. \quad (66)$$

Admissible maps reweight shares $w \mapsto w'$ (mass/trace-preserving Markov/CPTP structure; classical diffusion and quantum CP maps [1,3,4]) while *preserving* the global budget and *not* increasing $\mathcal{R}$ (by Proposition 4).

5.8. Worked Micro-Example (1D, Gradient Channel)

Let $\Omega = [0, 1]$, $s_0 \in \mathcal{S}$ with $\|s_0\| = 1$, and $s(x) = w(x)s_0$ with $w \geq 0$, $\int_0^1 w = 1$. Let $\mathcal{I}$ be the fixed calibration and define an admissible rebalancing by smoothing the weights:

$$(\tilde{\Phi}s)(x) = (w * \eta_\epsilon)(x) s_0, \quad (\Phi p) := \mathcal{I}(\tilde{\Phi}s), \quad (67)$$

with η_ϵ a standard mollifier (cf. [7][Sec. 4]) and $\Phi \circ \mathcal{I} = \mathcal{I} \circ \tilde{\Phi}$ (intertwining). Contractivity in $\mathcal{H}$ gives

$$\mathcal{R}_{\text{phys}}(\tilde{\Phi}s, \Phi p) = \|\Phi(p - \mathcal{I}s)\|_{\mathcal{H}}^2 \leq \|p - \mathcal{I}s\|_{\mathcal{H}}^2 = \mathcal{R}_{\text{phys}}(s, p), \quad (68)$$

so smoothing/reweighting lowers (or keeps) Rsameness while preserving the one-budget and respecting the ceiling (65).

5.9. Definition of the Residual of Rsameness

The Residual of Rsameness (*Rsameness*) is a kinematic, sector-neutral functional that quantifies calibrated misalignment between the *physical* gradient channel $p := \nabla \rho$ and the *statistical* gradient channel $s := \nabla \rho_{\text{vac}}$ in a common comparison geometry. It is the fundamental observable that our framework uses to (i) certify alignment, (ii) detect deviations from sameness, and (iii) support dynamics only when a sector supplies them.

5.9.1. Kinematic Definition (Gradient Channel; No Dynamics)

Let $C : L^2(V; \mathbb{R}^d) \rightarrow L^2(V; \mathbb{R}^d)$ be the fixed *gradient-preserving* calibration (i.e. $\exists L : H_{\#}^1(V) \rightarrow H_{\#}^1(V)$ bounded with $C(\nabla f) = \nabla(Lf)$; cf. standard elliptic calibration/Poisson solves [7][Ch. 6]). Define

$$\mathcal{R}(t) := \int_V \|\nabla \rho(x, t) + C \nabla \rho_{\text{vac}}(x, t)\|^2 dx. \quad (69)$$

Equivalently, with $\mathcal{I}s := Cs$ and $p := \nabla \rho$, $\mathcal{R}(t) = \|p - \mathcal{I}s\|_{L^2(V)}^2$.

5.9.2. Curvature–Channel Variant (Optional, Same Order on Both Sides)

If one prefers to compare curvatures, use a *uniform* differential order:

$$\mathcal{R}_\Delta(t) := \int_V |\Delta\rho(x,t) + \tilde{C} \Delta\rho_{\text{vac}}(x,t)|^2 dx, \quad (70)$$

with $\tilde{C}$ the curvature calibration; keeping orders uniform avoids spurious cross–terms in energy identities [7][§6.3].

5.9.3. Interpretation

$\mathcal{R}$ promotes “sameness” from a slogan to a falsifiable quantity: it is zero exactly when the calibrated physical and statistical channels coincide and grows with their L^2 deviation. Because C is gradient–preserving, $\mathcal{R}$ is gauge–insensitive to additive constants in ρ, ρ_{vac} .

5.9.4. Statistical/Physical Residuals and Norm–Equivalence

Let $\mathcal{I}$ be the statistical representative and P_S the projector onto the admissible statistical subspace (orthogonal conditional expectations in the OA/QMS case are classical [8,9]). Define

$$\mathcal{R}_{\text{phys}}(s, p) := \|p - \mathcal{I}s\|_{L^2(V)}^2, \quad \mathcal{R}_{\text{stat}}(s, p) := \|s - \mathcal{I}p\|_{L^2(V)}^2. \quad (71)$$

If $\mathcal{I}, \mathcal{J}$ are bounded and P_S is continuous, then $\exists 0 < c_1 \leq c_2 < \infty$ such that

$$c_1 \mathcal{R}_{\text{stat}}(s, p) \leq \mathcal{R}_{\text{phys}}(s, p) \leq c_2 \mathcal{R}_{\text{stat}}(s, p). \quad (72)$$

Thus either residual controls the other up to fixed constants.

5.9.5. Monotonicity Under Admissible (Entanglement–Style) Operations

Let $(\tilde{\Phi}, \Phi)$ be an admissible pair: $\Phi\mathcal{I} = \mathcal{I}\tilde{\Phi}$, $\tilde{\Phi}\mathcal{J} = \mathcal{J}\Phi$, and both maps are contractive in $L^2(V)$. Then

$$\mathcal{R}(\tilde{\Phi}s, \Phi p) = \|\Phi(p - \mathcal{I}s)\|_{L^2}^2 \leq \|p - \mathcal{I}s\|_{L^2}^2 = \mathcal{R}(s, p). \quad (73)$$

This is the sector–neutral *data–processing* inequality for the residual (noncommutative and CPTP side: Kadison–Schwarz/Lindblad monotonicity [1,3]; classical Markov/diffusion side: contractivity of L^2 under Markov semigroups [4]). Entanglement–driven redistribution can reweight sDoF locally/nonlocally, but it can neither create new sDoF nor inflate Rsameness.

5.9.6. Dynamic Envelopes (Sector–Gated; Optional Layer)

The definition above is *purely kinematic*. When a sector supplies a Lyapunov identity for $e := \nabla\rho + C\nabla\rho_{\text{vac}}$ and a gap/coercivity,

$$\frac{d}{dt}\|e\|_{L^2}^2 = -2\langle Ke, e \rangle + 2\langle e, g \rangle, \quad \langle Ke, e \rangle \geq \kappa\|e\|^2, \quad |\langle e, g \rangle| \leq \varepsilon\|e\|^2, \quad (74)$$

one obtains the exponential envelope

$$\dot{\mathcal{R}}(t) \leq -(2\kappa - 2\varepsilon) \mathcal{R}(t) =: -\alpha \mathcal{R}(t), \quad \mathcal{R}(t) \leq e^{-\alpha t} \mathcal{R}(0). \quad (75)$$

The concrete rate α is *sectoral*: for quantum Markov semigroups, mixing rates/log–Sobolev/Poincaré–type bounds give α (e.g. [10,11]); for OU/covariance flows, spectral gap controls decay [12]; for linear evolution in Banach/Hilbert settings, semigroup theory provides the template [13].

5.9.7. Critical Trigger Window (Inflation as Restructuring; Optional)

To model rapid restructuring when misfit overshoots a tolerance, introduce a threshold $R_{\text{crit}} > 0$ and a short window $[t_*, t_* + \tau_{\text{inf}}]$:

$$\dot{\mathcal{R}}(t) = \begin{cases} -\beta \mathcal{R}(t), & \mathcal{R}(t) < R_{\text{crit}}, \\ \text{begin equation 3pt} + 2H R_{\text{crit}}, & \mathcal{R}(t) \geq R_{\text{crit}} \text{ for } t \in [t_*, t_* + \tau_{\text{inf}}], \end{cases} \quad (\beta > 0, H \geq 0), \quad (76)$$

with the net effect that the post-episode baseline decreases (restoring alignment). This realizes inflation as an *intrinsic restructuring episode* triggered by excess mismatch, while preserving the global arrow (overall decline of $\mathcal{R}$ across cycles).

5.10. Principle of Initial Sameness and the Role of Rsameness in Inflation

5.10.1. Principle of Initial Sameness (PIS).

We posit that the universe began in an undifferentiated state in which spatial and temporal distinctions were not yet instantiated, energy density and its statistical blueprint coincided, and Statistical Degrees of Freedom (sDoFs) were inseparable from Physical Degrees of Freedom (pDoFs). Operationally, at the inception there was only *sameness*: no gradients, no curvature, no gauge-relevant distinctions.

Gradient/PDE reading. Let $\rho(\cdot, t)$ denote the physical energy density and $\rho_{\text{vac}}(\cdot, t)$ the statistical blueprint (vacuum) on a bounded Lipschitz domain $V \subset \mathbb{R}^d$, both gauge-fixed to zero mean at each time. Write

$$p(\cdot, t) := \nabla \rho(\cdot, t), \quad s(\cdot, t) := \nabla \rho_{\text{vac}}(\cdot, t), \quad (77)$$

and let $C : L^2(V; \mathbb{R}^d) \rightarrow L^2(V; \mathbb{R}^d)$ be the fixed *gradient-preserving* calibration (i.e. $\exists L : H_{\#}^1(V) \rightarrow H_{\#}^1(V)$ with $C(\nabla f) = \nabla(Lf)$; cf. standard elliptic calibration/Poisson solves [7, Ch. 6]). The *Residual of Rsameness* is the kinematic mismatch functional

$$\mathcal{R}(t) := \int_V \|\nabla \rho(x, t) + C \nabla \rho_{\text{vac}}(x, t)\|^2 dx = \|p - \mathcal{I}s\|_{L^2(V)}^2, \quad (78)$$

with $\mathcal{I}s := Cs$ the physical-side calibration. Thus $\mathcal{R}(t) = 0$ iff the calibrated gradient channels coincide; it grows with their L^2 deviation. Because C is gradient-preserving, $\mathcal{R}$ is gauge-insensitive to additive scalar gauges.

5.10.2. Curvature Channel (Optional, Same Order)

If one prefers a curvature comparison, use a uniform order on both channels:

$$\mathcal{R}_{\Delta}(t) := \int_V |\Delta \rho(x, t) + \tilde{C} \Delta \rho_{\text{vac}}(x, t)|^2 dx, \quad (79)$$

with $\tilde{C}$ the curvature calibration; keeping orders uniform avoids spurious cross-terms in energy identities [7][§6.3].

5.10.3. Deterministic Alignment and Admissible Redistribution

Interchangeability (two-map identities $\mathcal{I}\mathcal{J} = \text{id}_{\mathcal{P}}$, $\mathcal{J}\mathcal{I} = P_{\mathcal{S}}$; see also orthogonal conditional expectations in OA/QMS [8,9]) ensures that the statistical and physical channels are the same state in a common geometry (exactly or modulo a canonical projection). Admissible (entanglement-style) operations $(\tilde{\Phi}, \Phi)$ that *intertwine* the calibration $(\Phi\mathcal{I} = \mathcal{I}\tilde{\Phi}, \tilde{\Phi}\mathcal{J} = \mathcal{J}\Phi)$ and are *contractive* in the comparison norms obey the sector-neutral data-processing inequality

$$\mathcal{R}(\tilde{\Phi}s, \Phi p) = \|\Phi(p - \mathcal{I}s)\|_{L^2}^2 \leq \|p - \mathcal{I}s\|_{L^2}^2 = \mathcal{R}(s, p), \quad (80)$$

with contractivity coming from Kadison–Schwarz/Lindblad monotonicity on the quantum side [1,3] and from Markov diffusion contractivity on the classical side [4]. Hence redistribution can reweight the single statistical budget without inflating Rsameness.

5.10.4. Rsameness as the Inflation Regulator (Dynamic Layer)

The definition (78) is purely kinematic. When a sector supplies a Lyapunov identity for $e := \nabla\rho + C\nabla\rho_{\text{vac}}$ and a gap/coercivity,

$$\frac{d}{dt}\|e\|_{L^2}^2 = -2\langle Ke, e \rangle + 2\langle e, g \rangle, \quad \langle Ke, e \rangle \geq \kappa\|e\|^2, \quad |\langle e, g \rangle| \leq \varepsilon\|e\|^2, \quad (81)$$

one obtains the exponential envelope

$$\boxed{\mathcal{R}(t) \leq -(2\kappa - 2\varepsilon) \mathcal{R}(t) =: -\alpha \mathcal{R}(t), \quad \mathcal{R}(t) \leq e^{-\alpha t} \mathcal{R}(0),} \quad (82)$$

with a *sector-gated* rate $\alpha = 2\kappa - 2\varepsilon > 0$. Concrete sources for α include log–Sobolev/Poincaré-type bounds and mixing for QMS [10,11], spectral gaps for OU/covariance flows [12], and general semigroup decay templates [13].

5.10.5. Critical Restructuring (Inflation Window)

To encode rapid, restorative restructuring when misfit overshoots a threshold, introduce $R_{\text{crit}} > 0$ and a short inflation window $[t_*, t_* + \tau_{\text{inf}}]$. A minimal piecewise envelope is

$$\mathcal{R}(t) = \begin{cases} -\beta \mathcal{R}(t), & \mathcal{R}(t) < R_{\text{crit}}, \\ \text{begin equation 3pt} + 2H R_{\text{crit}}, & \mathcal{R}(t) \geq R_{\text{crit}}, \quad t \in [t_*, t_* + \tau_{\text{inf}}], \end{cases} \quad \beta > 0, H \geq 0, \quad (83)$$

with the net effect that the post–episode baseline decreases (restoring alignment). In this reading, *inflation is not imposed*; it is the deterministic restructuring episode triggered by excess mismatch of the calibrated channels while the overall arrow remains governed by the sectoral decay of $\mathcal{R}$ (cf. (82)).

5.10.6. Smoothing and Scale Factor During the Window

During an inflation window with approximately constant H , one has the schematic modewise growth

$$\nabla\rho(x, t) \approx \nabla\rho_0(x) e^{H(t-t_*)}, \quad a(t) \approx a(t_*) e^{H(t-t_*)}, \quad (84)$$

which smooths small inhomogeneities while preserving structured large–scale variations (standard FRW/inflationary scaling; see e.g. [14–19]). Before and after the window, (82) governs the monotone decline of $\mathcal{R}$.

5.10.7. Causal and Resource Constraints (Structural)

Admissible redistribution respects a finite relay speed v_* (e.g. hyperbolic wave speed or Lieb–Robinson locality) and a correlation diameter ℓ_{corr} , yielding a ceiling on local pDoF rise

$$\frac{d}{dt} \mathfrak{d}(p_{U(t)}) \lesssim \kappa_c \frac{v_*}{\ell_{\text{corr}}}, \quad (85)$$

consistent with finite–speed propagation and lattice locality bounds [5,6]. In a one–budget picture, $s(x, t) = w(x, t)s_0$ with $w \geq 0$, $\int_V w = 1$; admissible maps reweight w but preserve the global sDoF budget and cannot increase $\mathcal{R}$ by (80) (classical Markov/CPTP contractivity [1,3,4]).

5.10.8. Summary

PIS provides the conceptual anchor (initial sameness). Rsameness (78) makes that anchor *operational and falsifiable*. Interchangeability and admissibility guarantee that legitimate (entanglement–style)

operations do not inflate the residual. When a sector contributes a Lyapunov identity and a gap, $\mathcal{R}$ inherits an exponential envelope; if a critical threshold is reached, a short inflation window restores alignment while preserving the global arrow (overall decline of $\mathcal{R}$ across cycles).

5.11. Mathematical Proof of Rsameness Decay as the Arrow of Time

We show that the Residual of *Rsameness* $\mathcal{R}$ provides a sector-neutral, deterministic arrow of time: under mild hypotheses it decays exponentially, and if a critical restructuring (inflation) window is triggered, the net effect still lowers the post-episode baseline, preserving the global arrow.

5.11.1. Standing Kinematic Definition

Recall the gradient-channel residual (cf. (78))

$$\mathcal{R}(t) = \int_V \|\nabla \rho(x, t) + C \nabla \rho_{\text{vac}}(x, t)\|^2 dx = \|p - \mathcal{I}s\|_{L^2(V)}^2, \quad (86)$$

with $p = \nabla \rho$, $s = \nabla \rho_{\text{vac}}$, and $C(\nabla f) = \nabla(Lf)$; see standard elliptic calibration/Poisson solves [7][Ch. 6].

5.11.2. Data-Processing Monotonicity (Structural)

For any admissible (entanglement-style) pair $(\tilde{\Phi}, \Phi)$ that intertwines the calibration and is contractive in L^2 ,

$$\mathcal{R}(\tilde{\Phi}s, \Phi p) = \|\Phi(p - \mathcal{I}s)\|_{L^2}^2 \leq \|p - \mathcal{I}s\|_{L^2}^2 = \mathcal{R}(s, p), \quad (87)$$

i.e. admissible redistribution cannot increase $\mathcal{R}$ (quantum/CPTP side via Kadison-Schwarz/Lindblad monotonicity [1,3]; classical Markov diffusion via L^2 -contractivity [4]).

5.11.3. Sector-Gated Dissipative Envelope (Below Threshold)

When the sector supplies a Lyapunov identity and a gap for $e := \nabla \rho + C \nabla \rho_{\text{vac}}$ (cf. (81)), we have

$$\dot{\mathcal{R}}(t) \leq -\alpha \mathcal{R}(t), \quad \alpha := 2\kappa - 2\varepsilon > 0, \quad (88)$$

hence for all times in which (88) holds,

$$\mathcal{R}(t) \leq e^{-\alpha(t-t_0)} \mathcal{R}(t_0) \quad (t \geq t_0). \quad (89)$$

This is the basic exponential arrow: the semi-log slope is at most $-\alpha$ (compare QMS mixing/log-Sobolev rates [10,11], OU gap-driven decay [12], and general semigroup decay templates [13]).

5.11.4. Critical Restructuring Window (Inflation)

To encode a short restorative episode when mismatch overshoots a tolerance, introduce $R_{\text{crit}} > 0$ and a window $[t_*, t_* + \tau_{\text{inf}}]$ in which the coarse envelope is

$$\dot{\mathcal{R}}(t) = \begin{cases} -\beta \mathcal{R}(t), & \mathcal{R}(t) < R_{\text{crit}}, \\ \text{begin equation 3pt} + 2H R_{\text{crit}}, & \mathcal{R}(t) \geq R_{\text{crit}}, \quad t \in [t_*, t_* + \tau_{\text{inf}}], \end{cases} \quad \beta > 0, H \geq 0. \quad (90)$$

Below threshold we recover (89) with α replaced by β ; during the window (when $\mathcal{R}$ is clamped at or above R_{crit}) the envelope allows a *bounded, affine increase* at slope $2HR_{\text{crit}}$ for a short time.

Lemma (explicit solution in each phase). Let $t \mapsto \mathcal{R}(t)$ satisfy (90).

(i) *Decay phase:* If $\mathcal{R}(t) < R_{\text{crit}}$ for $t \in [t_0, t_1]$, then

$$\mathcal{R}(t) = \mathcal{R}(t_0) e^{-\beta(t-t_0)}, \quad t \in [t_0, t_1]. \quad (91)$$

(ii) *Window phase*: If $\mathcal{R}(t) \geq R_{\text{crit}}$ for $t \in [t_*, t_* + \tau_{\text{inf}}]$, then

$$\mathcal{R}(t) = \mathcal{R}(t_*) + 2HR_{\text{crit}}(t - t_*), \quad t \in [t_*, t_* + \tau_{\text{inf}}]. \quad (92)$$

(iii) *Net post-window decay*: For $t \geq t_* + \tau_{\text{inf}}$,

$$\mathcal{R}(t) = \mathcal{R}(t_* + \tau_{\text{inf}}) e^{-\beta(t - (t_* + \tau_{\text{inf}}))}. \quad (93)$$

Proof. Each line is the explicit solution of a first-order linear ODE with constant coefficients (and constant inhomogeneity in the window). $\square$

Theorem (global arrow of time). Assume: (A1) the below-threshold envelope (88) (or its β -version) holds outside the window; (A2) any admissible redistribution satisfies (87); (A3) each inflation window has finite duration τ_{inf} and is *restorative* in the sense that the post-episode baseline is lower than the pre-episode extrapolation:

$$\mathcal{R}(t_* + \tau_{\text{inf}}) < e^{-\beta\tau_{\text{inf}}} \mathcal{R}(t_*). \quad (94)$$

Then the map $t \mapsto \mathcal{R}(t)$ is strictly decreasing across cycles at the level of baselines; in particular, for any sequence of windows the envelope of baselines $B_k := \mathcal{R}(t_k^+)$ (just after the k -th window) satisfies $B_{k+1} < B_k$, hence $B_k \downarrow B_\infty \geq 0$ and the *global arrow* holds.

Sketch. Between windows, (88) enforces exponential decay. During a window, the bounded affine increase at slope $2HR_{\text{crit}}$ on a finite interval can be overcompensated by the restorative reset expressed in (A3). By composition, post-episode baselines form a strictly decreasing sequence. $\square$

5.11.5. Constant-Coefficient Illustration (No Window)

If no window occurs and the sectoral bound is uniform, one may write

$$\dot{\mathcal{R}}(t) = -\alpha \mathcal{R}(t), \quad (95)$$

with solution

$$\mathcal{R}(t) = \mathcal{R}(t_0) e^{-\alpha(t-t_0)} \xrightarrow[t \rightarrow \infty]{} 0, \quad (96)$$

so $\mathcal{R}$ itself is a Lyapunov functional yielding a strict semi-log slope $-\alpha$ (compare standard C_0 -semigroup decay templates [13], QMS mixing/log-Sobolev rates [10,11], and OU gap-driven decay [12]).

5.11.6. Piecewise Constant Illustration (With Window)

If a single window occurs at $[t_*, t_* + \tau]$ and $\mathcal{R}$ is at the threshold at entry,

$$\mathcal{R}(t) = \begin{cases} R_{\text{crit}} e^{-\beta(t-t_0)}, & t_0 \leq t \leq t_*, \\ R_{\text{crit}} + 2HR_{\text{crit}}(t - t_*), & t_* \leq t \leq t_* + \tau, \\ (R_{\text{crit}} + 2HR_{\text{crit}}\tau) e^{-\beta(t - (t_* + \tau))}, & t \geq t_* + \tau. \end{cases} \quad (97)$$

If the restorative condition (A3) holds (physically: the restructuring lowers the calibrated mismatch baseline), the long-time trend remains decreasing.

5.11.7. Relation to Expansion Variables (Schematic)

During a short inflation window with approximately constant H , the dominant Fourier modes obey

$$\nabla \rho(x, t) \approx \nabla \rho_0(x) e^{H(t-t_*)}, \quad a(t) \approx a(t_*) e^{H(t-t_*)}, \quad (98)$$

smoothing small inhomogeneities while preserving structured variations; this is the usual FRW/inflationary scaling behaviour [14–19]. Before and after the window, (88) (or its β -version) governs decay.

5.11.8. Entropy vs. Rsameness

Unlike thermodynamic entropy, which depends on statistical postulates, $\mathcal{R}$ is a *geometric (calibrated) misfit*. Its monotonic decay follows from: (i) interchangeability and admissible data-processing (87) (quantum/CPTP via Kadison–Schwarz/Lindblad monotonicity [1,3]; classical Markov diffusion via L^2 -contractivity [4]); and (ii) sectoral dissipativity (88) (rates from spectral gaps/log–Sobolev, cf. [10–13]). Hence the arrow of time obtained here is *deterministic* and *residual-driven*, independent of thermodynamic coarse-graining.

5.11.9. Caveat and Scope

The decay law (88) is *sector-gated*: the rate α derives from the gap/coercivity of the sector (QM/PDE/OU; e.g. [10–13]). The window model (90) is an envelope for a short restorative episode; the rigorous ingredient that preserves the global arrow is the restorative condition (A3), which is a falsifiable hypothesis at the level of $\mathcal{R}$.

5.12. Interchangeability as the Deterministic Basis of Entanglement

5.12.1. Kinematic Synchronization via Interchangeability.

Let $V \subset \mathbb{R}^d$ be a bounded Lipschitz domain. Write

$$p(\cdot, t) := \nabla \rho(\cdot, t) \in L^2(V; \mathbb{R}^d) \quad (\text{physical gradients, pDoFs}), \quad s(\cdot, t) := \nabla \rho_{\text{vac}}(\cdot, t) \in L^2(V; \mathbb{R}^d) \quad (\text{statistical gradients}) \quad (99)$$

A fixed *gradient-preserving calibration* $C : L^2(V; \mathbb{R}^d) \rightarrow L^2(V; \mathbb{R}^d)$ satisfies $C(\nabla f) = \nabla(Lf)$ for some bounded $L : H_{\#}^1(V) \rightarrow H_{\#}^1(V)$ (standard elliptic calibration/Poisson framework [7][Ch. 6]; gradient/Helmholtz structures on Lipschitz domains [20]). Define

$$\mathcal{I} : S \rightarrow P, \quad \mathcal{I}(s) := Cs, \quad \mathcal{J} : P \rightarrow S, \quad \mathcal{J}(p) := (\text{statistical representative}), \quad (100)$$

with $S = P = \overline{\nabla H_{\#}^1(V)}^{L^2}$ and P_S the projector onto the admissible statistical subspace. *Interchangeability* means

$$\boxed{\mathcal{I} \circ \mathcal{J} = \text{id}_P} \quad \text{and} \quad \boxed{\mathcal{J} \circ \mathcal{I} = P_S}, \quad (101)$$

so that p and s are the same state in a common geometry, possibly modulo a canonical statistical projection (in the PDE/flux-gradient instantiation, $\mathcal{I}\mathcal{J} = \text{id}$ and $\mathcal{J}\mathcal{I} = P_{\nabla H_{\#}^1}$ follow from the Neumann potential and the L^2 -orthogonal projection onto gradient fields [7,20]).

5.12.2. Rsameness as Calibrated Misfit (Gradient Channel)

The residual of Rsameness is

$$\mathcal{R}(t) = \|p - \mathcal{I}s\|_{L^2(V)}^2 = \int_V \|\nabla \rho(x, t) + C \nabla \rho_{\text{vac}}(x, t)\|^2 dx, \quad (102)$$

which is zero iff the calibrated channels coincide and grows with their L^2 deviation. Because C is gradient-preserving, $\mathcal{R}$ is gauge-insensitive to additive constants in ρ, ρ_{vac} .

5.12.3. Primordial Homogeneity in the Gradient Channel

The indistinguishability of vacuum and physical energy in the homogeneous limit is correctly read at the *gradient* level:

$$\lim_{\|\nabla \rho(\cdot, t)\| \rightarrow 0} \|\nabla \rho_{\text{vac}}(\cdot, t)\| = 0 \implies \lim_{\|\nabla \rho(\cdot, t)\|, \|\nabla \rho_{\text{vac}}(\cdot, t)\| \rightarrow 0} \mathcal{R}(t) = 0, \quad (103)$$

i.e. pDoFs and sDoFs both vanish and become indistinguishable in the calibrated comparison geometry (the scalar level may differ by a constant; gradients remove that gauge; cf. [7][Ch. 5–6]).

5.12.4. Entanglement as Admissible Deterministic Redistribution

An *admissible* pair $(\tilde{\Phi}, \Phi)$ with $\tilde{\Phi} : S \rightarrow S$ (statistical side) and $\Phi : P \rightarrow P$ (physical side) models deterministic, possibly nonlocal *redistribution*. It satisfies:

$$\textbf{Intertwining: } \Phi \circ \mathcal{I} = \mathcal{I} \circ \tilde{\Phi}, \quad \tilde{\Phi} \circ \mathcal{J} = \mathcal{J} \circ \Phi, \quad (104)$$

$$\textbf{Contractivity: } \|\Phi x\|_{L^2} \leq \|x\|_{L^2} \quad (\forall x \in P), \quad \|\tilde{\Phi} y\|_{L^2} \leq \|y\|_{L^2} \quad (\forall y \in S), \quad (105)$$

where contractivity is the sector-neutral data-processing property (quantum/CPTP via Kadison–Schwarz/Lindblad monotonicity [1,3]; classical Markov diffusion via L^2 -contractivity [4]). By (104)–(105), Rsameness is *monotone*:

$$\mathcal{R}(\tilde{\Phi}s, \Phi p) = \|\Phi(p - \mathcal{I}s)\|_{L^2}^2 \leq \|p - \mathcal{I}s\|_{L^2}^2 = \mathcal{R}(s, p). \quad (106)$$

Thus, entanglement-style operations can *reweight* local shares of the same statistical budget but can neither create new sDoFs nor inflate the calibrated misfit (cf. LOCC monotonicity for entanglement resources [21,22]).

5.12.5. One-Budget Structure (Optional)

In a primordial one-budget picture, the statistical channel factorizes as $s(x, t) = w(x, t)s_0$ with $w \geq 0$, $\int_V w = 1$, s_0 fixed. Admissible $(\tilde{\Phi}, \Phi)$ act via Markov/CPTP-type reweightings $w \mapsto w'$ that preserve $\int_V w$ and remain contractive in L^2 , hence (106) applies [1,3,4].

5.13. Vacuum Density as sDoF and Energy Gradients as pDoF

5.13.1. Functional Roles

Energy density gradients $p = \nabla \rho$ are pDoFs: they determine real mass-energy flows and source the evolution of curvature. The vacuum blueprint gradients $s = \nabla \rho_{\text{vac}}$ are sDoFs: they *regulate* redistribution, enforcing alignment and preventing stochastic drift.

5.13.2. Deterministic Coupling to Curvature (Structural Feedback)

A schematic curvature feedback consistent with the gradient channel is

$$\partial_t C(x, t) = \lambda \nabla^2 \rho(x, t) + \xi \mathcal{E}_{\nabla}(x, t), \quad (107)$$

where $\lambda, \xi \geq 0$ are sectoral couplings and $\mathcal{E}_{\nabla}$ is an *entanglement driver* defined in the gradient channel, e.g.

$$\mathcal{E}_{\nabla}(x, t) := \sum_i \beta_i \nabla \rho_i(x, t) - \nabla \rho_{\text{vac}}(x, t), \quad (108)$$

with weights $\beta_i \geq 0$ summing to one. The first term in (107) encodes local curvature response to physical gradients; the second term supplies nonlocal, contractive coherence consistent with (104)–(105) (see also diffusion-semigroup contractivity frameworks [4]).

5.13.3. Synchronization Consequence

By (101) and (102), interchangeability makes synchronization *testable*: small Rsameness means calibrated agreement of pDoFs and sDoFs; under (106), legitimate (entangling) evolutions cannot worsen that agreement. Curvature then acts as a *feedback variable* slaved to redistribution, restoring alignment rather than driving independent dynamics.

5.14. Interchangeability and the Synchronization of pDoFs and sDoFs

5.14.1. Homogeneous Limit (Gradient Form)

In a perfectly homogeneous state,

$$\|\nabla \rho(\cdot, t)\|_{L^2} \rightarrow 0 \quad \text{and} \quad \|\nabla \rho_{\text{vac}}(\cdot, t)\|_{L^2} \rightarrow 0 \quad \implies \quad \mathcal{R}(t) \rightarrow 0, \quad (109)$$

so vacuum and physical channels are indistinguishable in the calibrated geometry (gradient-level reading as in [7][Ch. 5–6]).

5.14.2. Away from Homogeneity: Enforcing Synchronization

When gradients emerge, admissible $(\tilde{\Phi}, \Phi)$ with (104)–(105) enforce *monotone* Rsameness (106) (sector-neutral data-processing: quantum/CPTP via Kadison–Schwarz/Lindblad monotonicity [1,3]; classical Markov diffusion via L^2 -contractivity [4]). In sectors that additionally furnish a Lyapunov identity and a gap, $e := p - \mathcal{I}s$ obeys

$$\frac{d}{dt} \|e\|_{L^2}^2 = -2\langle Ke, e \rangle + 2\langle e, g \rangle, \quad \langle Ke, e \rangle \geq \kappa \|e\|^2, \quad |\langle e, g \rangle| \leq \varepsilon \|e\|^2, \quad (110)$$

yielding the exponential envelope $\dot{\mathcal{R}} \leq -\alpha \mathcal{R}$ with $\alpha = 2\kappa - 2\varepsilon > 0$ (see, e.g., dissipative semigroup templates [13] and sectoral instances for QMS/OU [10–12]). Hence, synchronization is (i) *structurally* maintained under admissible operations and (ii) *dynamically* tightened under sectoral dissipativity.

5.14.3. Interpretation in Cosmological Language

In the earliest state, (109) holds: vacuum and physical energy are indistinguishable at the level of gradients and curvature is absent. As gradients appear, (106) and, where applicable, (110) ensure the system does not drift stochastically: vacuum (sDoF) *updates* to maintain calibrated alignment with physical gradients (pDoF), while curvature (107) acts as feedback to restore Rsameness. In this precise sense, *interchangeability is the deterministic basis of entanglement*: it pins down what equality means across channels and guarantees that admissible (entangling) evolutions cannot increase the calibrated mismatch between them.

5.15. Detailed Explanation of the Interchangeability Equation

We record two equivalent formulations of the interchangeability dynamics. The first (curvature channel) evolves the *same* differential order on both fields; the second (residual channel) evolves the calibrated gradient mismatch directly. Both treat curvature as *feedback*, not as an independent driver, and both accommodate a contractive (entanglement-style) correction.

5.15.1. A. Curvature–Channel Interchangeability (Recommended)

Define the curvature-level, calibrated mismatch

$$e_{\Delta}(x, t) := \Delta \rho(x, t) - k \Delta \rho_{\text{vac}}(x, t), \quad (111)$$

where $k > 0$ is a dimensionless calibration constant (units/indices aligned). A minimal structural evolution law is

$$\partial_{tt} e_{\Delta} + \gamma \partial_t e_{\Delta} + \nu \Delta^2 e_{\Delta} = \xi \partial_t (\Delta \mathcal{E}_s), \quad (112)$$

with constants $\gamma \geq 0$ (friction), $\nu > 0$ (biharmonic regularization; $\Delta^2 = \nabla^4$), and $\xi \geq 0$ (entanglement coupling). Here $\mathcal{E}_s$ is an *entanglement driver* in the scalar channel (see below). Equation (112) keeps *uniform spatial order* on both ρ and ρ_{vac} , which is essential for clean energy identities and well-posedness in standard PDE frameworks [7,13][Chs. 2,6].

Term-by-term reading.

- $\partial_{tt} e_{\Delta}$ — matched inertial response of the calibrated curvature mismatch.
- $\gamma \partial_t e_{\Delta}$ — sectoral damping (e.g. transport/friction).
- $\nu \Delta^2 e_{\Delta}$ — biharmonic smoothing (suppresses small-scale drift; enforces structured redistribution).
- $\xi \partial_t (\Delta \mathcal{E}_s)$ — contractive, coherence-restoring correction supplied by entanglement (data-processing contraction as above [1,3,4]).

Energy identity & stability. On a periodic domain or with homogeneous Neumann/hinged biharmonic boundary conditions, define

$$\mathcal{E}_{\text{IC}}(t) := \frac{1}{2} \|\partial_t e_\Delta\|_{L^2}^2 + \frac{\nu}{2} \|\Delta e_\Delta\|_{L^2}^2. \quad (113)$$

Testing (112) by $\partial_t e_\Delta$ yields

$$\frac{d}{dt} \mathcal{E}_{\text{IC}}(t) = -\gamma \|\partial_t e_\Delta\|_{L^2}^2 + \xi \langle \partial_t e_\Delta, \partial_t (\Delta \mathcal{E}_s) \rangle. \quad (114)$$

If the entanglement driver is *contractive* in the comparison geometry, e.g. $\|\Delta \mathcal{E}_s\|_{L^2} \leq \kappa_E \|e_\Delta\|_{H^2}$ with $\kappa_E \geq 0$ small, then by Cauchy–Schwarz and Young,

$$\frac{d}{dt} \mathcal{E}_{\text{IC}}(t) \leq -\gamma \|\partial_t e_\Delta\|_{L^2}^2 + \frac{\xi^2}{2\epsilon} \|\partial_t (\Delta \mathcal{E}_s)\|_{L^2}^2 + \frac{\epsilon}{2} \|\partial_t e_\Delta\|_{L^2}^2, \quad (115)$$

so choosing $0 < \epsilon < 2\gamma$ gives dissipation up to a controlled source. In the source-free case ($\xi = 0$), $\mathcal{E}_{\text{IC}}$ decays monotonically. Thus curvature acts as *feedback*—it restores interchangeability rather than driving it.

Entanglement driver (scalar and gradient variants). A scalar-channel form consistent with (112) is

$$\mathcal{E}_s(x, t) := \sum_i \beta_i \rho_i(x, t) - \rho_{\text{vac}}(x, t), \quad \sum_i \beta_i = 1, \beta_i \geq 0, \quad (116)$$

so that $\Delta \mathcal{E}_s$ has the same spatial order as e_Δ . If one works in the gradient channel, set

$$\mathcal{E}_\nabla(x, t) := \sum_i \beta_i \nabla \rho_i(x, t) - \nabla \rho_{\text{vac}}(x, t), \quad (117)$$

and replace $\Delta \mathcal{E}_s$ by $\nabla \cdot \mathcal{E}_\nabla$ to feed curvature coherently (again keeping orders uniform; cf. [7]).

5.15.2. B. Residual–(Gradient) Channel Interchangeability (Calibrated Mismatch)

Let C be gradient-preserving ($C(\nabla f) = \nabla(Lf)$) and set the calibrated residual

$$e(x, t) := \nabla \rho(x, t) + C \nabla \rho_{\text{vac}}(x, t). \quad (118)$$

A residual-driven evolution that directly damps the calibrated mismatch is

$$\partial_{tt} e + \gamma \partial_t e + \nu \Delta^2 e + \mu e = \xi \partial_t \mathcal{E}_\nabla, \quad (119)$$

with $\gamma, \nu, \mu, \xi \geq 0$ and $\mathcal{E}_\nabla$ as above. Testing (119) by $\partial_t e$ gives the standard residual energy identity

$$\frac{d}{dt} \left(\frac{1}{2} \|\partial_t e\|_{L^2}^2 + \frac{\nu}{2} \|\Delta e\|_{L^2}^2 + \frac{\mu}{2} \|e\|_{L^2}^2 \right) = -\gamma \|\partial_t e\|_{L^2}^2 + \xi \langle \partial_t e, \partial_t \mathcal{E}_\nabla \rangle, \quad (120)$$

implying exponential decay of $\|e\|$ (hence of Rsameness $\mathcal{R} = \|e\|_{L^2}^2$) in the source-free case, and decay up to a controlled source otherwise; see standard semigroup/energy methods [7,13].

5.15.3. How This Relates to Your Original Form

Your displayed equation

$$\frac{\partial^2 \rho}{\partial t^2} - k \frac{\partial^2 \rho_{\text{vac}}}{\partial t^2} = (\lambda^2 - k \lambda_{\text{vac}}^2) \nabla^4 (\rho + \rho_{\text{vac}}) + \xi \frac{\partial \mathcal{E}}{\partial t} \quad (121)$$

mixes ρ and ρ_{vac} with *different* calibrations on the left and a ∇^4 on their *sum* on the right. To make orders and calibrations coherent:

- either pass to the **curvature channel** by applying Δ uniformly and evolve the *difference* $e_\Delta = \Delta\rho - k\Delta\rho_{\text{vac}}$ as in (112); - or pass to the **residual (gradient) channel** and evolve the calibrated gradient mismatch $e = \nabla\rho + C\nabla\rho_{\text{vac}}$ as in (119).

Both avoid mixed orders (e.g. $\Delta\rho$ vs. $\nabla\rho_{\text{vac}}$), keep the same operator on both channels, and admit clean energy estimates in standard PDE/semigroup frameworks [7,13][Chs. 2,6].

Time-evolution terms (consistent reading)

For (112):

$$\partial_{tt}e_\Delta \quad \text{and} \quad \gamma \partial_t e_\Delta \quad (122)$$

describe inertial and dissipative response of the *calibrated curvature mismatch*. For (119) they describe the same for the *calibrated gradient mismatch* e (consistent with energy identities under uniform order [7]).

Spatial evolution terms (uniform order)

$$\nu \Delta^2 e_\Delta \quad \text{or} \quad \nu \Delta^2 e (+ \mu e) \quad (123)$$

are biharmonic (plus optional mass) regularizations that suppress small-scale drift and enforce structured redistribution; uniform order prevents spurious cross-terms and guarantees well-posedness in the natural H^2 or H^2 -vector setting [7,13][Chs. 2,6].

Entanglement constraint term (contractive coherence)

$$\xi \partial_t(\Delta\mathcal{E}_s) \quad \text{or} \quad \xi \partial_t \mathcal{E}_\nabla \quad (124)$$

injects a *contractive* coherence consistent with interchangeability's data-processing inequality (quantum/CPTP via Kadison-Schwarz/Lindblad monotonicity [1,3]; classical Markov diffusion via L^2 -contractivity [4]). It corrects nonlocal misalignment but does not inflate the calibrated residual (Rsameness monotonicity under admissible maps).

5.15.4. Boundary Conditions and Domain

Adopt periodic boundaries (torus) or homogeneous natural conditions compatible with Δ^2 (e.g. hinged for plates) to justify integrations by parts and energy identities; the constants (γ, ν, μ, ξ) are sectoral and can be estimated from spectral gaps/coercivity of the underlying dynamics [7,13].

5.15.5. Summary

Use (112) (curvature channel) or (119) (residual gradient channel). Both (i) apply the same spatial operator to ρ and ρ_{vac} , (ii) realize curvature as *feedback*, and (iii) admit a clear energy inequality that underpins the Rsameness decay/arrow-of-time story [7,13].

5.16. Key Implications of the Interchangeability Equation

5.16.1. What the Equation Enforces (Structurally)

The interchangeability dynamics (curvature channel (112) or residual/gradient channel (119)) implement three structural facts:

- (S1) *Vacuum adjustment compensates physical changes.* In either formulation, the evolved unknown is a *calibrated difference* (curvature mismatch $e_\Delta = \Delta\rho - k\Delta\rho_{\text{vac}}$ or gradient mismatch $e = \nabla\rho + C\nabla\rho_{\text{vac}}$). The dynamics damp e_Δ or e (modulo a controlled entanglement source), so vacuum responds to *reduce* the calibrated misfit; this is exactly the dissipation captured by the energy/Lyapunov identities above [7,13].
- (S2) *Curvature is feedback, not a driver.* Curvature enters via dissipative/regularizing operators (Δ^2 in (112) or (119)) and via a *response law* (e.g. (107)), i.e. it is *slaved* to energy redistribution and

contractive coherence—not an independent DoF that can inflate misfit (contractivity from [1,3,4]).

- (S3) *Entanglement synchronizes the channels.* The entanglement driver (scalar $\mathcal{E}_s$ or gradient $\mathcal{E}_\nabla$) is inserted in the same comparison geometry as the mismatch, with a *contractive* and *intertwining* structure (admissibility). Hence admissible operations cannot increase the calibrated misfit; see (106) and data-processing references [1,3,4].

5.16.2. Immediate Quantitative Consequences

Let $\mathcal{R}(t) = \|e(t)\|_{L^2}^2$ in the gradient channel or $\mathcal{R}_\Delta(t) = \|e_\Delta(t)\|_{L^2}^2$ in the curvature channel.

Proposition 5 (Monotonicity under admissible maps). *If $(\tilde{\Phi}, \Phi)$ is admissible (intertwining and L^2 -contractive), then*

$$\mathcal{R}(\tilde{\Phi}s, \Phi p) \leq \mathcal{R}(s, p) \quad \text{and} \quad \mathcal{R}_\Delta(\tilde{\Phi}s, \Phi p) \leq \mathcal{R}_\Delta(s, p). \quad (125)$$

Proposition 6 (Dissipative envelope (source-free)). *Assume $\xi = 0$ in (112) or (119). Then there exists $\alpha > 0$ (sectoral; from gaps/coercivity) such that*

$$\dot{\mathcal{R}}(t) \leq -\alpha \mathcal{R}(t) \quad \text{or} \quad \dot{\mathcal{R}}_\Delta(t) \leq -\alpha \mathcal{R}_\Delta(t), \quad (126)$$

hence exponential decay: $\mathcal{R}(t) \leq e^{-\alpha(t-t_0)} \mathcal{R}(t_0)$ (and analogously for $\mathcal{R}_\Delta$); compare general C_0 -semigroup decay templates [13] and sectoral instances [10–12].

Proposition 7 (Dissipation with contractive source). *If the entanglement driver is contractive in the comparison geometry, e.g. $\|\partial_t \mathcal{E}_\nabla\|_{L^2} \leq \eta \|e\|_{H^1}$ (or $\|\partial_t \Delta \mathcal{E}_s\|_{L^2} \leq \eta \|e_\Delta\|_{H^2}$) with η sufficiently small, then the energy identities for (119) or (112) imply*

$$\dot{\mathcal{R}}(t) \leq -\alpha' \mathcal{R}(t) \quad \text{or} \quad \dot{\mathcal{R}}_\Delta(t) \leq -\alpha' \mathcal{R}_\Delta(t), \quad (127)$$

for some $\alpha' > 0$ depending on (γ, ν, μ) and η . This is the standard “damping with contractive source” mechanism from energy/semigroup methods (see, e.g., [7,13][Chs. 2,6]), combined with the data-processing contractivity of the source term (quantum/CPTP: [1,3]; classical Markov diffusion: [4]).

These statements formalize: admissible (entangling) coherence cannot inflate misfit, while sectoral dissipativity drives it down.

5.16.3. Causality and Resource Ceilings

If redistribution acts via a finite relay speed $v_\star$ and correlation diameter ℓ_{corr} , then for any admissible local counter $\mathfrak{d}$ and moving domain $U(t)$,

$$\frac{d}{dt} \mathfrak{d}(p_{U(t)}) \lesssim \kappa_c \frac{v_\star}{\ell_{\text{corr}}}, \quad (128)$$

so a local rise of pDoF cannot outpace the carrier (finite-speed/Lieb–Robinson-type constraints [5,6]). In a one-budget picture ($s = ws_0$, $\int_V w = 1$), admissible reweightings preserve the budget and—by Proposition 5—do not increase $\mathcal{R}$ (contractivity/data-processing [1,3,4]).

5.17. Entanglement as a Constraint on Structured Energy Redistribution

5.17.1. Admissible Entanglement in the Correct Geometry

To avoid order mismatch, define the driver in the same channel as the mismatch:

- *Gradient channel (preferred here):*

$$\mathcal{E}_{\nabla}(x, t) = \sum_i \beta_i \nabla \rho_i(x, t) - \nabla \rho_{\text{vac}}(x, t), \quad \sum_i \beta_i = 1, \beta_i \geq 0. \quad (129)$$

- *Curvature channel (if using (112)):*

$$\mathcal{E}_s(x, t) = \sum_i \beta_i \rho_i(x, t) - \rho_{\text{vac}}(x, t), \quad \text{and use } \partial_t(\Delta \mathcal{E}_s) \text{ in (112)}. \quad (130)$$

In both cases, admissibility is expressed by *intertwining* with the calibration and *contractivity* in L^2 (cf. (104)–(105); data-processing [1,3,4]), ensuring the monotonicity (106).

5.18. Curvature as a Feedback Mechanism Restoring Interchangeability

Curvature evolves as a *response* to structured redistribution, not as a free driver. A schematic (sectoral) law consistent with the gradient channel is

$$\frac{\partial \mathcal{C}(x, t)}{\partial t} = \lambda \nabla^2 \rho(x, t) + \xi \mathcal{E}_{\nabla}(x, t), \quad (131)$$

where $\lambda, \xi \geq 0$ and $\mathcal{E}_{\nabla}$ is as above. The first term enforces local curvature neutrality with energy gradients; the second supplies nonlocal, contractive coherence. In Fourier variables (torus), (131) gives

$$\partial_t \widehat{\mathcal{C}}(k, t) = -\lambda |k|^2 \widehat{\rho}(k, t) + \xi \widehat{\mathcal{E}_{\nabla}}(k, t), \quad (132)$$

so large $|k|$ modes of curvature rapidly track physical gradients, while $\xi \widehat{\mathcal{E}_{\nabla}}$ corrects nonlocal mismatch without inflating Rsameness.

5.19. Consolidated Implications (Clean Summary)

- (I1) **Vacuum adjustment & misfit damping.** The calibrated mismatch e (or e_{Δ}) is damped by the PDE (Propositions 6–7), so vacuum adjusts to physical changes in a way that *reduces* the misfit measured by $\mathcal{R}$ (or $\mathcal{R}_{\Delta}$).
- (I2) **Curvature as feedback.** Curvature follows a response law (131) and the dissipative terms in (112)–(119); it is not an independent DoF that can grow mismatch. Hence “more curvature” is a *reaction* to redistribute energy coherently and restore interchangeability.
- (I3) **Entanglement is contractive (no inflation of Rsameness).** Admissible (intertwining, contractive) evolutions cannot increase the calibrated residual (Proposition 5), so entanglement enforces deterministic-statistical balance rather than injecting stochastic drift.
- (I4) **Arrow of time (when sectoral dissipativity holds).** If the sector supplies a Lyapunov identity and a gap, $\mathcal{R}$ obeys a strict exponential envelope below any trigger (and across cycles under restorative windows), giving a residual-driven arrow independent of thermodynamic entropy postulates.
- (I5) **Causality & resource ceilings.** In finite-speed settings, local pDoF growth is capped by a relay speed and correlation scale, compatible with admissible redistributions and with $\mathcal{R}$ monotonicity.

5.20. What to Remember (Checklist)

1. **Interchangeability (kinematics):** two maps, two identities (60). It defines *what equality means*, not how states move.
2. **Residual (metric):** one quadratic mismatch that is *monotone* under admissible operations (190).
3. **One global sDoF budget:** $s(x, t) = w(x, t) s_0$; local changes are *reweightings* (no new sDoF).

Table 3. Classical examples as special cases of interchangeability.

Setting	Maps $(\mathcal{I}, \mathcal{J})$	Residual & structural identity
PDE (flux-gradient)	$\mathcal{J}(\rho) = \nabla \rho, \mathcal{I}$ by $-\Delta \rho_{\text{tar}} = -\nabla \cdot s$ (zero mean)	$R = \int P - \nabla \rho ^2, \mathcal{I}\mathcal{J} = \text{id}, \mathcal{J}\mathcal{I} = P_{\nabla H^1_\#}$ [7,20].
OA/QMS ($L^2(\omega)$)	$\mathcal{I} = \iota$ (inclusion), $\mathcal{J} = E_{\mathcal{N}}$	$R = \ X - E_{\mathcal{N}}X\ _{2,\omega}^2; E_{\mathcal{N}}$ is the L^2 -orthogonal projector [8,9].
OU/free field	$\mathcal{I} = \mathcal{J} = \text{id}$	$R[f; \tau] = \langle f, (\Sigma_\tau - \Sigma_\infty)f \rangle $; covariance factorization gives decay at twice the gap [12,23].

4. **Entanglement (dynamics):** an admissible global map E that can *redistribute* shares coherently and nonlocally, but is *contractive* and *rank-preserving*.
5. **Causality ceiling:** local pDoF increases are speed-limited by the relay/carrier bound (no region can outrun the supply of the single sDoF share).

In short: *interchangeability* tells you *what it means* for s and p to be the same state; *entanglement-driven redistribution* tells you *how* the single sDoF budget can be reallocated across regions without creating new sDoF, without inflating mismatch, and without violating causality.

5.20.1. What the Picture Says

1. **Primordial Sameness** (Post. 10.1): initial shared reality $\Rightarrow$ sDoF/pDoF coincide in function.
2. **sDoF/pDoF identification & interchangeability** (Defs. 10, 11): one calibrated misfit—RoS (Def. 12).
3. **RoS dynamics** (Eq. (178)) plus a gap/coercivity $\Rightarrow$ monotone decay; a **critical threshold** triggers deterministic restructure (Def. 13) and an **arrow of time** (Prop. 12).
4. **Entanglement** implements nonlocal *but contractive* reweighting of the unique sDoF budget (no creation, causal ceiling); **curvature** obeys feedback law (186) to enforce alignment.

5.21. Vacuum Gradient and Energy Density as Interchangeable Degrees of Freedom

In this sector, we instantiate the general framework of statistical and physical degrees of freedom (sDoF/pDoF) by identifying:

- the *statistical channel* (sDoF) as the spatial gradient of a vacuum energy blueprint,

$$s(x, t) := \nabla \rho_{\text{vac}}(x, t),$$

(133)

- and the *physical channel* (pDoF) as the realized local energy density,

$$p(x, t) := \rho(x, t),$$

(134)

both defined over a spatial domain $V \subset \mathbb{R}^d$ and varying in time.

5.21.1. Hilbert Geometry

Let the comparison Hilbert space be $H = L^2(V)$, with inner product

$$\langle f, g \rangle := \int_V f(x) g(x) dx,$$

(135)

or, in the calibrated form, $H = H^{-1}(V)$ when residuals are expressed via elliptic comparison (e.g., inverse Laplacian in a DeTurck-type gauge).

5.21.2. Calibration Operator (Interchangeability Map)

We introduce a linear calibration operator $C : S \rightarrow P$ — typically a sectoral integration or regularization — that maps vacuum gradients to their compatible target densities. In a minimal (elliptic) formulation:

$$C(\nabla \rho_{\text{vac}}) := \rho_{\text{target}}, \quad (136)$$

where ρ_{target} solves the boundary-conditioned Neumann problem:

$$-\nabla \cdot \nabla \rho_{\text{target}} = -\nabla \cdot s = -\nabla \cdot \nabla \rho_{\text{vac}}, \quad \text{on } V, \quad (137)$$

with normalization (e.g., zero-mean) fixing the gauge.

5.21.3. Residual of Sameness

We define the sector-natural residual (R_{sameness}) as:

$$R_{\text{sameness}}(t) := \|\rho(x, t) - C(\nabla \rho_{\text{vac}}(x, t))\|_{L^2(V)}^2. \quad (138)$$

This residual quantifies the misalignment between the observed physical energy density and the target blueprint induced by the statistical vacuum gradient, as interpreted via the sectoral map C .

5.21.4. Coupled Constraint: Mutual Adaptivity

A key feature of this sector is that the degrees of freedom $s := \nabla \rho_{\text{vac}}$ and $p := \rho$ are not independent. Instead, the evolution of either field imposes a constraint on the other through the residual $R_{\text{sameness}}(t)$. Specifically:

$$\text{Any admissible change in } \rho_{\text{vac}} \Rightarrow \text{a required shift in } \rho, \quad \text{and vice versa}, \quad (139)$$

in order to maintain low residual and thus approximate interchangeability.

This implies:

$$\frac{\delta R}{\delta \rho} \neq 0, \quad \frac{\delta R}{\delta \rho_{\text{vac}}} \neq 0, \quad (140)$$

so that both fields act as functional drivers of residual geometry. Operationally, they act as a *residual-coupled pair*: changes in the blueprint demand compensatory physical responses, and observed shifts in energy require updated statistical attribution.

5.21.5. Equilibrium and Time-Asymptotic Behavior

If a coercive sectoral dynamics is supplied — e.g., a PDE or GR slice admitting a Lyapunov identity — then:

$$\frac{d}{dt} R_{\text{sameness}}(t) \leq -\alpha R_{\text{sameness}}(t) + \varepsilon(t), \quad \varepsilon(t) \rightarrow 0, \quad (141)$$

implying exponential realignment of ρ to the interchangeable image of $\nabla \rho_{\text{vac}}$, or vice versa, depending on which field is held fixed. In this sense, the vacuum-energy field system dynamically selects a shared DoF alignment path under residual decay.

Remark 5 (Interchangeability with feedback). *This sector exhibits feedback interchangeability: the residual not only diagnoses the misfit between vacuum blueprint and energy realization — it also governs their coupled re-alignment dynamics. The R_{sameness} residual here is both a metric and a driver.*

6. Entanglement as a Sector-Neutral Admissible Map

Throughout, let $(H, \langle \cdot, \cdot \rangle)$ be the comparison Hilbert geometry with norm $\|\cdot\|_H$, $P \subset H$ the physical channel space, and S the statistical channel space, both realized in the *same* comparison geometry. Let

$C : S \rightarrow P \subset H$ be the fixed calibration (units/gauge/index alignment). The physical-side residual of *Rsameness* is

$$\mathcal{R}_{\text{phys}}(s, p) := \|p - Cs\|_H^2, \quad (s, p) \in S \times P. \quad (142)$$

When interchangeability maps $(\mathcal{I}, \mathcal{J})$ are used explicitly, take $\mathcal{I} \equiv C$ and $\mathcal{R}_{\text{phys}}(s, p) = \|p - \mathcal{I}s\|_H^2$; the statistical-side residual $\mathcal{R}_{\text{stat}}(s, p) := \|s - \mathcal{J}p\|_H^2$ is *norm-equivalent* to $\mathcal{R}_{\text{phys}}$ under the usual boundedness assumptions.

6.1. One Global sDoF Budget and the Redistribution Layer

In the cosmological one-budget picture, there is a *single* global statistical resource that is *shared* across space. We encode this by the factorization

$$s(x, t) = w(x, t) s_0, \quad w(x, t) \geq 0, \quad \int_{\Omega} w(x, t) dx = 1, \quad s_0 \in S \text{ fixed.} \quad (143)$$

Interchangeability allows us to compare this shared statistical state with the physical one: $p \approx C(s)$.

To describe *redistribution* (“more pDoF here”), we act on the composite description by a sector-neutral map

$$E : (S \oplus P) \longrightarrow (S \oplus P), \quad (144)$$

that should be (i) *resource-preserving* (no new sDoF globally) and (ii) *contractive* in the comparison geometry so it cannot inflate the calibrated mismatch.

Definition 9 (Admissible map: intertwining + contractivity). *A pair of bounded linear maps $(\tilde{\Phi}, \Phi)$ with*

$$\tilde{\Phi} : S \rightarrow S, \quad \Phi : P \rightarrow P, \quad (145)$$

*is called **admissible** if the following identities and norm bounds hold:*

$$\text{(intertwine calibration)} \quad \Phi \circ C = C \circ \tilde{\Phi}, \quad (146)$$

$$\text{(intertwine representative)} \quad \tilde{\Phi} \circ \mathcal{J} = \mathcal{J} \circ \Phi, \quad (147)$$

$$\text{(contractivity)} \quad \|\Phi x\|_H \leq \|x\|_H \quad (\forall x \in P), \quad \|\tilde{\Phi} y\|_H \leq \|y\|_H \quad (\forall y \in S). \quad (148)$$

6.1.1. Explanation

(146) ensures “push-then-pull” is consistent with the calibration; (147) keeps statistical representatives aligned with physical actions; (190) is a sector-neutral data-processing inequality in the chosen geometry.

Proposition 8 (Admissible redistribution is *Rsameness*-contracting). *If $(\tilde{\Phi}, \Phi)$ is admissible, then for all $(s, p) \in S \times P$,*

$$\mathcal{R}_{\text{phys}}(\tilde{\Phi}s, \Phi p) = \|\Phi p - C \tilde{\Phi}s\|_H^2 = \|\Phi(p - Cs)\|_H^2 \leq \|p - Cs\|_H^2 = \mathcal{R}_{\text{phys}}(s, p). \quad (149)$$

Proof. Use (146) to write $\Phi p - C \tilde{\Phi}s = \Phi(p - Cs)$ and (190). $\square$

6.1.2. Dual (Statistical-Side) Statement

Under admissibility and boundedness of $\mathcal{J}$, the statistical residual is also contractive:

$$\mathcal{R}_{\text{stat}}(\tilde{\Phi}s, \Phi p) = \|\tilde{\Phi}s - \mathcal{J}\Phi p\|_H^2 = \|\tilde{\Phi}(s - \mathcal{J}p)\|_H^2 \leq \|s - \mathcal{J}p\|_H^2. \quad (150)$$

Proposition 9 (Equality conditions). *In (149), equality holds if and only if Φ acts isometrically on the residual direction: $\|\Phi z\|_H = \|z\|_H$ for $z = p - Cs$ (e.g. when Φ is an isometry on $\text{span}\{z\}$). If Φ is a strict contraction on the residual direction and $p \neq Cs$, the inequality is strict.*

Proposition 10 (Closure properties). *If $(\tilde{\Phi}_1, \Phi_1)$ and $(\tilde{\Phi}_2, \Phi_2)$ are admissible, then so are:*

1. *their composition $(\tilde{\Phi}_2\tilde{\Phi}_1, \Phi_2\Phi_1)$,*
2. *any convex combination $(\sum_j \theta_j \tilde{\Phi}_j, \sum_j \theta_j \Phi_j)$ with $\theta_j \geq 0, \sum_j \theta_j = 1$,*
3. *the strong-operator-topology (SOT) closure of the convex hull of a family of admissible pairs.*

Sketch. Intertwining identities are preserved under composition and convex combination; contractivity follows from $\|\sum \theta_j T_j\| \leq \sum \theta_j \|T_j\| \leq 1$ and $\|T_2 T_1\| \leq \|T_2\| \|T_1\| \leq 1$. $\square$

Proposition 11 (Characterization via Rsameness monotonicity). *Assume (146) and (147). Then the following are equivalent:*

1. *$(\tilde{\Phi}, \Phi)$ is admissible,*
2. *$\mathcal{R}_{\text{phys}}(\tilde{\Phi}s, \Phi p) \leq \mathcal{R}_{\text{phys}}(s, p)$ for all (s, p) ,*
3. *$\mathcal{R}_{\text{stat}}(\tilde{\Phi}s, \Phi p) \leq \mathcal{R}_{\text{stat}}(s, p)$ for all (s, p) .*

Idea. (1) $\Rightarrow$ (2) is Prop. 8. (2) $\Rightarrow$ contractivity of Φ : set $s = 0$ to get $\|\Phi p\|_{\text{H}} \leq \|p\|_{\text{H}} \forall p$; similarly (3) $\Rightarrow$ contractivity of $\tilde{\Phi}$ by setting $p = 0$. The intertwining is assumed. $\square$

6.1.3. Robustness to Small Modelling Errors

If $(\tilde{\Phi}, \Phi)$ is ϵ -admissible in the sense that

$$\|\Phi \circ C - C \circ \tilde{\Phi}\|_{\text{op}} \leq \epsilon, \quad \|\tilde{\Phi} \circ \mathcal{J} - \mathcal{J} \circ \Phi\|_{\text{op}} \leq \epsilon, \quad \|\Phi\|_{\text{op}} \leq 1, \quad \|\tilde{\Phi}\|_{\text{op}} \leq 1, \quad (151)$$

then

$$\mathcal{R}_{\text{phys}}(\tilde{\Phi}s, \Phi p) \leq \mathcal{R}_{\text{phys}}(s, p) + \epsilon^2 \|(s, p)\|^2, \quad (152)$$

so monotonicity holds up to an $O(\epsilon^2)$ defect; analogously for $\mathcal{R}_{\text{stat}}$.

6.2. No-Creation of the One-Budget (Resource Preservation)

Assume (143). Then *classical/Markov* and *quantum/CPTP* evolutions preserve the global sDoF budget.

Lemma 5 (Budget preservation). • **Classical/Markov.** *If $(\tilde{\Phi}s)(x) = \int_{\Omega} K(x, y) s(y) dy$ with $K \geq 0$ and $\int_{\Omega} K(x, y) dx = 1$ for all y , then $s' = \tilde{\Phi}s = w' s_0$ with $w'(x) := \int_{\Omega} K(x, y) w(y) dy \geq 0$ and $\int_{\Omega} w'(x) dx = \int_{\Omega} w(y) dy = 1$.*

• **Quantum/CPTP.** *If $\tilde{\Phi}$ is completely positive and trace-preserving on density operators, then $\mathfrak{B}(\sigma) := \text{Tr } \sigma$ satisfies $\mathfrak{B}(\tilde{\Phi}(\sigma)) = \mathfrak{B}(\sigma)$.*

6.2.1. Explanation

Markov kernels and CPTP maps can redistribute local shares but cannot create new sDoF. (In quantum resource theories, LOCC/CPTP cannot increase standard entanglement monotones; we only need trace/mass preservation and L^2 contractivity here.)

6.3. Causality Ceiling

Let $\mathfrak{d}$ be an admissible local complexity counter (e.g. L^2 -energy, effective rank) and $U(t)$ a moving control volume. If redistribution is mediated by a carrier/relay with finite speed v_* and correlation diameter ℓ_{corr} (hyperbolic finite-speed in PDE; Lieb–Robinson on lattices), then

$$65 \frac{d}{dt} \mathfrak{d}(p_{U(t)}) \lesssim \kappa \frac{v_*}{\ell_{\text{corr}}}. \quad (153)$$

Idea of proof. Domains of dependence in the continuum and Lieb–Robinson cones on lattices bound the influx across $\partial U(t)$, so no admissible operation can concentrate pDoF faster than the relay/front can supply it.

6.4. What Entanglement Can and Cannot Do

- **Can:** reweight the share field $w(x, t)$ coherently (possibly nonlocally), i.e. $w \mapsto w'$ with $\int w' = \int w = 1$ (Lemma 5); and do so *contractively*, hence $\mathcal{R}$ is monotone (Prop. 8).
- **Cannot:** create new sDoF (rank-1 budget preserved by Markov/CPTP); and *cannot outrun the relay* (Prop. 65).

Corollary 1 (Summary for practice). *Admissible, sector-neutral “entanglement” operations (i) preserve the one-budget globally, (ii) cannot increase Rsameness in either channel, and (iii) obey a causal ceiling. Thus they can redistribute local shares coherently—even nonlocally—but never create new sDoF, never inflate the calibrated mismatch, and never outrun the relay.*

6.4.1. Concrete Sectoral Instantiations (For Orientation)

- **PDE/transport.** $\tilde{\Phi}$ a Markov integral operator with mass-preserving kernel; Φ the induced contractive map on $\mathcal{P}$ obeying $\Phi C = C \tilde{\Phi}$.
- **OA/QMS pointer.** $\tilde{\Phi}$ a CPTP map on the pointer algebra; Φ the induced $L^2(\omega)$ -contractive map intertwining the conditional expectation; Kadison–Schwarz yields contractivity.
- **OU/free fields.** $\tilde{\Phi} = \Phi = e^{tA}$ with dissipative generator A (e.g. $-\Delta + m^2$); then $\|e^{tA}\| \leq 1$ and intertwining is trivial ($C = \text{id}$).

7. Diagnostics

What to test (overview).

Interchangeability is certified by three geometry-level checks: (i) *projection identities* (exact or up to tolerance), (ii) *data-processing monotonicity* under admissible (contractive, intertwining) maps, (iii) *angle geometry* (principal/Friedrichs angles) for near-separability. When time-resolved data are available, a *causality ceiling* complements these checks.

A. Projection test (exact / tolerance-aware)

Verify the interchangeability identities either exactly (symbolically) or numerically up to a prescribed tolerance:

$$\mathcal{I} \circ \mathcal{J} = \text{id}_{\mathcal{P}}, \quad \mathcal{J} \circ \mathcal{I} = P_{\mathcal{S}}. \quad (154)$$

Numerical acceptance. With test sets $\{\rho_k\} \subset \mathcal{P}$, $\{s_j\} \subset \mathcal{S}$,

$$\frac{\|\mathcal{I}\mathcal{J}\rho_k - \rho_k\|_{\mathcal{H}}}{\|\rho_k\|_{\mathcal{H}}} \leq \varepsilon_{\text{proj}}, \quad \frac{\|\mathcal{J}\mathcal{I}s_j - P_{\mathcal{S}}s_j\|_{\mathcal{S}}}{\|s_j\|_{\mathcal{S}}} \leq \varepsilon_{\text{proj}}, \quad (155)$$

for a tolerance $\varepsilon_{\text{proj}}$ commensurate with discretization error (e.g. 10^{-8} in double precision, or the observed solver error floor).

B. Data-processing (residual monotonicity) under admissible maps

For contractive coarse-grainings $(\tilde{\Phi}, \Phi)$ that *intertwine* the calibration $(\Phi\mathcal{I} = \mathcal{I}\tilde{\Phi}, \tilde{\Phi}\mathcal{J} = \mathcal{J}\Phi)$, check residual monotonicity on both sides:

$$\|\Phi p - \mathcal{I}(\tilde{\Phi}s)\|_{\mathcal{H}} \leq \|p - \mathcal{I}(s)\|_{\mathcal{H}}, \quad \|\tilde{\Phi}s - \mathcal{J}(\Phi p)\|_{\mathcal{S}} \leq \|s - \mathcal{J}(p)\|_{\mathcal{S}}. \quad (156)$$

Tolerance form. Accept if the left-minus-right is $\leq \varepsilon_{\text{dp}}$ with ε_{dp} at the measured noise floor.

7.1. Model Selection

7.1.1. Model Selection via Residual Deviation (Falsifiability)

Given a candidate calibration $\mathcal{I}$ and admissible $(\tilde{\Phi}, \Phi)$, (156) provides a falsifiable test. If a model (pointer/geometry) repeatedly violates residual monotonicity on held-out data or simulations (beyond tolerance), the model fails interchangeability and should be rejected or recalibrated.

C. Isometry Invariance (Sanity Check)

If $U : \mathcal{H} \rightarrow \mathcal{H}$ is an isometry with $U(P) = P$ and $U\mathcal{I} = \mathcal{I}U_S$, then

$$\|Up - U\mathcal{I}(s)\|_{\mathcal{H}} = \|p - \mathcal{I}(s)\|_{\mathcal{H}}. \quad (157)$$

Use. Apply a few random unitary/isometric transforms (finite dimension / FFT-based orthogonals) as a robustness check: residuals must be unchanged up to solver tolerance.

D. Principal-Angle Diagnostic (Near-Separability)

Build orthonormal bases Q_U for P and Q_V for $\overline{\text{ran } \mathcal{I}}$ (e.g. QR on snapshot matrices). Compute

$$Q_U^* Q_V = W \Sigma Z^*, \quad \Sigma = \text{diag}(\sigma_1, \dots, \sigma_m), \quad \sigma_k = \cos \theta_k, \quad (158)$$

so that $\cos \theta_F = \max_k \sigma_k$. Then the residual satisfies the sharp two-sided bounds

$$(1 - \cos \theta_F)(\|e_U\|^2 + \|e_V\|^2) \leq \|e\|^2 \leq (1 + \cos \theta_F)(\|e_U\|^2 + \|e_V\|^2), \quad (159)$$

implying *near-additivity* of channels when $\cos \theta_F \ll 1$. *Report.* Quote $\cos \theta_F$ and the induced condition number $\kappa(\theta_F) = \frac{1+\cos \theta_F}{1-\cos \theta_F}$.

E. Causality ceiling (time-resolved data; optional)

If redistribution acts through a finite-speed carrier v_* and a correlation diameter ℓ_{corr} (hyperbolic finite speed / Lieb–Robinson), local pDoF growth obeys

$$\frac{d}{dt} \mathfrak{d}(p_{U(t)}) \lesssim \kappa \frac{v_*}{\ell_{\text{corr}}}, \quad (160)$$

for an admissible complexity counter $\mathfrak{d}$ (e.g. energy or effective rank). *Diagnostic.* Estimate the left slope from time series and flag violations of the ceiling (beyond tolerance) as model/parameter inconsistencies.

F. Minimal reproducible recipe

1. Assemble paired samples (s_i, p_i) and compute $\mathcal{I}s_i, \mathcal{J}p_i$.
2. **Projection test:** evaluate relative errors for $\mathcal{I}\mathcal{J}\rho_k \approx \rho_k$ and $\mathcal{J}\mathcal{I}s_j \approx P_S s_j$.
3. **Residuals:** $R_i = \|p_i - \mathcal{I}s_i\|_{\mathcal{H}}^2$ (optionally also $R_i^* = \|s_i - \mathcal{J}p_i\|_{\mathcal{S}}^2$).
4. **Monotonicity:** choose a family of admissible $(\tilde{\Phi}, \Phi)$; verify $R(\tilde{\Phi}s_i, \Phi p_i) \leq R(s_i, p_i)$ for all i .
5. **Angles:** compute $\cos \theta_F$ from $Q_U^* Q_V$ and report $\kappa(\theta_F)$.
6. **(Optional) Causality:** if time-resolved, check the ceiling on $d\mathfrak{d}(p_{U(t)})/dt$.

G. Uncertainty and thresholds

- *Monte Carlo / bootstrap.* For noisy data, assess the fraction of trials where monotonicity holds; attach binomial CIs or bootstrap CIs on the mean drop $\Delta R = R_{\text{new}} - R_{\text{old}}$.
- *Tolerances.* Set $\varepsilon_{\text{proj}}, \varepsilon_{\text{dp}}$ from observed discretization/solver error (e.g. comparison of mesh refinement levels or unitary invariance spread).
- *Stability.* When subspaces are perturbed, use Davis–Kahan/Wedin-type bounds to interpret changes in θ_F as a function of operator/data perturbations.

8. Physical Example: A Toy Backreaction Residual

Consider $\rho(x, t)$ (energy density) and a statistical reference $\rho_{\text{vac}}(x, t)$ on a compact domain V . Let the calibration be $C = \lambda \nabla^2$ (curvature proxy), and define the residual of sameness

$$R_{\text{sameness}}(t) = \|\nabla^2 \rho(\cdot, t) + \lambda \nabla^2 \rho_{\text{vac}}(\cdot, t)\|_{L^2(V)}^2. \quad (161)$$

With a minimal energy identity $\frac{d}{dt}R = -2\langle Ke, e \rangle + 2\langle e, g \rangle$ and a gap $\langle Ke, e \rangle \geq \kappa \|e\|^2$ (e.g. elliptic response on a slice) one obtains $\dot{R} \leq -(2\kappa - 2\varepsilon)R$ for small remainders. Thus R acts as a gauge-invariant, slice-level mismatch measure for curvature-energy alignment; admissible coarse-grainings and entanglement-style redistributions can only *lower* R (Sec. 5).

9. Equivalence vs Identity: When Do Shared DoF Become the Same?

9.1. What Problem Are We Solving?

When we map a statistical channel s and a physical channel p into the same comparison space and link them by an interchangeable pair $(\mathcal{I}, \mathcal{J})$, we often say they “agree up to something.” This section turns that into a *clean, testable statement* on three fronts:

- **Exact agreement vs. agreement modulo redundancy.** Sometimes s and p are literally the *same degrees of freedom* written in two coordinate systems; other times they only agree after you factor out a gauge kernel or reconcile a dimension mismatch.
- **Interpretable residuals.** Even when the mismatch is numerically small, you need to know whether it’s a *real signal* or just a *geometric cross-term* that should be small by design.
- **Portable diagnostics.** Projection/data-processing tests certify interchangeability; principal-angle bounds certify that residuals are *meaningful and stable*.

9.2. Why Split “Identity” vs. “Equivalence up to Projection”?

9.3. Identity (No Gauge, Same Dimension)

If $P_S = \text{id}_S$ and $\mathcal{I}$ is bijective onto P with inverse $\mathcal{J} = \mathcal{I}^{-1}$ (bounded inverse in infinite dimension), then

$$p = \mathcal{I}(s) \iff s = \mathcal{J}(p). \quad (162)$$

Here “shared DoF” means *the very same DoF*, not merely correlated or comparable: same content, different coordinates.

9.3.1. Equivalence up to Projection (Gauge or Dimension Mismatch)

If $\ker P_S \neq \{0\}$ or $\dim S \neq \dim P$, the right statement lives on the quotient $S_q := S / \ker P_S$:

$$p = \mathcal{I}(s) \iff [s] = [\mathcal{J}(p)] \text{ in } S_q. \quad (163)$$

Now “shared DoF” means “the same after removing inessentials.” This prevents *false positives* (thinking the channels disagree when they only differ by pure gauge) and *false negatives* (discarding real differences masked by an unlucky parametrization).

9.4. Why Add the Principal-Angles Geometry?

Even in the identity/quotient case, any residual you compute lives in a sum of two subspaces. Decompose

$$e = e_U + e_V, \quad e_U \in P, \quad e_V \in \overline{\text{ran } \mathcal{I}}. \quad (164)$$

The cross-term $2\langle e_U, e_V \rangle$ can inflate/deflate norms unless it is controlled. The Friedrichs angle θ_F between P and $\overline{\text{ran } \mathcal{I}}$ gives exactly that control:

$$|\langle e_U, e_V \rangle| \leq \cos \theta_F \|e_U\| \|e_V\|. \quad (165)$$

If $\theta_F \approx \pi/2$ (the subspaces are nearly orthogonal), the residual is *nearly separable*:

$$\|e\|^2 \approx \|e_U\|^2 + \|e_V\|^2. \quad (166)$$

Thus you can safely read and compare the pieces (“statistical” vs “physical” contributions) without geometric bias. Moreover, Davis–Kahan ($\sin\Theta$) bounds guarantee that θ_F is *stable under small perturbations*, so your residual diagnostics are robust.

9.5. Why Is This Useful in Practice?

- You get a **decision procedure**: (i) test the identity case (full rank, no gauge); (ii) else pass to the quotient and claim equivalence mod projection; (iii) use principal angles to understand conditioning and near–separability of residuals.
- You get **portable diagnostics**: projection/data–processing certifies interchangeability; angle bounds certify residual interpretability.
- You avoid classic pitfalls: mistaking gauge/slack for physical discrepancy, or trusting residual numbers dominated by cross–terms rather than genuine mismatch.

9.6. Identity Case: Same Dimension, No Gauge

Assume $P_S = \text{id}_S$ and $\mathcal{I}$ is bijective onto P with inverse $\mathcal{J} = \mathcal{I}^{-1}$ (bounded inverse in the infinite–dimensional case). Then, for all (s, p) ,

$$p = \mathcal{I}(s) \iff s = \mathcal{J}(p), \quad \|p - \mathcal{I}(s)\| = 0 \iff \|s - \mathcal{J}(p)\|_S = 0. \quad (167)$$

In this case the two channels carry the *same degrees of freedom*: exact identity, different coordinates.

9.7. Equivalence up to Projection: Gauge or Dimension Mismatch

Let $\ker P_S$ denote the (statistical) gauge kernel and define the Hilbert–quotient

$$S_q := S / \ker P_S, \quad \|[s]\|_{S_q} := \|P_S s\|_S. \quad (168)$$

Under Def. 1, $\mathcal{I}$ descends to an isomorphism $\bar{\mathcal{I}} : S_q \rightarrow P$ via $\bar{\mathcal{I}}([s]) = \mathcal{I}(s)$ with inverse $p \mapsto [\mathcal{J}(p)]$. Thus equality holds *modulo* the projection:

$$p = \mathcal{I}(s) \iff [s] = [\mathcal{J}(p)]. \quad (169)$$

This is the precise way to say “they agree after removing inessentials.”

9.8. Geometry of Cross–Terms: Principal Angles & Near–Separability

Let $U = P$ and $V = \overline{\text{ran } \mathcal{I}}$ in the comparison Hilbert space H , with orthogonal projectors P_U, P_V . The *Friedrichs angle* $\theta_F \in [0, \pi/2]$ is defined by

$$\|P_U P_V\| = \cos \theta_F. \quad (170)$$

For any $e = e_U + e_V$ with $e_U \in U, e_V \in V$,

$$|\langle e_U, e_V \rangle| \leq \|P_U P_V\| \|e_U\| \|e_V\| = \cos \theta_F \|e_U\| \|e_V\|. \quad (171)$$

Hence if $\theta_F \approx \pi/2$ (subspaces nearly orthogonal), the mixed term is small and

$$\|e_U + e_V\|^2 = \|e_U\|^2 + \|e_V\|^2 + 2\langle e_U, e_V \rangle \approx \|e_U\|^2 + \|e_V\|^2, \quad (172)$$

i.e. the residual is *nearly separable*. Stability of θ_F under small perturbations is controlled by classical Davis–Kahan ($\sin\Theta$) bounds, so residual interpretability is *robust*.

9.9. Conclusion

The section formalizes when “shared DoF” truly means identical content and when it only means equal up to a canonical projection, and it adds the geometric controls (principal angles) needed to make residuals *meaningful and stable*. In short: identity if $P_S = \text{id}$ and $\mathcal{I}$ is an isomorphism; otherwise equivalence on the quotient $S / \ker P_S$; and in both cases principal-angle bounds keep your residual diagnostics honest.

10. The Primordial Sameness as Shared Reality

This section motivates and formalizes the hypothesis that the universe began in a state of *Primordial Sameness*: before any dynamical differentiation, the statistical and physical descriptions coincided in a common comparison geometry. We first state the Principle of Initial Sameness (PIS) and pin down which degrees of freedom are being identified (statistical vs. physical, with exclusivity and identity at an instant). We then introduce a calibration map that makes “sameness” operational and quantify deviations by a single observable—the *Residual of Sameness* (R_{sameness}). Finally, we explain how this residual underwrites two key narratives developed later: (i) an arrow-of-time envelope (including a thresholded “inflation” reset), and (ii) entanglement as admissible, contractive redistribution subject to causal ceilings. The goal is a minimal, testable kinematics of beginnings that the sectoral dynamics can build on.

10.1. Motivation ($EPR \rightarrow \text{Completeness}$; $\text{Beginnings} \rightarrow \text{Laws}$)

Einstein–Podolsky–Rosen highlighted that a complete theory should explain correlations without appealing to hidden, disconnected realities; Hawking argued that clarifying origins sharpens the laws that follow. We adopt the view that a *shared* physical reality at the beginning is the simplest way to underwrite completeness and unification. We call this *Primordial Sameness*. See [24,25].

Postulate 10.1 (Principle of Initial Sameness (PIS)). *At the inception, all physical distinctions collapse: space/time are not yet distinguished, the energy density field is everywhere the same, and statistical and physical descriptions coincide. Operationally: the only admissible global information is sameness.*

Detailed explanation and consequences. The postulate asserts a *kinematic* condition on admissible initial data, prior to any sector-specific dynamics:

- (P1) **Collapse of distinctions (kinematic content).** There exists a comparison Hilbert geometry $(H, \langle \cdot, \cdot \rangle)$, a statistical channel S , a physical channel $P \subset H$, and a calibration $C : S \rightarrow P$ (units/gauge alignment), such that the *calibrated cancellation law*

$$p_0 + C s_0 = 0 \quad (\text{in } H) \quad (173)$$

holds for the initial statistical/physical states (s_0, p_0) . Equivalently, the Residual of Sameness (R_{S}),

$$R_{\text{sameness}}(0) := \| p_0 + C s_0 \|_H^2,$$

vanishes at inception: $R_{\text{sameness}}(0) = 0$.

- (P2) **Space–time undistinguished (pre-metric regularity).** PIS is *agnostic* about any metric splitting into “space” and “time” at inception. Formally, no privileged foliation is assumed; all statements are made in the comparison geometry H (or, if convenient, in a curvature channel with matched differential order, cf. (174)).
- (P3) **Homogeneous blueprint at the correct order.** If the physical degrees of freedom are compared in the *gradient channel* (flux–gradient), then “everywhere the same” means

$$\nabla \rho(\cdot, 0) = 0 \quad \text{and} \quad \nabla \rho_{\text{vac}}(\cdot, 0) = 0,$$

so that $p_0 = \nabla \rho(\cdot, 0) = 0$ and, with a gradient-preserving C , $C s_0 = 0$. In a *curvature channel* (uniform order on both sides), the counterpart is

$$\Delta \rho(\cdot, 0) = k \Delta \rho_{\text{vac}}(\cdot, 0) \Rightarrow e_{\Delta}(\cdot, 0) := \Delta \rho - k \Delta \rho_{\text{vac}} = 0. \quad (174)$$

PIS always compares the *same differential order* on both channels to avoid spurious mismatch from order imbalance.

- (P4) **Information-theoretic reading (single global share).** Operationally, the *only* admissible global information at inception is sameness: the statistical carrier is rank-one,

$$s_0(x) = w_0(x) s_*, \quad w_0 \equiv 1 \text{ (normalized),}$$

so no meaningful partition of the global statistical budget exists yet. All local “labels” are gauge.

- (P5) **Symmetry and gauge.** PIS is invariant under any isometry $U : \mathcal{H} \rightarrow \mathcal{H}$ that preserves P and commutes with the calibration ($UC = CU_S$). Hence sameness is a geometric statement, not a coordinate artifact. Additive gauges (e.g., $\rho \mapsto \rho + c$) drop out in the gradient channel; in the curvature channel, uniform-order comparison removes constant and affine gauges.
- (P6) **What PIS does not assert.** PIS does not fix a microphysical law, a metric, or a specific potential/connection; it sets a *boundary condition* for the two-channel kinematics. It also does not assert minimal entropy in a probabilistic sense; instead, it asserts vanishing calibrated misfit $R = 0$ between the statistical blueprint and the physical response.
- (P7) **First departures from PIS (infinitesimal structure).** Let $e := p + Cs$. A small departure from PIS is a small $e(\cdot, t)$ at early times. If a sector contributes an energy/Lyapunov identity,

$$\frac{d}{dt} \|e\|_{\mathcal{H}}^2 = -2\langle Ke, e \rangle + 2\langle e, g \rangle,$$

with $\langle Ke, e \rangle \geq \kappa \|e\|^2$ and $|\langle e, g \rangle| \leq \varepsilon \|e\|^2$, then for $\alpha := 2\kappa - 2\varepsilon > 0$ one obtains the envelope $\dot{R} \leq -\alpha R$, so small deviations from sameness decay exponentially (unless a critical restructuring window is triggered later, cf. the “inflation” mechanism).

- (P8) **Equivalent formulations (useful in proofs).**

(E1) *Projection form.* If $\mathcal{I}, \mathcal{J}$ are the interchangeability maps with $\mathcal{I}\mathcal{J} = \text{id}_{\mathcal{P}}$, $\mathcal{J}\mathcal{I} = P_S$, then

$$p_0 = \mathcal{I}s_0 \iff s_0 = P_S \mathcal{J}p_0.$$

Thus equality is exact on the physical side and modulo the canonical statistical projection on the statistical side.

(E2) *RoS form.* $R_{\text{sameness}}(0) = 0 \iff \min_{s,p} \|p + Cs\|_{\mathcal{H}}^2 = 0$ with minimizer (s_0, p_0) in the interchangeability class.

(E3) *Angle form.* If $U = P$ and $V = \overline{\text{ran } C} \subset \mathcal{H}$, then PIS implies $e \in U \cap (-V)$ with $\|e\| = 0$; principal-angle bounds guarantee robust stability under small isometric perturbations.

- (P9) **Falsifiability (diagnostics at $t = 0$).** A candidate calibration C and channel identification (S, P) are *rejected* if any of the following fail at inception beyond numerical/experimental tolerance:

(F1) *Projection test:* $\|\mathcal{I}\mathcal{J}p_0 - p_0\|/\|p_0\|$ and $\|\mathcal{J}\mathcal{I}s_0 - P_S s_0\|/\|s_0\|$ are not small;

(F2) *RoS test:* $R_{\text{sameness}}(0) = \|p_0 + Cs_0\|_{\mathcal{H}}^2$ is not small;

(F3) *Isometry invariance:* residuals change under random $\mathcal{H}$ -isometries that preserve the setup (indicates coordinate dependence).

- (P10) **Cosmological reading (consistency, not commitment).** In a GR-like slice, reading ρ as (effective) energy density and using a curvature/gradient channel,

$$\nabla \rho(\cdot, 0) = 0, \quad \nabla \rho_{\text{vac}}(\cdot, 0) = 0 \Rightarrow R(0) = 0,$$

is consistent with homogeneous/isotropic initial data. PIS itself remains *sector-neutral*: it is a statement about *alignment* of channels, not about a specific metric or equation of state.

- (P11) **Compatibility with entanglement-as-redistribution.** With the one-budget model $s = w s_0$, $w \equiv 1$ at inception (no distinguished regions). Any admissible redistribution $(\tilde{\Phi}, \Phi)$ (intertwining + contractive) preserves global budget and cannot increase RoS:

$$R(\tilde{\Phi}s, \Phi p) = \|\Phi(p + Cs)\|_{\mathbf{H}}^2 \leq \|p + Cs\|_{\mathbf{H}}^2.$$

Thus PIS seeds a resource-preserving, contractive evolution of alignment.

- (P12) **Minimal perturbative model (first nontrivial order).** Let $e(\cdot, t) = p(\cdot, t) + Cs(\cdot, t)$ with $e(\cdot, 0) = 0$. Linearizing a residual-driven law (gradient channel)

$$\partial_{tt}e + \gamma \partial_t e + \nu \Delta^2 e + \mu e = \zeta \partial_t \mathcal{E}_{\nabla},$$

gives $\frac{d}{dt} \|e\|_{L^2}^2 = -2\gamma \|\partial_t e\|^2 - 2\nu \|\Delta e\|^2 - 2\mu \|e\|^2 + (\text{contractive source})$, hence small misfit remains controlled. In the curvature channel, replace e by $e_{\Delta} = \Delta \rho - k \Delta \rho_{\text{vac}}$.

Summary. PIS fixes the *zero* of the residual geometry: $R_{\text{sameness}}(0) = 0$. It is a kinematic boundary condition (not a dynamical law), stated at the correct matching order (gradient or curvature), invariant under isometries/gauges, and immediately falsifiable through projection/data-processing diagnostics. All later structure—rates, “inflation” resets, entanglement-style redistribution—acts on departures from this calibrated sameness.

Compatibility of PIS and Shared Ancestry.

PIS fixes the initial condition $R_{\text{sameness}}(0) = \|p_0 + Cs_0\|_{\mathbf{H}}^2 = 0$ with a rank-1 (one-budget) statistical carrier $s_0 = w_0 s_{\star}$ and $w_0 \equiv 1$. *Shared ancestry* is then modeled as a sequence of admissible redistributions $(\tilde{\Phi}_{t_k}, \Phi_{t_k})$ (intertwining the calibration and contractive in the comparison geometry) that *partition* the single budget into descendant branches while preserving the global mass:

$$s_{k+1} = \tilde{\Phi}_{t_k} s_k, \quad p_{k+1} = \Phi_{t_k} p_k, \quad \int_V w_{k+1} = \int_V w_k = 1.$$

Thus a rooted ancestry tree $\mathcal{T}$ forms from the PIS root (s_0, p_0) by repeated budget-preserving splits, each represented by a (Markov/CPTP) kernel on the statistical side and its intertwined physical action. Contractivity yields the global data-processing inequality

$$R_{\text{sameness}}(s_{k+1}, p_{k+1}) = \|\Phi_{t_k}(p_k - Cs_k)\|_{\mathbf{H}}^2 \leq \|p_k - Cs_k\|_{\mathbf{H}}^2 = R_{\text{sameness}}(s_k, p_k),$$

so the *global* Rsameness cannot inflate along the tree, while *regional* counters may diverge across branches, providing the observable content of ancestry.

10.1.1. Degrees of Freedom (DoF): Identification, Exclusivity, Identicality

Definition 10 (Statistical vs. physical channels). *Over a spatial domain V ,*

- the statistical DoF (**sDoF**) are the vacuum-density gradients $s(x, t) := \nabla \rho_{\text{vac}}(x, t)$;
- the physical DoF (**pDoF**) are the energy-density gradients $p(x, t) := \nabla \rho(x, t)$.

Principle 10.1 (Exclusivity (local uniqueness of sDoF)). *For any fixed t , the statistical encoding is nonredundant: $\nabla \rho_{\text{vac}}(x_1, t) \neq \nabla \rho_{\text{vac}}(x_2, t)$ whenever $x_1 \neq x_2$, and the statistical channel has full rank (no hidden duplication).*

Principle 10.2 (Identicality at an instant). *At each time t , all sDoF play the same role: they are gradients of the same scalar field and serve the same constraint. Formally $s(\cdot, t) = \nabla \rho_{\text{vac}}(\cdot, t)$ across V ; the play of values is spatial, but the function of sDoF is identical everywhere.*

Remark 6 (Temporal variation). Across times $t_1 \neq t_2$, the gradients change as the universe redistributes energy: $\nabla \rho(\cdot, t_1) \neq \nabla \rho(\cdot, t_2)$, and $\nabla \rho_{\text{vac}}(\cdot, t_1) \neq \nabla \rho_{\text{vac}}(\cdot, t_2)$, while Principles 10.1–10.2 still hold instantaneously.

10.2. Interchangeability and the Residual of Sameness (R_{sameness})

Definition 11 (Interchangeability map and calibrated equality). There exists a calibration operator C (units/gauge/indices) and a comparison geometry $(H, \langle \cdot, \cdot \rangle)$ such that the calibrated cancellation law

$$p(\cdot, t) + C s(\cdot, t) = 0 \quad (175)$$

expresses sameness between p DoF and s DoF. Deviations from sameness are measured in H by a single scalar residual.

Definition 12 (Residual of Sameness (R_{sameness})). The RoS is the quadratic misfit between physical and calibrated statistical channels,

$$R_{\text{sameness}}(t) := \|p(\cdot, t) + C s(\cdot, t)\|_H^2. \quad (176)$$

When a differential comparison is more natural, replace $p \mapsto Dp$ and $Cs \mapsto D(Cs)$ (e.g. Laplacian or Hessian to encode curvature proxies) and define

$$R_{\text{sameness}}^{(D)}(t) := \|D p(\cdot, t) + D(C s(\cdot, t))\|_H^2. \quad (177)$$

Remark 7 (Minimal dynamics for RoS). A sector–neutral energy identity gives

$$\frac{d}{dt} R_{\text{sameness}}(t) = -2 \langle Ke, e \rangle_H + 2 \langle e, g \rangle_H, \quad e := p + Cs, \quad (178)$$

with $K = K^* \succeq 0$ the instantaneous dissipative/elliptic channel (curvature or constitutive response), and g a controlled remainder (lower orders, background expansion, nonlocal corrections). If a gap/coercivity holds, $\langle Ke, e \rangle \geq \kappa \|e\|^2$ and $|\langle e, g \rangle| \leq \varepsilon \|e\|^2$, then

$$\dot{R}_{\text{sameness}}(t) \leq -\alpha R_{\text{sameness}}(t), \quad \alpha := 2\kappa - 2\varepsilon > 0, \quad (179)$$

so $R_{\text{sameness}}(t) \leq e^{-\alpha t} R_{\text{sameness}}(0)$ (a DSFL-type envelope).

10.3. Inflation and the Arrow of Time as RoS Mechanics

Definition 13 (Critical threshold and trigger). There exists $R_{\text{crit}} > 0$ such that if $R_{\text{sameness}}(t)$ overshoots R_{crit} , the system undergoes a deterministic restructuring episode (inflationary–like snap-back) that rapidly reduces the misfit by creating the required volume/curvature for gradients to realign.

Proposition 12 (Arrow of time from RoS). If $\dot{R}_{\text{sameness}} \leq -\beta R_{\text{sameness}}$ whenever $R_{\text{sameness}} < R_{\text{crit}}$, and if episodes at R_{crit} are restoring (net reduction over a cycle), then

$$S_R(t) := -\log \frac{R_{\text{sameness}}(t)}{R_{\text{sameness}}(0)} \quad (180)$$

is strictly increasing. Hence S_R is an operational arrow of time: it counts the accumulated reduction of calibrated mismatch independently of clocks or coordinates.

Remark 8 (Schematic evolution laws). A coarse rate model capturing decay and restructure reads

$$\frac{d}{dt} R_{\text{sameness}}(t) = -\beta R_{\text{sameness}}(t) \quad \text{for } R < R_{\text{crit}}, \quad \frac{d}{dt} R_{\text{sameness}}(t) = 2H R_{\text{crit}} \quad \text{at trigger}, \quad (181)$$

where H encodes the instantaneous expansion response during the restructuring window. Between episodes, $R(t) = R(0)e^{-\beta t}$; across an episode, the net is restoring when the post-trigger baseline is lower.

10.4. Entanglement and Curvature as Constraints/Feedback

10.4.1. Global sDoF Budget and Admissible Redistribution

Let the *global* statistical resource be of rank one (primordial sameness). Model the local statistical field as

$$s(x, t) = w(x, t) s_0, \quad w \geq 0, \quad \int_V w(x, t) dx = 1, \quad (182)$$

where s_0 encodes the global sDoF and w the local share. An *admissible* (entangling/coherent) redistribution is given by a pair of contractive maps $(\tilde{\Phi}, \Phi)$ on (S, P) that *intertwine* the calibration:

$$\Phi \circ C = C \circ \tilde{\Phi}, \quad \tilde{\Phi} \circ \mathcal{J} = \mathcal{J} \circ \Phi, \quad (183)$$

and satisfy *data-processing* (contractivity) for the residuals:

$$\|\Phi p - C \tilde{\Phi} s\|_H \leq \|p - Cs\|_H, \quad \|\tilde{\Phi} s - \mathcal{J}(\Phi p)\|_S \leq \|s - \mathcal{J}(p)\|_S. \quad (184)$$

Thus entanglement can *re-weight* w nonlocally (coherently) without creating new sDoF and without inflating the mismatch.

10.4.2. Causality Ceiling for Nonlocal Rebalancing

Assume a finite relay/carrier speed v_* (e.g. Lieb–Robinson bounds on lattices [26–28]) and a correlation diameter ℓ_{corr} . Then any local rise of pDoF due to redistribution obeys a ceiling of the form

$$\frac{d}{dt} \mathfrak{d}(p_{U(t)}) \lesssim \kappa \frac{v_*}{\ell_{\text{corr}}}, \quad (185)$$

for an admissible complexity counter $\mathfrak{d}$, a moving control volume $U(t)$, and a topology-dependent $\kappa \geq 1$. In words: “more pDoF here” cannot outpace the causal relay of the *same* global sDoF share.

10.4.3. Curvature as Feedback (Not a Driver)

In this programme curvature acts as a *feedback* variable slaved to energy redistribution and nonlocal coherence—its role is to *restore* interchangeability (reduce the RoS), not to trigger differentiation. A structural evolution law capturing this reads

$$\partial_t C(x, t) = \lambda \nabla^2 \rho(x, t) + \xi \mathcal{E}(x, t), \quad (186)$$

where the local elliptic part $\lambda \nabla^2 \rho$ ger omedelbar “krök-respons” på gradienter, medan $\xi \mathcal{E}$ representerar den kontraktiva, icke-lokala koherensen (admissibel entanglement-inducerad korrektion). Konsistent geometri (t.ex. Bianchi-identiteter i GR-liknande ramar) uppfylls genom valet av C som rätt variabel (t.ex. Lichnerowicz-liknande operator på fysikalisk subrymd); i denna artikel betraktas (186) som *strukturell*, ej som en fixerad mikromodell.

Remark 9 (Schematic interchangeability PDE). *Abstracting the synchronization requirement between ρ and ρ_{vac} one may write a structural relation*

$$\partial_{tt} \rho - k \partial_{tt} \rho_{\text{vac}} = (\lambda^2 - k \lambda_{\text{vac}}^2) \nabla^4 (\rho + \rho_{\text{vac}}) + \xi \partial_t \mathcal{E}, \quad (187)$$

which combines matched inertial response (left) with biharmonic spatial regularization and a nonlocal, contractive correction (right). Coefficients $(k, \lambda, \lambda_{\text{vac}}, \xi)$ are sectoral; the statement is structural (synchronization & regularization), not a fixed microphysical model.

10.5. Entanglement as Global Admissible Map

We model entanglement-driven redistribution as a *global, admissible* operation that (a) *reweights* local shares of a *single* global sDoF resource (the “one-budget”), (b) *does not create* new sDoF, and (c) *cannot increase* the residual of sameness.

10.5.1. Classical/Quantum Admissibility (Intertwining + Contractivity)

Let $C : S \rightarrow P \subset H$ be the calibration (units/gauge/index alignment) and $(H, \langle \cdot, \cdot \rangle)$ the comparison geometry. An operation on the composite statistical/physical description is said to be *admissible* if there exist linear maps

$$\tilde{\Phi} : S \rightarrow S, \quad \Phi : P \rightarrow P, \quad (188)$$

such that:

1. (Intertwining with calibration)

$$\Phi \circ C = C \circ \tilde{\Phi}, \quad \tilde{\Phi} \circ \mathcal{J} = \mathcal{J} \circ \Phi, \quad (189)$$

i.e. “push-through” identities hold on both sides.

2. (Contractivity/data processing) In the comparison norms,

$$\|\Phi p - \Phi C s\|_H \leq \|p - C s\|_H, \quad \|\tilde{\Phi} s - \tilde{\Phi} \mathcal{J} p\|_S \leq \|s - \mathcal{J} p\|_S. \quad (190)$$

The inequalities (190) are the *residual* form of Jensen/Kadison–Schwarz: for classical Markov maps (positivity, mass preservation) we have L^2 –kontraktivitet via Jensen; för kvantkanaler (CPTP) fås $L^2(\omega)$ –kontraktivitet via Kadison–Schwarz i GNS–geometrin när referenstillståndet bevaras [3]. Till-sammans med (189) ger detta den *direkta residualmonotonin*

$$R_{\text{sameness}}^{\text{new}} = \|\Phi p - C \tilde{\Phi} s\|_H^2 \leq \|p - C s\|_H^2 = R_{\text{sameness}}. \quad (191)$$

10.5.2. “One-Budget” (No Creation of New sDoF)

Let $s(x, t) = w(x, t) s_0$ with a fixed global s_0 (primordial sameness) and a nonnegative “share field” w normalised by $\int_V w = 1$. An admissible entanglement operation induces $w \mapsto w'$ with

$$w'(x) = \int_V K(x, y) w(y) dy, \quad K \geq 0, \quad \int_V K(x, y) dx = 1, \quad (192)$$

i.e. a Markov–typ kernel (eller CPTP–inducerad marginal) som *bevarar* massan av w och därmed den globala sDoF–budgeten. I kvantfall motsvaras “no new sDoF” av rank-/Schmidt–rank–bevarande under lokala/cPTP–operationer (LOCC kan inte öka global entanglement–rank; se t.ex. standardreferenser i kvantinformationsläran).

10.5.3. Causality Ceiling (Nonlocal but Not Instantaneous)

Antag en reläshastighet v_* (Lieb–Robinson–typ på gitter [26–28] eller ändlig våghastighet i PDE) och en korrelationsdiameter ℓ_{corr} . Då kan varje lokal ökning av pDoF genom omfördelning endast ske upp till ett tak

$$\frac{d}{dt} \mathfrak{d}(p_{U(t)}) \lesssim \kappa \frac{v_*}{\ell_{\text{corr}}}, \quad (193)$$

för en admissibel komplexitetsräknare $\mathfrak{d}$ (t.ex. effektiv rang/energi), en rörlig kontrollvolym $U(t)$ och en topologi–beroende $\kappa \geq 1$. Formel (193) kodar att “mer pDoF här” inte kan *överträffa* reläet som för över *samma* globala sDoF–andel.

Konsekvens. Entanglement ger tillåtna (icke–lokala) omviktningar $w \rightarrow w'$ som *aldrig* ökar residualen (191), *aldrig* skapar ny sDoF–budget, och *alltid* respekterar ett kausalitetstak (193).

10.5.4. A Schematic Interchangeability PDE

To encode synchronization of ρ and ρ_{vac} with leading-order curvature neutrality, we use the *schematic* relation

$$eq : interchangeability - schematic \partial_{tt} \rho - k \partial_{tt} \rho_{\text{vac}} = (\lambda^2 - k \lambda_{\text{vac}}^2) \nabla^4 (\rho + \rho_{\text{vac}}) + \xi \partial_t \mathcal{E}. \quad (194)$$

Left-hand side enforces matched inertial response (synchronization), the biharmonic term regularizes spatial structure (suppresses spurious small-scale drift), and the $\partial_t \mathcal{E}$ -term injects the admissible, contractive nonlocal correction (cf. (189)–(190)). Coefficients $(k, \lambda, \lambda_{\text{vac}}, \xi)$ are sector dependent; (187) is therefore a *structural* closure, not a microphysical law.

10.6. Geometry of Cross-Terms: Principal Angles, Two-Sided Bounds, and Stability

Let $U, V \subset \mathcal{H}$ be closed subspaces with orthogonal projectors P_U, P_V . The (canonical) principal angles $0 \leq \theta_1 \leq \dots \leq \theta_m \leq \frac{\pi}{2}$ ($m = \min\{\dim U, \dim V\}$ in finite dimension) quantify the relative position of U and V ; numerically they are the angles whose cosines are the singular values of a representative cross-Gramian (CS decomposition), see [29][Ch. I.4], [30][Ch. 2].

10.6.1. Friedrichs Angle (Use Reduced Projectors when $U \cap V \neq \{0\}$).

If $U \cap V = \{0\}$, one has

$$\|P_U P_V\| = \cos \theta_F, \quad \theta_F \in [0, \frac{\pi}{2}], \quad (195)$$

where θ_F is the Friedrichs angle between U and V . If $U \cap V \neq \{0\}$, then $\|P_U P_V\| = 1$ trivially; the correct definition uses the *reduced* subspaces $U_0 := U \ominus (U \cap V)$ and $V_0 := V \ominus (U \cap V)$:

$$\|P_{U_0} P_{V_0}\| = \cos \theta_F, \quad \theta_F \in [0, \frac{\pi}{2}], \quad (196)$$

see [31][Sec. 2], [29][Ch. I.4]. All statements below assume either $U \cap V = \{0\}$ or that U, V are first replaced by U_0, V_0 .

10.7. Cross-Term Control

Fix $U = \mathcal{P}$ and $V = \overline{\text{ran } \mathcal{I}} \subset \mathcal{H}$. For any decomposition $e = e_U + e_V$ with $e_U \in U$ and $e_V \in V$,

$$|\langle e_U, e_V \rangle| \leq \|P_U P_V\| \|e_U\| \|e_V\| = \cos \theta_F \|e_U\| \|e_V\|. \quad (197)$$

Hence, near orthogonality ($\theta_F \approx \frac{\pi}{2}$) suppresses the mixed term.

Theorem 5 (Near-separability: sharp two-sided bounds). *With $e = e_U + e_V$ as above,*

$$(1 - \cos \theta_F) (\|e_U\|^2 + \|e_V\|^2) \leq \|e\|^2 \leq (1 + \cos \theta_F) (\|e_U\|^2 + \|e_V\|^2). \quad (198)$$

In particular, if $\cos \theta_F \ll 1$ (subspaces nearly orthogonal), then $\|e\|^2 = \|e_U\|^2 + \|e_V\|^2 + O(\cos \theta_F)$ and the residual is nearly additive.

Proof. Use $2\langle e_U, e_V \rangle \leq \cos \theta_F (\|e_U\|^2 + \|e_V\|^2)$ and the reverse bound with sign, then expand $\|e\|^2 = \|e_U\|^2 + \|e_V\|^2 + 2\langle e_U, e_V \rangle$. $\square$

Corollary 2 (Angle-calibrated equivalence constants). *If a residual splits into U - and V -channels, (198) gives explicit equivalence constants with condition number $\kappa(\theta_F) = (1 + \cos \theta_F) / (1 - \cos \theta_F)$ whenever $\theta_F \in (0, \frac{\pi}{2})$.*

10.8. Oblique Projectors and Angle Blow-Up

If one calibrates by projecting *along* a closed complement W onto U (oblique projection $P_{U\|W}$), then

$$\|P_{U\|W}\| = \frac{1}{\sin \vartheta(U, W)}, \quad (199)$$

where $\vartheta(U, W)$ is the minimal principal angle between U and W (finite dimension), hence ill-conditioning as U approaches W ; for the orthogonal case $W = U^\perp$, the norm equals 1. See [30][Thm. 3.1], [32][Sec. 2.6].

10.9. Distance Between Subspaces via Principal Angles (Finite Dimension)

With principal angles $\{\theta_k\}_{k=1}^m$ one has

$$\|P_U - P_V\| = \sin \theta_{\max}, \quad \|P_U P_V^\perp\| = \sin \theta_{\min}, \quad (200)$$

where $\theta_{\max} = \theta_m$ and $\theta_{\min} = \theta_1$; see [29][Ch. I.4], [30][Ch. 2]. Thus small $\|P_U - P_V\|$ (small gap) certifies stability of decompositions and the bounds in (198).

10.10. Stability Under Perturbations (Davis–Kahan/Wedin)

If U and V arise as spectral subspaces of nearby self-adjoint operators with spectral gap $\gamma > 0$ separating target clusters, then

$$\sin \theta_{\max}(U, V) \leq \frac{\|E\|}{\gamma}, \quad (201)$$

(Davis–Kahan [33], [34][Ch. IV]). In the rectangular/SVD setting (Wedin’s theorem), if the cross-Gramian is perturbed by Δ and the singular value gap at the interface is $\delta > 0$, then $\sin \theta_{\max} \lesssim \|\Delta\|/\delta$ [35]. Consequently, the near-separability constants in (198) remain stable under small perturbations provided a gap persists.

10.11. Computing Angles (CS Decomposition)

If Q_U, Q_V have orthonormal columns spanning U, V , then the SVD $Q_U^* Q_V = W \Sigma Z^*$ yields $\Sigma = \text{diag}(\sigma_1, \dots, \sigma_m)$ with $\sigma_k = \cos \theta_k$. In particular,

$$\cos \theta_F = \|P_U P_V\| = \|Q_U^* Q_V\| = \max_k \sigma_k. \quad (202)$$

See [36], [30][Ch. 2].

Remark 10 (Summary). (i) Use reduced subspaces U_0, V_0 when $U \cap V \neq \{0\}$ to define θ_F . (ii) Near orthogonality ($\theta_F \approx \frac{\pi}{2}$) $\Rightarrow$ cross-terms are suppressed and residuals nearly decouple. (iii) Angle bounds are numerically robust and perturbation-stable under standard spectral/SVD gap hypotheses.

10.12. Nonlocal Transfer and DoF Accounting

Interchangeability does not only tell us *what* the statistical and physical channels encode; it constrains *how* the shared degrees of freedom (DoF) can be redistributed across space. In particular, local increases of physical DoF may be explained by nonlocal *conversion* of statistical DoF into physical ones elsewhere, with the *global* shared-DoF budget conserved (up to gauge).

10.12.1. Regionalization and Admissible Intertwiners

Let Ω be the domain, and let $R_U : \mathcal{H} \rightarrow \mathcal{H}_U$ denote restriction to a measurable region $U \subset \Omega$ (with a compatible trace/extension pair so that R_U is bounded and linear). Write $s_U := R_U s$, $p_U := R_U p$, and similarly for compositions with $\mathcal{I}, \mathcal{J}$.

We say that *regional intertwiners* $(\Phi_{U \leftarrow V}, \tilde{\Phi}_{U \leftarrow V})$ are *admissible* if they are bounded linear maps

$$\Phi_{U \leftarrow V} : \mathcal{P}_V \rightarrow \mathcal{P}_U, \quad \tilde{\Phi}_{U \leftarrow V} : \mathcal{S}_V \rightarrow \mathcal{S}_U, \quad (203)$$

which satisfy the *intertwining identities*

$$\Phi_{U \leftarrow V} R_V \mathcal{I} = R_U \mathcal{I} \tilde{\Phi}_{U \leftarrow V}, \quad \tilde{\Phi}_{U \leftarrow V} R_V \mathcal{J} = R_U \mathcal{J} \Phi_{U \leftarrow V}, \quad (204)$$

and are *contractive* in the regional norms (data-processing), i.e. $\|\Phi_{U \leftarrow V} x\|_{P_U} \leq \|x\|_{P_V}$ and $\|\tilde{\Phi}_{U \leftarrow V} y\|_{S_U} \leq \|y\|_{S_V}$ for all x, y . These maps model nonlocal relay/transport and $s \leftrightarrow p$ conversion that respects the interchangeability structure.

10.13. Shared-DoF Counter

Let $\mathfrak{d}(\cdot)$ denote a monotone, subadditive “complexity” counter on regional states, e.g. an L^2 -energy budget, an *effective rank* (nuclear norm over operator norm), a Kolmogorov n -width at a fixed tolerance, or a fixed-threshold entropy number. We only require:

1. **Monotonicity:** $\mathfrak{d}(Tx) \leq \mathfrak{d}(x)$ for contractive T (data-processing).
2. **Subadditivity on partitions:** $\mathfrak{d}(x_{U \cup V}) \leq \mathfrak{d}(x_U) + \mathfrak{d}(x_V)$ when U and V are essentially disjoint (up to null sets) and $x_{U \cup V} = x_U \oplus x_V$.

Define the global shared-DoF budget

$$\mathcal{B}_{\text{glob}} := \mathfrak{d}(P_S s) + \mathfrak{d}(p), \quad (205)$$

and regional budgets $\mathcal{B}(U) := \mathfrak{d}(R_U P_S s) + \mathfrak{d}(p_U)$.

Proposition 13 (Global budget conservation & local rebalancing). *Assume the interchangeability identities and the admissible intertwiners (204). Then:*

1. **Global conservation.** *For any contractive network of admissible intertwiners acting on $(P_S s, p)$, one has*

$$\mathcal{B}_{\text{glob}}^{\text{after}} \leq \mathcal{B}_{\text{glob}}^{\text{before}}, \quad (206)$$

with equality whenever all interconnections are isometries. In particular, the global shared-DoF budget cannot inflate under admissible (data-processing) rebalancing.

2. **Local rebalancing.** *Let $\Omega = U \cup V$ with $U \cap V$ of measure zero (disjoint cover). Then*

$$(\mathcal{B}(U) - \mathcal{B}(U)_{\text{before}}) = -(\mathcal{B}(V) - \mathcal{B}(V)_{\text{before}}) + \mathcal{E}(U \leftrightarrow V), \quad (207)$$

where $\mathcal{E}(U \leftrightarrow V)$ encodes the (nonnegative) loss due to contractivity and vanishes if all intertwiners are isometries (no dissipative coarse-graining). Consequently, a local increase of $\mathfrak{d}(p_U)$ can be balanced by a decrease of $\mathfrak{d}(R_V P_S s)$ via $s \rightarrow p$ conversion through admissible intertwiners.

Remark 11 (Operational reading). “More pDoF in U ” need not be created *ex nihilo*: interchangeability permits import of the global sDoF share and its conversion to pDoF elsewhere in the network. Globally the share is conserved (up to gauge); locally we only change which part of the budget is counted as sDoF versus pDoF and where it resides, with any deficit accounted for by the contraction term $\mathcal{E}(U \leftrightarrow V)$.

10.14. Causality/Entanglement Constraints (Nonlocal, but Not Instantaneous)

If the intertwiners obey a *finite-speed relay* or delay bound (e.g. Halanay-type delay inequality or Lieb–Robinson-type velocity on lattices), then local pDoF growth is *speed-limited*:

$$\frac{d}{dt} \mathcal{B}(U(t)) \leq \text{cap} \cdot \sup_{s \in [t - \tau_{\text{max}}, t]} \mathcal{B}(\partial U(s)), \quad (208)$$

[37] with cap a coupling capacity and $\tau_{\text{max}} \geq \ell_{\text{corr}}/v_*$ set by the relay/entanglement carrier (v_* a Lieb–Robinson or signal speed; ℓ_{corr} a correlation diameter). Thus entanglement can enable *nonlocal*

rebalancing (shares can move coherently), but admissibility (data-processing) forbids *creating* new sDoF; causality forbids *instantaneously* concentrating the global share.

11. Sector Mini-Cases (Standard Instantiations)

All three mini-cases below use well-known constructions and are included as *worked templates*. We state assumptions, the claim, and standard references; full proofs are omitted by design.

11.1. PDE (Flux-Gradient)

11.1.1. Assumptions.

Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain. Work in the zero-mean subspace $H_{\#}^1(\Omega) := \{\rho \in H^1(\Omega) : \int_{\Omega} \rho \, dx = 0\}$ equipped with the gradient norm $\|\rho\|_{H_{\#}^1} := \|\nabla \rho\|_{L^2(\Omega)}$, which is equivalent to the standard H^1 -norm on $H_{\#}^1$ by the Poincaré/Friedrichs inequality (the constants depend only on Ω). Define the statistical and physical spaces

$$S := \overline{\nabla H_{\#}^1(\Omega)}^{L^2(\Omega; \mathbb{R}^d)}, \quad P := H_{\#}^1(\Omega). \quad (209)$$

Let

$$\mathcal{J} : P \rightarrow S, \quad \mathcal{J}(\rho) = \nabla \rho, \quad \mathcal{I} : S \rightarrow P, \quad \mathcal{I}(s) = \rho_{\text{tar}}, \quad (210)$$

where ρ_{tar} solves the Neumann problem

$$-\Delta \rho_{\text{tar}} = -\nabla \cdot s \text{ in } \Omega, \quad \partial_{\nu} \rho_{\text{tar}} = 0 \text{ on } \partial\Omega, \quad \int_{\Omega} \rho_{\text{tar}} \, dx = 0. \quad (211)$$

By Lax–Milgram, this is well-posed whenever $s \in L^2(\Omega; \mathbb{R}^d)$ with $\nabla \cdot s \in H^{-1}(\Omega)$ (e.g. $s \in H(\text{div}, \Omega)$); see [7][Chap. 6].

11.1.2. Claim (Standard)

$\mathcal{I} \circ \mathcal{J} = \text{id}_{H_{\#}^1(\Omega)}$ and $\mathcal{J} \circ \mathcal{I} = P_{\nabla H_{\#}^1(\Omega)}$, the L^2 -orthogonal projector onto $\overline{\nabla H_{\#}^1(\Omega)}$. In particular,

$$\mathcal{R}_{\text{phys}}(s, \rho) := \|\rho - \mathcal{I}(s)\|_{H_{\#}^1(\Omega)}^2, \quad \mathcal{R}_{\text{stat}}(s, \rho) := \|s - \mathcal{J}(\rho)\|_{L^2(\Omega)}^2 \quad (212)$$

are equivalent up to constants that depend only on the Poincaré/Friedrichs constant of Ω .

11.1.3. Justification / References

The Helmholtz (gradient/solenoidal) decomposition in $L^2(\Omega; \mathbb{R}^d)$ and the characterization of gradient fields via Neumann potentials are classical; see [20][Chap. I.1, I.2]. The projector identity $\nabla \mathcal{I} = P_{\nabla H_{\#}^1(\Omega)}$ is the orthogonal projection of s onto the closure of gradients in L^2 (the Neumann potential is unique up to a constant fixed by the zero-mean condition). Norm-equivalence follows from Poincaré/Friedrichs; see [7][§5.8]. Principal-angle geometry used later to quantify cross-terms is in [29].[Ch. VIII]

11.2. OA/QMS (Pointer Algebra in $L^2(\omega)$)

11.2.1. Assumptions

Let $(\mathcal{M}, \omega)$ be a σ -finite von Neumann algebra with faithful normal state ω , GNS Hilbert space $L^2(\omega)$, and modular group $\{\sigma_t^{\omega}\}_{t \in \mathbb{R}}$. Let $\mathcal{N} \subset \mathcal{M}$ be *modular-invariant* ($\sigma_t^{\omega}(\mathcal{N}) = \mathcal{N}$ for all t). Then the ω -preserving conditional expectation $E_{\mathcal{N}} : \mathcal{M} \rightarrow \mathcal{N}$ exists, and its extension to $L^2(\omega)$ is the *orthogonal projector* onto $L^2(\mathcal{N}, \omega)$ (Takesaki–Tomiyama).

11.2.2. Claim (Standard)

Set

$$S = L^2(\mathcal{N}, \omega), \quad P \subset L^2(\omega), \quad \mathcal{I} = \iota : S \hookrightarrow L^2(\omega), \quad \mathcal{J} = E_{\mathcal{N}}|_P. \quad (213)$$

Then

$$\mathcal{I} \circ \mathcal{J} = \text{id}_P, \quad \mathcal{J} \circ \mathcal{I} = \text{id}_S, \quad (214)$$

i.e. the channels are *identical* on $L^2(\omega)$. The canonical residual $\mathcal{R}_{\text{phys}}(X) = \|X - E_N X\|_{2,\omega}^2$ is a metric projection distance. In reversible (self-adjoint) quantum Markov semigroups (QMS), Kadison–Schwarz and Dirichlet–form calculus yield L^2 –contractivity and gap $\Rightarrow$ rate implications for this residual; see, e.g., [3][Thm. 2.2], [38][Chs. 2–3].

11.2.3. Justification / References

Existence and properties of ω –preserving conditional expectations (including orthogonality in $L^2(\omega)$) under modular invariance are in [8,9]. For completely positive unital maps, Kadison–Schwarz and data–processing in noncommutative L^2 are standard [3]. Reversible QMS contractivity and entropy/Dirichlet–form techniques are classical [38].

11.3. OU / Free Field Covariance

11.3.1. Assumptions

Let $\Lambda = \mathbb{T}^d$ (torus) or $\mathbb{R}^d$ with $A := -\Delta + m^2 \geq 0$ (mass $m \geq 0$). Consider the Ornstein–Uhlenbeck covariance flow

$$\dot{\Sigma}_\tau = -A\Sigma_\tau - \Sigma_\tau A + 2I, \quad \Sigma_0 \geq 0, \quad (215)$$

on trace–class operators on $L^2(\Lambda)$. Assume either finite volume or $m > 0$ so that A^{-1} is bounded on the orthogonal complement of $\ker A$.

11.3.2. Claim (Standard)

The solution is

$$\Sigma_\tau - \Sigma_\infty = e^{-\tau A} (\Sigma_0 - \Sigma_\infty) e^{-\tau A}, \quad \Sigma_\infty = A^{-1} \quad (\text{on } \ker A^\perp). \quad (216)$$

Hence, for any test f with $Af = \lambda f$ ($\lambda > 0$),

$$\mathcal{R}_{\text{phys}}[f; \tau] := |\langle f, (\Sigma_\tau - \Sigma_\infty)f \rangle| \leq e^{-2\lambda\tau} \mathcal{R}_{\text{phys}}[f; 0]. \quad (217)$$

Equivalently, the (unsquared) two–point residual decays at twice the Hamiltonian gap; squaring the residual doubles the semi–log slope. In the massless infinite–volume case ($m = 0$, $\Lambda = \mathbb{R}^d$), no uniform exponential envelope exists; band–limiting $|\hat{f}| \subset \{|k| \geq k_0\}$ recovers a rate $2k_0^2$.

11.3.3. Justification / References

The Lyapunov semigroup solution, covariance convergence, and exponential envelopes via spectral bounds are standard in the stochastic PDE / OU literature; see [12][Chs. 5–7]. The stochastic quantization perspective (Parisi–Wu) provides physics background [23].

12. Equivalence vs Identity (Concise)

This section compresses when the statistical and physical channels are (i) *identical*, (ii) *equivalent modulo a projection* (quotient/gauge), or (iii) *exclusive* (no shared DoF). We rely on the interchangeability identities and residual results already proved.

12.1. What Changes (And What Doesn't)

- **Always:** two–way consistency and residual norm–equivalence hold (see the main text).
- **Identity:** if $P_S = \text{id}$ and $\mathcal{I}$ is an isomorphism with bounded inverse $\mathcal{J} = \mathcal{I}^{-1}$, then the channels carry the *same* DoF and both residuals vanish together (no gauge).
- **Quotient/gauge:** if $\ker P_S \neq \{0\}$ or $\dim S \neq \dim P$, equality holds *modulo* P_S on the quotient $S_q := S / \ker P_S$; residuals remain norm–equivalent.

- **Exclusivity:** if $\text{ran}(\mathcal{I}) \cap \mathcal{P} = \{0\}$ (on the admissible statistical subspace), there are no nontrivial shared DoF.

12.2. Quick Decision Table (Finite Dimension)

Let $\dim \mathcal{S} = m$, $\dim \mathcal{P} = n$, and A be the matrix of $\mathcal{I}$.

Case	Conditions	Verdict	Minimal test
Identity	$P_{\mathcal{S}} = \text{id}$, $\text{rank}(A) = n = m$	same DoF	A invertible
Quotient eqv.	$\text{rank}(A) = n$, $\ker P_{\mathcal{S}} \neq \{0\}$	equal mod $P_{\mathcal{S}}$	full column rank
Exclusivity	$\text{ran}(A) \cap \mathcal{P} = \{0\}$	no shared DoF	rank additivity / angle

12.3. Cross-Term Geometry and Near-Separability

Let $U = \mathcal{P}$ and $V = \overline{\text{ran } \mathcal{I}} \subset \mathcal{H}$ with orthogonal projectors P_U, P_V . Denote the Friedrichs angle by $\theta_F \in [0, \pi/2]$ defined via $\|P_U P_V\| = \cos \theta_F$ (see [29]; [Ch. VIII] [30] [Chs. I–II]). For $e = e_U + e_V$ with $e_U \in U, e_V \in V$,

$$|\langle e_U, e_V \rangle| \leq \cos \theta_F \|e_U\| \|e_V\|. \quad (218)$$

Thus

$$(1 - \cos \theta_F) (\|e_U\|^2 + \|e_V\|^2) \leq \|e\|^2 \leq (1 + \cos \theta_F) (\|e_U\|^2 + \|e_V\|^2), \quad (219)$$

i.e. if $\theta_F \approx \pi/2$ the residual is nearly additive; the condition number is $\kappa(\theta_F) = (1 + \cos \theta_F)/(1 - \cos \theta_F)$. If U, V are spectral/SVD subspaces of nearby operators with a spectral gap $\gamma > 0$, principal angles are Lipschitz in the perturbation norm: Davis–Kahan/Wedin bounds give $\sin \theta_{\max} \lesssim \|E\|/\gamma$, hence the bounds above are stable [30,33,35].

12.4. Canonical Examples

For OA/QMS L^2 , modular invariance yields an $L^2(\omega)$ -orthogonal conditional expectation and identity of channels on L^2 (pointer); for PDE on $H_{\#}^1$, potentials and gradients are identical after fixing the mean (before that, equality is modulo the Neumann gauge); for OU/free-field covariances, $\mathcal{I} = \mathcal{J} = \text{id}$ (identical channels).

13. Illustrations (Reproducible “Hello-Worlds”)

We record three compact tests—one per sector—that numerically (or algebraically) verify the identities and diagnostics above. Each isolates one dimension/channel to keep the geometry transparent.

13.1. 1D Heat (Flux–Gradient) on $\mathbb{T}^1$

13.1.1. Setup

$\Omega = \mathbb{T}^1 = [0, 1)$, periodic boundary. Single Fourier mode: $u(x, 0) = \sin(2\pi x)$; $u_t = u_{xx} \Rightarrow u(x, t) = e^{-4\pi^2 t} \sin(2\pi x)$. Physical channel $\rho(\cdot, t) = u(\cdot, t)$; statistical channel $s(\cdot, t) = \partial_x \rho(\cdot, t)$. Define $\mathcal{J}(\rho) = \partial_x \rho$, $\mathcal{I}(s) = \rho_{\text{tar}}$ solving $-\partial_{xx} \rho_{\text{tar}} = -\partial_x s$ with zero mean.

13.1.2. Exact Identities (Closed-Form)

Since $s = \partial_x \rho$, we have $-\partial_x s = -\partial_{xx} \rho$, so $\rho_{\text{tar}} = \rho$ (uniqueness by zero mean). Hence $\mathcal{I} \circ \mathcal{J} = \text{id}_{H_{\#}^1}$ pointwise in time and $\mathcal{J} \circ \mathcal{I}(s) = P_{\nabla H_{\#}^1} s = s$ because s is already a gradient. The residuals vanish:

$$\mathcal{R}_{\text{phys}}(s, \rho) = \|\rho - \mathcal{I}(s)\|_{H_{\#}^1}^2 = 0, \quad \mathcal{R}_{\text{stat}}(s, \rho) = \|s - \mathcal{J}(\rho)\|_{L^2}^2 = 0. \quad (220)$$

13.1.3. Discrete Check

On a grid $x_j = j/N$, centered difference D and periodic Laplacian L . Evolve $u^{n+1} = u^n + \Delta t L u^n$ ($\Delta t \lesssim N^{-2}$), set $\rho^n = u^n$, $s^n = D\rho^n$, solve $L\rho^{\text{tar},n} = Ds^n$ with zero-mean constraint. Verify $\|\rho^n - \rho^{\text{tar},n}\|_{H_h^1} \rightarrow 0$ and $\|s^n - D\rho^n\|_{2,h} \rightarrow 0$ as $N \rightarrow \infty$ (projection/residual tests). For mixed initial data (multi- k), modewise identities still hold. Principal angles between $\overline{\text{ran } \mathcal{I}}$ and P can be recovered via SVD of discrete bases [29,39].

13.2. Pointer Projection (OA/QMS) in Finite Dimension

13.2.1. Setup

$\mathcal{M} = M_d(\mathbb{C})$ with faithful state $\omega(\cdot) = \text{Tr}(\rho_\omega \cdot)$, and $\mathcal{N} = \text{diag}$ in a fixed basis. The ω -preserving conditional expectation (Lüders pinching) is $E_{\mathcal{N}}(X) = \sum_i P_i X P_i$ with $P_i = |i\rangle\langle i|$. In the GNS $L^2(\omega)$ norm, $\|X\|_{2,\omega}^2 = \omega(X^*X)$, $E_{\mathcal{N}}$ is the *orthogonal projector* onto $L^2(\mathcal{N}, \omega)$ [8,9].

13.2.2. Test

Draw random X and compute $\mathcal{R}_{\text{phys}}(X) = \|X - E_{\mathcal{N}}X\|_{2,\omega}^2$. Under any ω -preserving u.c.p. map Φ intertwining pointer algebras ($\Phi E_{\mathcal{N}_1} = E_{\mathcal{N}_2} \Phi$), Kadison–Schwarz gives $\|\Phi(X) - E_{\mathcal{N}_2} \Phi(X)\|_{2,\omega} \leq \|X - E_{\mathcal{N}_1} X\|_{2,\omega}$ [3]. Numerically, random convex combinations of unitary conjugations ($\omega \propto I$) Monte–Carlo the contraction.

13.3. OU/Free Field Two-Point Residual

13.3.1. Setup (Fourier Mode)

On $\mathbb{T}^1$, take $A = -\partial_{xx} + m^2$ with $m > 0$ so eigenvalues $\lambda_k = (2\pi k)^2 + m^2$. For $f_k(x) = \sin(2\pi kx)$,

$$\mathcal{R}_{\text{phys}}[f_k; \tau] = |\langle f_k, (\Sigma_\tau - \Sigma_\infty) f_k \rangle| = |\langle f_k, e^{-\tau A} (\Sigma_0 - \Sigma_\infty) e^{-\tau A} f_k \rangle| \leq e^{-2\lambda_k \tau} \mathcal{R}_{\text{phys}}[f_k; 0]. \quad (221)$$

Thus the semi-log slope equals $-2\lambda_k$ (unsquared residual). On a discrete grid, diagonalize by FFT and verify the envelope numerically. For $m = 0$ on $\mathbb{R}$, band-limit $|\hat{f}| \geq k_0$ to get $e^{-2k_0^2 \tau}$; without a band-limit there is no uniform exponential bound [12,23].

13.3.2. Takeaway

These tests exercise: (i) projector identities and norm-equivalence (PDE), (ii) orthogonal pointer projection and data processing (OA/QMS), (iii) gap-controlled envelopes (OU/free). All computations match the criteria and theorems in Secs. 11–12.

14. Concluding Discussion

This work develops a sector-neutral kinematics of *sameness* and a deterministic account of *entanglement* that does not rely on tensor-product postulates. The structural heart consists of (i) interchangeability maps $\mathcal{I}, \mathcal{J}$ yielding a two-way equivalence criterion $\mathcal{I}\mathcal{J} = \text{id}_P$, $\mathcal{J}\mathcal{I} = P_S$ (Thm. 3); (ii) a single observable residual $\mathcal{R}$ that is norm-equivalent across statistical and physical sides (Prop. 1); and (iii) a data-processing principle: any admissible (intertwining, contractive) redistribution cannot increase $\mathcal{R}$ (Lem./Prop. 1–4). Under an optional one-budget hypothesis, admissible maps conserve the global statistical resource (Lem. 2); with a finite-speed relay, local growth obeys a causal ceiling (Lem. 3); and when a sector supplies a Lyapunov identity and a gap, $\mathcal{R}$ becomes a Lyapunov functional with an explicit envelope and a robust “arrow-of-time” even in the presence of short restorative windows (§5.11).

What is structural vs. what is sectoral.

The interchangeability identities, the norm-equivalence of residuals, and the residual monotonicity under admissible maps are *purely kinematic*: they hold without invoking a dynamics. Resource conservation (one-budget) and the causality ceiling add structural constraints that are independent of microphysical detail. Rate statements and “arrow” results are *sector-gated*: they require a Lyapunov

identity and gap/coercivity (QMS Poincaré/log–Sobolev, OU spectral gap, PDE ellipticity). This separation clarifies precisely which conclusions travel across fields and which depend on sectoral hypotheses.

Implications.

(i) *Decision procedure*. The projector identities furnish a falsifiable test for “same state” across channels; $\mathcal{R}$ elevates that decision to a measurable diagnostic. (ii) *Deterministic entanglement*. Entanglement appears as admissible, contractive redistribution of a single statistical resource, with a built-in speed limit and a non-factorization witness (Thm. 4). (iii) *Geometry and conditioning*. Principal-angle bounds quantify cross-terms and certify when residuals are near-separable, providing stable conditioning constants for analysis and numerics (§9). (iv) *Portability*. The framework instantiates cleanly in three standard settings (PDE flux–gradient, OA/QMS pointer, OU/free), demonstrating that the same residual and monotonicity principles operate beyond any single formalism (§11).

Limitations and scope.

The theory assumes bounded $\mathcal{I}, \mathcal{J}$ and a continuous P_S ; approximate settings are handled in a tolerance form with $O(\epsilon^2)$ monotonicity defects (§7). Rate claims require the sectoral gap; massless/infinite-volume limits (OU) or strong gauge/geometry couplings (PDE/GR slices) may lack a uniform envelope unless band-limited or coercivity-reinforced. The one-budget picture is a modeling choice: useful for resource accounting and ancestry, but optional for the kinematic core.

For practice (adoption checklist).

(1) Specify S, P and construct $\mathcal{I}, \mathcal{J}$ at the *same differential order*. (2) Run the projection test $\mathcal{I}\mathcal{J} \approx \text{id}_P$, $\mathcal{J}\mathcal{I} \approx P_S$ at tolerance. (3) Compute $\mathcal{R}$ and verify monotonicity under a small battery of admissible maps (smoothing/Markov, conditional expectations, reversible steps). (4) Report principal angles $(\cos \theta_F, \kappa(\theta_F))$ for interpretability. (5) Where relevant, document sectoral gaps/coercivity used to claim rates.

Outlook.

Two directions look particularly promising. First, *data-driven calibrations*: learn $\mathcal{I}$ (with $\mathcal{J}$ constrained by $\mathcal{J}\mathcal{I} = P_S$) so that residual monotonicity becomes a validation loss; this ties the theory to robust ML pipelines with built-in no-inflation guarantees. Second, *networked redistribution*: extend the causality ceiling and non-factorization witness to graphs and multiplexed carriers, yielding quantitative ancestry trees backed by residual accounting. Both continue the paper’s theme: combining minimal kinematics (what “same” means) with testable constraints on how sameness can be redistributed—deterministically, contractively, and, when dynamics allow, with a clean arrow of time.

Executive Takeaways (Claims & Why They Matter)

- **C1 — Kinematics**. Interchangeability maps $\mathcal{I}, \mathcal{J}$ with $\mathcal{I}\mathcal{J} = \text{id}_P$, $\mathcal{J}\mathcal{I} = P_S \Rightarrow$ sector-neutral “same state” test (Thm. 3).
- **C2 — Observable**. One residual $\mathcal{R} = \|p - \mathcal{I}s\|_H^2$ (norm-equivalent to the statistical side) $\Rightarrow$ measurable, isometry-invariant misfit (Prop. 1).
- **C3 — Monotonicity**. Any admissible (intertwining, contractive) redistribution $\Rightarrow \mathcal{R}(\Phi s, \Phi p) \leq \mathcal{R}(s, p)$ (Lem./Prop. 1–4).
- **C4 — Resource & causality**. One-budget conservation (Lem. 2) and finite-speed ceiling (Lem. 3) $\Rightarrow$ no new sDoF; speed-limited local growth.
- **C5 — Intrinsic nonlocality**. Residual-reducing, share-moving steps across a cut are not simulable by local products (Thm. 4) $\Rightarrow$ deterministic “entanglement” witness.
- **C6 — Rates (sectoral)**. With a Lyapunov identity & gap, $\dot{\mathcal{R}} \leq -\alpha \mathcal{R} \Rightarrow$ Lyapunov envelope / arrow of time; trigger windows remain restorative (§5.11).

Consequences (practical & theoretical)

For theory.

Minimal kinematic foundation on which sector dynamics can be layered; when dynamics exist, R is a clean Lyapunov functional with rate α . Entanglement is recast as a structural property of admissible intertwiners: no increase of R , no new sDoF, finite-speed relay—matching LOCC monotonicity/no-signalling, but derived deterministically.

For computation & experiments.

Model selection & falsification: if a calibration $\mathcal{I}$ or pointer $\mathcal{J}$ fails projection/monotonicity on held-out data, discard/recalibrate. Robust reporting: principal-angle diagnostics ($\cos \theta_F$, $\kappa(\theta_F)$) certify near-separability. Reusable templates: PDE (Neumann potential + gradient residual), OA/QMS (pointer projection), OU (covariance gap).

For cross-domain translation.

A common yardstick to compare “agreement” claims between statistical blueprints and physical responses across quantum platforms, transport models, or Gaussian fields.

Limitations (explicit).

Rates α are sector-gated (need gaps/coercivity). Some systems admit only approximate projectors/intertwiners; small-defect variants are noted; a full treatment is future work.

One-paragraph elevator pitch

The paper builds a minimal, sector-neutral geometry in which a statistical description s and a physical description p are provably the same state via two identities, and it promotes their calibrated difference to a single observable residual R . Any admissible (intertwining, contractive, budget-preserving) operation can only *decrease* R , so the phenomenon usually called “entanglement” becomes a deterministic, resource-preserving redistribution—not a postulate. When a sector supplies a gap, R is a Lyapunov functional with an explicit decay rate. The framework yields portable diagnostics (projection & data-processing tests, angle bounds) and works out of the box in three standard sectors (PDE, OA/QMS, OU).

15. Discussion

15.1. What Interchangeability Buys

Interchangeability formalizes when two descriptions are the same state in different coordinates, and when they agree only up to a canonical projection. The two-map consistency $\mathcal{I}\mathcal{J} = \text{id}_p$ and $\mathcal{J}\mathcal{I} = P_S$ is a minimal, geometry-agnostic structure that: (i) supplies a single residual on either side (physical-side and statistical-side); (ii) yields norm-equivalence (either residual controls the other up to fixed constants); (iii) makes data-processing monotone by functoriality (admissible coarse-grainings that intertwine the pair cannot inflate the residual; cf. Sec. 7); and (iv) exposes cross-term geometry through principal angles, giving sharp near-separability bounds when the calibrated ranges are close to orthogonal (Sec. 9).

15.2. Identity, Exclusivity, and Identicality.

When $P_S = \text{id}$ and $\mathcal{I}$ is bijective with $\mathcal{J} = \mathcal{I}^{-1}$, the channels are *identical* degrees of freedom (Prop. 15.2): no gauge ambiguity survives and both residuals vanish simultaneously. When $P_S \neq \text{id}$ (gauge/quotient case), the channels are *exclusive* only up to the null directions of P_S : distinct statistical representatives that differ by $\ker P_S$ map to the same physical state. Passing to the gauge-reduced quotient $S_q := S / \ker P_S$ restores *identity*— $\mathcal{I}$ descends to an isomorphism $\bar{\mathcal{I}} : S_q \rightarrow P$ with inverse $p \mapsto [\mathcal{J}p]$ (Prop. 15.2). In short: equality holds exactly where it can (on P), and modulo the right identification where it should (on S).

15.3. Geometric Conditioning (Why Angles Matter)

Principal/Friedrichs angles between P and $\overline{\text{ran } \mathcal{I}}$ calibrate the cross-term size and the conditioning of any two-channel residual. The two-sided bounds in Sec. 15.2 show that $\cos \theta_F \ll 1$ yields near-additivity: $\|e_U + e_V\|^2 = \|e_U\|^2 + \|e_V\|^2 + O(\cos \theta_F)$, and provide explicit equivalence constants $\kappa(\theta_F) = (1 + \cos \theta_F)/(1 - \cos \theta_F)$ for separable Lyapunov functionals. This gives a clean, quantitative answer to “when can we decouple the channels?” and “how stable is the decomposition under perturbations?” (Davis–Kahan/Wedin-type stability).

15.4. Nonlocal Transfer and DoF Accounting

Beyond pointwise alignment, Sec. 10.12 shows how interchangeability constrains *redistribution* of shared DoF across space: regional intertwiners that are contractive and intertwine the pair conserve the global shared-DoF budget up to gauge, while permitting *local rebalancing*. Operationally, “more pDoF here” can be accounted for by $s \rightarrow p$ conversion elsewhere plus relay, without creating new statistical DoF. In physical terms, entanglement-like constraints act as admissible, nonlocal maps that enable coherent redistribution (subject to contractivity), while causality-type ceilings (e.g., Lieb–Robinson-style speed limits) cap how fast such rebalancing can raise local pDoF.

15.5. Portability (Why This Is Broadly Useful)

The framework is portable and testable with minimal structure: L^2 energy/projection in PDE (flux–gradient), $L^2(\omega)$ orthogonal conditional expectations in OA/QMS (pointer algebras), and operator semigroups/covariances in OU/free fields. Each sector supplies concrete $\mathcal{I}/\mathcal{J}$, a canonical residual, and diagnostics (projection test, data processing, angle geometry). This suffices to certify interchangeability and to quantify mismatch in a numerically reproducible manner.

15.6. Limitations and Next Steps

(i) *Approximate projectors*. Nonlinear/coarse models may only admit best-approximation maps (oblique or data-driven surrogates). Then P_S becomes approximate, and the identities hold up to controlled defects. Extending residual norm-equivalence and angle bounds to this setting is immediate future work. (ii) *Gauge and geometry*. In geometric/gauge problems one must specify the physical subspace (constraints/slicing) and ensure strict ellipticity before defining robust $\mathcal{I}, \mathcal{J}$. Gauge leakage appears as $\ker P_S$; quotienting by it is essential. (iii) *Infinite dimension and discretization*. The principal-angle calculus is classical in finite dimension and extends to closed subspaces under compactness or spectral gaps. Turning those bounds into stable, mesh-independent diagnostics on PDE/field discretizations (basis construction, randomized subspace probes, posterior angle error bars) is a practical numerical agenda. (iv) *Dynamics add-on*. This paper is geometric and static by design. If one couples interchangeability to a sector’s dynamics (e.g., a Lyapunov/contractive flow), the same residual becomes a natural Lyapunov functional (variance-type in OA/QMS; energy-type in PDE/OU), and rate statements follow under the sector’s gate (spectral gap/coercivity). Keeping the static and dynamic layers separate clarifies which conclusions are purely kinematic (interchangeability) and which are time-evolution (decay rates).

15.6.1. Operational Interpretation of Residuals

The residual $R(s, p) = \|p - \mathcal{I}s\|_H^2$ may be treated as an *effective observable*: a gauge-invariant (or projector-invariant) measure of statistical-physical alignment. In numerics, it can be monitored as a convergence or model-selection diagnostic (Sec. 7). In data pipelines, R can be bounded from above by fitting $\mathcal{I}$ and simple coarse-grainings Φ and checking monotonicity $R(\tilde{\Phi}s, \Phi p) \leq R(s, p)$. Low R over a family of admissible maps indicates that the chosen calibration captures the physical subspace well; sustained high R falsifies that identification and suggests a different pointer/geometry.

15.7. Takeaways

Interchangeability provides a small set of identities and tests that (a) certify “same state in two coordinates” vs “equivalent modulo projection”, (b) produce a single residual with norm–equivalence guarantees, (c) make coarse–graining safely contractive, and (d) quantify decoupling via angles. Those ingredients are lightweight, sector–neutral, and immediately implementable—useful both for theory (clean separation of identity vs equivalence) and for practice (robust diagnostics on real computations or data).

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Appendix A. Generic Two–Channel Application Template (Optional)

Let $s(t) \in S$ and $p(t) \in P \subset H$, and let a calibration $\mathcal{C} : S \rightarrow P$ align types/units. Define the misfit $e(t) := p(t) - \mathcal{C}s(t)$ and the residual $\mathcal{R}(t) := \|e(t)\|_H^2$. If the sector furnishes an identity

$$\dot{\mathcal{R}}(t) = -2\langle Ke, e \rangle_H + 2\langle e, g \rangle_H, \quad K = K^* \succeq 0, \quad |\langle e, g \rangle| \leq \varepsilon \|e\|_H^2, \quad (\text{A1})$$

and a gap $\langle Ke, e \rangle \geq \kappa \|e\|_H^2$, then

$$\dot{\mathcal{R}} \leq -(2\kappa - 2\varepsilon)\mathcal{R}, \quad \mathcal{R}(t) \leq e^{-(2\kappa - 2\varepsilon)t}\mathcal{R}(0). \quad (\text{A2})$$

This “propagation+gap” envelope is orthogonal to interchangeability, but pairs well with the residuals in this paper when one wishes to study dynamics.

Appendix B. Computing R and Principal Angles in Practice

Residual. Given $p \in P \subset H$ and $s \in S$, compute $\mathcal{I}s$ by your sector’s solver: Neumann solve (PDE), conditional expectation (OA/QMS), identity (OU). Then $R = \|p - \mathcal{I}s\|_H^2$.

Principal angles. Assemble orthonormal bases Q_U for P and Q_V for $\overline{\text{ran } \mathcal{I}}$ (e.g. QR on snapshot matrices). Compute the SVD $Q_U^* Q_V = W \Sigma Z^*$; the singular values are $\sigma_k = \cos \theta_k$ and $\cos \theta_F = \sigma_{\max}$. Use the two–sided bounds $(1 - \cos \theta_F)(\|e_U\|^2 + \|e_V\|^2) \leq \|e\|^2 \leq (1 + \cos \theta_F)(\|e_U\|^2 + \|e_V\|^2)$ to certify near–separability [29,39].

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