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Extensions of Derivations on JC-algebras

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Abstract: Extension of a derivation on a universally reversible JC-algebra $A \subseteq B(H)_{sa}$ to the C*-algebra $[A]$ generated by A in $B(H)$ was studied by Upmier [20, Theorem 2.5]. In this article we study the extension of a Jordan derivation on a universally JC-algebra A to its universal enveloping real and complex C*-algebras $\mathfrak{R}$ and $\mathfrak{A}$, respectively. Also, we establish the relationship between local derivations (resp., 2-local derivations, weak local derivations, weak-2-local derivations) of a universally JC-algebra A and the corresponding maps on its universal enveloping real and complex C*-algebras $\mathfrak{R}$ and $\mathfrak{A}$, respectively.

Keywords: JC-algebras; JW-algebras; C*-algebras; von Neumann aalgebra; Banach bimodules; derivations

MSC: Primary 46L05; 46L10; 46L57; 47B47; Secondary 15A86; 47C15

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1. Preliminaries

A uniformly closed Jordan subalgebra of the set of all continuous linear self adjoint operator $\mathfrak{B}(\mathfrak{H})_{sa}$ on a complex Hilbert space $\mathfrak{H}$ is called a *JC-algebra*, and its weak closure is called a *JW-algebra*. Let A be a JC-algebra, then there there is a unique (up to an isomorphism) a C*-algebra $\mathfrak{A}$, a Jordan isomrphism $\psi : A \to \mathfrak{A}$ such that $\psi(A)$ generates $\mathfrak{A}$ as a C*-algebra, and $\mathfrak{A}$ has a unique * antiautomrphism Φ of order 2 keeping the points of $\psi(A)$ fixed. The set $\mathfrak{R} = \{x \in \mathfrak{A} : \Phi(x) = x^*\}$ is a real C*-sublgebra of $\mathfrak{A}$ which satisfies $\mathfrak{R} \cap i\mathfrak{R} = \{0\}$, and $\mathfrak{A} = \mathfrak{R} \oplus i\mathfrak{R}$ [**1,** Proposition 4.40, Lemma 4.41].

We refer the reader to [**1, 4, 8, 13, 16, 17, 19**] for the needed background of the theory of JC-algebras and JW-algebras, and to [**10, 11, 12, 18**] for the prerequisite background of C*-algebras and von Neumann algebras. Sufficient information about derivations can be found in [**3, 5, 6**]. Throughout this paper, we identify A with $\psi(A)$ in $\mathfrak{A}$, and assume that A is a universally reversible JC-algebra.

2. Derivations

The proof of the following theorem is almost as that of [**20,** Theorem 2.5], except we make the extension of the Jordan derivation on a universally reversible JC-algebra to its universal enveloping C*-algebra (see [**1,** Proposition 4.36], [**8,** 7.1.8] for its existance and propertites). Since this theorem plays an essential role in our results, we include the proof for completeness. But first note that if $\varphi : A \to B$, where A and B are JC-algebras, and if $\delta : A \to A$ is a Jordan derivation (i.e. $\delta(a \circ b) = \delta(a) \circ b + a \circ \delta(b)$, for all $a, b \in A$), then $\widetilde{\delta} : \varphi(A) \to \varphi(A)$ defined by $\widetilde{\delta}(\varphi(a)) = \varphi(\delta(a))$, for all $a \in A$, is a Jordan derivation.

Theorem 2.1. *Suppose that $D : A \to A$ is a Jordan derivation on a universall reversible JC-algebra A. Then D can be extended to a *-derivation $\hat{D} : \mathfrak{A} \to \mathfrak{A}$ of the C*-algebra $\mathfrak{A}$.*

Proof. Let ρ be a pure state of $\mathfrak{A}$, and let $\{\pi_\rho, H_\rho, \zeta_\rho\}$ be the cyclic reprentation of $\mathfrak{A}$ induced by ρ, by the Gelfand-Neumark-Segal construction (GNS construction). By [**18,** Theorem 9.22], [**10**, Theorm 4.5.5], $\pi_\rho : \mathfrak{A} \to B(H_\rho)$ is irreducible, so that, $\overline{\pi_\rho(\mathfrak{A})}^w = (\pi_\rho(\mathfrak{A}))'' = B(H_\rho)$, where $\overline{\pi_\rho(\mathfrak{A})}^w$ is the σ-weak

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closure of $\pi_\rho(\mathfrak{A})$ in $B(H_\rho)$, and $\pi_\rho(\mathfrak{A})''$ is the double commutant of $\pi_\rho(\mathfrak{A})$ [**12,** Theorem 3.13.2], [**15,** Proposition 1.21.9]. Having $\mathfrak{A}^{**}$ and A^{**} as the second duals of $\mathfrak{A}$ and A, respectively, and deducing that π_ρ is a type I factor representation, then by [**18,** Lemma 2.2], [**15,** Theorem 1.9.1], π_ρ extends to a normal representation $\pi_\rho^{**} : \mathfrak{A}^{**} \to B(H_\rho)$, which is irreducible, since $\pi_\rho^{**}(\mathfrak{A}^{**}) = \overline{\pi_\rho(\mathfrak{A})}^{w} = B(H_\rho)$. Put $M = \pi_\rho^{**}(A^{**}) = \pi_\rho(A)'' = \overline{\pi_\rho(A)}^{w}$, then the restriction $\pi_\rho\mid_A$: $A \to B(H_\rho)$ is an irreducible representation. Note that A^{**} is universally reversible, since A is universally reversible, [**1,** Lemma 4.33], and so, M is a reversible JW-factor of type I_n, for some $n \geq 3$ [**8,** Corollary 5.3.7], which implies that the self-adjoint part $Z(\mathfrak{M})_{sa}$ of the center $Z(\mathfrak{M})$ of the The universal enveloping von Neumann algebra $\mathfrak{M}$ of of M equals to the center $Z(M)$ of M by [**8,** Theorem 7.3.5]. Therefore, $\mathfrak{M}$ is a factor of type I, by [**8,** Theorem 7.4.2 (i)]. By [**2,** Theorem 3.1], (see also [**9,** Theorem 7.5.11]), either $M \cong B(H_\rho)_{sa}^{\alpha}$ with α *a real flip* on $B(H_\rho)$, or $M \cong B(H_\rho)_{sa}^{\beta}$ with β a quaternionian flip on $B(H_\rho)$.This means that $M \cong B(K)_{sa}$ for some $\mathbb{K}$-Hilbert space on H_ρ, where $\mathbb{K} = (\mathbb{R}$, or $\mathbb{C}$, or $\mathbb{H})$. Note that $D_1 : \pi_\rho(A) \to \pi_\rho(A)$ defined by $D_1(\pi_\rho(a)) = \pi_\rho(D(a))$ for all $a \in A$, is a Jordan derivation, which extends to a normal Jordan derivation $D_2 : \overline{\pi_\rho(A)}^{w} \to \overline{\pi_\rho(A)}^{w}$, by [**15,** Lemma 4.1.4], then $\pi_\rho(D(a)) = D_2(\pi_\rho(a))$ for all $a \in A$. Therefore, $D_2(d) = [w_\rho, d]$ for all $d \in \overline{\pi_\rho(A)}^{w} = \pi_\rho^{**}(A^{**})$, and for some element $w_\rho \in B(H_\rho)$, by [**20,** Lemma 2.6]. Let $H = \sum^{\oplus}_{\rho\in P(\mathfrak{A})} H_\rho$, $\pi = \sum^{\oplus}_{\rho\in P(\mathfrak{A})} \pi_\rho$, $w = \sum^{\oplus}_{\rho\in P(\mathfrak{A})} w_\rho$, then, for all $a \in A$, we have

$$\begin{aligned}
\pi(D(a)) &= (\sum^{\oplus}_{\rho\in P(\mathfrak{A})} \pi_\rho)(D(a)) \\
&= \sum^{\oplus}_{\rho\in P(\mathfrak{A})} (\pi_\rho(D(a)) \\
&= \sum^{\oplus}_{\rho\in P(\mathfrak{A})} [w_\rho, \pi_\rho(a)] = \sum^{\oplus}_{\rho\in P(\mathfrak{A})} (w_\rho \pi(a) - \pi_\rho(a) w_\rho) \\
&= (\sum^{\oplus}_{\rho\in P(\mathfrak{A})} w_\rho)(\sum^{\oplus}_{\rho\in P(\mathfrak{A})} \pi_\rho(a)) - (\sum^{\oplus}_{\rho\in P(\mathfrak{A})} \pi_\rho(a))(\sum^{\oplus}_{\rho\in P(\mathfrak{A})} w_\rho) \\
&= w\pi(a) - \pi(a)w = [w, \pi(a)].
\end{aligned}$$

That is, $[w, \pi(a)] = \pi(D(a)) \in \pi(A) \subseteq \pi(\mathfrak{A})$. Since A generates $\mathfrak{A}$ as a C*-algebra, we have $[w, \pi(x)] \in \pi(\mathfrak{A})$ for all $x \in \mathfrak{A}$. It is clear that $\overline{D} : \pi(\mathfrak{A}) \to \pi(\mathfrak{A})$ defined by $\overline{D}(\pi(x)) = [w, \pi(x)], x \in \mathfrak{A}$, is a *-derivation. Since $\pi : \mathfrak{A} \to B(H)$ is a faithful representation of $\mathfrak{A}$ (see [**10,** Proposition 4.5.5 and Theorem 4.5.6]), we can easily see that $\hat{D} : \mathfrak{A} \to \mathfrak{A}$ defined by $\hat{D}(x) = (\pi^{-1} \circ \overline{D} \circ \pi)(x)$ is a *-derivation on $\mathfrak{A}$ extending D. □

Remark 2.2. *Given a *-derivation $\delta : \mathfrak{R} \to \mathfrak{R}$ of the real C*-algebra $\mathfrak{R}$. Then $\tilde{\delta} : \mathfrak{A} \to \mathfrak{A}$ defined by $\tilde{\delta}(x + iy) = \delta(x) + i\delta(y)$, $x, y \in \mathfrak{R}$, is a *-derivation of $\mathfrak{A}$. The linearity of $\tilde{\delta}$ is obvious. Let $z_j \in \mathfrak{A}$, $j = 1, 2$. Then $z_j = x_j + iy_j$ for some $x_j, y_j \in \mathfrak{R}$. Then $z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2)$, and so,*

$$\begin{aligned}
\tilde{\delta}(z_1z_2) &= \tilde{\delta}((x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2)) \\
&= \delta(x_1x_2 - y_1y_2) + i\delta(x_1y_2 + y_1x_2) \\
&= \delta(x_1x_2) - \delta(y_1y_2) + i\delta(x_1y_2) + i\delta\gamma(y_1x_2) \\
&= \delta(x_1)x_2 + x_1\delta(x_2) - \delta(y_1)y_2 - y_1\delta(y_2) \\
&\quad + i\delta(x_1)y_2 + ix_1\delta(y_2) + i\delta(y_1)x_2 + iy_1\delta(x_2)
\end{aligned}$$

On the othe hand, we have,

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$$\begin{aligned}\widetilde{\not\delta}(z_1)z_2 + z_1\widetilde{\delta}(z_2) &= (\widetilde{\delta}(x_1+iy_1))(x_2+iy_2) + (x_1+iy_1)(\widetilde{\delta}(x_2+iy_2)) \\ &= (\delta(x_1)+i\delta(y_1))(x_2+iy_2) + (x_1+iy_1)(\delta(x_2)+i\delta(y_2)) \\ &= \delta(x_1)x_2 + i\delta(x_1)y_2 + i\delta(y_1)x_2 - \delta(y_1)y_2 \\ &\quad + x_1\delta(x_2) + ix_1\delta(y_2) + iy_1\delta(x_2) - y_1\delta(y_2)),\end{aligned}$$

from which we see that $\widetilde{\delta}(z_1z_2) = \widetilde{\delta}(z_1)z_2 + z_1\widetilde{\delta}(z_2)$*, and hence* $\widetilde{\delta}$ *is a derivation on* $\mathfrak{A}$*. Now, if* $z \in \mathfrak{A}$*,* $z = u + iw$ *for some* $u, w \in \mathfrak{R}$*, then* $z^* = u^* - iw^*$*, and so,* $\widetilde{\delta}(z^*) = \delta(u^*) - i\delta(w^*) = \delta(u)^* - i\delta(w)^* = (\delta(u) + i\delta(w))^* = (\widetilde{\delta}(z))^*$*. That is,* $\widetilde{\delta}$ *is a *-derivation, which clearly extends* D *to* $\mathfrak{A}$*.*

Theorem 2.3. *Let* A *be a universall reversible JC-algebra , and let* $D : \mathfrak{A} \to \mathfrak{A}$ *be a *-derivation of the C*-algebra* $\mathfrak{A}$*. Then:*

(i) $D\mid_A$*, is a Jordan derivation of* A*.*

(ii) $D\mid_{\mathfrak{R}}$ *is *-derivation of the real C*-algebra* $\mathfrak{R}$*.*

Proof. Since A is universally reversible, we have $A = \mathfrak{R}_{sa}$, and $A = \mathfrak{R}_{sa} = \mathfrak{A}^{\Phi}_{sa}$ [**8**, Proposition 7.3.3], and $\mathfrak{R} = \mathfrak{R}_{s.a} \oplus \mathcal{B}$, where $\mathcal{B} = \{b \in R^*(A) : b^* = -b\}$ (see [**7**, p. 103]).

(i) Note that $D\mid_{\mathfrak{A}_{sa}}: \mathfrak{A}_{sa} \to \mathfrak{A}_{sa}$ defined by $D(x \circ y) = x \circ D(y) + D(x) \circ y$, for all $x, y \in \mathfrak{A}_{sa}$, is a Jordan derivation, where $x \circ y = \frac{xy+yx}{2}$ [**6**, Remark 2.2], and $D(x) = (\Phi \circ D \circ \Phi)(x)$, for all $x \in \mathfrak{A}$. Hence, for all $a \in A = \mathfrak{A}^{\Phi}_{sa}$, we have $D(a) = (\Phi \circ D \circ \Phi)(a) = \Phi(D(a))$, since $D(a)^* = D(a^*) = D(a)$. That is, $D(a) \in \mathfrak{A}^{\Phi}_{sa} = \mathfrak{R}_{sa} = A$, implying that $D\mid_A$, is a Jordan derivation of A.

(ii) Let $b \in \mathcal{B}$, since $\Phi(b) = b^* = -b$, and $D(b) = (\Phi \circ D \circ \Phi)(b)$, we have

$$\begin{aligned}\Phi(D(b)) &= -(\Phi \circ D)(-b) \\ &= -(\Phi \circ D)(\Phi(b)) \\ &= -(\Phi \circ D \circ \Phi)(b) \\ &= -D(b) = D(-b) = D(b^*) = (D(b))^*,\end{aligned}$$

implying that $D(b) \in \mathcal{B} \subseteq \mathfrak{R}$. Therefore,

$$\begin{aligned}\Phi(D(x)) &= \Phi(D(a+b)) \\ &= \Phi(D(a)) + \Phi(D(a)) \\ &= D(a) + (D(b))^* \\ &= (D(a+b))^* = (D(x))^*,\end{aligned}$$

and so, $D(x) \in \mathfrak{R}$ for all $x \in \mathfrak{R}$. Hence, $D\mid_{\mathfrak{R}}$ is *-derivation. □

Corollary 2.4. *Every Jordan derivation on a universal reversible JC-algebra* A *extends to a *-derivation on the real enveloping C*-algebra* $\mathfrak{R}$ *of* A*.*

Proof. Let $D : A \to A$ be a Jordan derivation. By Theorem 2.1, D extends to *-derivation $\hat{D} : \mathfrak{A} \to \mathfrak{A}$ of the C*-algebra $\mathfrak{A}$, such that $\hat{D}|_A = D$, and by Theorem 2.3, $\hat{D}|_{\mathfrak{R}}$ is *-derivation, which is obiously an extension of D to $\mathfrak{R}$. □

The following Corllary is immediate by Theorems 2.1 and [**14,** Theorem 2].

Corollary 2.5. *Every Jordan derivation on a universal reversible JC-algebra* A *is continuous.*

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Before establishing the relation between local derivations (resp., 2-local derivations, weak-local derivations , weak-2-local derivations) on a universally reversible JC-algebra A, and their extensions to its universal inveloping real and complex C*-algebras $\mathfrak{R}$ and $\mathfrak{A}$, let us introduce a notion of a set-local (resp. set-2-local, set-weak-local, set-weak-2-local) derivation.

Definition 1. *Let $\mathcal{A}$ be a Banach algebra, and $T : \mathcal{A} \to \mathcal{A}$ be a linear map. We call T a set-local derivation if for every $x \in \mathcal{A}$, there exists a derivation, $D_x : \mathcal{A} \to \mathcal{A}$, depending on x, and a subset $\mathcal{C}$ of $\mathcal{A}$ containing x, satisfying $T(y) = D_x(y)$ for all $y \in \mathcal{C}$. It is called a set-2-local derivation if for every $x, y \in \mathcal{A}$, there exists a derivation, $D_{x,y} : \mathcal{A} \to \mathcal{A}$, depending on x, and y, and a subset $\mathcal{C}$ of $\mathcal{A}$ containing x and y such that $T(z) = D_{x,y}(z)$ and for all $z \in \mathcal{C}$. If for every $x \in \mathcal{A}$, and every $\varphi \in \mathcal{A}^*$, there exists a derivation, $D_{x,\varphi} : \mathcal{A} \to \mathcal{A}$, depending on x and φ, and a subset $\mathcal{C}$ of $\mathcal{A}$ containing x, satisfying $\varphi T(y) = \varphi D_{x,\varphi}(y)$ for all $y \in \mathcal{C}$, then it is called a set-weak-local derivation. If for every $x, y \in \mathcal{A}$, and every $\varphi \in \mathcal{A}^*$, there exists a derivation, $D_{x,y,\varphi} : \mathcal{A} \to \mathcal{A}$), depending on x, y and φ, and a subset $\mathcal{C}$ of $\mathcal{A}$ containing x and y, satisfying $\varphi T(z) = \varphi D_{x,y,\varphi}(z)$ for all $z \in \mathcal{C}$, then it is called a set-weak-2-local derivation*

Theorem 2.6. *Every local derivation $T : A \to A$ on a universally reversible JC-algebra A extends to a set-local *-derivation $\widetilde{T}$ on its universal inveloping real C*-algebra $\mathfrak{R}$.*

Proof. Let $x \in \mathfrak{R}$ be a fixed element. then $x = a + b$, for some $a \in A = \mathfrak{R}_{sa}$ and $b \in \mathcal{B}$. Since T be a local derivation on A, there exists a Jordan derivation $D_a : A \to A$, depending on a, such that $T(a) = D_a(a)$. Let $\widetilde{D}_a$ be the extension of D_a to a *-derivationon $\mathfrak{R}$, by Corollary 2.4. It is then clear that the mapping $\widetilde{T} : \mathfrak{R} \to \mathfrak{R}$ defined by $\widetilde{T}(y) = T(c) + \widetilde{D}_a(d), y = c + d$ in $\mathfrak{R}_{sa} \oplus \mathcal{B} = \mathfrak{R}$ is a linear extension of T. Let $\mathcal{E}_a = \{a + d : d \in \mathcal{B}\}$, and let $y \in \mathcal{E}_a$, then $y = a + d$ for some $d \in \mathcal{B}$, so we have,

$$\begin{aligned}\widetilde{T}(a+d) &= T(a) + \widetilde{D}_a(d)\\ &= D_a(a) + \widetilde{D}_a(d)\\ &= \widetilde{D}_a(a) + \widetilde{D}_a(d)\\ &= \widetilde{D}_a(a+d).\end{aligned}$$

That is, for $y \in \mathcal{E}_a$, $\widetilde{T}(y) = \widetilde{D}_a(y)$. Hence, $\widetilde{T}$ is a set-local *-derivation on $\mathfrak{R}$, since $\mathcal{E}_a$ is a subset of $\mathfrak{R}_{sa} \oplus \mathcal{B}$ containing x. □

Theorem 2.7. *Every local derivation T on a universally reversible JC-algebra A extends to a set-local derivation S on the self-adjoint part $\mathfrak{A}_{sa}$ of its universal inveloping C*-algebra $C^*(A)$.*

Proof. Note first that $\mathfrak{A}_{sa} = A \oplus i\mathcal{B}$, since A is universally reversible, and $A = \mathfrak{R}_{sa}$. Let $T : A \to A$ be a local derivation on A, and let x be a fixed element in $\mathfrak{A}_{sa}$, say, $x = a \oplus ib$, for some $a \in A$ and $b \in \mathcal{B}$. Since T is a local derivation on A, then for a, there exists a Jordan derivation $D_a : A \to A$ depending on a such that $T(a) = D_a(a)$. By Theorem 2.1, D_a extends to a *-derivations $\hat{D}_a$ on $\mathfrak{A}$. It is then clear that $S : y \mapsto T(c) + i\hat{D}_a(d)$, where $y \in \mathfrak{A}_{sa}$, $y = c + id$, $c \in A$, $d \in \mathcal{B}$ defines a linear map S on $\mathfrak{A}_{sa}$ which extends T. Let $\mathcal{F}_a = \{a + id : d \in \mathcal{B}\}$, then for any $y \in \mathcal{F}_a$, where $y = a + id$, for some $d \in \mathcal{B}$, we have

$$\begin{aligned}S(y) &= S(a+id)\\ &= T(a) + i\hat{D}_a(d)\\ &= D_a(a) + i\hat{D}_a(d)\\ &= \hat{D}_a(a) + i\hat{D}_a(d)\\ &= \hat{D}_a(a+id) = \hat{D}_a(y).\end{aligned}$$

Hence, S is a set-local derivation on $\mathfrak{A}_{sa}$ that extends T. □

Theorem 2.8. *Every local derivation T on a universally reversible JC-algebra A extends to a set-local *-derivation $\hat{T}$ on its universal inveloping C*-algebra $\mathfrak{A}$.*

Proof. Let $T : A \to A$ be a local derivation on a universally reversible JC algebra A, and let x be a fixed element in $\mathfrak{A}$, then $x = y + iz$, for some $y, z \in \mathfrak{R}$. Let $y = a + b$, where $a \in A = \mathfrak{R}_{sa}$, and $b \in \mathcal{B}$. Since T is a local derivation on A, then for a, there exists a Jordan derivation $D_a : A \to A$ depending on a such that $T(a) = D_a(a)$. By Theorem 3.1, D_a extends to a *-derivation $\hat{D}_a$ on $\mathfrak{A}$. Let $\mathcal{E}_a = \{a + d : d \in \mathcal{B}\}$, and let $\tilde{T}$ be the extention of T to a set-local *- derivation on $\mathfrak{R}$ arises in Theorem 2.6, then $\tilde{T}(y) = \hat{D}_a(y)$ for all $y \in \mathcal{E}_a = \{a + d : d \in \mathcal{B}\}$. Let $\hat{T} : \mathfrak{A} \to \mathfrak{A}$ be the map defined by $\hat{T}(w) = \tilde{T}(u) + i\hat{D}_a(v)$, $w \in \mathfrak{A}$, $w = u + iv$, $u, v \in \mathfrak{R}$. It is clear that $\hat{T}$ is a linear map extending $\tilde{T}$, and hence extends T. Note that, for all $u \in \mathcal{E}_a$ and all $v \in \mathfrak{R}$, we have,

$$\begin{aligned}\hat{T}(u + iv) &= \tilde{T}(u) + i\hat{D}_a(v)\\ &= \hat{D}_a(u) + i\hat{D}_a(v)\\ &= \hat{D}_a(u + iv).\end{aligned}$$

Since it is clear that x belongs to the subset $\mathcal{E}_a + i\mathfrak{R}$ of $\mathfrak{A}$, we see that $\hat{T}$ is a set-local *-derivation on $\mathfrak{A}$. □

The following Corllary is immediate by Theorems 2.1, Theorems 2.8, and [**9**, Theorem 5.3, p.318]

Corollary 2.9. *Every local derivation T on a universally reversible JC-algebra A is contiuous.*

The relation between 2-local derivations (resp., weak-2-local derivations) on a universally reversible JC-algebra, and their extensions on its universal inveloping real and complex C*-algebras $\mathfrak{R}$ and $\mathfrak{A}$ is established in the following two results.

Theorem 2.10. *Let A be a universally reversible JC-algebra A, and let $T : A \to A$ be a 2-local derivation. Then T extends to a set-local *-derivation $\tilde{T}$ on its universal inveloping real C*-algebra $\mathfrak{R}$.*

Proof. Let $x, y \in \mathfrak{R}$ be two fixed elements, then $x = a + c$ and $y = b + d$, for some $a, b \in A = \mathfrak{R}_{sa}$ and $c, d \in \mathcal{B}$. Since T be a 2-local derivation on A, there exists a Jordan derivation $D_{a,b} : A \to A$, depending on a and b, such that $T(a) = D_{a,b}(a)$ and $T(b) = D_{a,b}(b)$. Let $\tilde{D}_{a,b}$ be the extension of $D_{a,b}$ to a *-derivationon $\mathfrak{R}$, by Corollary 3.4. It is then clear that the mapping $\tilde{T} : \mathfrak{R} \to \mathfrak{R}$ defined by $\tilde{T}(w) = T(u) + \tilde{D}_{a,b}(v), w = u + v$, $u \in R^*(A)_{sa}$ and $v \in \mathcal{B}$, is a linear extension of T. Let $\mathcal{E}_a = \{a + v : v \in \mathcal{B}\}$ and $\mathcal{E}_b = \{b + v : v \in \mathcal{B}\}$, then $x \in \mathcal{E}_a$, and $y \in \mathcal{E}_b$. Note that for any $v \in \mathcal{B}$, we have,

$$\begin{aligned}\tilde{T}(a + v) &= T(a) + \tilde{D}_{a,b}(v)\\ &= D_{a,b}(a) + \tilde{D}_{a,b}(v)\\ &= \tilde{D}_{a,b}(a) + \tilde{D}_{a,b}(v)\\ &= \tilde{D}_{a,b}(a + v),\end{aligned}$$

and so, $\widetilde{T}(w) = \widetilde{D}_{a,b}(w)$ for all $w \in \mathcal{E}_a$. Also, we can see that $\widetilde{T}(w) = \widetilde{D}_{a,b}(w)$, for all $w \in \mathcal{E}_b$. That is, $\widetilde{T}(w) = \widetilde{D}_{a,b}(w)$ for all $w \in \mathcal{E}_a \cup \mathcal{E}_b$. Hence, $\widetilde{T}$ is a set-local *-derivation on $\mathfrak{R}$. □

Theorem 2.11. *Every 2-local derivation T on a universally reversible JC-algebra A extends to a set-2-local *-derivation $\hat{T}$ on its universal inveloping C*-algebra $\mathfrak{A}$.*

Proof. Let $T : A \to A$ be a weak 2-local derivation on a universally reversible JC algebra A, and let x_j, $j = 1, 2$ a fixed element in $\mathfrak{A}$, then $x_j = y_j + iz_j$, for some $y_j, z_j \in \mathfrak{R}$, since $\mathfrak{A} = \mathfrak{R} \oplus i\mathfrak{R}$. Let $y_j = a_j + c_j$ and $z_j = b_j + d_j$, where $a_j, b_j \in A$, and $c_j, d_j \in \mathcal{B}$. Since T is a 2-local derivation on A, then for a_1, a_2, there exists a Jordan derivation $D_{a_1,a_2} : A \to A$ depending on a_1, a_2 such that $T(a_1) = D_{a_1,a_2}(a_1)$ and $T(a_2) = D_{a_1,a_2,}(a_2)$. By Theorem 2.1, D_{a_1,a_2} extends to a *-derivation $\hat{D}_{a_1,a_2}$on $\mathfrak{A}$. Let $\widetilde{T}$ be the extention of T to a set-2-local *- derivation on $\mathfrak{R}$ arises in Theorem 2.10, where $\widetilde{T}(w) = \widetilde{D}_{a_1,a_2}(w)$ for all $w \in \mathcal{E}_{a_1} \cup \mathcal{E}_{a_2}$. Then $\hat{T} : \mathfrak{A} \to \mathfrak{A}$ defined by $\hat{T}(w) = \widetilde{T}(u) + i\hat{D}_{a_1,a_2}(v)$, $w \in \mathfrak{A}$, $w = u + iv$, $u, v \in \mathfrak{R}$ is an extention of $\widetilde{T}$, and hence of T. As in the proof of Theorem 2.10, we can easily see that for all $u \in \mathcal{E}_{a_j}$, $j = 1, 2$, and all $v \in \mathfrak{R}$, we have,

$$\begin{aligned}\hat{T}(u + iv) &= \widetilde{T}(u) + i\hat{D}_{a_1,a_2}(v) \\ &= \hat{D}_{a_1,a_2}(u) + i\hat{D}_{a_1 a_2}(v) \\ &= \hat{D}_{a_1,a_2}(u + iv),\end{aligned}$$

that is, for all $w \in ((\mathcal{E}_{a_2} \cup \mathcal{E}_{a_2}) + \mathfrak{R})$, we have, $\hat{T}(w) = \hat{D}_{a_1,a_2}(w)$. Hence, $\hat{T}$ is a set-2-local *-derivation on $\mathfrak{A}$. □

Theorem 2.12. *Every weak-local derivation T on a universally reversible JC-algebra A extends to a set-weak-local *-derivation $\widetilde{T}$ on its universal inveloping real C*-algebra $\mathfrak{R}$.*

Proof. Let $T : A \to A$ be a weak-local derivation on a universally reversible JC algebra A, and let $x \in \mathfrak{R}$ be a fixed element, then $x = a + b$, for some $a \in A$, and $b \in \mathcal{B}$. Let $\theta \in \mathfrak{R}^*$, then $\varphi = \theta|_A \in A^*$. Since T is a weak-local derivation on A, then for a and φ, there exists a derivation $D_{a,\varphi} : A \to A$, depending on a and φ, such that $\varphi T(a) = \varphi D_{a,\varphi}(a)$. Let $\widetilde{D}_{a,\varphi}$be the extention of D to a *-derivation on $\mathfrak{R}$, by Corollary 2.4, and define $\widetilde{T} : \mathfrak{R} \to \mathfrak{R}$ by $\widetilde{T}(y) = T(c) + \widetilde{D}_{a,\varphi}(d)$ for each $y = c + d$ in $\mathfrak{R} = \mathfrak{R}_{sa} \oplus \mathcal{B} = A \oplus \mathcal{B}$. It is then clear that $\widetilde{T}\,|_A = T$, and for all $y \in \mathcal{E}_a = \{a + d : d \in \mathcal{B}\}$. A similar argument as in the the proof of Theorem 2.6. shows that

$$\theta(\widetilde{T}(y)) = \theta(\widetilde{D}_{a,\varphi}(y)), \text{for all } y \in \mathcal{E}_a.$$

So that $\widetilde{T}$ is a set-weak-local *-derivation on $\mathfrak{R}$. □

Theorem 2.13. *Every weak local derivation T on a universally reversible JC-algebra A extends to a set-weak-local *-derivation $\hat{T}$ on its universal inveloping C*-algebra $\mathfrak{A}$.*

Proof. Let $T : A \to A$ be a weak local derivation on a universally reversible JC algebra A, and let x be a fixed element in $\mathfrak{A}$, and $\sigma \in \mathfrak{A}^*$, then Put $\varphi = \sigma|_A$, then $\varphi \in A^*$, and $x = y + iz$, for some $y, z \in R^*(A)$. Let $y = a + b$ where $a \in A$, and $b \in \mathcal{B}$. Since T is a weak-local derivation on A, then for a and φ, there exists a Jordan derivation $D_{a,\varphi} : A \to A$ depending on a and φ such that $\varphi T(a) = \varphi D_{a,\varphi}(a)$. By

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Theorem 2.1, $D_{a,\varphi}$ extends to a *-derivation $\hat{D}_{a,\varphi}$on $\mathfrak{A}$. Let $\tilde{T}$ be the extention of T to a set-weak-local *-derivation on $\mathfrak{R}$ arises in Theorem 2.7, where $\tilde{T}(y) = \hat{D}_{a,\varphi}(y)$ for all $y \in \mathcal{E}_a = \{a + d : d \in \mathcal{B}\}$. Then $\hat{T} : \mathfrak{A} \to \mathfrak{A}$ defined by $\hat{T}(w) = \tilde{T}(u) + i\hat{D}_{a,\varphi}(v)$, $w \in \mathfrak{A}$, $w = u + iv$, $u, v \in \mathfrak{R}$ is an extention of $\tilde{T}$. As in the proof of Theorem 2.8, we can easily see that for all $u \in \mathcal{E}_a$, and all $v \in \mathfrak{R}$, we have,

$$\begin{aligned}
\sigma\hat{T}(u + iv) &= \sigma(\tilde{T}(u) + i\hat{D}_{a,\varphi}(v)) \\
&= \sigma(\hat{D}_{a,\varphi}(u) + i\hat{D}_{a,\varphi}(v)) \\
&= \sigma(\hat{D}_{a,\varphi}(u + iv)),
\end{aligned}$$

that is, for all $w \in (\mathcal{E}_a + iR^*(A))$, $\psi\hat{T}(w) = \psi\hat{D}_{a,\varphi}(w)$. Hence, $\hat{T}$ is a set-weak-local *-derivation on $\mathfrak{A}$. □

The following Corllary is immediate by Theorems 2.1 and Theorem 2.8, and [**6**, Theorem 2.1].

Corollary 2.14. *Every weak-local derivation T on a universally reversible JC-algebra A is contiuous.*

Theorem 2.15. *Every weak-2-local derivation T on a universally reversible JC-algebra A extends to a set-weak-2-local *-derivation $\tilde{T}$ on its universal inveloping real C*-algebra $\mathfrak{R}$.*

Proof. Let $T : A \to A$ be a weak-local derivation on a universally reversible JC algebra A, and let $x, y \in \mathfrak{R}$ be a fixed elements, then $x = a + c$ and $y = b + d$ for some $a, b \in \mathfrak{R}_{sa} = A$, and $c, d \in \mathcal{B}$. Let $\overset{\sim}{\rho} \in \mathfrak{R}^*$, then $\rho = \overset{\sim}{\rho}|_A \in A^*$. Since T is a weak–2-local derivation on A, then for a, b and ρ, there exists a Jordan derivation $D_{a,b,\rho} : A \to A$, depending on a, b and ρ, such that $\rho T(a) = \rho D_{a,b,\rho}(a)$ and $\rho T(b) = \rho D_{a,b,\rho}(b)$. Let $\tilde{D}_{a,b,\rho}$be the extention of $D_{a,b,\rho}$ to a *-derivation on $\mathfrak{R}$, by Corollary 2.4, and define $\tilde{T} : \mathfrak{R} \to \mathfrak{R}$ by $\tilde{T}(w) = T(u) + \tilde{D}_{a,b,\rho}(v)$ for each $w = u + v$ in $\mathfrak{R}$, where $u \in A, v \in \mathcal{B}$. It is then clear that $\tilde{T}|_A = T$. A similar argument in the proof of Theorem 2.10 and Theorem 2.13, we see that for all $z \in \mathcal{E}_a = \{a + v : v \in \mathcal{B}\}$ or $z \in \mathcal{E}_b = \{b + v : v \in \mathcal{B}\}$,

$$\overset{\sim}{\rho}(\tilde{T}(z)) = \overset{\sim}{\rho}(\tilde{D}_{a,b,\rho}(y)).$$

That is, $\overset{\sim}{\rho}(\tilde{T}(z)) = \overset{\sim}{\rho}(\tilde{D}_{a,b,\varphi}(y))$, for all $z \in \mathcal{E}_a \cup \mathcal{E}_b$. So that $\tilde{T}$ is a set-weak-local *-derivation on $\mathfrak{R}$. □

The proof of the next Theorem is similar to the proof of Theorem 2.15, and using a similar argument in the proof of Theorem 2.13.

Theorem 2.16. *Every weak-2-local derivation T on a universally reversible JC-algebra A extends to a set-weak-2-local *-derivation $\hat{T}$ on its universal inveloping C*-algebra $\mathfrak{A}$.*

Proof. Let $T : A \to A$ be a weak 2-local derivation on a universally reversible JC algebra A, and let x_j, $j = 1, 2$ a fixed element in $\mathfrak{A}$, then $x_j = y_j + iz_j$, for some $y_j, z_j \in \mathfrak{R}$. Let $\sigma \in \mathfrak{A}^*$, then $\rho = \sigma|_A \in A^*$. Let $y_j = a_j + c_j$ and $z_j = b_j + d_j$, where $a_j, b_j \in A$, and $c_j, d_j \in \mathcal{B}$. Since T is a weak-2-local derivation on A, then for a_1, a_2 and ρ, there exists a Jordan derivation $D_{a_1,a_2,\rho} : A \to A$ depending on a_1, a_2 and ρ such that $\rho T(a_1) = \rho D_{a_1,a_2,\rho}(a_1)$ and $\rho T(a_2) = \rho D_{a_1,a_2,\rho}(a_2)$. By Theorem 2.1, $D_{a_1,a_2,\rho}$ extends to a *-derivation $\hat{D}_{a_1,a_2,\rho}$on $\mathfrak{A}$. Let $\tilde{T}$ be the extention of T to a set-weak 2-local *- derivation on $\mathfrak{R}$ arises in Theorem 2.14, where $\tilde{T}(w) = \tilde{D}_{a,b}(w)$ for all $w \in \mathcal{E}_a \cup \mathcal{E}_b$. Then $\hat{T} : \mathfrak{A} \to \mathfrak{A}$ defined by $\hat{T}(w) = \tilde{T}(u) + i\hat{D}_{a_1,a_2,\rho}(v)$, $w \in \mathfrak{A}$, $w = u + iv$, $u, v \in \mathfrak{R}$ is an extention of T. As in the proof of

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Theorem 2.11, we can easily see that for all $z \in ((\mathcal{E}_a \cup \mathcal{E}) + \mathfrak{R}$, we have, $\sigma\hat{T}(z) = \sigma\hat{D}_{a_1 a_2, \rho}(z)$. Hence, $\hat{T}$ is a set-2-weak local *-derivation on $\mathfrak{A}$. □

Theorem 2.17. *Let A be a universally reversible JC algebra, and $\hat{T} : \mathfrak{A} \to \mathfrak{A}$ be a local *-derivation (resp. 2-local *-derivation, weak-local *-derivation, weak-2-local *-derivation).*

(i) If $\hat{T}(A) \subseteq A$, then $\hat{T}\,|_A : A \to A$ is a local derivation (resp. 2-local derivation, weak-local derivation, weak-2-local derivation).

*(ii) If $\hat{T}(\mathfrak{R})) \subseteq \mathfrak{R}$, then $\hat{T}\,|_{\mathfrak{R}} : \mathfrak{R} \to \mathfrak{R}$ is a local *-derivation (resp. 2-local *-derivation, weak-local *-derivation, weak-2-local *-derivation).*

Proof. It immediate by Theorem 2.3. □

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