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Keywords: Akash distribution; Convolution; Maximum likelihood

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Akash Convolution Distribution with Applications

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Abstract: In this article, we propose a new distribution based on the convolution of two independent Akash (AK) random variables with the same parameter, which we call the Akash convolution (AKC) distribution. We study its density and some basic properties, and perform parameter estimation using the methods of moments and maximum likelihood (ML), along with the Fisher information. We evaluate the performance of the ML estimator through a simulation study. Additionally, we present two applications with real data, demonstrating a better performance of the AKC distribution compared to two other distributions.

Keywords: Akash distribution; Convolution; Maximum likelihood

MSC: 62E99, 62P99, 62F10

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1. Introduction

A widely used distribution is the Lindley model (see Lindley [1]), we say that a random variable X has a Lindley distribution (L) if its probability density function (pdf) is given by

$$f_X(x;\theta) = \frac{\theta^2}{\theta+1}(1+x)\exp(-\theta x), \quad x > 0, \tag{1}$$

where $\theta > 0$ is a shape parameter. We denote this by $X \sim L(\theta)$. The cumulative distribution function (cdf) corresponding to (1) is

$$F_X(x;\theta) = 1 - \left[1 + \frac{\theta x}{\theta+1}\right]\exp(-\theta x), \quad x > 0. \tag{2}$$

The L distribution has been used in various settings such as engineering, demography, reliability, and medicine, among others. Some researchers who have carried out these studies are: Hussain [2], Ghitany et al. [3], Zakerzadeh and Dolati [4], Gómez-Déniz and Calderin-Ojeda [5], Krishna and Kumar [6], Bakouch et al. [7], Shanker et al. [8], Ghitany et al. [9], Al-Mutairi et al. [10], Oluyede and Yang [11], Shanker et al. [12] and Abouammoh et al. [13], among others.

Shanker [14] introduced the Akash distribution and applied it to real lifetime data sets from medical science and engineering. Thus we say that a random variable Y has an Akash distribution (AK) with shape parameter θ if its pdf is given by

$$f_Y(y;\theta) = \frac{\theta^3}{(\theta^2+2)}(1+y^2)\exp(-\theta y), \quad y > 0, \tag{3}$$

where $\theta > 0$ is a shape parameter, and it is denoted by $Y \sim AK(\theta)$.

In Figure 1, we show the pdf of the AK distribution for several values of θ.

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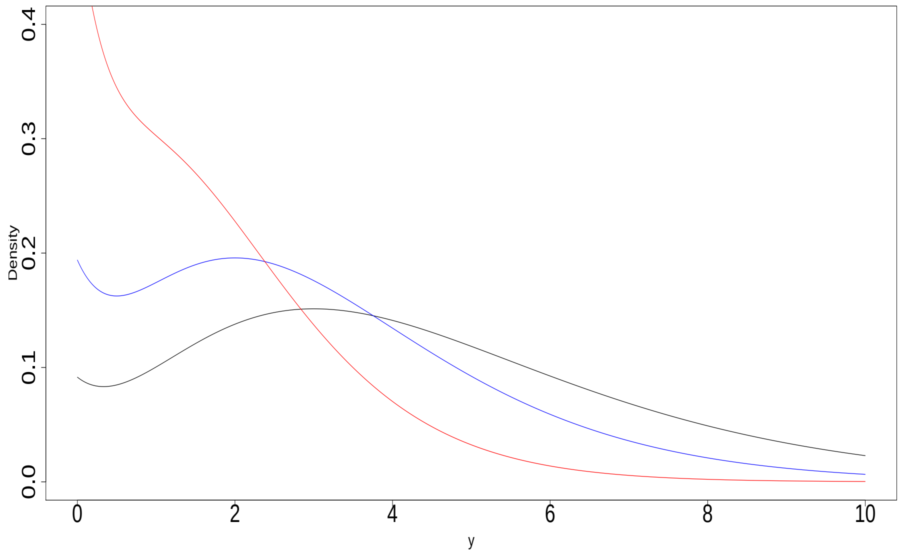

Figure 1. Examples of the $AK(0.6)$ (black color), $AK(0.8)$ (blue color) and $AK(1.2)$ (red color).

Some properties of this pdf are:

a) The cdf of Y is
$$F_Y(y;\theta) = 1 - \left[1 + \frac{\theta y(\theta y + 2)}{\theta^2 + 2}\right] \exp(-\theta y), \quad y > 0. \tag{4}$$

b) For $r = 1, 2, 3, \ldots$, the r-th moment of Y is
$$\mathbb{E}(Y^r) = \frac{r!\left[\theta^2 + (r+1)(r+2)\right]}{\theta^r(\theta^2 + 2)}. \tag{5}$$

c) The moment generating function ($M_Y(t)$) is given by
$$M_Y(t) = \mathbb{E}(\exp(tY)) = \frac{\theta^3}{\theta^2 + 2}\left[\frac{(\theta - t)^2 + 2}{(\theta - t)^3}\right], \quad t < \theta \tag{6}$$

Extensions AK distribution are given by Shanker and Shukla [15,16] and Gómez et al. [17], among others.

In this article, we consider two independent AK random variables with the same parameter and study their convolution. We show that the convolution distribution has a closed form and several interesting properties. Applications show that it can be an alternative to the AK and L distributions.

The paper develops as follows: In Section 2, we furnish the AKC distribution and its properties. In Section 3, we perform inference by the methods of moments and maximum likelihood; the Fisher information is obtained, and a simulation study is also carried out. In Section 4, applications are made to two real data sets and compared with the AK and L distributions. In Section 5, we provide some conclusions.

2. AKC Distribution

In this section we introduce the density function properties of the AKC distribution.

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2.1. Density Function

Definition 1. *We say that a random variable Z has a AKC distribution with shape parameter θ, $Z \sim AKC(\theta)$, if it has the following pdf:*

$$f_Z(z;\theta) = \frac{\theta^6 z}{30(\theta^2+2)^2}(30+20z^2+z^4)\exp(-\theta z), \quad z > 0, \tag{7}$$

where $\theta > 0$.

In Figure 2, we show the pdf of the AKC distribution for several values of θ.

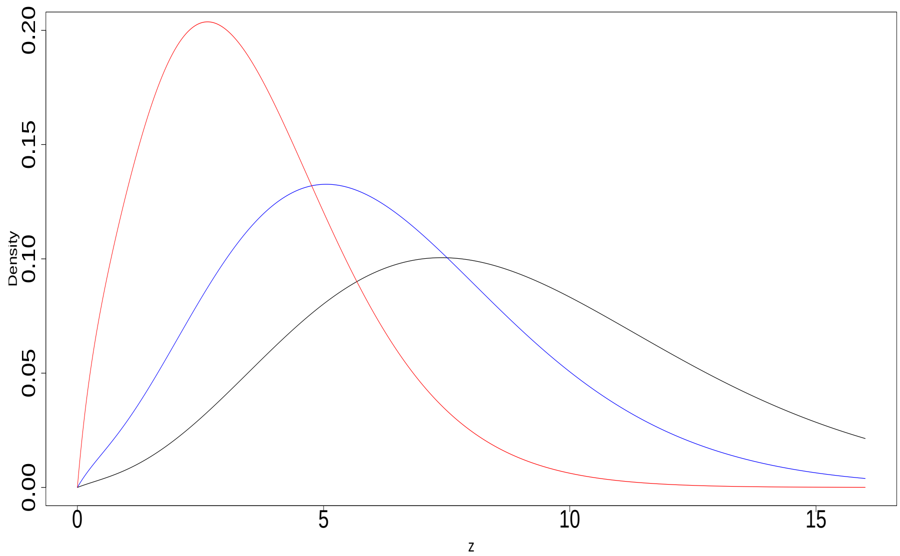

Figure 2. Examples of the $AKC(0.6)$ (black color), $AKC(0.8)$ (blue color) and $AKC(1.2)$ (red color).

2.2. Properties

The following proposition shows that the pdf of the AKC distribution can be obtained from a convolution of two AK distributions.

Proposition 1. *Let $X_1 \sim AK(\theta)$, $X_2 \sim AK(\theta)$ be two independent random variables and define $Z = X_1 + X_2$, then $X \sim AKC(\theta)$.*

Proof. Since X_1 and X_2 are independent, the pdf of Z may be obtained from the following convolution product: for $x > 0$,

$$f_Z(z;\theta) = \int_0^z f_{X_1}(z-t;\theta) f_{X_2}(t;\theta)dt = \frac{\theta^6\exp(-\theta z)}{(\theta^2+2)^2}\int_0^z (1+z^2-2zt+t^2)(1+t^2)dt.$$

By calculating the right-side integral, the result follows. □

The following proposition shows the cdf of the AKC distribution.

Proposition 2. *Let $Z \sim AKC(\theta)$. Then, the cdf of Z is given by*

$$F_Z(z;\theta) = \frac{1}{30(\theta^2+2)^2}\left\{30\theta^4\gamma(2,\theta z) + 20\theta^2\gamma(4,\theta z) + \gamma(6,\theta z)\right\}, \quad z > 0, \tag{8}$$

where $\theta > 0$ and $\gamma(a,v) = \int_0^v u^{a-1}\exp(-u)du$ is the incomplete gamma function.

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Proof. By directly calculating the cdf of Z we have

$$F_Z(z;\theta) = \int_0^z \frac{\theta^6 t}{30(\theta^2+2)^2}(30+20t^2+t^4)\exp(-\theta t)dt,$$

and making the following change of variable: $u = \theta t$, the result is obtained. □

2.3. Reliability Analysis

The reliability function $r(t)$ and the hazard function $h(t)$ of the AKC distribution are given in the following corollary.

Corollary 1. *Let $T \sim AKC(\theta)$. Then, the $r(t)$ and $h(t)$ of T are given by*

1. $r(t) = \frac{1}{30(\theta^2+2)^2}\left\{30(\theta^2+2)^2 - 30\theta^4\gamma(2,\theta t) - 20\theta^2\gamma(4,\theta t) - \gamma(6,\theta t)\right\},$
2. $h(t) = \frac{\theta^6 t(30+20t^2+t^4)\exp(-\theta t)}{30(\theta^2+2)^2 - 30\theta^4\gamma(2,\theta t) - 20\theta^2\gamma(4,\theta t) - \gamma(6,\theta t)},$

where $\theta, t > 0$.

In Figure 3, we show the hazard function of AKC distribution for several values of θ.

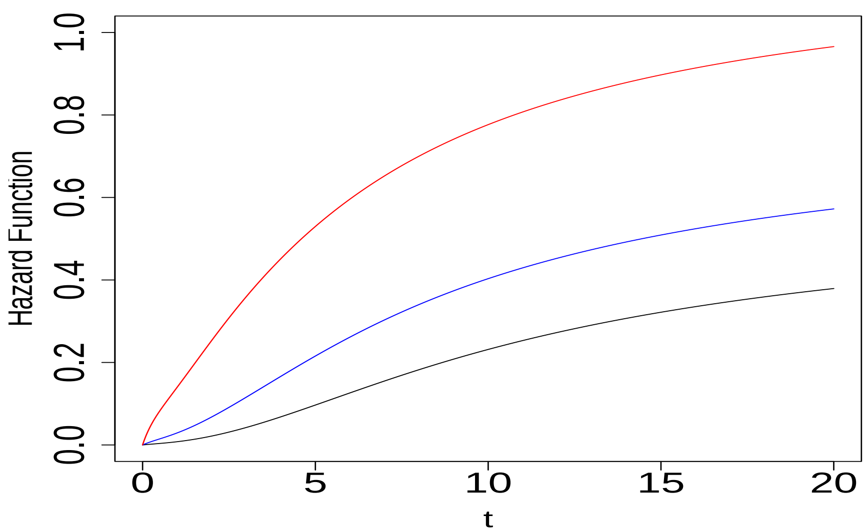

Figure 3. Hazard function of AKC distribution for selected values of θ: $\theta = 0.6$ (black color), $\theta = 0.8$ (blue color) and $\theta = 1.2$ (red color)

Proposition 3. *Let $T \sim AKC(\theta)$. Then, the hazard function of T is increasing for all $t > 0$.*

Proof. Using item (a) of a theorem given in Glaser [18] we have that

$$\eta(t) = -\frac{f'(t)}{f(t)} = \theta - \frac{1}{t} - \frac{40t+4t^3}{30+20t^2+t^4},$$

where $f(t)$ is the pdf given in (7). Then differentiating the function $\eta(t)$ with respect to t

$$\eta'(t) = \frac{900+900t^4+80t^6+5t^8}{t^2(30+20t^2+t^4)^2} > 0, \ \forall t > 0,$$

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the result follows. □

2.4. Order Statistics

Let $Z_1, ..., Z_n$ be a random sample of a random variable $Z \sim AKC(\theta)$. Let us denote by $Z_{(j)}$ the $jth-$order statistics, $j \in \{1, \ldots, n\}$.

Proposition 4. *The pdf of $Z_{(j)}$ is*

$$\begin{aligned} f_{Z_{(j)}}(z) &= \frac{n!\theta^6 z\exp(-\theta z)(30+20z^2+z^4)}{(j-1)!(n-j)!30^n(\theta^2+2)^{2n}}\Big\{30\theta^4\gamma(2,\theta z)+20\theta^2\gamma(4,\theta z)+\gamma(6,\theta z)\Big\}^{j-1} \\ &\times \Big\{30(\theta^2+2)^2-30\theta^4\gamma(2,\theta z)-20\theta^2\gamma(4,\theta z)-\gamma(6,\theta z)\Big\}^{n-j}, \end{aligned}$$

where $z > 0$ and $\theta > 0$. In particular, the pdf of the minimum, $Z_{(1)}$, is

$$f_{Z_{(1)}}(z) = \frac{n\theta^6 z\exp(-\theta z)(30+20z^2+z^4)}{30^n(\theta^2+2)^{2n}}\Big\{30(\theta^2+2)^2-30\theta^4\gamma(2,\theta z)-20\theta^2\gamma(4,\theta z)-\gamma(6,\theta z)\Big\}^{n-1},$$

and the pdf of the maximum, $Z_{(n)}$, is

$$f_{Z_{(n)}}(z) = \frac{n\theta^6 z\exp(-\theta z)(30+20z^2+z^4)}{30^n(\theta^2+2)^{2n}}\Big\{30\theta^4\gamma(2,\theta z)+20\theta^2\gamma(4,\theta z)+\gamma(6,\theta z)\Big\}^{n-1},$$

Proof. Since we are dealing with an absolutely continuous model, the pdf of the $jth-$order statistic is obtained by applying

$$f_{Z_{(j)}}(x) = \frac{n!}{(j-1)!(n-j)!}f(z)\,[F(z)]^{j-1}\,[1-F(z)]^{n-j}, \quad , \quad j \in \{1, \ldots, n\}$$

where F and f denote the cdf and pdf of the parent distribution, $Z \sim AKC(\theta)$ in this case. □

2.5. Moments

In this subsection we provide the moments, the moment generating function and skewness and kurtosis coefficients.

Proposition 5. *Let $Z \sim AKC(\theta)$. Then, for $r = 1, 2, 3, \ldots$, the r-moment of Z is given by*

$$\mathbb{E}(Z^r) = \frac{r!}{\theta^r(\theta^2+2)^2}\sum_{k=0}^{r}\Big[\theta^2+(k+1)(k+2)\Big]\Big[\theta^2+(r-k+1)(r-k+2)\Big]. \quad (9)$$

Proof. Using the representation of Z given in Proposition 1 and the binomial theorem, we have that the r-th moment is

$$\mathbb{E}(Z^r) = \mathbb{E}((X_1+X_2)^r) = \sum_{k=0}^{r}\binom{r}{k}\mathbb{E}(X_1^k)\mathbb{E}(X_2^{r-k}),$$

then using the moments of the AK random variable given in (5), the result is obtained. □

Proposition 6. *Let $Z \sim AKC(\theta)$. Then, the moment generating function of the random variable Z is given by*

$$M_Z(t) = \mathbb{E}(\exp(tZ)) = \frac{\theta^6}{(\theta^2+2)^2}\left[\frac{(\theta-t)^2+2}{(\theta-t)^3}\right]^2, \quad t < \theta. \quad (10)$$

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Proof. Using the representation given in Proposition 1, we get:

$$M_Z(t) = \mathbb{E}(\exp(tZ)) = \mathbb{E}(\exp(t(X_1 + X_2))) = M_{X_1}(t)M_{X_2}(t),$$

and using the moment generating function of the AK random variable given in (6), the result follows. □

Corollary 2. *Let $Z \sim AKC(\theta)$. Then, the mean and variance Z are given respectively by:*

$$\mathbb{E}(Z) = \frac{2(\theta^2 + 6)}{\theta(\theta^2 + 2)} \quad \text{and} \quad Var(Z) = \frac{2(\theta^4 + 16\theta^2 + 12)}{\theta^2(\theta^2 + 2)^2}.$$

The skewness and kurtosis coefficients are respectively:

$$\sqrt{\beta_1} = \frac{\sqrt{2}(\theta^6 + 30\theta^4 + 36\theta^2 + 24)}{(\theta^4 + 16\theta^2 + 12)^{3/2}},$$

$$\beta_2 = \frac{6(\theta^8 + 40\theta^6 + 172\theta^4 + 240\theta^2 + 96)}{(\theta^4 + 16\theta^2 + 12)^2}.$$

Figure 4 depicts plots of skewness and kurtosis coefficients of the AKC distribution.

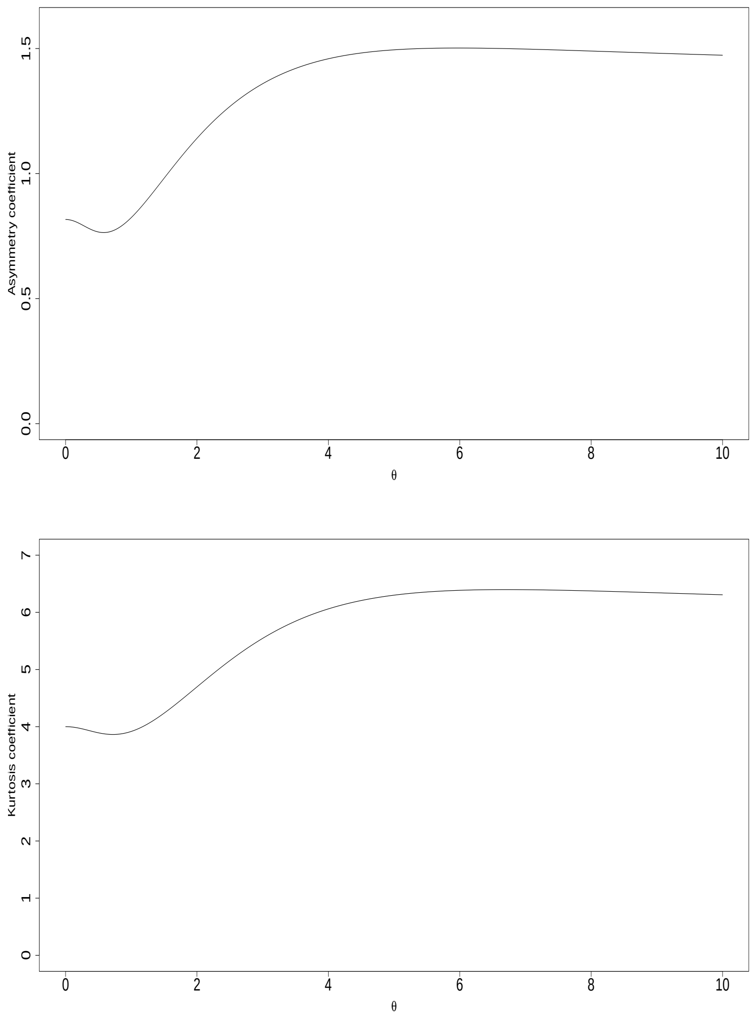

Figure 4. Plots of the skewness and kurtosis coefficients of the AKC model.

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3. Inference

In this section we estimate the θ parameter of the AKC model using the method of moments (MM) and the method of maximum likelihood (ML). Asymptotic properties of the ML estimator are discussed and a simulation study is presented.

3.1. Method of Moments

Let $z_1, \ldots, z_n$ be a random sample from $Z \sim AKC(\theta)$. Let $\bar{z} = \frac{\sum_{i=1}^n z_i}{n}$ be the first sample moment.

Proposition 7. *Given $z_1, \ldots, z_n$, a random sample from $Z \sim AKC(\theta)$, the method of moment estimator of θ ($\hat{\theta}_M$) provides the following estimator:*

$$\begin{aligned}\hat{\theta}_M &= \frac{2}{3\bar{z}} + \frac{-4+6\bar{z}^2}{3 \times 2^{1/3}\bar{z}(-4-72\bar{z}^2+3\sqrt{6}\sqrt{12\bar{z}^2+94\bar{z}^4+\bar{z}^6})^{1/3}} \\ &\quad - \frac{2^{1/3}(-4-72\bar{z}^2+3\sqrt{6}\sqrt{12\bar{z}^2+94\bar{z}^4+\bar{z}^6})^{1/3}}{3\bar{z}}.\end{aligned}$$

Proof. The equation for the method of moments is given by

$$\mathbb{E}[Z] = \frac{2(\theta^2+6)}{\theta(\theta^2+2)} = \bar{z}. \tag{11}$$

Solving equation (11) for the parameter θ, we obtain the result. □

3.2. Maximum Likelihood Estimation

For a random sample, $z_1, \ldots, z_n$, derived from the $AKC(\theta)$ distribution, the log-likelihood function can be written as

$$\ell(\theta) = 6n\log(\theta) + \sum_{i=1}^{n}\log(z_i) - 2n\log(\theta^2+2) - n\bar{z}\theta + \sum_{i=1}^{n}\log\left(30+20z_i^2+z_i^4\right) - n\log(30). \tag{12}$$

The score equation are given by

$$\frac{d\ell(\theta)}{d\theta} = \frac{6n}{\theta} - \frac{4n\theta}{\theta^2+2} - n\bar{z} = 0,$$

so we get this equation

$$-\bar{z}\theta^3 + 2\theta^2 - 2\bar{z}\theta + 12 = 0. \tag{13}$$

The ML estimator for θ ($\hat{\theta}$) is obtained by solving equation (13). Note that this equation is identical to the equation derived for the method of moments. Consequently, the ML estimator of θ coincides with the moments estimator given in the Proposition 7.
Hence, for large samples, the ML estimator, $\widehat{\theta}$, is asymptotically normal. That is,

$$\sqrt{n}\left(\widehat{\theta}-\theta\right) \xrightarrow{\mathcal{L}} N(0,\ I_F^{-1}(\theta)),$$

where the asymptotic variance of the ML estimator $\hat{\theta}$ is the inverse of Fisher's information:

$$Var(\widehat{\theta}) \approx \frac{\theta^2(\theta^2+2)^2}{2n(\theta^4+16\theta^2+12)}, \qquad I_F(\theta) = \frac{2(\theta^4+16\theta^2+12)}{\theta^2(\theta^2+2)^2}.$$

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3.3. Simulation Study

To examine the performance of the ML estimator of the parameter θ of the AKC distribution, a simulation study is carried out. An algorithm is available to generate random numbers from the AKC distribution. The simulation analysis is performed generating 1000 samples of sizes $n = 20, 50$ and 100 from the AKC distribution. The algorithm used to generate random numbers from the AKC distribution is shown below. The Algorithm 1 is based on the representation given in Proposition 1.

Algorithm 1 for simulating from the $Z \sim AKC(\theta)$ can proceed as follows.

Step 1: Generate $S_i \sim Exp(\theta), \ \ i = 1, 2.$
Step 2: Generate $T_i \sim Gamma(3, \theta), \ \ i = 1, 2.$
Step 3: Generate $V_i \sim Bernoulli\left(\frac{\theta}{1+\theta}\right), \ \ i = 1, 2.$
Step 4: Compute $X_i = V_i S_i + (1 - V_i)T_i, \ \ i = 1, 2.$
Step 5: Compute $Z = X_1 + X_2$

Table 1 shows the empirical bias (B), the average of the standard errors (SE), the empirical root mean squared error (RMSE) and the coverage probability (CP). It is to be noted that the CP converge reasonably well to the nominal value used in their construction (95%), suggesting that the normal distribution is a reasonable asymptotic distribution for the the ML estimators in the AKC model. Also, Table 1 shows that, the performance of the estimates is very good even when n is small.

Table 1. B, SE, RMSE, and CP for the AKC model with $n = 20, 50$ and 100.

θ	n	B	SE	RMSE	CP
0.2	20	0.001	0.018	0.018	0.942
	50	0.001	0.012	0.012	0.949
	100	0.0002	0.008	0.008	0.955
0.5	20	0.002	0.046	0.046	0.952
	50	0.001	0.029	0.029	0.952
	100	0.001	0.019	0.019	0.946
0.8	20	0.004	0.073	0.073	0.945
	50	0.002	0.045	0.045	0.957
	100	0.002	0.031	0.031	0.953
1	20	0.007	0.091	0.091	0.955
	50	-0.001	0.056	0.056	0.954
	100	0.004	0.040	0.040	0.941
2	20	0.013	0.204	0.204	0.954
	50	0.007	0.126	0.126	0.943
	100	0.008	0.088	0.088	0.948
3	20	0.051	0.366	0.369	0.943
	50	0.021	0.223	0.224	0.952
	100	0.009	0.155	0.155	0.950
4	20	0.067	0.537	0.540	0.946
	50	0.025	0.323	0.324	0.953
	100	0.028	0.238	0.240	0.949
5	20	0.052	0.688	0.690	0.955
	50	0.049	0.441	0.444	0.954
	100	0.016	0.309	0.309	0.948

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4. Applications

In this Section the *AKC* distribution is fitted to two engineering science data sets and compared with the *AK* and *L* distributions.

4.1. Application 1

In this application the data set consists of the strength of glass of the aircraft window reported by Fuller et al. [19]. The data are shown in Table 2.

Table 2. Strength of glass of aircraft window reported by Fuller et al. (1994).

18.83	20.80	21.657	23.03	23.23	24.05	24.321	25.50	25.52	25.80
26.69	26.77	26.78	27.05	27.67	29.90	31.11	33.20	33.73	33.76
33.89	34.76	35.75	35.91	36.98	37.08	37.09	39.58	44.045	45.29
45.381									

Descriptive statistics are given in Table 3, where CS is the sample skewness coefficient and CK is the sample kurtosis coefficient.

Table 3. Descriptive statistics for strength of glass data.

n	Median	Mean	Variance	CS	CK
31	29.90	30.81	52.61	0.405	2.287

Table 4 shows the ML estimates of the parameters of the *AKC*, *AK* and *L* models together with their standard errors in parentheses, Also the values of the AIC and BIC criteria are given for each model, (Akaike information criterion (AIC) introduced by Akaike [20] and the Bayesian information criterion (BIC) proposed by Schwarz [21]).

Table 4. ML Estimates of the *AKC*, *AK* and *L* Models for Glass Strength Data.

Model	ML estimate	AIC	BIC
AKC(θ)	$\widehat{\theta} = 0.192(0.014)$	**225.868**	**227.302**
AK(θ)	$\widehat{\theta} = 0.097(0.010)$	242.682	244.116
L(θ)	$\widehat{\theta} = 0.063(0.008)$	255.988	257.422

Observe that the smallest AIC and BIC values correspond to the AKC model, meaning that this model produces a better fit than the *AK* and *L* models.
For the fitted *AKC* distribution we calculated the quantile residuals (QR). If the model is appropriate for the data, the QRs should be a sample from the standard normal model (see Dunn and Smyth [22]). This assumption can be validated with traditional normality tests, such as the Anderson-Darling (AD), Cramèr-von Mises (CVM) and Shapiro-Wilkes (SW) tests.

In Figure 5 the QRs for the AKC distribution are shown together with the p-values of the AD, CVM and SW normality tests. It is seen that the AKC distribution verifies the assumption that the QRs come from the standard normal distribution, validating the fit of the *AKC* distribution to the strength of glass data.

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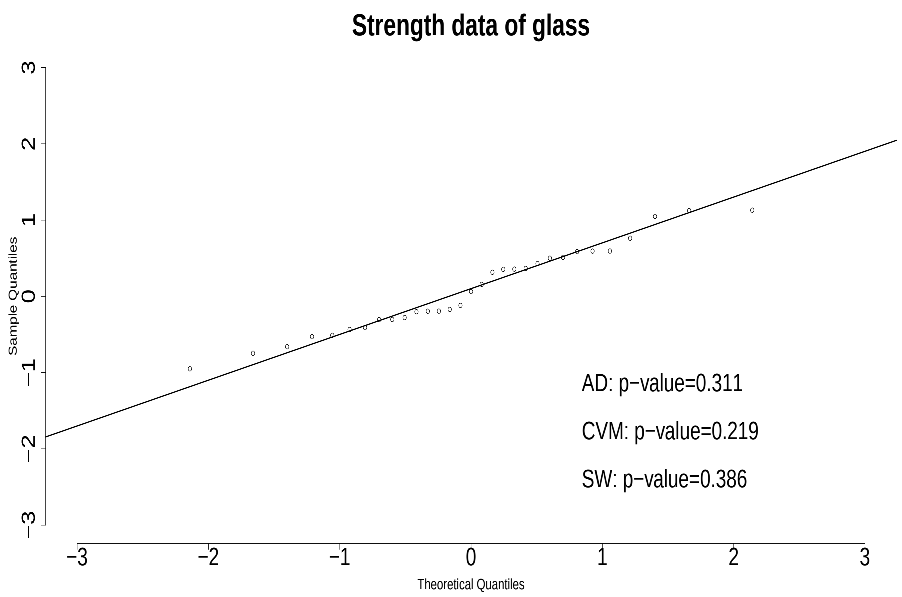

Figure 5. Q-Q plots of the QRs for AKC distribution

4.2. Application 2

In this application we model a data set collected by the Department of Mines of the University of Atacama, Chile. The data, consisting of Yttrium measurements in 86 samples of mineral, are shown in Table 5.

Table 5. Yttrium data collected by the Mines Department, University of Atacama, Chile.

27	20	17	12	33	19	27	27	25	26
17	13	15	22	24	20	9	20	13	26
8	27	29	7	33	28	15	36	31	24
22	12	26	35	15	45	10	27	24	42
27	23	16	18	28	17	30	18	19	10
30	30	12	20	17	26	23	35	33	18
12	22	30	15	19	32	19	19	50	24
17	34	29	34	32	12	44	39	34	16
31	22	20	17	18	25				

Summary statistics are reported in Table 6. ML estimates of the parameter of the AKC, AK and L models, together with their SE and the values of the AIC and BIC are given in Table 7.

Table 6. Descriptive statistics for Yttrium data.

n	Median	Mean	Variance	CS	CK
86	23.00	23.53	79.264	0.489	3.033

Table 7. ML Estimates of AKC, AK, and L Models for Yttrium data.

Model	ML estimate	AIC	BIC
AKC(θ)	$\widehat{\theta} = 0.250(0.011)$	**617.956**	**620.411**
AK(θ)	$\widehat{\theta} = 0.127(0.008)$	640.645	643.100
L(θ)	$\widehat{\theta} = 0.082(0.006)$	669.327	671.781

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We observe that the smallest values of the AIC and BIC criteria correspond to the AKC model, meaning that the AKC model fits the data better than the AK and L models.
The residual quantiles of the AKC distribution are shown in Figure 6. Also the p-values for the AD, CVM and SW normality tests are given to verify if the QRs come from a standard normal distribution. These indicate that the AKC distribution provides a good fit for the Yttrium data.

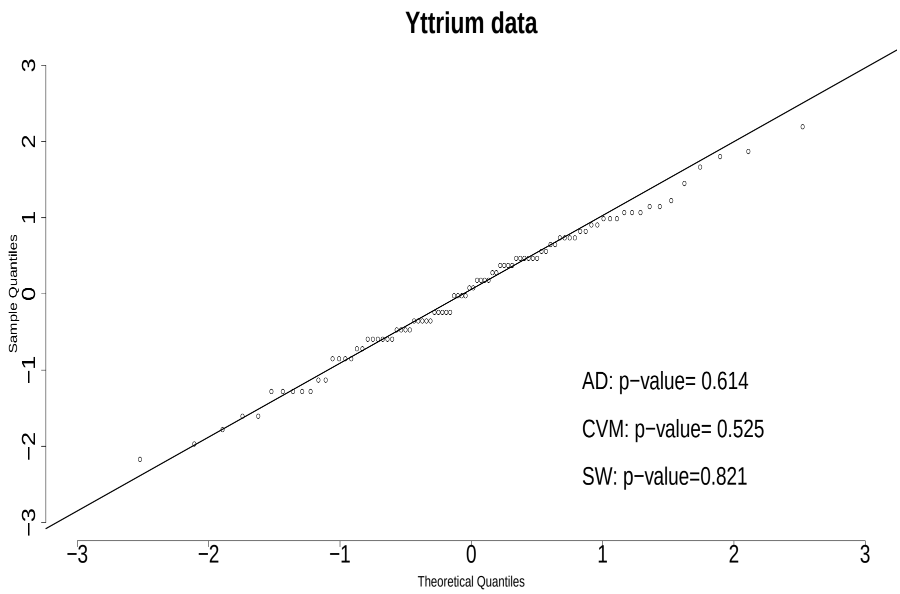

Figure 6. Q-Q plots of the QRs for AKC distribution

5. Conclusions

In this article, we have studied the convolution of two independent AK random variables with a parameter in common. We have given some of the properties of the new AKC distribution and developed parameter estimates by the methods of moments and maximum likelihood. A simulation study is carried out to investigate the performance of the ML estimator. Also, the ability of the AKC distribution is demonstrated in two applications made to real data sets. It consistently produces better fits than the AK and L distributions.

Some additional features of the AKC are:

- The AKC distribution has a simple representation.
- The cumulative distribution and risk functions are explicit and represented by known functions.
- The Moments and Maximum Likelihood estimators coincide and have closed form solutions.
- The Maximun Likelihood estimator performs very well, even when samples are small.
- Applications show that the AKC distribution is a good alternative to model positively skewed data when kurtosis is not too high. This is confirmed by the AIC and BIC model selection criteria and statistical tests such as Anderson-Darling, Cramèr-von Mises and Shapiro-Wilkes.

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References

1. Lindley, D.V. Fiducial distributions and Bayes' theorem. *J. R. Stat. Soc. Ser. B* **1958**, *20*, 102–107.
2. Hussain, E. The non-linear functions of Order Statistics and Their Properties in selected probability models, Ph.D thesis, Department of Statistics, University of Karachi, Pakistan, 2006.
3. Ghitany, M.E.; Atieh, B.; Nadarajah, S. Lindley distribution and its applications. *Math. Comput. Simul.* **2008**, *78(4)*, 493–506.
4. Zakerzadeh, H.; Dolati, A. Generalized Lindley distribution. *J. Math. Ext.* **2009**, *3(2)*, 13–25.
5. Gómez-Déniz, E.; Calderin-Ojeda, E. The discrete Lindley distribution: properties and application, *J. Stat. Comput. Simul.* **2011**, *81(11)*, 1405–1416.
6. Krishna, H.; Kumar, K. Reliability estimation in Lindley distribution with progressively type II right censored sample. *Math. Comput. Simulat.* **2011** *82(2)*, 281–294.
7. Bakouch, H.S.; Al-Zaharani, B.; Al-Shomrani, A.; Marchi, V.; Louzada, F. An extended Lindley distribution. *J. Korean Stat. Soc.* **2012**, *41*, 75–85.
8. Shanker, R.; Sharma, S.; Shanker, R. A two-parameter Lindley distribution for modeling waiting and survival times data. *Appl. Math.* **2013**, *4*, 363–368.
9. Ghitany, M.; Al-Mutairi, D.; Balakrishnan, N.; Al-Enezi, I. Power Lindley distribution and associated inference. *Comput. Stat. Data Anal.* **2013**, *64*, 20–33.
10. Al-Mutairi, D.K.; Ghitany, M.E.; Kundu, D. Inferences on stress-strength reliability from Lindley distributions. *Commun. Stat. – Theory Methods* **2013**, *42(8)*, 1443–1463.
11. Oluyede, B.O.; Yang, T. A new class of generalized Lindley distribution with applications. *J. Stat. Comput. Simul.* **2014**, *85(10)*, 2072–2100.
12. Shanker, R.; Hagos, F.; Sujatha, S. On modeling of Lifetimes data using exponential and Lindley distributions. *Biom. Biostat. Int. J.* **2015**, *2(5)*, 1–9.
13. Abouammoh, A.M.; Alshangiti, A.M.; Ragab, I.E. A new generalized Lindley distribution. *J. Stat. Comput. Simul.* **2015**, *85(18)*, 3662–3678.
14. Shanker, R. Akash Distribution and Its Applications. *International Journal of Probability and Statistics* **2015**, *4(3)*, 65–75.
15. Shanker, R.; Shukla, K.K. On Two-Parameter Akash Distribution. *Biom. Biostat. Int. J.* **2017**, *6(5)*, 00178.
16. Shanker, R.; Shukla, K.K. Power Akash Distribution and Its Application. *J. Appl. Quant. Methods* **2017**, *12(3)*, 1–10.
17. Gómez, Y.M.; Firinguetti-Limone, L.; Gallardo, D.I.; Gómez, H.W. An Extension of the Akash Distribution: Properties, Inference and Application. *Mathematics* **2014**, *12*, 31. https://doi.org/10.3390/math12010031
18. Glaser, R.E. Bathtub and Related Failure Rate Characterizations. *J. Am. Stat. Assoc.* **1980**, *75(371)*, 667–672.
19. Fuller, E.J.; Frieman, S.; Quinn, J.; Quinn, G.; Carter, W. Fracture mechanics approach to the design of glass aircraft windows: A case study, SPIE Procceding, 1994, Vol. 2286, 419–430. https://doi.org/10.1117/12.187363
20. Akaike, H. A new look at the statistical model identification. *IEEE Trans. Automat. Contr.* **1974**, *19*, 716–723.
21. Schwarz, G. Estimating the dimension of a model. *Ann. Stat.* **1978**, *6*, 461–464.
22. Dunn, P.K.; Smyth, G.K. Randomized Quantile Residuals. *J. Comput. Graph. Stat.* **1996**, *5(3)*, 236–244.