The SquAd derivation: a Square Additive Approach to the Turbulent Prandtl Number

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Abstract

A formula for the turbulent Prandtl number is derived in terms of local variables available from two-equations turbulence models. The derivation is a direct consequence of the expected square additivity of the molecular and turbulent parameters defining the effective viscosity and the effective conductivity. The formula does not degenerate and leads to a Kays like formulation if approximated. Its utility and its limitation are discussed.

1 Introduction

Low Prandtl number liquid metals serve as primary coolant for MYRRHA and ALFRED, two Gen-IV reactors under development [1]. The low Prandtl number induces discrepancies in the modeling of the turbulent heat transfer when directly treated according to the Reynolds analogy with a constant turbulent Prandtl number. The thermal boundary layer is considerably larger than the velocity one, up to the point that, while we can clearly define a bulk velocity, the temperature profile may not exhibit any almost constant bulk temperature plateau. Many correlations have been derived to tackle the issue [2, 3, 4]. Almost all of them use global parameters such as the Reynolds or Peclet numbers which are not well defined in complex geometries and by such not suitable for the related CFD simulations. To the author knowledge, only Kays' correlation is using only local parameters. Several different variants of the correlations have been used with significant success, leading to some perplexity. In a former paper [5], we showed that the correlations can be simply derived on a basic assumption with regards to the non-linear combination of stochastic effects and the variants then come from different approximations of a mother formula. A defect of the mother formula, which is transferred to the variants is that the turbulent Prandtl number becomes infinite at vanishing turbulence. In this paper, we proceed further with the constructive hypothesis to refine the formula, extending the principle of square additivity not only to the thermal conductivity

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but also to the viscosity. The derived formula has the merit to be simple and to not degenerate anymore at vanishing turbulence.

2 Derivation of Pr_t and its approximation

Viscous effects and heat diffusion modelling and implementation are represented as the direct sum of two contributions, the molecular one and the turbulent one. This is a convenient and simple representation based on the fact that for most turbulent flows, the turbulent viscosity and conductivity are much higher than their molecular counterparts. The turbulent quantities were also not expected, at least historically, to be known at a high level of precision.

With regard to the heat diffusion, both the molecular and the turbulent contributions act in the same direction, proportionally to the local temperature gradient. Each contribution is also proportional to a conductivity coefficient. In other words, the effective conductivity coefficient k_e is given as

$$k_e = k + k_t \tag{1}$$

where k is the molecular conductivity coefficient observed in static or laminar flows and k_t is the increment due to turbulence. With ρ the fluid density and C_p its specific heat, this can be rewritten in terms of thermal diffusivity:

$$\alpha_e = \alpha + \alpha_t \tag{2}$$

in which $\alpha_e = k_e/(\rho C_p)$ is the effective diffusivity, $\alpha = k/(\rho C_p)$ is the molecular part and $\alpha_t = k_t/(\rho C_p)$ is the turbulent part.

Similarly, with regards to the viscosity the effective kinematic viscosity ν_e, is the sum of a laminar contribution ν and a turbulent contribution ν_t:

$$\nu_e = \nu + \nu_t \tag{3}$$

The argument developed in precedence [5] is the following. Conduction in general is originated from a stochastic process. Molecular conduction and turbulent conduction look like different unrelated mechanisms operating at different scales. While molecular conduction is in fact a molecular process, turbulent conduction is rather based on a convective process. The cumulative effect is a convolution rather than a mere juxtaposition. In turn, one would expect the combined effects to be according to:

$$\alpha_e = \sqrt{\alpha^2 + \alpha_0^2}, \tag{4}$$

where α_0 is the turbulent thermal diffusivity expected for a similar fluid with (almost) zero molecular conductivity or similarly when the turbulence is extremely large.

We now extend the argument to the viscous process and state that the effective viscosity comes from two independent processes whose intensity should be square additive:

$$\nu_e = \sqrt{\nu^2 + \nu_0^2}. \tag{5}$$

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By simple substitution, we have just defined two quantities:

$$\nu_0 = \nu_t \sqrt{1 + \frac{2\nu}{\nu_t}}. \tag{6}$$

and

$$\alpha_0 = \alpha_t \sqrt{1 + \frac{2\alpha}{\alpha_t}}. \tag{7}$$

We can also redefine ν_t and α_t in terms of these quantities:

$$\nu_t = \sqrt{\nu^2 + \nu_0^2} - \nu \tag{8}$$

$$\alpha_t = \sqrt{\alpha^2 + \alpha_0^2} - \alpha \tag{9}$$

The (molecular) Prandlt numbers is defined as $Pr = \frac{\nu}{\alpha}$. Similarly, the turbulent Prandtl number is defined as $Pr_t = \frac{\nu_t}{\alpha_t}$ and the asymptotic turbulent Prandtl number as $Pr_0 = \frac{\nu_0}{\alpha_0}$. This later number has been conceived in order to be the most possible independent of the fluid. Noting that $\frac{\alpha}{\alpha_0} = \frac{Pr_0 \nu}{Pr \nu_0}$, simple algebraic manipulations from the former equations give:

$$Pr_t = \frac{Pr_0^2}{Pr} \frac{\sqrt{1 + (\frac{Pr\nu_0}{Pr_0\nu})^2} + 1}{\sqrt{1 + (\frac{\nu_0}{\nu})^2} + 1}. \tag{10}$$

This specific form is chosen to show that is cannot degenerate to zero or infinity. In effect, because ν_0 goes to zero if ν_t does, then Pr_t tend to $\frac{Pr_0^2}{Pr}$ while forge large turbulence, Pr_t tends to Pr_0. In turn, if Pr_0, built for this purpose, do not degenerate, then neither do Pr_t.

The former expression can be rewritten in terms of ν/ν_0:

$$Pr_t = Pr_0 \frac{\sqrt{1 + (\frac{Pr_0\nu}{Pr\nu_0})^2} + \frac{Pr_0\nu}{Pr\nu_0}}{\sqrt{1 + (\frac{\nu}{\nu_0})^2} + \frac{\nu}{\nu_0}}. \tag{11}$$

This formula is quite complicated and not easy to interpret at first glance, but considering ν/ν_0 as a small parameter, a brutal simplification of 11 at first order in this parameter gives:

$$Pr_t \simeq Pr_0[1 + (\frac{Pr_0}{Pr} - 1)\frac{\nu}{\nu_0}] \tag{12}$$

This formulation is practical only if ν_0 is readily available. This would be the case if we had a transport equation for ν_0 or a related variable similarly to what is done with the usual 2-equations turbulence models. Exploring the potential of this possibility is beyond the scope of the current argument and we need to express Pr_t in terms of known parameters.

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Thus, expressing Pr_t in terms of ν_t instead of ν_0, the formula becomes:

$$Pr_t = \frac{Pr_0^2}{Pr} \frac{\sqrt{1 + (\frac{Pr}{Pr_0})^2 \frac{\nu_t}{\nu}(\frac{\nu_t}{\nu} + 2)} + 1}{2 + \frac{\nu_t}{\nu}}, \tag{13}$$

this form being useful for interpretation at vanishing turbulent viscosity. Dividing this last formula by Pr_0, we have a relation between the three adimentional numbers $\frac{Pr_t}{Pr_0}$, $\frac{Pr}{Pr_0}$ and $\frac{\nu_t}{\nu}$ in the form:

$$\frac{Pr_t}{Pr_0} = f(\frac{Pr}{Pr_0}, \frac{\nu_t}{\nu}). \tag{14}$$

In view of a development in ν/ν_t, the formula becomes:

$$Pr_t = Pr_0 \frac{\sqrt{1 + \frac{2\nu}{\nu_t} + (\frac{Pr_0\nu}{Pr\nu_t})^2)} + \frac{Pr_0\nu}{Pr\nu_t}}{1 + \frac{2\nu}{\nu_t}} \tag{15}$$

Here again, a brutal first order development in terms of ν/ν_t, valid only when both ν_t/ν and $Pr\nu_t/Pr_0\nu$ are large, gives:

$$Pr_t \simeq Pr_0[1 + (\frac{Pr_0}{Pr} - 1)\frac{\nu}{\nu_t}], \tag{16}$$

meaning that changing from ν_0 to ν_t brings only second order terms approximation. In particular, this last expression has the same form as the Kays correlation [2].

3 Discussion

Taking into consideration that Pr_0 is usually taken as $Pr_0 = 0.85$, we get for $Pr \simeq 0.025$ typical of heavy liquid metal:

$$Pr_t \simeq 0.85 + \frac{0.70}{Pr\frac{\nu_t}{\nu}} \tag{17}$$

For the second coefficient (here 0.7), Kays indicated two values, 0.7 and 2 discussing without reaching a conclusion in favor of one or the other value. However, a less brutal approximation could lead to a different coefficient. For example, the value of 1.46 is used by [6] to better fit heat transfer in a tube by direct analytical integration. Moreover, the formula is not much different than the one derived previously [5] and a more precise approximation, while more complex to derive would most probably also lead to an increased second coefficient about 1.45.

There are indications [2] that Pr_t and therefore also Pr_0 is about 0.85 for very large ν_t/ν. We do not know the behavior of Pr_0 for lower values. Nevertheless, all this construction makes sense only under the hypothesis of a basically

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constant Pr_0. A particular case is when $Pr = Pr_0$. Then $Pr_t = Pr_0$ too. It is also interesting to observe that for medium and high Pr, then Pr_t in formula (11) to not significantly departs from Pr_0, except for strongly vanishing ν_t/ν.

With a little algebra, we found that Pr_t is a decreasing function of ν_t for $Pr \leq Pr_t$ and increasing function of ν_t for $Pr \geq Pr_t$, with values spanning the interval $[Pr_0; Pr_0^2/Pr]$ and $[Pr_0^2/Pr; Pr_0]$. The approximation 16, while with the same monotonousness, fails to meet the correct bound for vanishing viscosity, for which it degenerates. Interestingly, we can see that the second coefficient depends critically on the Prandtl number, to the point that it vanishes for $Pr_t = Pr_0$ and change sign afterwards, like for the complete expression. This is a clear indication that the Kays correlation should be used as is only for low Prandtl number (say < 0.1) fluids.

While the derived formula is thought to be used within a turbulence model, if correct it would make sense only if the turbulence model correctly predicts the turbulent viscosity, not only in the viscous boundary layer but also and principally in the bulk, in which the thermal boundary layer is still in development. The problem is that the turbulence models mainly focus on the correct near-wall boundary layer turbulent viscosity profile, as it is the place where almost all the pressure drop is built, with the main aim to capture the correct wall shear stress. The turbulent viscosity profile in the bulk is normally of no practical importance, except for thermal flows of low Prandtl fluids. Looking at the profile of ν_t compared with DNS data even in the simpliest 2D channel flow, see figure 5 in [7], we can see that the turbulence models fail to give a correct profile for a wall Y+ above 30. In particular, ν_t is under predicted below Y+ up to 100-150 and over predicted afterward. The difference between the simply additive and the square additive approaches is mainly concentrated and felt around the values where both contributions are similar: $\nu/Pr \sim \nu_t/Pr_0$ or equivalently $\nu_t/\nu \sim Pr_0/Pr$. For a low Prandtl fluids with $Pr = 0.025$ we have $Pr_0/Pr = 34$, so all the region where ν_t/ν lies between say 10 and 100 is concerned, that is precisely for Y+ above 30 for the case analysed in [7]. The balance between the thermal effects of both Y+ region is shifted towards an artificially increased diffusion. To counteract this effect, the turbulent Prandtl number can be increased artificially for a better fit. This could be an explanation for the use of an augmented second coefficient in the Kays correlation as indicated previously.

The main effect of applying the square additivity to the effective viscosity is that it removes the degeneration of the Pr_t formula for vanishing viscosity that was still present in [5]. It is not clear nor sure that it is of practical importance anywhere else. It seems that, within the level of approximation given by the 2-equations turbulence models, ν_t and ν_0 can be used indifferently in the formula. In other words, the square additivity could be used solely for the energy equation.

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4 Conclusion

We consider that turbulent viscosity and molecular viscosity are originated from two independent stochastic processes whose intensity is square additive. We make the same consideration for the turbulent conductivity and the thermal conductivity. By stating that the asymptotic Prandtl number is in fact a constant under the square additive approach, comes an expression for the Prandtl number which first order approximation is very similar to the first variant of Kays' correlation, the only non-trivial correlation exclusively based on local parameters and thus suited for arbitrary CFD simulations. The derivation sheds light on the Kays correlation and indicates that the second coefficient depends critically on the Prandtl number to the point that it vanishes when $Pr = 0.85$. Under the condition that the classical 2-equations turbulence models become able to capture correctly the turbulent viscosity profile, we expect that directly implementing the square additivity of the components of the effective conductivity could give improved thermal results independently of the Prandtl number and particularly for the low Prandtl liquid Lead and Lead alloys.

The Reynolds analogy and its extension to thermal flows could have a much wider domain of validity than expected by combining it with the square additivity of the coefficients.

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