Article

Not peer-reviewed version

Preservation of Mean-Square Lyapunov Exponents for Nonautonomous Stochastic Evolution Equations

Dmytro Shtefan * , Oleksandr Stanzhytskyi , Svitlana Kushnirenko

Posted Date: 17 March 2026

doi: 10.20944/preprints202603.1278.v1

Keywords: dichotomy; Wiener process; spectrum; stability; Lyapunov exponents; Q-Wiener process

Article

Preservation of Mean-Square Lyapunov Exponents for Nonautonomous Stochastic Evolution Equations

Dmytro Shtefan *, Oleksandr Stanzhytskyi and Svitlana Kushnirenko

Taras Shevchenko National University of Kyiv, Faculty of Mechanics and Mathematics, 64/13 Volodymyrska Street, Kyiv, 01601, Ukraine

* Correspondence: dmitrijstefan396@gmail.com

Abstract

We study the long-time behavior of nonlinear stochastic evolution equations in a separable Hilbert space driven by a Q-Wiener process. The linear part of the equation is generated by a strongly continuous semigroup with an exponential dichotomy, which provides fixed rates of decay and growth. The nonlinear drift and diffusion terms are globally Lipschitz and become small as time tends to infinity. Our main result shows that under these conditions, the mean-square Lyapunov exponents of the nonlinear system coincide with those of the linear part. In other words, nonlinear stochastic perturbations that **decay in time** do not change the main growth or decay rates of solutions in the mean-square sense. This result provides simple and verifiable criteria ensuring that the long-time Lyapunov behavior of the nonlinear stochastic equation is fully determined by the linear semigroup, even in the presence of time-dependent stochastic perturbations.

Keywords: dichotomy; Wiener process; spectrum; stability; Lyapunov exponents; Q-Wiener process

---

1. Introduction

The asymptotic behavior of infinite-dimensional evolution equations under deterministic and stochastic perturbations is a fundamental problem with deep connections to stability theory, invariant manifold theory, and the qualitative analysis of stochastic partial differential equations. In many concrete models arising in physics, biology, and engineering, the linear part of the dynamics provides a natural splitting of the phase space into exponentially stable and unstable directions; this splitting is conveniently formalized by an *exponential dichotomy* of the linear semigroup. Understanding whether and how the asymptotic growth rates dictated by the linear part persist in the presence of (possibly multiplicative) stochastic forcing and nonlinear perturbations is the central question studied in this paper.

We consider stochastic evolution equations on a separable Hilbert space H of the form

$$\mathrm{d}y(t) = \big(Ay(t) + F(t, y(t))\big)\,\mathrm{d}t + G(t, y(t))\,\mathrm{d}W(t), \tag{1}$$

where $A : D(A) \subset H \to H$ generates a strongly continuous semigroup $\{e^{At}\}_{t \geq 0}$ and W denotes a Q-Wiener process. The unperturbed linear semigroup is assumed to admit an exponential dichotomy on $[0, \infty)$: there is a bounded projection that splits $H = H^- \oplus H^+$ and a rate $\alpha > 0$ so that solutions originating in the negative subspace H^- decay exponentially at least at a rate α, while those associated with the positive subspace H^+ grow at least at a rate α. This linear dichotomy provides the reference exponents $\pm\alpha$ against which the perturbed dynamics will be measured.

The nonlinear terms are taken to satisfy two structural conditions. First, a global Lipschitz condition (assumption (H1)) ensures well-posedness and permits mean-square estimates for solutions. Second, an *asymptotic smallness* condition (assumption (H2)) requires that the nonlinear drift and diffusion coefficients decay to zero, as $t \to \infty$, in a uniform, multiplicative manner. Intuitively, (H2)

 doi:10.20944/preprints202603.1278.v1

means that for large times the nonlinear stochastic perturbation becomes negligible in comparison with the linear dynamics; such a regime is natural in models with temporally vanishing forcing or coefficients and is crucial for comparing long-time growth rates with the linear reference.

Our main result (Theorem 1) states that, under (H1) and (H2), the asymptotic mean-square growth rates of nontrivial mild solutions coincide with the reference exponents provided by the linear dichotomy: any solution that remains bounded in mean-square decays with upper mean-square Lyapunov exponent at most $-\alpha$, while any solution whose mean-square norm diverges grows with lower mean-square Lyapunov exponent at least $+\alpha$. Equivalently, there is no intermediate mean-square growth rate: every nontrivial trajectory is either mean-square exponentially decaying or mean-square exponentially growing, and the corresponding mean-square Lyapunov exponent equals $\pm\alpha$. This preservation of mean-square Lyapunov exponents demonstrates a robust dichotomy of asymptotic dynamics for the full stochastic system.

From a conceptual viewpoint, the result highlights a form of spectral rigidity for mean-square growth rates: when nonlinear and stochastic perturbations decay in time they cannot create new asymptotic mean-square growth rates distinct from those already present in the linear part. Practically, this gives a simple and computable criterion to classify long-time behaviour of a large class of stochastic evolution equations by inspecting the linear generator and verifying the mild decay of perturbations.

The hypotheses adopted here strike a balance between generality and technical transparency. The global Lipschitz assumption could be relaxed in future work (for instance to local Lipschitz with suitable dissipativity), and the asymptotic smallness condition might be weakened to integrability-type assumptions; both directions raise interesting technical challenges, especially in controlling stochastic convolutions without uniform Lipschitz estimates. We briefly discuss such extensions and open problems in the conclusion.

The paper is organized as follows. Immediately after this introduction, we present precise definitions, the standing hypotheses (H1)-(H2), the notion of mild solution and mean-square Lyapunov exponents, and we state the main preservation theorem together with remarks on the interpretation of the driving noise (see Section 2). In Section 3, we give a detailed proof of Theorem 1. Section 4 contains two concrete examples that illustrate the applicability of the abstract result to simple stochastic evolution models. The manuscript concludes with a discussion in Section 5, outlining possible generalizations and further research directions.

Notation. Throughout the paper $\|\cdot\|$ denotes the norm on H, $L(H)$ denotes the space of bounded linear operators on H, and $\|\cdot\|_{HS}$ is the Hilbert–Schmidt norm for operators where it appears. We work in the mean-square framework and use standard notions for stochastic integrals in Hilbert spaces.

2. Materials and Methods

2.1. Exponential Dichotomy of the Linear Part

We assume that the linear semigroup generated by A admits an exponential dichotomy on $[0,\infty)$ (see, e.g., [1,2]). More precisely, there exists a bounded projection $P_- \in L(H)$ inducing a decomposition

$$H = H^- \oplus H^+,$$

where $H^- := P_- H$, and $H^+ := (I - P_-)H$, and constants $N \geq 1$, $\alpha > 0$ such that

$$\|P_- e^{As}\| \leq N e^{-\alpha s}, \qquad \|P_+ e^{-As}\| \leq N e^{-\alpha s}, \qquad s \geq 0, \tag{2}$$

where $P_+ := I - P_-$.

This dichotomy provides a reference splitting into exponentially stable and unstable directions for the unperturbed linear dynamics.

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 17 March 2026 doi:10.20944/preprints202603.1278.v1

2.2. *Assumptions on the Nonlinear Terms*

Throughout the paper we impose the following standing hypotheses on the nonlinearities.

(H1) *Global Lipschitz continuity.* There exists a constant $L > 0$ such that for all $t \geq 0$ and all $y_1, y_2 \in H$

$$\|F(t, y_1) - F(t, y_2)\| + \|G(t, y_1) - G(t, y_2)\|_{HS} \leq L\|y_1 - y_2\|.$$

(H2) *Asymptotic smallness.* There exists a scalar function $\varepsilon : [0, \infty) \to [0, \infty)$ with $\varepsilon(t) \to 0$ as $t \to \infty$ such that for all $t \geq 0$ and all $y \in H$

$$\|F(t, y)\| + \|G(t, y)\|_{HS} \leq \varepsilon(t)\|y\|.$$

Assumption (H1) ensures global well-posedness (see, e.g., [2]), while (H2) expresses that the nonlinear perturbation becomes negligible along large-time trajectories, allowing comparison with the linear dynamics.

2.3. *Q-Wiener Processes*

Let U be a separable Hilbert space, and let $Q \in L(U)$ be a self-adjoint, nonnegative operator with finite trace $\operatorname{Tr} Q < \infty$. Then there exists an orthonormal basis $\{e_j\}_{j \geq 1} \subset U$ and a sequence of nonnegative eigenvalues $\{\lambda_j\}_{j \geq 1}$ such that

$$Qe_j = \lambda_j e_j, \qquad \lambda_j \geq 0, \qquad \sum_{j=1}^{\infty} \lambda_j = \operatorname{Tr} Q < \infty.$$

Definition 1 (*Q*-Wiener process)**.** *Let $\{\beta_j(t)\}_{j \geq 1}$ be a collection of mutually independent standard one-dimensional Wiener processes defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$, adapted to the filtration $\{\mathcal{F}_t\}$. The U-valued process*

$$W(t) = \sum_{j=1}^{\infty} \sqrt{\lambda_j}\, \beta_j(t)\, e_j \tag{3}$$

is called a Q-Wiener process. If $\operatorname{Tr} Q < \infty$, then for each fixed $t \geq 0$ the series (3) converges in $L^2(\Omega; U)$, and W admits a version with continuous trajectories in U almost surely (see, e.g., [2]).

In the sequel, we shall treat the driving noise in (1) as a Q-Wiener process taking values in U, and interpret $G(t, y)$ as an operator in the space $L_{HS}(U, H)$ of Hilbert–Schmidt operators from U to H; consequently $\|G\|_{HS}$ denotes the Hilbert–Schmidt norm.

2.4. *Mild Solutions*

Definition 2. *A mild solution to the equation (1) at $t \geq 0$, with initial data $y(0) = y_0 \in H$ is a $\{\mathcal{F}_t\}$-adapted process $y(\cdot) \in H$ that satisfies the integral representation*

$$y(t) = S(t)\, y_0 + \int_0^t S(t - s)F\big(s, y(s)\big)\, \mathrm{d}s + \int_0^t S(t - s)G\big(s, y(s)\big)\, \mathrm{d}W(s). \tag{4}$$

The first integral is understood as a Bochner integral in H, and the second as the stochastic Itô integral.

Remark 1. *Under assumptions **(H1)** and **(H2)** on F and G, the mild solution in Definition 2 exists, is pathwise unique, and extends for all $t \geq 0$. Moreover, the function*

$$m(t) := \mathbb{E}\, \|y(t)\|^2$$

is finite and continuous for $t \geq 0$. These facts follow from standard results on stochastic evolution equations; see e.g., [1,2] (and further discussion in [5]).

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 17 March 2026 doi:10.20944/preprints202603.1278.v1

2.5. Mean-Square Lyapunov Exponents

Definition 3. *Let $y(t; y_0)$ be the unique mild solution of (1) with deterministic initial data $y_0 \in H$. Define the* upper *and* lower mean-square Lyapunov exponents *of y_0 by*

$$\overline{\lambda}_2(y_0) := \frac{1}{2} \limsup_{t\to\infty} \frac{1}{t} \ln \mathbb{E}\|y(t; y_0)\|^2, \qquad \underline{\lambda}_2(y_0) := \frac{1}{2} \liminf_{t\to\infty} \frac{1}{t} \ln \mathbb{E}\|y(t; y_0)\|^2.$$

If $\overline{\lambda}_2(y_0) = \underline{\lambda}_2(y_0)$, their common value is denoted by

$$\lambda_2(y_0) := \overline{\lambda}_2(y_0) = \underline{\lambda}_2(y_0),$$

and is called the mean-square Lyapunov exponent *of the solution starting at y_0.*

Remark 2. *For deterministic linear systems the above definition coincides with the usual Lyapunov exponent (see, e.g., [1]). Indeed, if*

$$y'(t) = Ay(t) \tag{5}$$

and $y(t) = S(t)y_0$, then

$$\lambda_2(y_0) = \frac{1}{2} \lim_{t\to\infty} \frac{1}{t} \ln \|S(t)y_0\|^2 = \lim_{t\to\infty} \frac{1}{t} \ln \|S(t)y_0\|,$$

which is the classical Lyapunov exponent of the trajectory $S(\cdot)y_0$. Thus, for (5) the mean-square exponent reduces to the standard exponent of the semigroup generated by A.

3. Results

The following result shows that the nonlinear stochastic perturbation preserves the asymptotic mean-square growth rates given by the linear dichotomy.

Theorem 1 (Preservation of mean-square Lyapunov exponents)**.** *Assume that the generator A of the linear semigroup admits an exponential dichotomy (see (2)) with projection P_- and rate $\alpha > 0$. Let F and G satisfy assumptions* (H1)–(H2)*. Let $y(t; y_0)$ be the mild solution of (1) with initial data $y_0 \in H$. Then:*

1. *If the solution has uniformly bounded second moment,*

$$\sup_{t\geq 0} \mathbb{E}\|y(t; y_0)\|^2 < \infty,$$

then the upper mean-square Lyapunov exponent satisfies

$$\overline{\lambda}_2(y_0) \leq -\alpha.$$

2. *If the solution has unbounded second moment,*

$$\sup_{t\geq 0} \mathbb{E}\|y(t; y_0)\|^2 = +\infty,$$

then the lower mean-square Lyapunov exponent satisfies

$$\underline{\lambda}_2(y_0) \geq \alpha.$$

In particular, every nontrivial solution of (1) is either mean-square exponentially decaying or mean-square exponentially growing, and its mean-square Lyapunov exponent coincides with the corresponding linear exponent $\pm\alpha$.

Proof of Theorem 1. We split the proof into several parts.

Stable projection.

Apply the projection P_- to the mild solution (4). We have

$$P_- y(t_2) - P_- e^{A(t_2-t_1)} y(t_1) = \int_{t_1}^{t_2} P_- e^{A(t_2-\tau)} F(\tau, y(\tau))\,\mathrm{d}\tau + \int_{t_1}^{t_2} P_- e^{A(t_2-\tau)} G(\tau, y(\tau))\,\mathrm{d}W(\tau). \tag{6}$$

Set $C_0 := 3N^2$. From (6) we obtain the moment estimate

$$\begin{aligned}\mathbb{E}\|P_- y(t_2)\|^2 &= \mathbb{E}\Bigg\| P_- e^{A(t_2-t_1)} y(t_1) + \int_{t_1}^{t_2} P_- e^{A(t_2-\tau)} F(\tau, y(\tau))\,\mathrm{d}\tau \\ &\quad + \int_{t_1}^{t_2} P_- e^{A(t_2-\tau)} G(\tau, y(\tau))\,\mathrm{d}W(\tau)\Bigg\|^2 \\ &\le C_0 e^{-2\alpha(t_2-t_1)} \mathbb{E}\|y(t_1)\|^2 + 3\,\mathbb{E}\Big\| \int_{t_1}^{t_2} P_- e^{A(t_2-\tau)} F(\tau, y(\tau))\,\mathrm{d}\tau \Big\|^2 \\ &\quad + 3\,\mathbb{E}\Big\| \int_{t_1}^{t_2} P_- e^{A(t_2-\tau)} G(\tau, y(\tau))\,\mathrm{d}W(\tau) \Big\|^2.\end{aligned} \tag{7}$$

Let $\beta = \beta(t_0) = \sup_{t \ge t_0} \varepsilon(t)$. Obviously, $\beta \to 0$, as $t_0 \to \infty$. By assumption **(H2)**, we have

$$\|F(t, y)\| \le \beta \|y\| \tag{8}$$

for all $t \ge t_0,\ y \in H$.

In particular, if $t_1 \ge t_0$ then (8) holds for every $\tau \in [t_1, t_2]$, and hence, may be used to estimate the deterministic term in (7).

$$\begin{aligned}&\mathbb{E}\left(\left\| \int_{t_1}^{t_2} P_- e^{A(t_2-\tau)} F(\tau, y)\,\mathrm{d}\tau \right\|^2 \right) \\ &\le N^2 \beta^2 \mathbb{E}\left(\left(\int_{t_1}^{t_2} \|y(\tau)\| e^{-\alpha(t_2-\tau)}\,\mathrm{d}\tau \right)^2 \right) \\ &= N^2 \beta^2 \mathbb{E}\left(\left(\int_{t_1}^{t_2} \|y(\tau)\| e^{-\alpha'(t_2-\tau)} e^{-(\alpha-\alpha')(t_2-\tau)/2} e^{-(\alpha-\alpha')(t_2-\tau)/2}\,\mathrm{d}\tau \right)^2 \right) \\ &\le N^2 \beta^2 \int_{t_1}^{t_2} \mathbb{E}(\|y(\tau)\|^2) e^{-2\alpha'(t_2-\tau)} e^{-(\alpha-\alpha')(t_2-\tau)}\,\mathrm{d}\tau \int_{t_1}^{t_2} e^{-(\alpha-\alpha')(t_2-\tau)}\,\mathrm{d}\tau \\ &\le \frac{N^2\beta^2}{(\alpha-\alpha')^2} \max_{t_1 \le \tau \le t_2} \mathbb{E}(\|y(\tau) e^{-\alpha'(t_2-\tau)}\|^2)\end{aligned} \tag{9}$$

for any $\alpha' \in (0; \alpha)$.

Next, we estimate the stochastic integral using Itô's isometry and the smallness assumption $\|G(\tau, y)\|_{HS} \leq \beta\|y\|$. We obtain

$$\begin{aligned}
&\mathbb{E}\Big\| \int_{t_1}^{t_2} P_- e^{A(t_2-\tau)} G(\tau, y(\tau))\, \mathrm{d}W(\tau) \Big\|^2 \\
&\quad = \mathbb{E} \int_{t_1}^{t_2} \|P_- e^{A(t_2-\tau)} G(\tau, y(\tau))\|_{HS}^2 \, \mathrm{d}\tau \\
&\quad \leq N^2\beta^2 \int_{t_1}^{t_2} e^{-2\alpha(t_2-\tau)} \mathbb{E}\|y(\tau)\|^2 \, \mathrm{d}\tau \\
&\quad \leq N^2\beta^2 \sup_{t_1 \leq \tau \leq t_2} \big(\mathbb{E}\|y(\tau)\|^2 e^{-2\alpha'(t_2-\tau)}\big) \int_{t_1}^{t_2} e^{-2(\alpha-\alpha')(t_2-\tau)} \, \mathrm{d}\tau \\
&\quad \leq \frac{N^2\beta^2}{2(\alpha-\alpha')} \sup_{t_1 \leq \tau \leq t_2} \big(\mathbb{E}\|y(\tau)\|^2 e^{-2\alpha'(t_2-\tau)}\big).
\end{aligned} \tag{10}$$

Combining the inequality (10) with (9) yields

$$\mathbb{E}\|P_- y(t_2)\|^2 \leq C_0 e^{-2\alpha(t_2-t_1)} \mathbb{E}\|y(t_1)\|^2 + \beta_- \sup_{t_1 \leq \tau \leq t_2} \big(\mathbb{E}\|y(\tau)\|^2 e^{-2\alpha'(t_2-\tau)}\big), \tag{11}$$

where

$$\beta_- := \frac{3N^2\beta^2}{\alpha-\alpha'} + \frac{3N^2\beta^2}{(\alpha-\alpha')^2}.$$

Therefore, $\beta(t_0) \to 0$, as $t_0 \to \infty$, and we have $\beta_- \to 0$.

Unstable projection.

Apply the projection P_+ to the mild solution (4). We obtain

$$\begin{aligned}
P_+ y(t_1) = P_+ e^{A(t_1-t_2)} y(t_2) - \int_{t_1}^{t_2} P_+ e^{A(t_1-\tau)} F(\tau, y(\tau))\, \mathrm{d}\tau \\
- \int_{t_1}^{t_2} P_+ e^{A(t_1-\tau)} G(\tau, y(\tau))\, \mathrm{d}W(\tau).
\end{aligned} \tag{12}$$

Using the estimate $\|P_+ e^{A(t_1-\tau)}\| = \|P_+ e^{-A(\tau-t_1)}\| \leq N e^{-\alpha(\tau-t_1)}$ for $\tau \in [t_1, t_2]$, and repeating the same steps as in the stable case, we obtain

$$\mathbb{E}\|P_+ y(t_1)\|^2 \leq C_0 e^{-2\alpha(t_2-t_1)} \mathbb{E}\|y(t_2)\|^2 + \beta_+ \sup_{t_1 \leq \tau \leq t_2} \big(\mathbb{E}\|y(\tau)\|^2 e^{-2\alpha'(\tau-t_1)}\big), \tag{13}$$

where

$$\beta_+ := \frac{3N^2\beta^2}{\alpha-\alpha'} + \frac{3N^2\beta^2}{(\alpha-\alpha')^2}.$$

Moreover, $\beta_+ \to 0$, as $t_0 \to 0$.

Main inequality.

Set $\beta' := \beta_- = \beta_+$. Let $t_1 \leq t' \leq t_2$ be arbitrary, with $t_1, t_2 \geq t_0$. Applying (11) on $[t_1, t']$ and (13) on $[t', t_2]$, and using the identity $\|y\|^2 = \|P_- y\|^2 + \|P_+ y\|^2$, we obtain

$$\begin{aligned}
\mathbb{E}\|y(t')\|^2 \leq C_0 e^{-2\alpha(t_2-t')} \mathbb{E}\|y(t_2)\|^2 + C_0 e^{-2\alpha(t'-t_1)} \mathbb{E}\|y(t_1)\|^2 \\
+ \beta' \sup_{t_1 \leq \tau \leq t'} \big(\mathbb{E}\|y(\tau)\|^2 e^{-2\alpha'(t'-\tau)}\big) + \beta' \sup_{t' \leq \tau \leq t_2} \big(\mathbb{E}\|y(\tau)\|^2 e^{-2\alpha'(\tau-t')}\big).
\end{aligned} \tag{14}$$

Next, we state and prove the following lemma.

 doi:10.20944/preprints202603.1278.v1

Lemma 1. *For any γ with $\alpha' > \gamma > -\alpha'$, there exists a time $T > 0$ such that if, for an arbitrary nontrivial solution, the estimate*

$$\mathbb{E}\|y(t_1)\|^2 < \mathbb{E}\left\|y(t_1+T)e^{-\gamma T}\right\|^2$$

holds, then for any $t > 2T$, the inequality

$$\mathbb{E}\|y(t_1)\|^2 < \mathbb{E}\left\|y(t_1+t)e^{-\gamma t}\right\|^2$$

is satisfied.

Proof. We argue by contradiction. Suppose there exists $t_2 > 2T + t_1$ such that

$$\mathbb{E}\|y(t_1)\|^2 \geq \mathbb{E}\left\|y(t_1+t_2)e^{-\gamma(t_2-t_1)}\right\|^2.$$

Consider the set

$$A = \{t \in [t_1, t_2] \mid \mathbb{E}\|y(t_1+T)\|^2 e^{2\alpha'|t-(t_1+T)|} \leq \mathbb{E}\|y(t)\|^2\}.$$

This set is closed because $\mathbb{E}|y(t)|^2$ is continuous, it is also bounded, and hence compact. Since $t_1 + T \in A$, the set A is nonempty. Therefore, there exists a point $t' \in A$ at which the maximum of $\mathbb{E}|y(t)|^2$ on A is attained. By the definition of A, for every $t \in [t_1, t_2]$, we have

$$\mathbb{E}\|y(t')\|^2 e^{2\alpha'|t-t'|} \geq \mathbb{E}\|y(t)\|^2. \tag{15}$$

Using the assumption for $t_1 + T$ and the relation above, we obtain the following estimate at the point t_2:

$$\begin{aligned}
\mathbb{E}\|y(t')\|^2 &\geq e^{2\alpha'|t'-(t_1+T)|}\mathbb{E}\|y(t_1+T)\|^2 \\
&\geq e^{2\alpha'|t'-(t_1+T)|+2\gamma(t_1+T-t_2)}\mathbb{E}\|y(t_2)\|^2 \\
&\geq e^{2(\alpha'-|\gamma|)|t'-(t_1+T)|-2|\gamma||t'-t_2|}\mathbb{E}\|y(t_2)\|^2 \\
&= e^{2(\alpha'-|\gamma|)|t'-(t_1+T)|+2(\alpha-|\gamma|)|t'-t_2|}e^{-2\alpha(t_2-t')}\mathbb{E}\|y(t_2)\|^2 \\
&\geq e^{2(\alpha'-|\gamma|)(t_2-(t_1+T))}e^{-2\alpha(t_2-t')}\mathbb{E}\|y(t_2)\|^2.
\end{aligned}$$

Hence

$$\mathbb{E}\|y(t')\|^2 e^{-2(\alpha'-|\gamma|)T} \geq e^{-2\alpha(t_2-t')}\mathbb{E}\|y(t_2)\|^2. \tag{16}$$

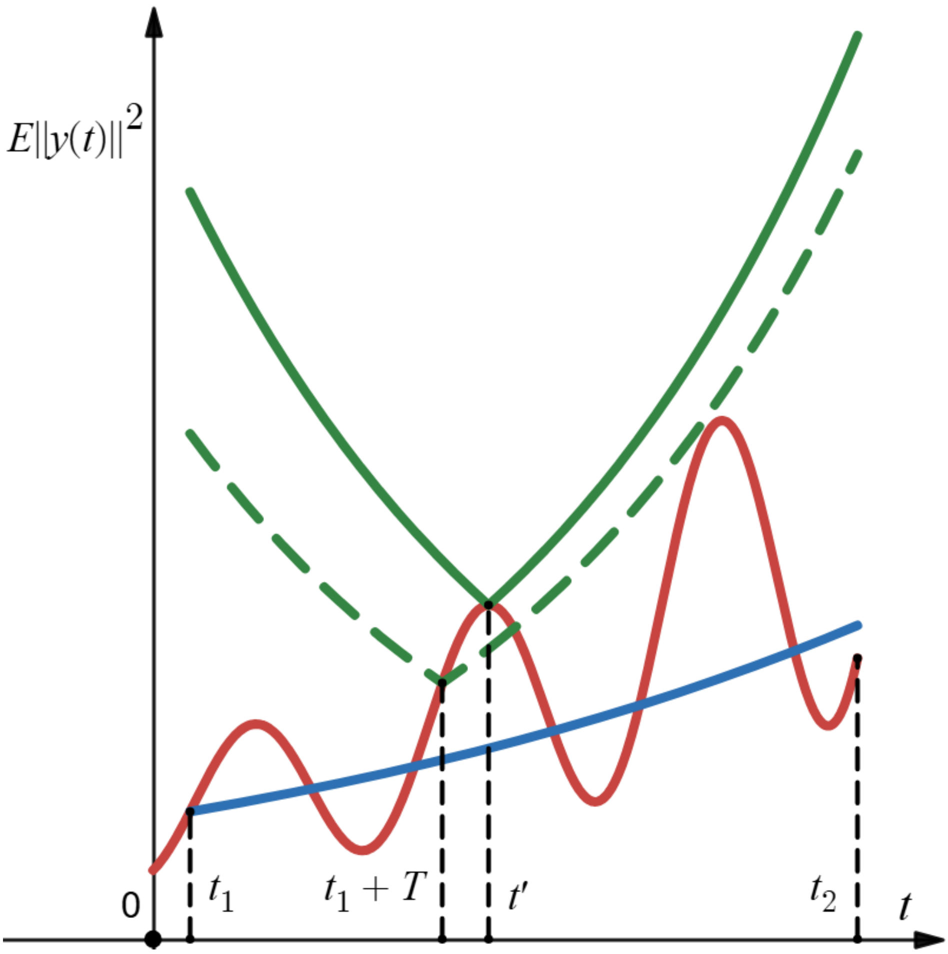

Figure 1. Illustration of Lemma 1. The red line is $\mathbb{E}\|y(t)\|^2$, and the blue line is the threshold $\mathbb{E}\|y(t_1)\|^2 e^{2\gamma(t-t_1)}$. Green curves illustrate the choice of t' such that $\mathbb{E}\|y(t')\|^2 e^{2\alpha'|t-t'|} \geq \mathbb{E}\|y(t)\|^2$.

A similar argument yields the estimate at t_1:

$$\mathbb{E}\|y(t')\|^2 e^{-2(\alpha'-|\gamma|)T} \geq e^{-2\alpha(t'-t_1)}\mathbb{E}\|y(t_1)\|^2. \tag{17}$$

From (15), we also get

$$\max_{t'\leq\tau\leq t_2} \mathbb{E}\big(\|y(\tau)e^{-\alpha'(\tau-t')}\|^2\big) = \max_{t_1\leq\tau\leq t'} \mathbb{E}\big(\|y(\tau)e^{-\alpha'(t'-\tau)}\|^2\big) = \mathbb{E}\big(\|y(t')\|^2\big). \tag{18}$$

Using estimates (17) and (18), together with the inequality (14) applied on appropriate subintervals, we obtain

$$\mathbb{E}\big(\|y(t')\|^2\big) \leq 2C_0\mathbb{E}\|y(t')\|^2 e^{-2(\alpha'-|\gamma|)T} + 2\beta'\mathbb{E}\big(\|y(t')\|^2\big). \tag{19}$$

Now, choose β' and t_0 such that

$$\beta' < \frac{1}{8},$$

and take T satisfying

$$T > \frac{\ln(16N^2)}{2(\alpha' - |\gamma|)}.$$

Then the inequality (19) contradicts the nontriviality of $\mathbb{E}\|y(t')\|^2$, which proves the lemma. □

Growth or Decay in Mean-Square

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 17 March 2026 doi:10.20944/preprints202603.1278.v1

Assume that $y(t)$ is a nontrivial solution to (1). Then, by Lemma 1, for any fixed γ with $\alpha' > \gamma > -\alpha'$, there exists t_0' such that the graph of $\mathbb{E}(\|y(t)\|^2)$ does not intersect the graph of $\mathbb{E}(\|y(t_1)e^{\gamma(t-t_1)}\|^2)$ for $t \geq t_0'$. By continuity, for $t \geq t_0'$ one of the inequalities

$$\mathbb{E}(\|y(t_1)e^{\gamma(t-t_1)}\|^2) < \mathbb{E}(\|y(t)\|^2) \quad \text{or} \quad \mathbb{E}(\|y(t_1)e^{\gamma(t-t_1)}\|^2) > \mathbb{E}(\|y(t)\|^2)$$

holds for all $t \geq t_0'$. The first case with $\gamma > 0$ yields exponential growth, the second case with $\gamma < 0$ yields exponential decay. It remains to rule out the possibility that the solution $y(t)$ is trapped between two exponentials with negative and positive exponents.

Fix γ with $0 < \gamma < \alpha'$, and choose t_0' large enough so that the double inequality

$$\mathbb{E}(\|y(t_1)e^{-\gamma(t-t_1)}\|^2) < \mathbb{E}(\|y(t)\|^2) < \mathbb{E}(\|y(t_1)e^{\gamma(t-t_1)}\|^2), \quad \text{for all } t > t_0', \tag{20}$$

holds.

For any $\tau > t_0'$ define

$$A_\tau = \{t \geq t_0' : \mathbb{E}\|y(\tau)\|^2 \leq \mathbb{E}\|y(t)\|^2 e^{-2\alpha'|t-\tau|}\}.$$

Using (20), one shows that A_τ is bounded, and by continuity it is closed, hence compact. Therefore, there exists $\tau' \in A_\tau$ at which the maximum of $\mathbb{E}\|y(t)\|^2$ on A_τ is attained; for this τ', one has

$$\tau' : \ A_{\tau'} = \{\tau'\}. \tag{21}$$

Substituting $t_1 = t_0'$ and $t' = \tau'$ into the projection estimate (14), using (20) and (21), and letting $t_2 \to \infty$, we obtain

$$\begin{aligned} \mathbb{E}(\|y(\tau')\|^2) &\leq C_0 e^{-2\alpha(\tau'-t_0')}\mathbb{E}(\|y(t_0')\|^2) + 2\beta'\mathbb{E}(\|y(\tau')\|^2) \\ &< (C_0 e^{-2(\alpha-\alpha')(\tau'-t_0')} + 2\beta')\mathbb{E}(\|y(\tau')\|^2). \end{aligned} \tag{22}$$

Choosing τ' sufficiently large yields a contradiction. This completes the proof.

Final step: translation into Lyapunov characteristics.

By the continuity of $\mathbb{E}\|y(t)\|^2$ and Lemma 1, for any fixed $\gamma \in (0, \alpha')$, one of the following mutually exclusive alternatives holds, for all large t, for each nontrivial solution:

- $\mathbb{E}\|y(t)\|^2$ grows at least like $e^{2\gamma t}$, as $t \to \infty$; consequently $\underline{\lambda}_2(y_0) \geq \gamma$, which implies $\underline{\lambda}_2(y_0) \geq \alpha'$,
- or $\mathbb{E}\|y(t)\|^2$ decays at least like $e^{-2\gamma t}$, as $t \to \infty$; consequently $\overline{\lambda}_2(y_0) \leq -\gamma$, which implies $\overline{\lambda}_2(y_0) \leq -\alpha'$.

Given an arbitrary $\alpha^* \in (0, \alpha)$, we obtain the following bounds for the Lyapunov exponent:

- For unbounded solutions, $\overline{\lambda}_2(y_0) \geq \alpha$.
- For bounded solutions, $\overline{\lambda}_2(y_0) \leq -\alpha$.

This completes the proof of the theorem.

□

Remark 3. *The assumption that the exponential dichotomy of A is given with a single rate $\alpha > 0$ can be relaxed to admit distinct rates for the stable and unstable parts. More precisely, suppose there exist constants $\alpha_-, \alpha_+ > 0$ and a bounded projection P_- such that*

$$\|P_- e^{As}\| \leq N e^{-\alpha_- s}, \qquad \|P_+ e^{-As}\| \leq N e^{-\alpha_+ s}, \qquad s \geq 0.$$

 doi:10.20944/preprints202603.1278.v1

Then, under the remaining hypotheses of Theorem 1 (with the smallness condition in (H2) verified for sufficiently large times and with the perturbation amplitudes chosen small relative to α_- and α_+), the conclusions of Theorem 1 hold with the rates applied separately to each spectral part. That is,

$$\overline{\lambda}_2(y_0) \leq -\alpha_- \quad \textit{(for solutions bounded in mean-square)},$$

and

$$\underline{\lambda}_2(y_0) \geq \alpha_+ \quad \textit{(for solutions with unbounded second moment)}.$$

4. Examples

Here are examples illustrating Theorem 1.

Example 1. Let $D \subset \mathbb{R}^d$ be a bounded domain with sufficiently smooth boundary,

$$Au := \sum_{i,j=1}^{d} (a_{ij}(\xi)u_{\xi_i})_{\xi_j} = \mathrm{div}(a(\xi)\nabla u),$$

where $a(\xi) = (a_{ij}(\xi))$ is a symmetric and bounded matrix, and $a_{ij}(\xi)$ are Hölder continuous with the Hölder exponent $\beta \in (0,1)$. Furthermore, assume there exists a constant $a_0 > 0$ such that

$$|\langle a(\xi)h, h\rangle| \geq a_0\|h\|^2,$$

for all $\xi \in D$ and $h \in \mathbb{R}^d$. The boundedness of $A(\xi)$ implies that there exists $a_1 > 0$ such that

$$|\langle a(\xi)h, h\rangle| \leq a_1\|h\|^2.$$

Set $H = L^2(D)$. It is well known that the operator A generates a C_0-semigroup $S(t) = e^{At} \in H$, and has a discrete spectrum: a sequence of eigenvalues

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \ldots, \qquad \lambda_n \to \infty, \text{as } n \to \infty.$$

Let $\{e_n\}$ be the corresponding orthonormal eigenbasis in H, such that

$$\sup_{n\in N} \|e_n\|_{L^\infty(D)} < \infty.$$

According to standard results (see, e.g., [8]), such a basis always exists.

Let $\mu_n \geq 0$, $\sum_{n=1}^{\infty} \mu_n < \infty$, and we introduce the covariance operator $Q \in \mathcal{L}(H)$ such that Q is non-negative, $\mathrm{Tr}(Q) < \infty$, and $Qe_n = \mu_n e_n$.

This allows us to define an H-valued stochastic process

$$W(t) := \sum_{n=1}^{\infty} \sqrt{\mu_n}\, e_n(\xi)\, \beta_n(t), \qquad t \geq 0,$$

which is a Q-Wiener process. Here, $\beta_n(t)$ are standard, scalar, mutually independent Wiener processes. Denote $U := Q^{\frac{1}{2}}(L^2(D))$. It follows from [9, Lemma 2.2], that $U \subset L^\infty(D)$. For any fixed $\phi \in L^2(D)$, we may now introduce the multiplication operator $\Phi : U \to H$ defined by

$$\Phi(\phi) := \phi \cdot \psi, \qquad \forall\, \psi \in U.$$

Since $\psi \in L^2(D)$ and $\phi \in L^\infty(D)$, the operator Φ is well-defined. Hence $\Phi \circ Q^{1/2} : L^2(D) \to L^2(D)$ is a Hilbert-Schmidt operator.

 doi:10.20944/preprints202603.1278.v1

In the sequel, we denote by $L_2^0 = L_2(Q^{1/2}H, H)$ the space of Hilbert–Schmidt operators with the norm $\|\cdot\|_{HS}$. Then

$$\begin{aligned}\|\Phi \circ Q^{1/2}\|_{HS}^2 &:= \sum_{i=1}^{\infty} \|\Phi \circ Q^{1/2} e_i\|^2 = \sum_{i=1}^{\infty} \mu_i \int_D \phi(\xi)^2 e_i^2(\xi)\, d\xi \\ &\leq \mathrm{Tr}(Q) \sup_i \|e_i\|_{L^\infty(D)}^2 \cdot \|\phi\|_{L^2(D)}^2.\end{aligned}$$

Hence, if $\Phi : \Omega \times [0,T] \to L(U,H)$ is a predictable process such that

$$\mathbb{E} \int_0^T \|\Phi(s) Q^{1/2}\|_{HS}^2\, ds < \infty,$$

then, following [9], we can define the stochastic integral

$$\int_0^t \Phi(s)\, dW(s) \in H,$$

$$\int_0^t \Phi(s)\, dW(s) = \sum_{i=1}^{\infty} \sqrt{\mu_i} \int_0^t \Phi(s,\cdot) e_i(\cdot) d\beta_i(s),$$

and

$$\mathbb{E}\left| \int_0^t \Phi(s)\, dW(s) \right|^2 \leq \mathrm{Tr}(Q) \sup_i \|e_i\|_{L^\infty(D)}^2 \int_0^t \mathbb{E}|\phi(s,\cdot)|^2\, ds.$$

We consider the following stochastic linear partial differential equation

$$\begin{cases} dX(t,\xi) = \left(\sum_{i,j=1}^{d} \left(a_{ij}(\xi) X_{\xi_i}\right)_{\xi_j} + \lambda X(t,\xi) \right) dt + \sum_{n=1}^{\infty} \sqrt{\mu_n}\, p_n(t,\xi,X)\, d\beta_n(t), \\ X(t,\xi)\big|_{\xi \in \partial D} = 0. \end{cases} \tag{23}$$

where the function $p(t,\xi,X) = \sqrt{\sum_{n=1}^{\infty} {p_n}^2(t,\xi,X)}$ is Lipschitz continuous for $t \geq 0$, $\xi \in D$, and $\sup_{\xi \in D} p(t,\xi,X) \to 0$, as $t \to \infty$. The parameter $\lambda > 0$ is chosen such that $\lambda \neq \lambda_j$, for all $j = 1, 2, \ldots$.

Then, obviously, all the conditions of Theorem 1 are fulfilled, and the mean-square Lyapunov exponents of equation (23) are preserved (i.e., they coincide with the reference exponents of the linear part).

In the following example, we consider the one-dimensional problem (23) with $A = \Delta$. In this case, the spectrum is given explicitly.

Example 2. Let $D = (0, \pi)$. Consider the problem

$$\begin{cases} dX(t,\xi) = \left(\Delta_\xi X(t,\xi) + \lambda X(t,\xi)\right) dt \\ \qquad\qquad + \sum_{n=1}^{\infty} \sqrt{\mu_n}\, p_n(t,\xi,X) e_n(\xi)\, d\beta_n(t), \\ X(t,0) = X(t,\pi) = 0, \qquad t \geq 0, \end{cases} \tag{24}$$

$\lambda > 0$, $\lambda \neq n^2$, $n \in \mathbb{N}$. The function $p(t,\xi,X)$ satisfies the previous conditions. In this case, $H = L^2(0,\pi)$, and the operator $A = \frac{d^2}{d\xi^2} + \lambda I$, with $D(A) = H^2(0,\pi) \cap H_0^1(0,\pi)$.

Then, as is well known [10], the spectrum

$$\sigma(A) = \{\lambda - n^2 :\ n \in \mathbb{N}\}$$

is real and $0 \notin \sigma(A)$. Set

$$\alpha_+ := \min_{n^2 < \lambda} (\lambda - n^2), \qquad \alpha_- := \min_{n^2 > \lambda} (n^2 - \lambda).$$

 doi:10.20944/preprints202603.1278.v1

Then $\alpha_+, \alpha_- > 0$ and $\alpha = \min\{\alpha_+, \alpha_-\}$.

For any mild solution $y(t; y_0)$ of (24), Theorem 1 implies:

1. If the solution has an unbounded second moment, then
$$\underline{\lambda}_2(y_0) \geq \alpha.$$
2. If the solution has a bounded second moment, then
$$\overline{\lambda}_2(y_0) \leq -\alpha.$$

Thus, on the unstable spectral subspace, the mean-square Lyapunov exponent is not smaller than the minimal positive eigenvalue α_+, while on the stable subspace, it does not exceed the negative of the minimal modulus of the negative eigenvalues, $-\alpha_-$.

Remark. Inspecting the proof of Theorem 1, one can see that, if necessary, the exact rates α_+ and α_- can be taken instead of the common value α.

5. Discussion

This paper studies the Lyapunov structure of nonlinear stochastic evolution equations in a separable Hilbert space driven by a Q-Wiener process. The linear operator A generates a C_0-semigroup admitting an exponential dichotomy, which induces a splitting associated with positive and negative Lyapunov exponents. The nonlinear drift and diffusion terms are assumed to be globally Lipschitz and to decay in time.

The main result shows that the Lyapunov exponents of the nonlinear stochastic equation coincide with those of the linear part. In particular, no new mean-square growth rates appear due to the nonlinear perturbations, and the splitting induced by the operator A remains unchanged in the mean-square sense.

The analysis confirms that the linear exponential dichotomy provides a correct reference structure for the nonlinear stochastic system and that this structure is robust under small fading perturbations in both drift and diffusion terms. The proof relies on the spectral splitting of the semigroup generated by A and on mean-square estimates based on the Itô isometry.

The obtained results give a simple and verifiable criterion for the invariance of Lyapunov exponents in nonlinear stochastic evolution equations. Possible extensions include weaker decay conditions and almost-sure versions of the obtained statements.

Author Contributions: Conceptualization, Dmytro Shtefan; Methodology, Oleksandr Stanzhytskyi; Formal analysis, Dmytro Shtefan and Oleksandr Stanzhytskyi; Investigation and Oleksandr Stanzhytskyi, Dmytro Shtefan; Validation, Oleksandr Stanzhytskyi and Svitlana Kushnirenko; Resources, Oleksandr Stanzhytskyi and Svitlana Kushnirenko; Data curation, Oleksandr Stanzhytskyi and Svitlana Kushnirenko; Writing–original draft preparation, Dmytro Shtefan and Svitlana Kushnirenko; Writing–review and editing, Oleksandr Stanzhytskyi and Svitlana Kushnirenko; Visualization, Dmytro Shtefan; Project administration, Oleksandr Stanzhytskyi. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: The data that support the findings of this study are available from the corresponding author, Dmytro Shtefan, upon reasonable request.

Acknowledgments: The authors are grateful for helpful discussions with colleagues and for the careful reading of preliminary drafts.

Conflicts of Interest: The authors declare no conflicts of interest.

 doi:10.20944/preprints202603.1278.v1

References

1. Ju. L. Daleckii and M. G. Krein, Stability of Solutions of Differential Equations in Banach Space, *J. Amer. Math. Soc.*, **43**, 1974; 386.
2. G. Da Prato and J. Zabczyk and S. Ball, *Encyclopedia of Mathematics and its Applications*, 2^{nd} edition, Cambridge University Press, Cambridge, 2014; 512.
3. O.Stanzhytskyi, Investigation of exponential dichotomy of Ito stochastic systems by using quadratic forms, *Ukrainian Mathematical Journal*, **53(11)**, 2001; 1882–1894.
4. O.Stanzhytskyi, Exponential dichotomy and mean-square bounded solutions of linear stochastic Ito systems, *Nonlinear Oscillations*, **4(3)**, 2001.
5. A.M. Samoilenko, O.M. Stanzhytskyi, *Qualitative and asymptotic analysis of differential equations with random perturbations*, Singapore:World Scientific, 2011; 322.
6. V. M. Ungureanu, Representation theorem for stochastic differential equations in Hilbert spaces and its applications, *Surveys in Mathematics and its Applications*, **1**, 2006; 117–134.
7. D. Stoica, Uniform exponential dichotomy of stochastic cocycles, *Stochastic Processes and their Applications*, **120**, 2010; 1920–1928.
8. R.I. Ovsepian and A. Pelczynski, On the existence of a fundamental total bi orthogonal sequence in every separable Banach space and related constructions of uniformly bounded orthonormal systems in L^2, *Stud. Math.*, Paris, France, 1975; 149–159.
9. R. Manthey and T. Zausinger, Stochastic evolution in L^{2v}_{ρ}, *Stochastic, Stochastics Rep*, **66**, 1999; 37–85.
10. L. C. Evans, *Partial Differential Equations*, 2^{nd} edition, American Mathematical Society, **19**, 2010.