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Review

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Seokcheon Lee ⓘD

Department of Physics, Institute of Basic Science, Sungkyunkwan University, Suwon, 16419, Korea; skylee@skku.edu

Abstract: It has been known that dimensional constants such as $\hbar$, c, G, e, and k are merely human constructs whose values and units vary depending on the chosen system of measurement. Therefore, the time variation of dimensional constants lacks operational significance due to their dependence on them. It is well-structured and represents a valid discussion. However, this fact only becomes a meaningful debate within the context of a static or present universe. As well-established theoretically and observationally, the current universe is undergoing accelerated expansion, wherein dimensional quantities, like the wavelength of light, also experience redshift phenomena elongating over cosmic time. In other words, in an expanding universe, dimensional quantities of physical parameters vary with cosmic time. From this perspective, there exists the possibility that dimensional constants, such as the speed of light, could vary with the expansion of the universe. In this review paper, we contemplate under what circumstances the speed of light may change or remain constant over cosmic time, and discuss the potential for distinguishing these cases observationally.

Keywords: varying speed of light models; standard cosmological model; Robertson-Walker metric; physical constants; Hubble parameter

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1. Introduction

The laws of physics should be invariant under changes in units or measurement tools. It is achievable when expressed with dimensionless quantities like the fine structure constant, α, as seen in the Standard Model of particle physics. Dimensional constants such as $\hbar$, c, G, e, and k are human constructs whose values vary with the choice of units. In this sense, only dimensionless constants are fundamental. Thus, the potential time variation of dimensionless fundamental constants is a valid subject of inquiry, but that of dimensional constants like c or G is unit-dependent and may lead to disagreement among observers [1–4]. However, the above arguments hold only within the context of a static universe or the one at the present epoch [5,6].

The contemporary standard cosmological model, known as the ΛCDM model, is based on the Robertson-Walker (RW) metric, which assumes spatial homogeneity and isotropy on large scales (*i.e.*, cosmological principle (CP)). Evidence for isotropy is found in the uniformity of the cosmic microwave background (CMB) temperature [7,8]. Although proving homogeneity is more challenging, support comes from the uniform matter distribution on scales of more than 100 million light-years as large-scale structures (LSS) [9,10]. The ΛCDM model incorporates an expanding metric space, evidenced by the redshift of spectral lines in light from distant galaxies. This expansion causes objects not under shared gravitational influence to move apart, but it does not increase their size, such as galaxies. The cosmological redshift, often explained as the result of photon wavelengths stretching due to space expansion, can be understood using equations from general relativity (GR) describing a homogeneous and isotropic universe. This redshift, formulated as a function of the time-varying cosmic scale factor $a(t)$, yields positive values for z in our expanding universe. This phenomenon causes distant galaxies to exhibit redshift as time advances, where $1 + z = 1/a$, utilizing the present value of the scale factor as $a_0 = 1$. Therefore, the redshift of a galaxy can be estimated by examining the emission lines emitted by glowing gas within the galaxy. For instance, the Hα line, a red Balmer line of neutral hydrogen, has a rest wavelength of 6562Å. If the observed wavelength of this line presently measures 8100Å, it indicates that the galaxy is positioned at $z = 0.234$ (*i.e.*, $a = 0.81$). Therefore, in an expanding universe, the value of a dimensional quantity, such as wavelength, *does* vary depending on the time of

observation (*i.e.*, cosmic time). Additionally, it has been observed that the temperature of the CMB decreases with the age of the universe, scaling inversely with the scale factor $T = T_0 a^{-1}$.

The Lorentz transformation (LT) between inertial frames (IFs) stems from special relativity (SR), which hinges on the speed of light, c, as its sole parameter with a constant value. SR's universal Lorentz covariance, rooted in Minkowski spacetime, adequately satisfies its principles [11]. In contrast, in GR, an IF refers to a freely falling one. While Lorentz invariant (LI) spacetime intervals can be established between events, defining a global time in GR is hindered by the absence of a universal IF. However, a global time can be defined for the universe satisfying CP, allowing for a foliation of spacetime into non-intersecting spacelike 3D surfaces. It is the universe described by the RW metric [12–15]. The LI varying speed of light (VSL) model is feasible if c remains locally constant (i.e., at each given epoch) but varies on cosmic time [5,6]. In other words, in an expanding universe, if the speed of light is given as a function of the scale factor, $c[a]$, then although its value varies like wavelengths at different epochs, say a_1 and a_2, it attains a constant local value at each epoch, ensuring LI and thus maintaining the validity of quantum mechanics and electromagnetism satisfying SR every epoch. However, testing simultaneous variations in c and Newton's gravitational constant G is crucial to prevent trivial rescaling of units, given their combination in the Einstein action [5,16].

The RW metric starts by positing that all galaxies exist on a hypersurface, where the surface of simultaneity of their local Lorentz frame (LF) aligns with this hypersurface. This conceptualization allows for the hypersurface to be visualized as a composite of the smoothly meshed LF of all galaxies, with each galaxy's four-velocity being orthogonal to the hypersurface [12,13]. Assigning a parameter t to this sequence of hypersurfaces serves as the proper time of any galaxy, establishing a universal time reference. This cosmic time corresponds to the measurement by a comoving observer, who perceives the universe expanding uniformly around her. Therefore, in the RW metric, the proper time is equivalent to cosmic time [12,17,18].

In the traditional RW metric, the assumption of the constancy of the speed of light is contingent upon a specific hypothesis regarding cosmological time dilation (TD), rather than being directly derived from the foundational principles of the metric [19–21]. There have been several projects to measure cosmological time dilation. Direct observation of the TD measures the decay time of distance supernova (SN) light curves and spectra [22–26]. Another method is measuring TD by searching the stretching of peak-to-peak timescales of gamma-ray bursters (GRBs) [27–35]. There has been a search for the TD effect in the light curves of quasars (QSOs) located at cosmological distances [36,37]. So far, it seems fair to say that no convincing detection has been made for cosmic time dilation with the conflict between different measurements. Without explicit laws governing TD, the speed of light in the RW metric can vary with cosmological time, similar to other physical properties such as mass density, temperature, and fundamental constants like the Planck constant [38]. This variation presents a plausible scenario known as the VSL with cosmic time. When delineating the background of the Friedmann-Lemaître-Robertson-Walker (FLRW) universe, a hypersurface of constant time can be delineated based on physical quantities such as temperature or density, owing to the Universe's homogeneity, which ensures uniform temperature and density at each cosmic time. Nevertheless, it is crucial to recognize that temperature and mass density can undergo redshift due to the Universe's expansion.

In Section 2., we review the conventional derivation of the RW metric using CP principles and Weyl's postulate. Section 3 explores the possibility of the VSL model in the RW metric by considering the various implications of TD relationships. Subsequently, Section 4 comprehensively investigates the repercussions of VSL on a range of physical quantities and fundamental constants. In Section 5, we compare the Friedmann equations of various models. In Section 6, we will summarize the main points and draw conclusions based on the insights gained throughout the document, emphasizing the potential ramifications of the cosmological time-varying speed of light across various dimensions of its definitions and implications.

2. Review of the Robertson-Walker Metric

The contemporary standard model of cosmology, the ΛCDM model, relies on the RW metric, which assumes spatial homogeneity and isotropy in the Universe on its largest scales. Through a synthesis of observational data from LSS and CMB, there is a consensus that the Universe demonstrates nearly perfect homogeneity and isotropy on a large scale. In this section, we offer a comprehensive examination of the derivation of the RW metric, employing the CP and Weyl's postulate.

2.1. Isotropic and Homogeneous Space

First, it's crucial to highlight that the isotropy and homogeneity of space are defined at each moment in time. We present a review of a method utilizing Killing vectors (KVs) to translate the CP into a geometric condition explicitly satisfied by the spacetime metric, particularly its spatial component [39]. Initially, we note that the Lie derivatives of the KVs, responsible for generating isotropy symmetry in a 3-dimensional space (*i.e.*, the metric remains invariant under spatial rotations), must vanish. These three KVs are in spherical polar coordinates $x^\mu = (ct', r, \theta, \phi)$ as

$$\bar{\xi}_i = \xi^\mu_{i(\text{sp})} \bar{e}^{(\text{sp})}_{(\mu)} \,, \text{where} \ \ \bar{e}^{(\text{sp})}_{(\mu)} = \left(\frac{\partial}{c\partial t}, \frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}, \frac{\partial}{\partial \phi} \right), \xi^\mu_{1(\text{sp})} = (0, 0, -\sin\phi, -\cot\theta \cos\phi) \,,$$

$$\xi^\mu_{2(\text{sp})} = (0, 0, \cos\phi, -\cot\theta \sin\phi) \,, \xi^\mu_{3(\text{sp})} = (0, 0, 0, 1) \,. \tag{1}$$

Therefore, by applying these KVs to the condition of rotational isometry ($\mathcal{L}_{\bar{\xi}} g_{\mu\nu} = 0$), we derive the most comprehensive metric component for an isotropic space at a specific time, represented as $t' = t'_l$

$$g^{(\text{iso})}_{\mu\nu}(t'_l) = \begin{pmatrix} g_{00}(t'_l, r) & g_{01}(t'_l, r) & 0 & 0 \\ g_{10}(t'_l, r) & g_{11}(t'_l, r) & 0 & 0 \\ 0 & 0 & g_{22}(t'_l, r) & 0 \\ 0 & 0 & 0 & g_{33}(t'_l, r) \end{pmatrix}. \tag{2}$$

On cosmological scales, space is also homogeneous, so the above metric must satisfy the translation isometry for three translation KVs as given by

$$\bar{\eta}_i = \eta^\mu_{i(\text{sp})} \bar{e}^{(\text{sp})}_{(\mu)} \,, \eta^\mu_{1(\text{sp})} = \left(0, \sin\theta \cos\phi, \frac{1}{r} \cos\theta \cos\phi, -\frac{1}{r} \frac{\sin\phi}{\sin\theta} \right) \,,$$

$$\eta^\mu_{2(\text{sp})} = \left(0, \sin\theta \sin\phi, \frac{1}{r} \cos\theta \sin\phi, \frac{1}{r} \frac{\cos\phi}{\sin\theta} \right) \,, \eta^\mu_{3(\text{sp})} = \left(0, \cos\theta, -\frac{1}{r} \sin\theta, 0 \right) \,. \tag{3}$$

The most general metric component for an isotropic and homogenous space at t'_l is obtained by employing these KVs to the translation isometry for the isotropic metric provided in (2) (*i.e.*, $\mathcal{L}_{\bar{\eta}} g^{(\text{iso})}_{\mu\nu} = 0$)

$$g^{(\text{CP})}_{\mu\nu}(t'_l) = \text{diag}\left(g_{00}(t'_l), A(t'_l), A(t'_l)r^2, A(t'_l)r^2 \sin^2\theta \right). \tag{4}$$

Consequently, the general form of the 4-dimensional line element for the homogeneous and isotropic space at a particular time t'_l can be expressed as

$$ds^2(t'_l) = g_{\mu\nu}(t'_l)dx^\mu dx^\nu = -c^2_l g_{00}(t'_l)dt'^2_l + A(t'_l)\left[dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \right]. \tag{5}$$

By introducing the transformation $dt_l = \sqrt{g_{00}(t'_l)} dt'_l$ (*i.e.*, normalizing the lapse function as 1), the homogeneous and isotropic metric in (4) can be rewritten as

$$g_{\mu\nu}(t_l) = \text{diag}\left(-1, a^2(t_l), a^2(t_l)r^2, a^2(t_l)r^2 \sin^2 \theta\right),\tag{6}$$

where $a^2(t_l) = A(t'_l)$. Consequently, the 4-dimensional line element for the homogeneous and isotropic space at a particular time t_l is given by

$$\begin{aligned}ds_l^2 \equiv ds^2(t_l) &= -c_l^2 dt^2 + a^2(t_l)\left[dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2\right]\\&\equiv -c_l^2 dt^2 + a^2(t_l)\left[dr^2 + r^2 d\Omega^2\right].\end{aligned}\tag{7}$$

It is crucial to emphasize that both the scale factor a_l and the speed of light c_l in Eq. (7) must remain constant to maintain homogeneity at a specific time t_l.

2.2. Fundamental Observers

In the preceding Section 2.1, we derived the most general metric for an isotropic and homogeneous space at a specific time, denoted as t_l. To effectively utilize this metric in cosmology, it becomes necessary to introduce a global time parameter (*i.e.*, referred to as generalized t_l as t). In SR, one can define a globally valid time within a selected IF. In GR, the absence of a global IF renders the concept of an instantaneous moment ambiguous. Instead, three-dimensional spacelike hypersurfaces replace this notion. By introducing non-intersecting spacelike hypersurfaces labeled by t_ls, a global time parameter can be defined, representing a universal time [12,14].

To specify the preferred slicing, fundamental observers, assumed to have no motion relative to the overall cosmological fluid, are introduced. If Weyl's postulate is adopted, the timelike worldlines of these observers form a bundle in spacetime, diverging from or converging to a point in the past or future. Thus, $t_l = \text{constant}$ hypersurfaces are constructed where the four-velocity of any fundamental observer is orthogonal to it. Consequently, each hypersurface can be seen as the amalgamation of all the local LFs of fundamental observers. [12].

In relativistic cosmology, Weyl's postulate dictates that the worldlines of fluid particles (*i.e.*, galaxies) should be hypersurface orthogonal, meaning they should be everywhere orthogonal to a family of spatial hyperslices. In essence, fundamental observers move with the cosmic fluid, remaining at rest in the comoving frame.

2.3. Synchronous Coordinates

The parameter t_l assigned to hypersurfaces is termed the *synchronous time coordinate* when considered as the proper time along the worldline of any fundamental observer [12–14,39]. Additionally, the so-called comoving coordinates can be adopted, where each fundamental observer possesses fixed spatial coordinates $x^i = (x^1, x^2, x^3)$. Consequently, the worldline of a fundamental observer is expressed as $x^\mu(\tau)$

$$x^\mu(\tau) = \left(x^0, x^i\right) = \left(c_l \tau_l, x^1 = \text{constant}, x^2 = \text{constant}, x^3 = \text{constant}\right),\tag{8}$$

where τ_l represents the proper time along the fundamental observer. Since $dx^i = 0$ along the worldline, this yields $ds_l = c_l d\tau_l = c_l dt_l$. Hence, the proper time along the worldline equals the coordinate time t_l, commonly referred to as cosmic time [12,14]. The four-velocity of a fundamental observer in comoving coordinates, denoted as $u^\mu \equiv dx^\mu/d\tau_l = (c_l, 0, 0, 0)$, is orthogonal to any vector $B^\mu = (0, dx^1, dx^2, dx^3)$ lying in the hypersurface $t_l = \text{constant}$, meaning $g_{\mu\nu}u^\mu B^\nu = 0$.

Let us consider two nearby fundamental observers located at comoving coordinates (x^1, x^2, x^3) and $(x^1 + dx^1, x^2 + dx^2, x^3 + dx^3)$ at a specific time $t = t_l$. Their proper (physical) distances at this time are given by

$$
\begin{aligned}
(r^1, r^2, r^3) &= a(t_l)(x^1, x^2, x^3), \\
(r^1 + \Delta r^1, r^2 + \Delta r^2, r^3 + \Delta r^3) &= a(t_l)(x^1 + dx^1, x^2 + dx^2, x^3 + dx^3).
\end{aligned}
\tag{9}
$$

Considering the triangles formed by these observers at t_l and at some later time, due to homogeneity and isotropy of space, both triangles must be similar. the triangle formed by these same observers at some later [12]. Moreover, the magnification factor must be constant regardless of the triangle's position in the three-dimensional space. Therefore, the spatial separation on the same hypersurface $t = t_l$ between two nearby fundamental observers is expressed as

$$
d\sigma_l^2 \equiv g_{ij}(t_l)\Delta r^i \Delta r^j = a^2(t_l)\gamma_{ij}dx^i dx^j \equiv a^2(t_l)dl^2,
\tag{10}
$$

where the γ_{ij} depends solely on (x^1, x^2, x^3) [12,14].

2.4. Curved Spatial Hypersurface

The spatial section in the line element in Eq.(7) is flat, as it is derived using translational isometry in Section 2.1. A curved spatial hypersurface can be embedded into a flat 4-dimensional Euclidean space $\mathbb{E}^4$. Closed and open spaces correspond to spherical and hyperbolic spaces, respectively, with a curvature radius of b

$$
\left(x^1\right)^2 + \left(x^2\right)^2 + \left(x^3\right)^2 + \left(x^4\right)^2 \equiv r'^2 + \left(x^4\right)^2 = b^2, \text{where}
\tag{11}
$$

$$
x^1 = b \sin\chi \sin\theta \cos\phi, \ x^2 = b \sin\chi \sin\theta \sin\phi, \ x^3 = b \sin\chi \cos\theta, \ x^4 = b \cos\chi,
$$

$$
-\left(x^1\right)^2 - \left(x^2\right)^2 - \left(x^3\right)^2 + \left(x^4\right)^2 \equiv -r'^2 + \left(x^4\right)^2 = b^2, \text{where}
\tag{12}
$$

$$
x^1 = b \sinh\chi \sin\theta \cos\phi, \ x^2 = b \sinh\chi \sin\theta \sin\phi, \ x^3 = b \sinh\chi \cos\theta, \ x^4 = b \cosh\chi.
$$

To consider a curved space, three translation KVs given in Eq. (3) should be replaced with KVs for rotations along the x^4 axis

$$
\bar{\eta}_1 = x^4 \frac{\partial}{\partial x^1} - x^1 \frac{\partial}{\partial x^4}, \ \bar{\eta}_2 = x^4 \frac{\partial}{\partial x^2} - x^2 \frac{\partial}{\partial x^4}, \ \bar{\eta}_3 = x^4 \frac{\partial}{\partial x^3} - x^3 \frac{\partial}{\partial x^4}.
\tag{13}
$$

Then, the line element for the homogeneous and isotropic spaces at $t = t_l$ for maximally symmetric closed 3-space ($\mathbb{S}^3$) can be expressed

$$
ds_l^2 = -c_l^2 dt_l^2 + a_l^2 \left(\frac{dr'^2}{1 - Kr'^2} + r'^2 \left(d\theta^2 + \sin^2\theta d\phi^2 \right) \right), \text{where} \quad K = \frac{1}{b^2}.
\tag{14}
$$

Also, the line element for the open 3-space (hyperbolic) is expressed as

$$
ds_l^2 = -c_l^2 dt_l^2 + a_l^2 \left(\frac{dr'^2}{1 - Kr'^2} + r'^2 \left(d\theta^2 + \sin^2\theta d\phi^2 \right) \right), \text{where} \quad K = -\frac{1}{b^2}.
\tag{15}
$$

Therefore, the spacetime interval for a curved spatial hypersurface adhering to the CP at a specific time t_l can be expressed by

$$
ds_l^2 = -c_l^2 dt^2 + a_l^2 \left[\frac{dr^2}{1 - Kr^2} + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2 \right) \right] \quad \text{at } t = t_l.
\tag{16}
$$

2.5. Robertson Walker Metric

By adopting Weyl's postulate to extend the metric described in Eq. (16) to cosmic time t, we can represent the line element as

$$ds^2 = -c(t)^2 dt^2 + a(t)^2 \left[\frac{dr^2}{1 - Kr^2} + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2 \right) \right] \equiv -c(t)^2 dt^2 + a(t)^2 dl_{3D}^2 . \qquad (17)$$

In this equation (17), the speed of light is expressed as a function of time, deviating from the conventional RW metric. Initially, this equation may seem incorrect or counterintuitive. However, as illustrated in Figure 1, the original RW metric implies that on hypersurfaces defined by t_l or $t_k =$ constants, various quantities such as the scale factor $a_l = a(t_l)$, mass density $\rho_l = \rho(t_l)$, pressure $P_l = P(t_l)$, temperature $T_l = T(t_l)$, speed of light $c_l = c(t_l)$, Boltzmann constant $k_l = k(t_l)$, and Planck constant $\hbar_l = \hbar(t_l)$ remain constant regardless of the 3-D spatial position. However, according to Weyl's postulate, these quantities or constants can be expressed as functions of cosmic time t, accounting for cosmological redshift, as depicted in Figure 1. Traditionally, it has been assumed that physical constants, including the speed of light, remain constant over cosmic time. This additional assumption, namely that the speed of light remains constant ($c_l = c_k$) regardless of cosmic time, is not directly tied to the two conditions necessary to derive the RW metric: the CP and Weyl's postulate. The constancy of the speed of light relies on cosmological TD, and it is crucial to recognize that GR does not specify any particular physical laws governing this constancy, as we will elucidate shortly. As the Universe progresses from t_k to t_l, physical quantities such as $a(t)$, $\rho(t)$, $P(t)$, and $T(t)$ undergo changes over cosmic time t. The precise functional expressions for these quantities are obtained through the solution of Einstein's Field Equations (EFEs) and Bianchi's identity (BI), considering the equation of state of fluids [5,6].
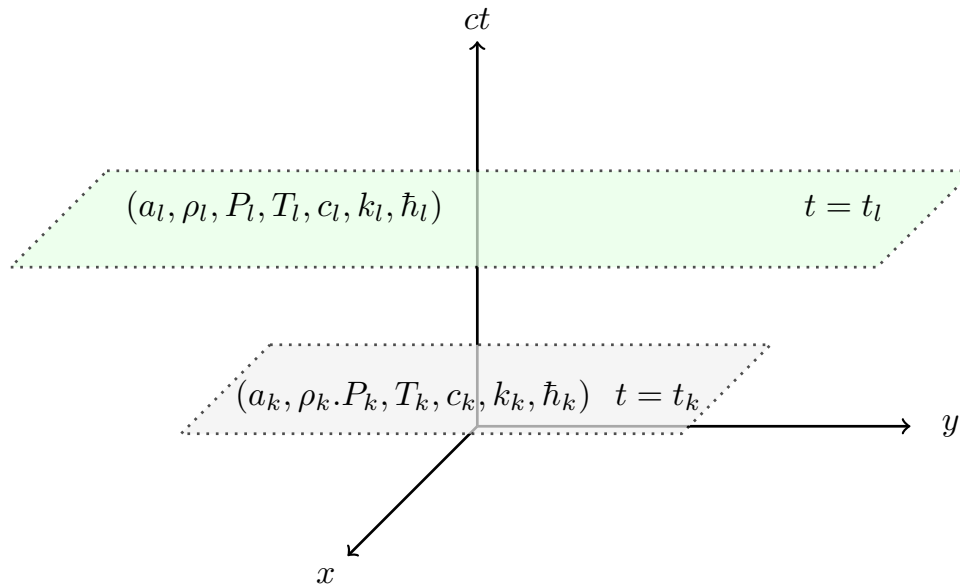

Figure 1. At $t = t_k$, the values of physical quantities and constants, such as a_k, ρ_k, P_k, T_k, c_k, k_k, and $\hbar_k$, are fixed and independent of spatial position on the $t = t_k$ hypersurface. As the universe expands, these quantities and constants transition to a_l, ρ_l, P_l, T_l, c_l, k_l, and $\hbar_l$. The CP and Weyl's postulate do not restrict c_k to be equal to c_l; its value is determined by the cosmological TD relation.

2.6. Rescale Time

Some argue that debating whether the speed of light varies is meaningless when transitioning from equation (16) to equation (17) by substituting ct with x^0. However, as demonstrated in equation (19), when defining the 3D comoving distance that light traverses along its path, this equation becomes

$$dl_{3D} = \frac{x^0(t_i)}{a(t_i)} \quad \Rightarrow \quad \frac{x^0(t_1)}{a_1} = \frac{x^0(t_2)}{a_2} \quad \Rightarrow \quad \frac{\lambda_1}{a_1} = \frac{\lambda_2}{a_2}, \tag{18}$$

where x^0 represents distance related to the wavelength of light. Thus, this equation illustrates the cosmological redshift of the wavelength, mirroring Eq. (22). Dividing it by the clock rate ν and the speed of light c reveals both possibilities: the speed of light either remains constant or varies with cosmic time, contingent upon the cosmological TD outlined in Eq. (19).

3. The Possibility of Varying Speed of Light Theory in the Robertson-Walker Metric

The derivation of redshift involves employing the geodesic equation for a light wave, where $ds^2 = 0$ as Eq. (17). The consistency of dl_{3D} over time is ensured by the exclusive use of comoving coordinates. Expanding upon this groundwork, we reach the expression for outgoing light signals as

$$dl_{3D} = \frac{c(t_i)dt_i}{a(t_i)} \quad : \quad \frac{c_1 dt_1}{a_1} = \frac{c_2 dt_2}{a_2} \Rightarrow \begin{cases} c_1 = c_2 = c & \text{if} \quad \frac{dt_1}{a_1} = \frac{dt_2}{a_2} & \text{SMC} \\ c_1 = \frac{f(a_2)}{f(a_1)}\frac{a_1}{a_2}c_2 & \text{if} \quad \frac{dt_1}{f(a_1)} = \frac{dt_2}{f(a_2)} & \text{VSL} \\ c_1 = \left(\frac{a_1}{a_2}\right)^{\frac{b}{4}}c_2 & \text{if} \quad \frac{dt_1}{a_1^{1-\frac{b}{4}}} = \frac{dt_2}{a_2^{1-\frac{b}{4}}} & \text{meVSL} \end{cases}, \tag{19}$$

where $dt_i = \nu(t_i)$ represents the time interval between successive crests of light at t_i (*i.e.*, the inverse of the frequency ν_i at t_i), and $f(a_i)$ denotes an arbitrary function of $a(t_i)$ [40].

In the standard model of cosmology (SMC), an additional assumption is introduced, positing that the cosmological TD between two hypersurfaces at $t = t_1$ and $t = t_2$ is proportional to the inverse of the scale factors $a(t)$ at each respective time. This assumption leads to the conclusion that the speed of light on these hypersurfaces remains constant, $c(t_1) = c(t_2) = c$. However, this assumption lacks derivation from any physical laws. Moreover, within the framework of GR, there exists no inherent physical basis for the constancy of the speed of light across cosmic time, as it holds solely for the local inertial observer.

Conversely, in an expanding universe, the progression from one hypersurface to another results in an increase in the scale factor, naturally leading to the cosmological redshift of various physical quantities, including mass density and temperature. However, it is impossible to conclude about cosmological TD based solely on the CP and Weyl's postulate in the RW metric. Instead, establishing such relationships relies on experimental observations. Efforts to measure cosmological time dilation have included direct observations of SN light curves and spectra to evaluate decay times of distance [20,22–24,26]. Another avenue to explore cosmological TD involves analyzing the elongation of peak-to-peak timescales observed in GRBs [27–35]. Additionally, researchers have investigated TD effects within the light curves of cosmologically distant QSOs [36,37]. However, current observational evidence does not definitively confirm an exact correspondence between cosmological TD and predictions made by the SMC. Moreover, the RW model lacks a mechanism to determine cosmological TD conclusively. Thus, it remains valuable to explore the possibility of VSL in these observations, provided that the findings are consistent with those predicted by the SMC.

Given the theoretical absence of cosmological TD, considering this relationship as a general function $f(a)$ of the scale factor, the speed of light can be expressed as

$$c(t_1) = \frac{f(a_2)}{f(a_1)}\frac{a(t_1)}{a(t_2)}c(t_2). \tag{20}$$

This underscores that while we cannot assert the generality of the VSL model within the framework of GR, it appears to be a natural consequence in an expanding Universe as described by the RW metric. The minimally extended varying speed of light (meVSL) model is a specific instance of VSL, characterized by $f(a) = a^{1-b/4}$ [5,6].

4. The Consequences of the Varying Speed of Light

In the previous Section 3, we illustrated how the speed of light may change over cosmic time within an expanding Universe according to the RW metric. However, for this concept to form a coherent model, the variable nature of the speed of light must be integrated into EFEs and solved for solutions. Our previous works have addressed such scenarios, particularly in the context of a model known as meVSL [5,6]. In this section, we delve into extending the cosmological evolution of physical quantities and constants to encompass a broader range of VSL models.

4.1. Stress Energy Tensor

In cosmology, matter is treated as a perfect fluid, characterized by its rest frame mass density ρ and isotropic pressure P. In the preceding Section 2.2, we introduced the notion of a fundamental observer at rest relative to this fluid. Within the framework of GR, the stress-energy tensor describes this perfect fluid

$$T^{\mu\nu} = \left(\rho + \frac{P}{c^2} \right) u^\mu u^\nu + P g^{\mu\nu} , \tag{21}$$

where u^μ represents its four-velocity. When the fluid is in motion, a set of fundamental observers is deemed comoving with it, characterized by a four-velocity denoted as $u^\mu = (c, 0, 0, 0)$, as discussed in Section 2.3. Once we have established the metric and the stress-energy tensor, the subsequent step entails solving EFEs to elucidate the dynamics of the scale factor in the metric. These equations govern the dynamics of expansion, encompassing the speed and acceleration of the Universe's expansion as observed between two fundamental observers.

4.2. Cosmological Redshift in the RW Metric

We obtain the cosmological redshift through the geodesic equation to a light wave ($ds^2 = 0$). When analyzing electromagnetic waves traversing through a vacuum, it is essential to highlight the maintenance of a linear dispersion relation, expressed as $\lambda_i \nu_i = c_i$. It is noteworthy that the SMC, VSL, and meVSL models all yield identical predictions for a given wavelength's cosmological redshift, as depicted in Eq. (19)

$$\frac{c(t_i)dt_i}{a(t_i)} \equiv \frac{c_1}{a_1 \nu_1} = \frac{c_2}{a_2 \nu_2} \quad \Rightarrow \quad \lambda_1 = \frac{a_1}{a_2} \lambda_2 . \tag{22}$$

Determining the redshift of a distant source entails scrutinizing its light spectrum, particularly absorption or emission lines, and variations in light intensity. Cosmological redshift, attributed to the Universe's expansion, is defined by the relative discrepancy between the wavelengths observed and emitted by an object [40]. By observing redshifted wavelengths, we can gain insights into the characteristics of photons beyond their speed and frequency. Consequently, VSL models find a natural framework within the RW metric, especially during periods of universal expansion.

4.3. Hilbert-Einstein Action

The EFEs stem from the principle of least action via the Einstein-Hilbert (EH) action

$$S \equiv \int \left[\frac{1}{2\kappa} (R - 2\Lambda) + \mathcal{L}_m \right] \sqrt{-g} dt d^3 x , \tag{23}$$

where κ represents the Einstein gravitational constant. Introducing a variation exclusively in the speed of light as a function of cosmic time presents a challenge in deriving EFEs due to the impact of the Palatini identity term on the varying speed of light. Hence, allowing for a variation in the gravitational constant becomes essential to ensure that the combination of these constants ($\kappa = 8\pi G/c^4$) within the EH action remains unaffected by cosmic time

$$\kappa = \text{const} \quad \Rightarrow \quad \begin{cases} G_1 = G_2 = G & \text{SMC} \\ G_1 = \frac{f(a_2)^4}{f(a_1)^4}\frac{a_1^4}{a_2^4} G_2 & \text{VSL} \\ G_1 = \left(\frac{a_1}{a_2}\right)^b G_2 & \text{meVSL} \end{cases} . \tag{24}$$

EFEs, which incorporate the cosmological constant, can be expressed as

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} \equiv G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} , \tag{25}$$

where $G_{\mu\nu}$ represents the Einstein tensor. The structure of EFEs closely resembles that of the SMC.

4.4. Adiabatic Expansion and Cosmological Evolution of the Planck Constant

As outlined in Section 3, the RW metric accommodates cosmological VSL while upholding the CP. Preserving adiabatic conditions is crucial to maintain homogeneity and isotropy. If the speed of light varies over time while leaving other physical constants unchanged, it could potentially disrupt the isotropy of the Universe. An uneven energy flux might undermine isotropy if there is a preferential direction for energy flow. Adiabaticity also has the potential to promote homogeneity if the outward (or inward) energy flow remains isotropic. For a VSL model reliant on an expanding universe to be considered viable, it must adhere to the requirement of adiabatic expansion. This condition also leads to the cosmological evolution of the Planck constant within VSL models [38].

The first law of thermodynamics embodies energy conservation, illustrating that a fluid element within a momentarily comoving reference frame can exchange energy with its surroundings through heat conduction (absorption of heat) and work (exertion of work). Given the significant contribution of photons to entropy, our focus is primarily on photons. Consequently, the first law of thermodynamics can be expressed as

$$dQ = dE + PdV \equiv d(\varepsilon_\gamma V) + P_\gamma dV \quad , \text{ where}$$
$$\varepsilon_\gamma = \frac{\pi^2}{15}\frac{(k_B T_\gamma)^4}{(\hbar c)^3} \equiv \sigma_\gamma T^4 \quad , \quad p_\gamma = \frac{1}{3}\varepsilon_\gamma , \tag{26}$$

where σ_γ represents the black-body constant. Processes in which $dQ = 0$ are termed adiabatic processes, with the adiabatic expansion of the Universe leaving its entropy unaffected. The following equation describes the change in heat concerning the variation of the speed of light

$$dQ = 4\sigma_\gamma T_\gamma^4 V_0 a^3 \left[d\ln T_\gamma + d\ln a + \frac{1}{4}d\ln \sigma_\gamma \right] = 0 . \tag{27}$$

To maintain the observed behavior of the CMB temperature's time evolution, $T_\gamma = T_{\gamma 0}a^{-1}$, the term $d\ln\sigma_\gamma$ must vanish. This observation implies the absence of time evolution for the Boltzmann constant while inducing the reduced Planck constant as

$$\sigma_\gamma = \text{const} \quad \Rightarrow \quad \begin{cases} \hbar_1 = \hbar_2 = \hbar & \text{SMC} \\ \hbar_1 = \frac{f(a_1)}{f(a_2)}\frac{a_2}{a_1}\hbar_2 & \text{VSL} \\ \hbar_1 = \left(\frac{a_2}{a_1}\right)^{\frac{b}{4}}\hbar_2 & \text{meVSL} \end{cases} . \tag{28}$$

4.5. Einstein Tensors

We initiate the derivation of Einstein's tensors for VSL models within the framework of the RW metric, as depicted in Eq. (17). The Christoffel symbols associated with the RW metric are defined by

$$\Gamma^0_{ij} = \frac{a\dot{a}}{c}\gamma_{ij} \quad , \quad \Gamma^i_{0j} = \frac{1}{c}\frac{\dot{a}}{a}\delta^i_j \quad , \quad \Gamma^i_{jk} =^s \Gamma^i_{jk}, \tag{29}$$

where $^s\Gamma^i_{jk}$s represent the Christoffel symbols related to the spatial metric γ_{ij}. In Eq. (29), the Christoffel symbols for VSL models take forms akin to those of the SMC. However, in VSL models, the value of c varies with the scale factor. While the Christoffel symbols of the RW metric display resemblances between the meVSL and SMC models, distinctions arise in the Ricci curvature tensors due to the time-varying speed of light. These differences stem from the derivatives of the Christoffel symbols involving the changing speed of light concerning cosmic time t (or the scale factor a). These derivatives depict distortions of shapes along geodesics in space. Once more, due to temporal fluctuations in the speed of light, correction terms impact both R_{00} and R_{ij}

$$R_{00} = -\frac{3}{c^2}\left(\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2}\frac{d\ln c}{d\ln a}\right), R_{ij} = \frac{\gamma_{ij}}{c^2}a^2\left(2\frac{\dot{a}^2}{a^2} + \frac{\ddot{a}}{a} + 2k\frac{c^2}{a^2} - \frac{\dot{a}^2}{a^2}\frac{d\ln c}{d\ln a}\right). \tag{30}$$

One can trace the Ricci tensors to determine the Ricci scalar

$$R = \frac{6}{c^2}\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + k\frac{c^2}{a^2} - \frac{\dot{a}^2}{a^2}\frac{d\ln c}{d\ln a}\right), \tag{31}$$

where the VSL effect becomes apparent in the final term. By working through these equations, one can derive the components of the Einstein tensor

$$G_{00} = \frac{3}{c^2}\left[\frac{\dot{a}^2}{a^2} + k\frac{c^2}{a^2}\right] \quad , \quad G_{ij} = -\frac{\gamma_{ij}a^2}{c^2}\left[\frac{\dot{a}^2}{a^2} + 2\frac{\ddot{a}}{a} + k\frac{c^2}{a^2} - 2\frac{\dot{a}^2}{a^2}\frac{d\ln c}{d\ln a}\right]. \tag{32}$$

4.6. Bianchi Identity and Cosmological Evolution of Rest Mass

Even with the incorporation of time-varying terms for the speed of light in the Einstein tensors of VSL models, depicted in Eq. (32), it can be shown that these models retain the characteristic that the covariant derivatives of both the Einstein tensors $G_{\mu\nu}$ and the metric $g_{\mu\nu}$ remain zero

$$\nabla^\nu G_{\mu\nu} = 0 \quad , \quad \nabla^\nu g_{\mu\nu} = 0. \tag{33}$$

This essential characteristic is known as the Bianchi Identity (BI). By utilizing the BI along with the constancy of the Einstein gravitational constant κ, it is possible to derive the local conservation law for energy and momentum, as illustrated in Eq. (25)

$$\nabla^\nu\left(\kappa T_{\mu\nu}\right) = \kappa\nabla^\nu T_{\mu\nu} = 0 \quad \Rightarrow \quad d\ln\left(\rho_i c^2\right) + 3(1+\omega_i)d\ln a = 0, \tag{34}$$

where $\omega_i = P_i/(\rho_i c^2)$ denotes the equation of state (EOS) for the i-component. Consequently, one can infer the cosmological evolution of the mass density for the i-component in various models as

$$\rho_i = \rho_{i0}\left(c_0^2/c^2\right)a^{-3(1+\omega_i)} = \begin{cases} \rho_{i0}a^{-3(1+\omega_i)} & \text{SMC} \\ \rho_{i0}\left(\frac{f(a)}{f(1)a}\right)^2 a^{-3(1+\omega_i)} & \text{VSL} \\ \rho_{i0}a^{-\frac{b}{2}}a^{-3(1+\omega_i)} & \text{meVSL} \end{cases}, \tag{35}$$

where we denote the subscript 0 as the present values of corresponding quantities and utilize Eq. (19). Additionally, we set $a_0 = 1$. By interpreting Eq. (35) as the dynamic cosmological rest mass within VSL

models, we can achieve a coherent derivation of a fluid's covariant rest mass energy with a constant eos. Therefore, understanding the cosmological evolution of rest mass is crucial for establishing consistent VSL models

$$E = mc^2 = m_0 c_0^2 \quad \text{for SMC ,VSL ,and meVSL .} \tag{36}$$

5. Friedmann Equation and Huuble Tension

We express the 00-component of the EFEs as

$$\frac{\dot{a}^2}{a^2} + k\frac{c^2}{a^2} - \frac{\Lambda c^2}{3} = \frac{8\pi G}{3}\sum_i \rho_i \quad \Rightarrow \quad \frac{\dot{a}^2}{a^2} \equiv H^2 = \begin{cases} H^2_{(\text{SMC})} & \text{SMC} \\ H^2_{(\text{SMC})}\left(\frac{a}{f(a)}\right)^2 & \text{VSL} \\ H^2_{(\text{SMC})} a^{\frac{b}{2}} & \text{meVSL} \end{cases},$$

$$\text{where} \quad H^2_{(\text{SMC})} \equiv \left[\frac{8\pi G_0}{3}\sum_i \rho_{i0} a^{-3(1+\omega_i)} + \frac{\Lambda \tilde{c}_0^2}{3} - k\frac{\tilde{c}_0^2}{a^2}\right] \equiv H_0^2 E^2 . \tag{37}$$

In the context of the VSL (meVSL) model, the expansion rate of the Universe denoted as H, incorporates an additional factor of $a/f(a)$ ($a^{b/4}$) compared to the expansion rate of the SMC, $H^{(\text{SMC})}$. Consequently, the present-day values of the Hubble parameter coincide for both the SMC and VSL (meVSL) models. However, a notable difference arises in the Hubble parameter's value between VSL (meVSL) and SMC within an expanding Universe framework. This straightforward observation offers potential insights into addressing the Hubble tension [5].

5.1. Hubble Radius

The primary aim of earlier VSL models has been to offer an alternative framework for elucidating cosmic inflation. These models introduce the concept of a diminishing comoving Hubble radius over time, indicated by the condition $d(c/aH)/dt < 0$. The evolution of the comoving Hubble radius in VSL models relies on Equations (19) and (37). Upon closer examination of equation (38), it becomes apparent that the Hubble radius in the context of VSL models is equivalent to that of the SMC

$$\frac{\tilde{c}}{aH} = \frac{\tilde{c}_0}{aH^{(\text{SMC})}} . \tag{38}$$

This raises a crucial consideration: to maintain the viability of a VSL model, it is imperative to avoid postulating variations solely at the speed of light. There is a concern that the Hubble radius could align with that of the SMC, potentially compromising the uniqueness of the proposal.

5.2. Local Physics Laws

In the meVSL model, we take into account local thermodynamics, energy conservation, and other local physical phenomena. These factors lead to the temporal evolution of various physical constants and quantities, as outlined in Table 1.

Table 1. Summary for cosmological evolutions of physical constants and quantities of the meVSL model. These relations satisfy all known local physics laws, including special relativity, thermodynamics, and electromagnetic force [5,6,38].

local physics laws	Special Relativity	Electromagnetism		Thermodynamics
quantities	$\tilde{m} = \tilde{m}_0 a^{-b/2}$	$\tilde{e} = \tilde{e}_0 a^{-b/4}$, $\tilde{\lambda} = \tilde{\lambda}_0 a$, $\tilde{\nu} = \tilde{n}u_0 a^{-1+b/4}$		$\tilde{T} = \tilde{T}_0 a^{-1}$
constants	$\tilde{c} = \tilde{c}_0 a^{b/4}$, $\tilde{G} = \tilde{G}_0 a^b$	$\tilde{\epsilon} = \tilde{\epsilon}_0 a^{-b/4}$, $\tilde{\mu} = \tilde{\mu}_0 a^{-b/4}$		$\tilde{k}_{B0}$, $\tilde{\hbar} = \tilde{\hbar}_0 a^{-b/4}$
energies	$\tilde{m}\tilde{c}^2 = \tilde{m}_0 \tilde{c}_0^2$	$\tilde{\hbar}\tilde{\nu} = \tilde{\hbar}_0 \tilde{\nu}_0 a^{-1}$		$\tilde{k}_B \tilde{T} = \tilde{k}_B \tilde{T}_0 a^{-1}$

6. Discussion

The ΛCDM cosmological model relies on the Robertson-Walker metric. To comprehend this model, we must examine the role of the cosmological principle and Weyl's postulate in constructing this metric. Fundamental observers in this context are in free fall, and their proper time coincides with coordinate time because their spatial position remains constant along their worldlines. The surface of simultaneity for these observers establishes a link between coordinate and cosmic time. When defining fundamental observers as comoving with matter, we can define cosmic time utilized in the global metric and energy stress tensor. In the traditional Robertson-Walker metric, the constancy of the speed of light arises from a specific assumption of cosmological time dilation, rather than being a direct consequence of the cosmological principle and Weyl's postulate. Without explicit physics laws governing time dilation, the speed of light in this metric can vary with cosmic time, akin to other physical quantities like mass density, temperature, and fundamental constants such as the Planck constant. It suggests the possibility of a varying speed of light with cosmic time. Essentially, the assumption that the speed of light remains constant for cosmic time in the traditional Robertson-Walker metric was an additional postulate introduced through the assumed cosmological time dilation. This theoretical concept implies that, unless experimentally validated, the speed of light, like other physical quantities, undergoes cosmological redshift in an expanding universe. This model does not contradict the original principles of the Robertson-Walker metric. Therefore, we should check its validity through observations [19,20,41].

Data Availability Statement: No new data were created and used in this reserach.

Conflicts of Interest: The authors declare no conflicts of interest.

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