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Article

Gröbner-Shirshov Bases Theory for Trialgebras†

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Abstract: We establish a Gröbner-Shirshov bases theory for trialgebras and show that every ideal of a free trialgebra has a unique reduced Gröbner-Shirshov basis. As applications, we give a method for the construction of normal forms of elements of an arbitrary trisemigroup, in particular, A.V. Zhuchok's (2019) normal forms of the free commutative trisemigroups are rediscovered and some normal forms of the free abelian trisemigroups are first constructed. Moreover, the Gelfand-Kirillov dimension of finitely generated free commutative trialgebra and free abelian trialgebra are calculated respectively.

Keywords: Gröbner-Shirshov basis, normal form, Gelfand-Kirillov dimension, trialgebra, trisemigroup

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1. Introduction

The notion of a trialgebra (trioid also known as trisemigroup) was introduced by Loday [24] and investigated in many papers (see, for example,[24,31,32,34,35]). Originally, trioids and trialgebras arose in algebraic topology. They have close relationships with Hopf algebras [26], Leibniz 3-Algebras [18] and Rota-Baxter operators [20]. There are two main motivations for the study of trialgebras. The first one is that many results obtained for trialgebras can be applied to trioids. A trioid [24] is a nonempty set equipped with three binary associative operations satisfying some axioms (Definition 2.1). A trialgebra is a linear analog of a trioid. Therefore, many results obtained for trioids also can be applied to trialgebras. The other reason for our interest in trialgebras (trioids) is their connection with the notions of dialgebras (dimonoids also known as disemigroups) and associative algebras (semigroups). If the operation $\perp$ coicides with $\dashv$ or $\vdash$, then we obtain a dialgebra (dimonoid) [23]. So trialgebras are generalizations of dialgebras. The classes of dialgebras and dimonoids were studied by various authors (see, for example, [7,8,29,30,33]). If all operations of a trialgebra (trioid) coindide, we obtain an associative algebra (semigroup). Loday [24] constructed a free one generator trialgebra and a free trioid of rank 1. A.V. Zhuchok [31,32] constructed the free trioids of an arbitrary rank and the free commutative trioids.

Gröbner bases and Gröbner-Shirshov bases were invented independently by A.I. Shirshov for ideals of free (commutative, anti-commutative) non-associative algebras [27,28], free Lie algebras [28] and implicitly free associative algebras [28] (see also [2,4]), by H. Hironaka [21] for ideals of the power series algebras (both formal and convergent), and by B. Buchberger [14] for ideals of the polynomial algebras. Gröbner bases and Gröbner-Shirshov bases theories have been proved to be very useful in different branches of mathematics, including commutative algebra and combinatorial algebra. It is a powerful tool for solving the following classical problems about: normal form; word problem; conjugacy problem; rewriting system; automaton; embedding theorem; PBW theorem; extension; homology; growth function; Dehn function; complexity; etc. See, for example, the books [1,13,15–17,19] and the surveys [5,6,9–12].

The key in establishing Gröbner-Shirshov bases theory for centain algebras is to establish the "Composition-Diamond lemma (CD lemma)" for such algebras. The name "CD lemma" combines the Neuman Diamond Lemma [25], the Shirshov Composition Lemma [27] and the Bergman Diamond Lemma [2].

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Trialgebras are closely connected with associative algebras and dialgebras, so it is natural to ask what kind of properties of associative algebras and dialgebras remain valid for trialgebras. For instance, CD lemma for dialgebras has been established by Bokut, Chen and Liu in 2010 [7] and by Zhang and Chen in 2017 [8]. They gave a method for constructing normal forms of elements of an arbitrary disemigroup in [8]. So we shall establish the CD lemma for trialgebras and thus offers a way of constructing normal forms of elements of an arbitrary trisemigroup. For associative algebras, polynomial algebras and dialgebras, it is known that every ideal has a unique reduced Gröbner-Shirshov basis. Here we shall prove an analog for trialgebras.

We shall develop a theory of Gröbner–Shirshov bases for trialgebras and prove that there exists a unique reduced Gröbner-Shirshov basis for trialgebras. Moreover, we offers a method of constructing normal forms of elements of an arbitrary trisemigroup. The principle of our construction is directly reminiscent of what was done for dialgebras in [7,8]. However, the extension is not obvious, because more operations are involved and the difficulty increases. First, we must ensure that the considered well ordering on the standard linear basis of a free trialgebra is more or less compatible with the products. Second, a trialgebra has one more operation than dialgebra, so difficulty in the proof of some critical lemmas increases naturally. This will make the proof of the CD lemma much more complicated in the trialgebra case than in the dialgebra case.

The article is organized as follows. In Section 2, we first recall the linear basis constructed by Loday and Ronco [24] of the free trialgebra $Tri\langle X\rangle$ over a field $\mathbf{k}$ generated by X. In Section 3, we establish Gröbner–Shirshov bases theory for trialgebras. We show that for an arbitrary monomial-centers ordering on the linear basis, each ideal of free trialgebra has a unique reduced Gröbner–Shirshov basis. In section 4, we give a method for constructing normal forms of elements of an arbitrary trisemigroup, in particular, we give short proofs of some results in [32]. Moreover, we apply the method of Gröbner–Shirshov bases for centain trialgebras and trisemigroups to obtain normal forms and their Gelfand-Kirillov dimensions.

2. Preliminaries

Throughout the paper, we fix a field $\mathbf{k}$. For a nonempty set X, we denote by X^* the free monoid generated by X, which consists of all associative words on X including the empty word (denoted by ε). Then we denote by $X^+ = X^*\backslash\{\varepsilon\}$ the free semigroup generated by X. For every $u = x_1x_2...x_n \in X^+$, where $x_1, ..., x_n \in X$, we define the *length* $u)$ of u to be n. For convenience, we define $\ell(\varepsilon) = 0$.

Definition 2.1. [24] An *associative trialgebra* (resp. *trisemigroup*), trialgebra for short, is a $\mathbf{k}$-*module* T (resp. a set T) equipped with 3 binary associative operations: $\dashv$ called left, $\vdash$ *called* right, and $\perp$ called middle, satisfying the following 8 identities:

$$\begin{cases} a\dashv(b\vdash c)=a\dashv(b\dashv c),\\ (a\dashv b)\vdash c=(a\vdash b)\vdash c,\\ a\vdash(b\dashv c)=(a\vdash b)\dashv c,\\ a\dashv(b\perp c)=a\dashv(b\dashv c),\\ (a\perp b)\vdash c=(a\vdash b)\vdash c,\\ a\vdash(b\perp c)=(a\vdash b)\perp c,\\ a\perp(b\dashv c)=(a\perp b)\dashv c,\\ a\perp(b\vdash c)=(a\dashv b)\perp c \end{cases} \tag{1}$$

for all $a,\ b,\ c\in T$.

Note that in [24,31,32], the authors call trisemigroups as trioids, and in [3], they are called trisemigroups. Here we follow the terminology of [3].

Definition 2.2. For an arbitrary set X, the *triwords* over X are defined inductively as follows:

(i) For every $x \in X$, the expression (x) is a triword over X of length 1;

(ii) For all triwords (v) and (w) of lengths n and m respectively, all monomials $((v) \dashv (w))$, $((v) \vdash (w))$ and $((v) \perp (w))$ are triwords over X of length $n+m$.

Recall that for every trialgebra T, for all $b_1, ..., b_m \in T$, every parenthesizing of

$$(b_1 \vdash \cdots \vdash b_{m_1-1}) \vdash (b_{m_1} \dashv \cdots \dashv b_{m_2-1}) \perp (b_{m_2} \dashv \cdots \dashv b_{m_3-1}) \perp \cdots \perp (b_{m_r} \dashv \cdots \dashv b_m)$$

gives the same element in T [24], we denote such an element by $[b_1...b_m]_U$, where U is defined to be the set $\{m_i \mid 1 \le i \le r\}$. In particular, assume that T is the free trialgebra generated by X. Then the triword (with an arbitrary bracketing way)

$$(x_{i_1} \vdash \cdots \vdash x_{i_{m_1-1}}) \vdash (x_{i_{m_1}} \dashv \cdots \dashv x_{i_{m_2-1}}) \perp \cdots \perp (x_{i_{m_r}} \dashv \cdots \dashv x_{i_{m_r+t_r}})$$

over X can be determined by the sequence $u := x_{i_1} \dots x_{i_{m_r+t_r}}$ and the set of index

$$U := \{m_i \mid 1 \le i \le r\};$$

therefore, we call such a triword a *normal triword* over X and denote it by $[u]_U$, and call $x_{m_1}, x_{m_2}, ..., x_{m_r}$ the middle entries of $[u]_U$. In case we would like to emphasize the middle entries, we also denote

$$[u]_U := x_{i_1}...x_{i_{m_1-1}}\dot{x}_{i_{m_1}}...x_{i_{m_2-1}}...\dot{x}_{i_{m_r}}...x_{i_{m_r+t_r}}.$$

We call u the *associative word* of the triword $[u]_U$. Let $\mathcal{P}(\mathbb{N})$ be the power set of the positive integers $\mathbb{N}$. We define

$$[X^+]_{\mathcal{P}(\mathbb{N})} := \{[u]_U \mid u \in X^+, \emptyset \neq U \subseteq \{1, ..., \ell(u)\}\}$$

to be the set of all normal triwords on X.

In [24], Loday and Ronco constructed a linear basis for a one-generated free trialgebra, which can be easily generalized for the construction of a linear basis for an arbitrary free trialgebra, see also [31].

Proposition 2.3. [24] *The set $[X^+]_{\mathcal{P}(\mathbb{N})}$ of all normal triwords over X forms a linear basis of the free trialgebra generated by X.*

For every integer $k \in \mathbb{Z}$ and $\emptyset \neq U \in \mathcal{P}(\mathbb{N})$, we define

$$U + k = \{m + k \mid m \in U\}$$

and define $[\varepsilon]_\emptyset = \varepsilon$. For convenience, when we write a set $U = \{m_1, m_2, ..., m_r\} \in \mathcal{P}(\mathbb{N})$, we always assume $m_1 < m_2 < ... < m_r$. Moreover, the cardinality of the set U is denoted by $|U|$; and we simply denote $[u]_{\{m\}}$ by $[u]_m$.

Let $Tri\langle X\rangle$ be the free trialgebra generated by X. Then by [24], $Tri\langle X\rangle$ is the free $\mathbf{k}$-module with a $\mathbf{k}$-basis $[X^+]_{\mathcal{P}(\mathbb{N})}$ and for all $[u]_U, [v]_V \in [X^+]_{\mathcal{P}(\mathbb{N})}$, we have

$$[u]_U \vdash [v]_V = [uv]_{\ell(u)+V}, \quad [u]_U \dashv [v]_V = [uv]_U, \quad [u]_U \perp [v]_V = [uv]_{U \cup (\ell(u)+V)}.$$

Moreover, with the above products, $([X^+]_{\mathcal{P}(\mathbb{N})}, \dashv, \vdash, \perp)$ forms the free trisemigroup generated by X [31]. Though $[\varepsilon]_\emptyset$ is not an element in $[X^+]_{\mathcal{P}(\mathbb{N})}$, we still extend the operations $\vdash$ and $\dashv$ involving $[\varepsilon]_\emptyset$ to make formulas in the sequel simplified. More precisely, we extend them with the following convention:

$$[\varepsilon]_\emptyset \vdash [u]_U = [u]_U \dashv [\varepsilon]_\emptyset = [u]_U \perp [\varepsilon]_\emptyset = [\varepsilon]_\emptyset \perp [u]_U = [u]_U$$

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for every $[u]_U \in [X^+]_{\mathcal{P}(\mathbb{N})}$.

The following lemma shows that every triword can be written as a leftnormed product of triwords.

Lemma 2.4. *Let $[u]_U = [u_1u_2...u_n]_U \in [X^+]_{\mathcal{P}(\mathbb{N})}$ with $u_1,...,u_n \in X^+$. Then there exist some operations $\delta_1,...,\delta_{n-1} \in \{\dashv,\vdash,\perp\}$ such that*

$$[u]_U = (\dots(([u_1]_{U_1}\delta_1[u_2]_{U_2})\delta_2[u_3]_{U_3})...)\delta_{n-1}[u_n]_{U_n} \qquad \textit{(leftnormed bracketing)}.$$

Proof. We use induction on n to prove the claim. For $n=1$, there is nothing to prove. Assume $n>1$ and $U=\{m_1,...,m_r\}$. There are several subcases to consider:

Case 1. If $\ell(u_1...u_{n-1}) \geq m_r$, then we have $[u]_U = [u_1...u_{n-1}]_U \dashv [u_n]_1$. By induction hypothesis, we obtain

$$[u]_U = (...(([u_1]_{U_1}\delta_1[u_2]_{U_2})\delta_2[u_3]_{U_3})...) \dashv [u_n]_1.$$

Case 2. If $\ell(u_1...u_{n-1}) < m_1$, then we have $[u]_U = [u_1...u_{n-1}]_1 \vdash [u_n]_{-\ell(u_1...u_{n-1})+U}$. By induction hypothesis, we obtain

$$[u]_U = (...(([u_1]_{U_1}\delta_1[u_2]_{U_2})\delta_2[u_3]_{U_3})...) \vdash [u_n]_{-\ell(u_1...u_{n-1})+U}.$$

Case 3. If $m_i \leq \ell(u_1...u_{n-1}) < m_{i+1}$ for some $i \in \{1,...,r-1\}$, then we have

$$[u]_U = [u_1...u_{n-1}]_{\{m_1,...,m_i\}} \perp [u_n]_{-\ell(u_1...u_{n-1})+\{m_{i+1},...,m_r\}}.$$

By induction hypothesis, we obtain

$$[u]_U = (...(([u_1]_{U_1}\delta_1[u_2]_{U_2})\delta_2[u_3]_{U_3})...) \perp [u_n]_{-\ell(u_1...u_{n-1})+\{m_{i+1},...,m_r\}}.$$

The proof is completed. □

3. Composition-Diamond lemma for Trialgebras

For constructing normal forms of certain trisemigroups or linear bases of certain trialgebras, we shall apply the method of Gröbner-Shirshov bases for trialgebras. The main idea for the Gröbner-Shirshov bases for some algebras, trialgebras for instance, is to find some tractable conditions under which the elements of the ideal $Id(S)$ of $Tri\langle X\rangle$ generated by S can be written as linear combinations of certain polynomials of $Tri\langle X\rangle$ (hereafter called normal S-polynomials), whose leading monomials are pairwise distinct. Our aim in this section is to prove that, for a quotient trialgebra $Tri\langle X|S\rangle := Tri\langle X\rangle/Id(S)$, if the set S of relations satisfies some tractable conditions, then a linear basis of $Tri\langle X|S\rangle$ can be effectively constructed.

We first introduce a well ordering on X^+. Let X be a well-ordered set. We define the *deg-lex ordering* on X^+ as the following: for $u = x_{i_1}x_{i_2}...x_{i_m}$, $v = x_{j_1}x_{j_2}...x_{j_n} \in X^+$, where x_{i_l}, $x_{j_t} \in X$, we define

$$u > v \quad \text{if } (\ell(u), x_{i_1}, x_{i_2},..., x_{i_m}) > (\ell(v), x_{j_1}, x_{j_2},..., x_{j_n}) \text{ lexicographically}.$$

A well ordering $>$ on X^+ is called *monomial* if for all $u,v,w \in X^+$, we have

$$u > v \Rightarrow uw > vw \ \text{ and } \ u > v \Rightarrow wu > wv.$$

Clearly, the above deg-lex ordering on X^+ is monomial.

We proceed to define a well ordering on $\mathcal{P}(\mathbb{N})\backslash\{\emptyset\}$. For all $U=\{m_1,...,m_r\}$ and $V=\{n_1,...,n_t\} \in \mathcal{P}(\mathbb{N})\backslash\{\emptyset\}$, we define

$$U > V \quad \text{if } (r, m_1,...,m_r) > (t, n_1,...,n_t) \text{ lexicographically}.$$

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Fix a monomial ordering $>$ on X^+. Then we define the *monomial-centers ordering* $>$ on $[X^+]_{\mathcal{P}(\mathbb{N})}$ as follows: for all $[u]_U$, $[v]_V \in [X^+]_{\mathcal{P}(\mathbb{N})}$,

$$[u]_U > [v]_V \quad \text{if } (u, U) > (v, V) \text{ lexicographically,} \tag{2}$$

where we compare u and v by the fixed ordering on X^+. Though we use the same notation $>$ for orderings on X^+, $\mathcal{P}(\mathbb{N})\backslash\{\emptyset\}$ and $[X^+]_{\mathcal{P}(\mathbb{N})}$, no confusion will arise because the monomials under consideration are always clear.

It is clear that a monomial-centers ordering is a well ordering on $[X^+]_{\mathcal{P}(\mathbb{N})}$. Finally, if $>$ is the deg-lex ordering on X^+, then we call the ordering defined by (2) the *deg-lex-centers ordering* on $[X^+]_{\mathcal{P}(\mathbb{N})}$.

For all $[u]_U, [v]_V, [u']_{U'}, [v']_{V'} \in [X^+]_{\mathcal{P}(\mathbb{N})}$ and $\delta, \delta' \in \{\dashv, \vdash, \perp\}$, assume $[u]_U \delta [v]_V = [w]_W$ and $[u']_{U'} \delta' [v']_{V'} = [w']_{W'}$. Then by $[u]_U \delta [v]_V > [u']_{U'} \delta' [v']_{V'}$ we mean $[w]_W > [w']_{W'}$.

From now on, we always assume that $>$ is a monomial-centers ordering $>$ on $[X^+]_{\mathcal{P}(\mathbb{N})}$.

We observe that the monomial-centers ordering $>$ on $[X^+]_{\mathcal{P}(\mathbb{N})}$ is monomial in the following sense:

Lemma 3.1. *Let $[u]_U$, $[v]_V$ and $[w]_W \in [X^+]_{\mathcal{P}(\mathbb{N})}$ with $[u]_U > [v]_V$. Then we have*

$$\begin{aligned} &[w]_W \vdash [u]_U > [w]_W \vdash [v]_V, && [u]_U \dashv [w]_W > [v]_V \dashv [w]_W, \\ &[u]_U \perp [w]_W > [v]_V \perp [w]_W, && [w]_W \perp [u]_U > [w]_W \perp [v]_V, \\ &[u]_U \vdash [w]_W \geq [v]_V \vdash [w]_W, && [w]_W \dashv [u]_U \geq [w]_W \dashv [v]_V. \end{aligned}$$

Moreover, if $u > v$, then $[u]_U \vdash [w]_W > [v]_V \vdash [w]_W$ and $[w]_W \dashv [u]_U > [w]_W \dashv [v]_V$.

For every polynomial $f = \sum_{i=1}^{n} \alpha_i [u_i]_{U_i} \in Tri\langle X\rangle$, where $0 \neq \alpha_i \in \mathbf{k}$, $[u_i]_{U_i} \in [X^+]_{\mathcal{P}(\mathbb{N})}$ and $[u_1]_{U_1} > [u_2]_{U_2} > \ldots > [u_n]_{U_n}$, we call $[u_1]_{U_1}$ the *leading monomial* of f, denoted by $\overline{f}$; α_1 the *leading coefficient* of $\overline{f}$, denoted by $lc(f)$; finally, we denote by $\widetilde{f}$ the associative word of $\overline{f}$. A polynomial f is called *monic* if $lc(f) = 1$, and a nonempty subset S of $Tri\langle X\rangle$ is called *monic* if every element in S is monic. We call a nonzero polynomial $f \in Tri\langle X\rangle$ *strong* if $\widetilde{f} > \widetilde{r_f}$, where $r_f := f - lc(f)\bar{f}$.

For convenience, we define $\bar{0} = \widetilde{0} = 0$, $\widetilde{0} < u$ and $\bar{0} < [u]_U$ for any $[u]_U \in [X^+]_{\mathcal{P}(\mathbb{N})}$.

From Lemma 3.1 it follows that

Lemma 3.2. *Let $0 \neq f \in Tri\langle X\rangle$ and $[u]_U \in [X^+]_{\mathcal{P}(\mathbb{N})}$. Then we have*

$$\begin{aligned} &\overline{([u]_U \vdash f)} = [u]_U \vdash \overline{f}, && \overline{(f \dashv [u]_U)} = \overline{f} \dashv [u]_U, \\ &\overline{([u]_U \perp f)} = [u]_U \perp \overline{f}, && \overline{(f \perp [u]_U)} = \overline{f} \perp [u]_U, \\ &\overline{([u]_U \dashv f)} \leq [u]_U \dashv \overline{f}, && \overline{(f \vdash [u]_U)} \leq \overline{f} \vdash [u]_U. \end{aligned}$$

Moreover, if f is strong, then we obtain $\overline{([u]_U \dashv f)} = [u]_U \dashv \overline{f}$ and $\overline{(f \vdash [u]_U)} = \overline{f} \vdash [u]_U$.

Now we begin to study elements of an ideal generated by a subset of $Tri\langle X\rangle$. We begin with the following notation. For every $[u]_U = [x_{i_1} \ldots x_{i_t} \ldots x_{i_n}]_U \in [X^+]_{\mathcal{P}(\mathbb{N})}$ such that $x_{i_1}, \ldots, x_{i_n}$ lie in X, by Lemma 2.4, we may assume that $[u]_U = ([v]_V \delta_1 x_{i_t}) \delta_2 [w]_W$. Then for every polynomial $f \in Tri\langle X\rangle$, we define

$$[u]_U|_{x_{i_t} \mapsto f} = ([v]_V \delta_1 f) \delta_2 [w]_W, \tag{3}$$

where by convention, if exactly one of $[u]_U$ and $[v]_V$ is $[\varepsilon]_\emptyset$, then we define $[u]_U \delta [v]_V = [uv]_{U \cup V}$, in particular, the formula (3) makes sense. Clearly, the resulting polynomial $([v]_V \delta_1 f) \delta_2 [w]_V$

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is independent of the choice of $[v]_V, [w]_W$ and δ_1, δ_2. For simplicity, we usually denote by (vfw) a polynomial of the form (3).

Definition 3.3. Let S be a monic subset of $Tri\langle X\rangle$. Then for every $[u]_U = [x_{i_1}...x_{i_t}...x_{i_n}]_U$ in $[X^+]_{\mathcal{P}(\mathbb{N})}$ such that $x_{i_1}, \ldots, x_{i_n}$ lie in X and for every $s \in S$, $[u]_U|_{x_{i_t}\mapsto s}$ is called an *s-polynomial* or *S-polynomial*, and it is called *normal* if either $t \in U$ or s is strong.

Remark 3.4. By Lemma 2.4 and Definition 3.3 , it follows that
(i) Every S-polynomial (asb) has an expression:

$$(asb) = ([a]_A\delta_1 s)\delta_2[b]_B \tag{4}$$

for some $\delta_1, \delta_2 \in \{\dashv, \vdash, \perp\}$ and $a, b \in X^*$. In (4), by convention, we always assume $\delta_1 \in \{\vdash, \perp\}$ (resp. $\delta_2 \in \{\dashv, \perp\}$) in case $[a]_A = [\varepsilon]_\emptyset$ (resp. $[b]_B = [\varepsilon]_\emptyset$). Then $([a]_A\delta_1 s)\delta_2[b]_B$ is a normal S-polynomial if and only if one of the following conditions hlods:
(a) $\delta_1 \in \{\vdash, \perp\}$ and $\delta_2 \in \{\dashv, \perp\}$ hold;
(b) s is strong.
Moreover, if $([a]_A\delta_1 s)\delta_2[b]_B$ is normal and $\overline{([a]_A\delta_1 s)\delta_2[b]_B} = [w]_W$, then we denote

$$[asb]_W := ([a]_A\delta_1 s)\delta_2[b]_B.$$

(ii) If $(asb) = ([a]_A\delta_1 s)\delta_2[b]_B$ is a normal S-polynomial, then $([u]_U\delta_1 s)\delta_2[v]_V$ is still a normal S-polynomial for all $[u]_U, [v]_V \in [X^+]_{\mathcal{P}(\mathbb{N})}$.
(iii) Let $([a]_A\delta_1 s)\delta_2[b]_B$ be a normal S-polynomial and assume that s is not strong. Then both

$$[a']_{A'}\delta_3(([a]_A\delta_1 s)\delta_2[b]_B) \text{ and } ([a]_A\delta_1 s)\delta_2[b]_B)\delta_4[b']_{B'}$$

are normal S-polynomials if and only if $\delta_3 \in \{\vdash, \perp\}$ and $\delta_4 \in \{\dashv, \perp\}$.

The following lemma follows from the definition of normal S-polynomials.

Lemma 3.5. *Let $(asb) = ([a]_A\delta_1 s)\delta_2[b]_B$ be a normal S-polynomial. Assume $\bar{s} = [u]_U$ and $\overline{(asb)} = [w]_W$. Then we have*

$$(\ell(a) + U) \subseteq W \subseteq (\{1, ..., \ell(a), \ell(a\bar{s}) + 1, ..., \ell(a\bar{s}) + \ell(b)\} \cup (\ell(a) + U)),$$

or

$$\emptyset \neq W \subseteq \{1, ..., \ell(a), \ell(a\bar{s}) + 1, ..., \ell(a\bar{s}) + \ell(b)\}.$$

Moreover, if W is a nonempty subset of $\{1, ..., \ell(a), \ell(a\bar{s}) + 1, ..., \ell(a\bar{s}) + \ell(b)\}$, then s is strong. Finally, for every such a set W satisfying the above conditions, there exists a normal S-polynomial (asb) such that $\overline{(asb)} = [w]_W$.

In view of Lemma 3.5, for every normal S-polynomial $(asb) = ([a]_A\delta_1 s)\delta_2[b]_B$ with $\bar{s} = [u]_U$ and $\overline{(asb)} = [w]_W$, we define $P([asb])$ to be the set of all the possible W for a normal S-polynomial of the form (asb) as in Lemma 3.5; in other words, we have

$$P([asb]) = \begin{cases} \{(\ell(a) + U) \cup W, W \mid W \subseteq \{1, ..., \ell(a), \ell(a\bar{s}) + 1, ..., \ell(a\bar{s}) + \ell(b)\}\} \setminus \{\emptyset\}, \\ \text{if } s \text{ is strong;} \\ \\ \{(\ell(a) + U) \cup W \mid W \subseteq \{1, ..., \ell(a), \ell(a\bar{s}) + 1, ..., \ell(a\bar{s}) + \ell(b)\}\}, \\ \text{if } s \text{ is not strong.} \end{cases}$$

In particular, we have $P(s) = U$.
By Lemma 3.1, we immediately obtain the following lemma.

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Lemma 3.6. *Let (asb) be a normal s-polynomial and $[u]_U, [v]_V \in [X^+]_{\mathcal{P}(\mathbb{N})}$. Then*

$$[u]_U \vdash [asb]_C \dashv [v]_V = [uasbv]_{\ell(u)+C}, \quad [u]_U \vdash [asb]_C \perp [v]_V = [uasbv]_{(\ell(u)+C)\cup\ell(ua\widetilde{s}b)+V)}$$

$$[u]_U \perp [asb]_C \dashv [v]_V = [uasbv]_{U\cup(\ell(u)+C)}, \quad [u]_U \perp [asb]_C \perp [v]_V = [uasbv]_{U\cup(\ell(u)+C)\cup(\ell(ua\widetilde{s}b)+V)}.$$

The following lemma shows that the set

$Irr(S) := \{[u]_U \in [X^+]_{\mathcal{P}(\mathbb{N})} \mid [u]_U \neq \overline{[asb]_U}$ for any normal S-polynomial $[asb]_U\}$

is a linear generating set of the quotient trialgebra $Tri\langle X|S\rangle$.

Lemma 3.7. *Let S be a monic subset of $Tri\langle X\rangle$. Then for every nonzero polynomial $f \in Tri\langle X\rangle$, we have*

$$f = \sum \alpha_i[u_i]_{U_i} + \sum \beta_j[a_js_jb_j]_{C_j},$$

for some $[u_i]_{U_i} \in Irr(S)$, $\alpha_i, \beta_j \in \mathbf{k}$, $a_j, b_j \in X^*$, $s_j \in S$, $[u_i]_{U_i} \leq \overline{f}$ and $\overline{[a_js_jb_j]_{C_j}} \leq \overline{f}$.

Proof. Let $f = lc(f)\overline{f} + r_f$. If $\overline{f} \in Irr(S)$, then we define $f_1 = f - lc(f)\overline{f}$. If $\overline{f} \notin Irr(S)$, then we obtain $\overline{f} = \overline{[asb]_C}$ for some normal S-polynomial $[asb]_C$. And we define $f_1 = f - lc(f)[asb]_C$. In both cases, we have $\overline{f_1} < \overline{f}$ and the result follows by induction on $\overline{f}$. □

We shall show in Theorem 3.15 that, if S is a Gröbner–Shirshov basis in $Tri\langle X\rangle$, then the set $Irr(S)$ is a linear basis of $Tri\langle X|S\rangle$. So what we need at this point is a family of tractable conditions to recognize that S is (possibly) a Gröbner–Shirshov basis.

Our first step is to introduce compositions (Definition 3.8), which are special polynomials in the ideal $Id(S)$ playing an important role in deciding whether the set S of relations is a Gröbner–Shirshov basis in $Tri\langle X\rangle$ or not.

Definition 3.8. Let S be a monic subset of $Tri\langle X\rangle$. For all $f, g \in S$, $f \neq g$, we define *compositions* as follows:

(i) If f is not strong, then for all $x \in X$ and $[u]_{\ell(u)} \in [X^+]_{\mathcal{P}(\mathbb{N})}$, we call $x \dashv f$ a *left multiplication composition* of f and call $f \vdash [u]_{\ell(u)}$ a *right multiplication composition* of f.

(ii) Suppose that $w = \widetilde{f} = a\widetilde{g}b$ for some words $a, b \in X^*$ and let (agb) be a normal S-polynomial.

(a) If $P(f) \in P([agb])$, then we call

$$(f, g)_{\overline{f}} = f - [agb]_{P(f)}$$

an *inclusion composition* of S.

(b) If $P(f) \notin P([agb])$ and both f and g are strong, then for every $x \in X$ we call

$$(f, g)_{[xw]_1} = [xf]_1 - [xagb]_1$$

a left multiplicative inclusion composition of S, and call

$$(f, g)_{[wx]_{\ell(wx)}} = [fx]_{\ell(wx)} - [agbx]_{\ell(wx)}$$

a *right multiplicative inclusion composition* of S.

(iii) Suppose that there exists a word $w = \widetilde{f}b = a\widetilde{g}$ for some words $a, b \in X^*$ such that $|\widetilde{f}| + |\widetilde{g}| > \ell(w)$. Let (fb) be a normal f-polynomial and let (ag) be a normal g-polynomial.

(a) If $P([fb]) \cap P([ag]) \neq \emptyset$, then for every $W \in P([fb]) \cap P([ag])$, we call

$$(f, g)_{[w]_W} = [fb]_W - [ag]_W$$

an *intersection composition* of S.

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(b) If $P([fb]) \cap P([ag]) = \emptyset$ and both f and g are strong, then for every $x \in X$ we call

$$(f,g)_{[xw]_1} = [xfb]_1 - [xag]_1$$

a *left multiplicative intersection composition* of S, and call

$$(f,g)_{[wx]_{\ell(wx)}} = [fbx]_{\ell(wx)} - [agx]_{\ell(wx)}$$

a *right multiplicative intersection composition* of S.

A polynomial $h \in Tri\langle X \rangle$ is called *trivial modulo* S (resp. $(S, [w]_W)$), denoted by

$$h \equiv 0 \; mod \; (S) \;\; (\text{resp. } mod \; (S, [w]_W)),$$

if $h = \sum \alpha_i [a_i s_i b_i]_{C_i}$, where each $\alpha_i \in \mathbf{k}$, $a_i, b_i \in X^*$, $s_i \in S$ and $\overline{[a_i s_i b_i]_{C_i}} \leq \overline{h}$ (resp. $\overline{[a_i s_i b_i]_{C_i}} < [w]_W$). Denote by $h \equiv h' \; mod \; (S)$ (resp. $mod \; (S, [w]_W)$) if $h - h' \equiv 0 \; mod \; (S)$ (resp. $mod \; (S, [w]_W)$).

A monic set S is said to be *closed under left* (resp. *right*) *multiplication compositions* if every left (resp. right) multiplication composition $x \dashv f$ (resp. $f \vdash [u]_{\ell(u)}$) of S is trivial modulo S. A monic set S is called a *Gröbner-Shirshov basis* in $Tri\langle X \rangle$ if S is closed under left and right multiplication compositions and every composition $(f,g)_{[u]_U}$ of S is trivial modulo S.

We shall prove that, to some extent, the ordering $<$ is compatible with the normal S-polynomials and normal triwords.

Lemma 3.9. *Let S be a monic subset of $Tri\langle X \rangle$ that is closed under left multiplication compositions and assume $f \in S$. If f is not strong, then for every $[u]_1 \in [X^+]_{\mathcal{P}(\mathbb{N})}$, we have $[u]_1 \dashv f \equiv 0 \; mod \; (S)$.*

Proof. We shall use induction on $(u\widetilde{f}, \ell(u))$ to prove the claim. If $\ell(u) = 1$, then it is clear. Assume $\ell(u) \geq 2$ and $[u]_1 = [vx]_1$, $v \in X^+$, $x \in X$. Then $[u]_1 \dashv f = [v]_1 \dashv (x \dashv f)$ can be written as a linear combination of S-polynomials of the form $[v]_1 \dashv [asb]_C$, where $s \in S$ and $[a\widetilde{s}b]_C \leq \overline{(x \dashv f)}$. So we obtain

$$\overline{([v]_1 \dashv [asb]_C)} \leq [v]_1 \dashv [a\widetilde{s}b]_C \leq [v]_1 \dashv \overline{(x \dashv f)} = \overline{([u]_1 \dashv f)} \text{ and } a\widetilde{s}b \leq x\widetilde{f}.$$

If s is strong, then $[v]_1 \dashv [asb]_C$ is already a normal S-polynomial, and we are done. Now we assume that s is not strong. If a is the empty word, then we have

$$[v]_1 \dashv [asb]_C = ([v]_1 \dashv s) \dashv [b]_1$$

and $(v\widetilde{s}, \ell(v)) < (u\widetilde{f}, \ell(u))$. If a is not the empty word, then we have $[asb]_C = [a]_1 \delta [sb]_{-\ell(a)+C}$, where δ lies in $\{\vdash, \perp\}$. So we obtain

$$[v]_1 \dashv [asb]_C = ([va]_1 \dashv s) \dashv [b]_1$$

and $C > \{1\}$. Since $[a\widetilde{s}b]_C \leq [x\widetilde{f}]_1$, we obtain $a\widetilde{s}b < x\widetilde{f}$ and $(va\widetilde{s}, \ell(va)) < (u\widetilde{f}, \ell(u))$. By induction, $[v]_1 \dashv [asb]_C$ is a linear combination of S-polynomials of the form $[cs'd]_L \dashv [b]_1$, where $s' \in S$ and $[c\widetilde{s'}d]_L \leq \overline{([va]_1 \dashv s)}$. By Lemma 3.6, $[cs'd]_L \dashv [b]_1$ is a normal S-polynomial. So we deduce

$$[c\widetilde{s'}d]_L \dashv [b]_1 \leq \overline{([va]_1 \dashv s)} \dashv [b]_1 = \overline{([v]_1 \dashv [asb]_C)} \leq \overline{([u]_1 \dashv f)}.$$

The proof is completed. $\square$

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Let $f \in S$ be a polynomial that is not strong, and assume that $f \vdash x$ is trivial modulo S for every $x \in X$. Then the following example shows that $f \vdash [u]_{\ell(u)}$ may be not trivial modulo S for some $u \in X^+$.

Example 3.10. ([8, Example 3.12]) Let $X = \{x_1, x_2\}, x_1 > x_2$. Assume that the characteristic of the underlying field $\mathbf{k}$ is not 2. Let $S = \{f, g, h\}$, where $f = [x_1x_2]_2 + [x_1x_2]_1$, $g = [x_1x_2x_1]_3 - \frac{1}{2}[x_1x_2x_1]_2 - \frac{1}{2}[x_1x_2x_1]_1$, $h = [x_1x_2x_2]_3 - \frac{1}{2}[x_1x_2x_2]_2 - \frac{1}{2}[x_1x_2x_2]_1$. These three polynomials are not strong. By a direct calculation, we have $g \vdash x_i = 0$, $h \vdash x_i = 0$, $i = 1, 2$, and $f \vdash x_1 = 2g + f \dashv x_1 \equiv 0 \ mod\ (S)$, $f \vdash x_2 = 2h + f \dashv x_2 \equiv 0 \ mod\ (S)$. However, $f \vdash [x_1x_1]_2 = 2[x_1x_2x_1x_1]_4$ is not trivial modulo S, because $f \vdash [x_1x_1]_2$ is not normal, and for every polynomial $f' \in \{g, h\}$, we have $f' \vdash x_1 = 0$.

Lemma 3.11. *Let S be a monic subset of $Tri\langle X\rangle$ that is closed under left and right multiplication compositions. Then for all normal S-polynomial $[asb]_C$ and normal triword $[u]_U \in [X^+]_{\mathcal{P}(\mathbb{N})}$, we have*

$$[u]_U\delta[asb]_C \equiv 0 \ mod\ (S) \ \ and \ \ [asb]_C\delta[u]_U \equiv 0 \ mod\ (S),$$

where $\delta \in \{\dashv, \vdash, \perp\}$. Moreover, for every normal triword $[w]_W \in [X^+]_{\mathcal{P}(\mathbb{N})}$, if $a\bar{s}b < w$, then we have

$$[u]_U\delta[asb]_C \equiv 0 \ mod\ (S,\ [u]_U\delta[w]_W) \ \ and \ \ [asb]_C\delta[u]_U \equiv 0 \ mod\ (S,\ [w]_W\delta[u]_U),$$

where $\delta \in \{\dashv, \vdash, \perp\}$.

Proof. By Lemma 3.6, it suffices to show that $[u]_U \dashv [asb]_C$ and $[asb]_C \vdash [u]_U$ are trivial modulo S, where s is not strong. So we assume that s is not strong.

We first prove that $[u]_U \dashv [asb]_C$ is trivial modulo S. By Lemma 2.4, obviously we have

$$\begin{aligned}[u]_U \dashv [asb]_C &= (([u]_U \dashv [a]_A) \dashv s) \dashv [b]_B = (([u_1]_{U_1}\delta_1[u_2]_1) \dashv s)) \dashv [b]_B \\ &= ([u_1]_{U_1}\delta_1([u_2]_1 \dashv s)) \dashv [b]_B,\end{aligned}$$

where $\delta_1 \in \{\vdash, \perp\}$ and $ua = u_1u_2$ with $u_1 \in X^*$ and $u_2 \in X^+$. If $[u_1]_{U_1} = [\varepsilon]_\emptyset$, then we have $\delta_1 = \vdash$ by convention. By Lemmas 3.9, 3.6 and 3.1, the result follows. Moreover, if $a\bar{s}b < w$, then we obtain

$$ua\bar{s}b < uw \text{ and } \overline{([u]_U\delta[asb]_C)} < [u]_U\delta[w]_W,$$

where $\delta \in \{\dashv, \vdash, \perp\}$. Therefore, we deduce $[u]_U\delta[asb]_C \equiv 0 \ mod\ (S,\ [u]_U\delta[w]_W)$.

The proof for the case of $[asb]_C \vdash [u]_U$ is similar to the above case. More precisely, by Lemma 2.4, we have

$$[asb]_C \vdash [u]_U = [a]_A \vdash (s \vdash ([b]_B \vdash [u]_U)) = [a]_A \vdash ((s \vdash [u_1]_{\ell(u_1)})\delta_2[u_2]_{U_2}),$$

where $\delta_2 \in \{\dashv, \perp\}$ and $bu = u_1u_2$ with $u_1 \in X^+$, $u_2 \in X^*$. Since S is closed under right multiplication compositions, the result follows by Lemmas 3.6 and 3.1. Moreover, if $a\bar{s}b < w$, then we have

$$a\bar{s}bu < wu \text{ and } \overline{([asb]_C\delta[u]_U)} < [w]_W\delta[u]_U,$$

where $\delta \in \{\dashv, \vdash, \perp\}$. Therefore, we deduce $[asb]_C\delta[u]_U \equiv 0 \ mod\ (S,\ [w]_W\delta[u]_U)$. □

The following corollary is useful in the sequl, which shows that, if we replace certain "subtriword" in a triword with a "small" normal S-polynomial, then we shall obtain a linear combination of "small" normal S-polynomials.

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Corollary 3.12. *Let S be a monic subset of $Tri\langle X\rangle$ that is closed under left and right multiplication compositions. Let $[w]_W$ be a normal triword such that $([a]_A\delta_1[u]_U)\delta_2[b]_B = [w]_W$, and let f be a normal S-polynomial with $\overline{f} < [u]_U$. If $\widetilde{f} < u$, or if $\delta_1 \in \{\vdash, \perp\}$ and $\delta_2 \in \{\dashv, \perp\}$, then we have*

$$([a]_A\delta_1 f)\delta_2[b]_B \equiv 0 \ mod\ (S,\ [w]_W).$$

Proof. If $\delta_1 \in \{\vdash, \perp\}$ and $\delta_2 \in \{\dashv, \perp\}$, then by Lemmas 3.6 and 3.1, $([a]_A\delta_1 f)\delta_2[b]_B$ is a normal S-polynomial with

$$\overline{([a]_A\delta_1 f)\delta_2[b]_B} < ([a]_A\delta_1[u]_U)\delta_2[b]_B = [w]_W,$$

the result follows.

Now we assume $\widetilde{f} < u$. By Lemma 3.11, $([a]_A\delta_1 f)\delta_2[b]_B$ can be written as a linear combination of S-polynomials of the form $[cs'd]_L\delta_2[b]_B$, where $s' \in S$ and $c\widetilde{s'}d < au$. And for every S-polynomial $[cs'd]_L\delta_2[b]_B$, by Lemma 3.11 and by the fact that $c\widetilde{s'}db < aub = w$, we have $[cs'd]_L\delta_2[b]_B \equiv 0 \ mod\ (S,\ [w]_W)$. □

Now we show that, if a monic set S is closed under left and right multiplication compositions, then the elements of the ideal $Id(S)$ of $Tri\langle X\rangle$ can be written as linear combinations of normal S-polynomials.

Corollary 3.13. *Let S be a monic subset of $Tri\langle X\rangle$ that is closed under left and right multiplication compositions. Then every S-polynomial (asb) has an expression of the form:*

$$(asb) = \sum \alpha_i[a_is_ib_i]_{C_i},$$

where each $\alpha_i \in \mathbf{k}$, $s_i \in S$, $a_i, b_i \in X^*$.

Proof. Let $[u]_U$ be a triword such that $\bar{s} < [u]_U$. And assume $(asb) = ([a]_A\delta_1 s)\delta_2[b]_B$ and $[w]_W = ([a]_A\delta_1[u]_U)\delta_2[b]_B$. Then by Corollary 3.12, we obtain

$$(asb) = ([a]_A\delta_1 s)\delta_2[b]_B \equiv 0 \ mod\ (S,\ [w]_W).$$

The proof is completed. □

The following lemma shows that, under certain assumptions, if two normal S-polynomials have the same leading monomial, then their difference is nonessential.

Lemma 3.14. *Let S be a Gröbner-Shirshov basis in $Tri\langle X\rangle$. Suppose that $[a_1s_1b_1]_{C_1}$, $[a_2s_2b_2]_{C_2}$ are two normal S-polynomials with $\overline{[a_1s_1b_1]_{C_1}} = \overline{[a_2s_2b_2]_{C_2}} = [w]_W$. Then we have*

$$[a_1s_1b_1]_{C_1} - [a_2s_2b_2]_{C_2} \equiv 0 \ mod\ (S, [w]_W).$$

Proof. Since $[w]_W = \overline{[a_1s_1b_1]_{C_1}} = \overline{[a_2s_2b_2]_{C_2}}$, we obtain $w = a_1\widetilde{s_1}b_1 = a_2\widetilde{s_2}b_2$ and $W = C_1 = C_2$. We have to consider the following three cases:

Case 1. Without loss of generality, we can assume $b_1 = a\widetilde{s_2}b_2$ and $a_2 = a_1\widetilde{s_1}a$, here a may be the empty word. Assume $s_1 = \overline{s_1} + \sum \beta_i[u_i]_{U_i}$ and $s_2 = \overline{s_2} + \sum \beta'_j[v_j]_{V_j}$. Then by Lemma 2.4, we have

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$$\begin{aligned}
& [a_1s_1b_1]_{C_1} - [a_2s_2b_2]_{C_2} \\
= \; & [a_1s_1a\widetilde{s_2}b_2]_W - [a_1\widetilde{s_1}as_2b_2]_W \\
= \; & ((([a_1]_{A_1}\delta_1 s_1)\delta_2[a]_A)\delta_3\overline{s_2})\delta_4[b_2]_{B_2} - ((([a_1]_{A_1}\delta_1\overline{s_1})\delta_2[a]_A)\delta_3 s_2)\delta_4[b_2]_{B_2} \\
= \; & ((([a_1]_{A_1}\delta_1 s_1)\delta_2[a]_A)\delta_3(\overline{s_2}-s_2))\delta_4[b_2]_{B_2} - ((([a_1]_{A_1}\delta_1(\overline{s_1}-s_1))\delta_2[a]_A)\delta_3 s_2)\delta_4[b_2]_{B_2} \\
= \; & -\sum\beta_j'((([a_1]_{A_1}\delta_1 s_1)\delta_2[a]_A)\delta_3[v_j]_{V_j})\delta_4[b_2]_{B_2} \\
& +\sum\beta_i((([a_1]_{A_1}\delta_1[u_i]_{U_i})\delta_2[a]_A)\delta_3 s_2)\delta_4[b_2]_{B_2}
\end{aligned}$$

for some $\delta_1, \delta_2, \delta_3, \delta_4 \in \{\dashv, \vdash, \perp\}$.

If s_1 and s_2 are both strong, then all the resulting polynomials

$$((([a_1]_{A_1}\delta_1 s_1)\delta_2[a]_A)\delta_3[v_j]_{V_j})\delta_4[b_2]_{B_2} \text{ and } ((([a_1]_{A_1}\delta_1[u_i]_{U_i})\delta_2[a]_A)\delta_3 s_2)\delta_4[b_2]_{B_2}$$

are normal S-polynomials; If neither s_1 nor s_2 is strong, then by Remark 3.4, we deduce $\delta_3 = \perp$, $\delta_1 \in \{\vdash, \perp\}$ and $\delta_2, \delta_4 \in \{\dashv, \perp\}$, which implies the above resulting S-polynomials are normal; If only one of s_1 and s_2 is not strong, say, s_1 is not strong, then by Remark 3.4, we deduce $\delta_1 \in \{\vdash, \perp\}$ and $\delta_2, \delta_3, \delta_4 \in \{\dashv, \perp\}$. It follows that the resulting S-polynomials are normal. In all subcases, by Lemmas 3.1 and 3.2, the leading monomials of the resulting normal S-polynomials are less than $[w]_W$.

Case 2. Without loss of generality, we may assume $\widetilde{s_1} = a\widetilde{s_2}b$, $a_2 = a_1a$ and $b_2 = bb_1$. If $P(s_1) \in P([as_2b])$, then since S is a Gröbner-Shirshov basis, we may assume

$$s_1 - [as_2b]_{P(s_1)} = \sum\alpha_i'[c_i's_i'd_i']_{L_i'}$$

satisfying $\overline{[c_i's_i'd_i']_{L_i'}} \leq \overline{s_1 - [as_2b]_{P(s_1)}} < \overline{s_1}$ for every i. So we have

$$\begin{aligned}
[a_1s_1b_1]_{C_1} - [a_2s_2b_2]_{C_2} &= ([a_1]_{A_1}\delta_1(s_1 - [as_2b]_{P(s_1)}))\delta_2[b_1]_{B_1} \\
&= \sum\alpha_i'([a_1]_{A_1}\delta_1[c_i's_i'd_i']_{L_i'})\delta_2[b_1]_{B_1}.
\end{aligned}$$

If one of s_1 and s_2 is not strong, then we deduce $\delta_1 \in \{\vdash, \perp\}$ and $\delta_2 \in \{\dashv, \perp\}$; and if s_1 and s_2 are strong, then we obtain $c_i'\widetilde{s_i'}d_i' < \widetilde{s_1}$ for all i. In either of these subcases, by Corollary 3.12, we obtain

$$[a_1s_1b_1]_{C_1} - [a_2s_2b_2]_{C_2} = \sum\alpha_i'([a_1]_{A_1}\delta_1[c_i's_i'd_i']_{L_i'})\delta_2[b_1]_{B_1} \equiv 0 \; mod \; (S, [w]_W).$$

If $P(s_1) \notin P([as_2b])$, then we deduce that s_1, s_2 are strong and $\ell(a_1) + \ell(b_1) \geq 1$. So we have either

$$[a_1s_1b_1]_{C_1} - [a_2s_2b_2]_{C_2} = ([a_1']_{A_1'}\delta_1'([xs_1]_1 - [xas_2b]_1))\delta_2[b_1]_{B_1}$$

or

$$[a_1s_1b_1]_{C_1} - [a_2s_2b_2]_{C_2} = ([a_1]_{\ell(a_1)}\delta_1([s_1y]_{\ell(\widetilde{s_1}y)} - [as_2by]_{\ell(\widetilde{s_1}y)}))\delta_2'[b_1']_{B_1'},$$

where we have $a_1 = a_1'x$ and $b_1 = yb_1'$ for some words $a_1', b_1' \in X^*$ and $x, y \in X$. Then by the fact that S is a Gröbner-Shirshov basis and by Lemmas 3.6 and 3.11, we deduce

$$[a_1s_1b_1]_{C_1} - [a_2s_2b_2]_{C_2} \equiv 0 \; mod \; (S, [w]_W).$$

Case 3. Without loss of generality, we assume $a_2 = a_1a$, $b_1 = bb_2$ and $w' = \widetilde{s_1}b = a\widetilde{s_2}$. If $P([s_1b]) \cap P([as_2]) \neq \emptyset$, then since S is a Gröbner-Shirshov basis, we may assume

$$[s_1b]_C - [as_2]_C = \sum\alpha_i'[c_i's_i'd_i']_{L_i'}$$

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satisfying $\overline{[c_i' s_i' d_i']_{L_i'}} \leq \overline{[s_1 b]_C - [a s_2]_C} < [w']_C$ for every i. So we have

$$\begin{aligned}[a_1 s_1 b_1]_{C_1} - [a_2 s_2 b_2]_{C_2} &= ([a_1]_{A_1} \delta_1([s_1 b]_C - [a s_2]_C))\delta_2 [b_2]_{B_2} \\ &= \sum \alpha_i' ([a_1]_{A_1} \delta_1 [c_i' s_i' d_i']_{L_i'})\delta_2 [b_2]_{B_2}.\end{aligned}$$

If one of s_1 and s_2 is not strong, then we deduce $\delta_1 \in \{\vdash, \perp\}$ and $\delta_2 \in \{\dashv, \perp\}$; and if s_1 and s_2 are strong, then we obtain $c_i' \widetilde{s_i'} d_i' < \widetilde{w'}$ for all i. In either of these subcases, by Corollary 3.12, we obtain

$$[a_1 s_1 b_1]_{C_1} - [a_2 s_2 b_2]_{C_2} = \sum \alpha_i' ([a_1]_{A_1} \delta_1 [c_i' s_i' d_i']_{L_i'})\delta_2 [b_2]_{B_2} \equiv 0 \; mod \; (S, [w]_W).$$

If $P([s_1 b]) \cap P([a s_2]) = \emptyset$, then we deduce that s_1, s_2 are strong and $\ell(a_1) + \ell(b_1) \geq 1$. So we have either

$$[a_1 s_1 b_1]_{C_1} - [a_2 s_2 b_2]_{C_2} = ([a_1']_{A_1'} \delta_1' ([x s_1 b]_1 - [x a s_2]_1))\delta_2 [b_2]_{B_2}$$

or

$$([a_1]_{\ell(a_1)} \delta_1 ([s_1 b y]_{\ell(w' y)} - [a s_2 y]_{\ell(w' y)}))\delta_2' [b_2']_{B_2'},$$

where we have $a_1 = a_1' x$, $b_2 = y b_2'$ for some $a_1', b_2' \in X^*$ and $x, y \in X$. Then by the fact that S is a Gröbner-Shirshov basis and by Lemmas 3.6 and 3.11, we deduce

$$[a_1 s_1 b_1]_{C_1} - [a_2 s_2 b_2]_{C_2} \equiv 0 \; mod \; (S, [w]_W).$$

The proof is completed. $\square$

With the previous results, the proof of the CD lemma is now a routine job, almost the same as in the case of associative algebras or dialgebras. What is really different is what we have done so far. For the convenience of the reader, we quickly repeat the argument.

Theorem 3.15. *(Composition-Diamond lemma for trialgebras) Let S be a monic subset of $Tri\langle X\rangle$, $>$ a monomial-center ordering on $[X^+]_{\mathcal{P}(\mathbb{N})}$ and $Id(S)$ the ideal of $Tri\langle X\rangle$ generated by S. Then the following statements are equivalent.*

(i) *S is a Gröbner-Shirshov basis in $Tri\langle X\rangle$.*

(ii) *$0 \neq f \in Id(S) \Rightarrow \overline{f} = \overline{[asb]_C}$ for some normal S-polynomial $[asb]_C$.*

(iii) *$Irr(S) = \{[u]_U \in [X^+]_{\mathcal{P}(\mathbb{N})} \mid [u]_U \neq \overline{[asb]_C}$ for any normal S-polynomial $[asb]_C\}$ is a $\mathbf{k}$-basis of the quotient trialgebra $Tri\langle X|S\rangle = Tri\langle X\rangle / Id(S)$.*

Proof. (i) $\Rightarrow$ (ii) Let $0 \neq f \in Id(S)$. Then by Corollary 3.13, we may assume $f = \sum_{i=1}^{n} \alpha_i [a_i s_i b_i]_{C_i}$, where each $\alpha_i \in \mathbf{k}$, $a_i, b_i \in X^*$, $s_i \in S$. Define $[w_i]_{C_i} = \overline{[a_i s_i b_i]_{C_i}} = [a_i \widetilde{s_i} b_i]_{C_i}$, $1 \leq i \leq n$. Then we may assume without loss of generality that

$$[w_1]_{C_1} = [w_2]_{C_2} = ... = [w_l]_{C_l} > [w_{l+1}]_{C_{l+1}} \geq [w_{l+2}]_{C_{l+2}} \geq ....$$

Now we use induction on $[w_1]_{C_1}$ to show $\overline{f} = \overline{[asb]_C}$ for some normal S-polynomial $[asb]_C$. For $[w_1]_{C_1} = \overline{f}$, there is nothing to prove. For $[w_1]_{C_1} > \overline{f}$, we have $\sum_{i=1}^{l} \alpha_i = 0$ and

$$\begin{aligned} f &= \sum_{i=1}^{l} \alpha_i [a_i s_i b_i]_{C_i} + \sum_{i=l+1}^{n} \alpha_i [a_i s_i b_i]_{C_i} \\ &= (\sum_{i=1}^{l} \alpha_i)[a_1 s_1 b_1]_{C_1} - \sum_{i=2}^{l} \alpha_i ([a_1 s_1 b_1]_{C_1} - [a_i s_i b_i]_{C_i}) + \sum_{i=l+1}^{n} \alpha_i [a_i s_i b_i]_{C_i} \\ &= 0 + \sum \beta_j [c_j s_j' d_j]_{L_j} + \sum_{i=l+1}^{n} \alpha_i [a_i s_i b_i]_{C_i}, \end{aligned}$$

where each $[c_j s'_j d_j]_{L_j}$ is a normal S-polynomial and $\overline{[c_j s'_j d_j]_{L_j}} < [w_1]_{C_1}$ by Lemma 3.14. Thus the result follows by induction hypothesis.

(ii) $\Rightarrow$ (iii) By Lemma 3.7, the set $Irr(S)$ generates $Tri\langle X|S\rangle$ as a linear space. Suppose that $h = \sum \alpha_i [u_i]_{U_i} = 0$ in $Tri\langle X|S\rangle$, where $\alpha_i \in \mathbf{k}$, $[u_i]_{U_i} \in Irr(S)$ for every i and $[u_1]_{U_1} > [u_2]_{U_2} > \cdots$. This implies that $h \in Id(S)$. Then all α_i must be equal to zero. Otherwise, $\overline{h} = [u_j]_{U_j}$ for some j, which is a contradiction.

(iii) $\Rightarrow$ (i) Suppose that h is a composition of elements of S. Clearly, $h \in Id(S)$. By Lemma 3.7, $h = \sum_i \alpha_i [u_i]_{U_i} + \sum_j \beta_j [a_j s_j b_j]_{C_j}$, where each $[u_i]_{U_i} \in Irr(S)$, $\alpha_i, \beta_j \in \mathbf{k}$, $a_j, b_j \in X^*$, $s_j \in S$, and $[u_i]_{U_i} \leq \overline{h}$, $\overline{[a_j s_j b_j]_{C_j}} \leq \overline{h}$. Then $\sum_i \alpha_i [u_i]_{U_i} \in Id(S)$. By (iii), we have $\alpha_i = 0$ for every i, and thus we obtain $h \equiv 0 \ mod\ (S)$. $\square$

Shirshov algorithm If a monic subset $S \subset Tri\langle X\rangle$ is not a Gröbner-Shirshov basis then one can add to S all nontrivial compositions. Continuing this process repeatedly, we finally obtain a Gröbner-Shirshov basis S^{comp} that contains S and generates the same ideal, that is, $Id(S^{comp}) = Id(S)$.

Similarly, we may introduce the Gröbner-Shirshov bases for trirings, which may be useful when one would like to construct an R-basis for some trisemigroup-trirings over an associative and commutative ring R with unit.

Definition 3.16. A *triring* is a quinary $(T, +, \dashv, \vdash, \perp)$ such that all of $(T, +, \vdash)$, $(T, +, \dashv)$ and $(T, +, \perp)$ are associative rings such that the identities in (1) hold in T.

Let $(T, \dashv, \vdash, \perp)$ be a trisemigroup, and E the free left R-module with R-basis T. Then $(E, +, \dashv, \vdash, \perp)$ is a triring with a natural way: for all $f = \sum_i r_i u_i$, $g = \sum_j r'_j v_j \in E$, $r_i, r'_j \in R$, $u_i, v_j \in T$,

$$f \vdash g := \sum_{i,j} r_i r'_j (u_i \vdash v_j), \quad f \dashv g := \sum_{i,j} r_i r'_j (u_i \dashv v_j), \quad f \perp g := \sum_{i,j} r_i r'_j (u_i \perp v_j).$$

Such a triring, denoted by $Tri_R(T)$, is called a *trisemigroup-triring* of T over R.

We denote by $Tri_R\langle X\rangle$ the trisemigroup-triring of $Trisgp\langle X\rangle$ over R which is also called the *free triring* over R generated by X, where $Trisgp\langle X\rangle$ is the free trisemigroup generated by X. In particular, $Tri_{\mathbf{k}}\langle X\rangle = Tri\langle X\rangle$ is the free trialgebra generated by X when $\mathbf{k}$ is a field.

An *ideal* I of $Tri_R\langle X\rangle$ is an R-submodule of $Tri_R\langle X\rangle$ such that $f \vdash g$, $g \vdash f$, $f \dashv g, g \dashv f$, $f \perp g$, $g \perp f \in I$ for every $f \in Tri_R\langle X\rangle$ and $g \in I$.

Similar to the proof of Theorem 3.15, we have the following Composition-Diamond lemma for trirings.

Theorem 3.17. *(Composition-Diamond lemma for trirings) Let R be an associative and commutative ring with unit. Let S be a monic subset of $Tri_R\langle X\rangle$, $>$ a monomial-centers ordering on $[X^+]_{\mathcal{P}(\mathbb{N})}$ and $Id(S)$ the ideal of $Tri_R\langle X\rangle$ generated by S. Then the following statements are equivalent.*

(i) S is a Gröbner-Shirshov basis in $Tri_R\langle X\rangle$.

(ii) $0 \neq f \in Id(S) \Rightarrow \overline{f} = \overline{[asb]_C}$ for some normal S-polynomial $[asb]_C$.

(iii) $Irr(S) = \{[u]_U \in [X^+]_{\mathcal{P}(\mathbb{N})} \mid [u]_U \neq \overline{[asb]_C}$ for any normal S-polynomial $[asb]_C$ $\}$ is an R-basis of the quotient triring $Tri_R\langle X|S\rangle := Tri_R\langle X\rangle/Id(S)$, i.e. $Tri_R\langle X|S\rangle$ is a free R-module with R-basis $Irr(S)$.

Remark 3.18. Shirshov algorithm does not work generally in $Tri_R\langle X\rangle$.

We now turn to the question on how to recognize whether two ideals of $Tri\langle X\rangle$ is the same or not. We begin with the notion of a minimal (resp. reduced) Gröbner-Shirshov basis.

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Definition 3.19. A Gröbner-Shirshov basis S in $Tri\langle X\rangle$ is called *minimal* if for every $s \in S$, we have $\bar{s} \in Irr(S\backslash\{s\})$. A Gröbner-Shirshov basis S in $Tri\langle X\rangle$ is called *reduced* if for every $s \in S$, we have $supp(s) \subseteq Irr(S\backslash\{s\})$, where

$$supp(s) := \{[u_1]_{U_1}, \ldots, [u_n]_{U_n}\}$$

for $s = \alpha_1[u_1]_{U_1} + \cdots + \alpha_n[u_n]_{U_n}$, $0 \neq \alpha_i \in \mathbf{k}$, $[u_i]_{U_i} \in [X^+]_{\mathcal{P}(\mathbb{N})}$.

Suppose that I is an ideal of $Tri\langle X\rangle$ and $I = Id(S)$. If S is a (reduced) Gröbner-Shirshov basis in $Tri\langle X\rangle$, then we call S is a (reduced) Gröbner-Shirshov basis for the ideal I or for the quotient dialgebra $Tri\langle X\rangle/I$.

For associative algebras, polynomial algebras and dialgebras, it is known that every ideal has a unique reduced Gröbner-Shirshov basis. Now we show that an analogous result holds for trialgebras.

Lemma 3.20. *Let I be an ideal of $Tri\langle X\rangle$ and S a Gröbner-Shirshov basis for I. For every $T \subseteq S$, if $Irr(T) = Irr(S)$, then T is also a Gröbner-Shirshov basis for I.*

Proof. For every $f \in I$, since $Irr(T) = Irr(S)$ and S a Gröbner-Shirshov basis for I, by Theorem 3.15, we obtain $\bar{f} = \overline{[asb]_C} = \overline{[cgd]_C}$ for some $s \in S$, $g \in T$, $a, b, c, d \in X^*$. So we obtain $f_1 = f - lc(f)[cgd]_C \in I$ and $\bar{f_1} < \bar{f}$. By induction on $\bar{f}$, we deduce that f is a linear combination of normal T-polynomials, i.e. $f \in Id(T)$. This shows that $I = Id(T)$. Now the result follows from Theorem 3.15. □

Let S be a subset of $Tri\langle X\rangle$ and $[w]_W \in [X^+]_{\mathcal{P}(\mathbb{N})}$. We set

$$\bar{S} := \{\bar{s} \in [X^+]_{\mathcal{P}(\mathbb{N})} \mid s \in S\},\ \ S^{[w]_W} := \{s \in S \mid \bar{s} = [w]_W\},\ \ S^{<[w]_W} := \{s \in S \mid \bar{s} < [w]_W\}.$$

Theorem 3.21. *Let I be an ideal of $Tri\langle X\rangle$ and $>$ a monomial-centers ordering on $[X^+]_{\mathcal{P}(\mathbb{N})}$. Then there is a unique reduced Gröbner-Shirshov basis for I.*

Proof. We first prove the existence of a reduced Gröbner-Shirshov basis for I. Define $S = \{lc(f)^{-1}f \mid 0 \neq f \in I\}$. Then clearly S is a Gröbner-Shirshov basis for I. For each $[w]_W \in \bar{S}$, we fix a polynomial $f^{[w]_W}$ in S such that $\overline{f^{[w]_W}} = [w]_W$. Define

$$S_0 = \{f^{[w]_W} \in S \mid [w]_W \in \bar{S}\}.$$

Then the leading monomials of elements in S_0 are pairwise different. Since $I \supseteq S \supseteq S_0$ and $\bar{I} = \bar{S} = \bar{S_0}$, we have $Irr(S_0) = Irr(S) = [X^+]_{\mathcal{P}(\mathbb{N})}\backslash\bar{S}$. By Lemma 3.20, S_0 is a Gröbner-Shirshov basis for I.

Moreover, we may assume that for every $s \in S_0$, we have

$$supp(s - \bar{s}) \subseteq Irr(S_0), \tag{5}$$

i.e. $supp(s - \bar{s}) \subseteq [X^+]_{\mathcal{P}(\mathbb{N})}\backslash\bar{S_0}$. If $supp(s - \bar{s}) \cap \bar{S_0} \neq \emptyset$ for some $s \in S_0$, then set $[u]_U = \max\{supp(s - \bar{s}) \cap \bar{S_0}\}$. Then there exists an element $f \in S_0$ such that $\bar{f} = [u]_U$. Note that $\bar{s} > [u]_U = \bar{f}$ and $\overline{s - \alpha f} = \bar{s}$, where α is the coefficient of $[u]_U$ in s. Replace s by $s - \alpha f$ in S_0. Then $supp(s - \alpha f - \overline{s - \alpha f}) \cap \bar{S_0} = \emptyset$ or $\max\{supp(s - \alpha f - \overline{s - \alpha f}) \cap \bar{S_0}\} < [u]_U$. Since $>$ is a well ordering on $[X^+]_{\mathcal{P}(\mathbb{N})}$, this process will terminate.

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Noting that for every $[w]_W \in \overline{S_0}$, there exists a unique $f \in S_0$ such that $[w]_W = \overline{f}$. Set $\min\{\overline{S_0}\} = \overline{s_0}$ with $s_0 \in S_0$. Define $S_{\overline{s_0}} := \{s_0\}$. Suppose that $f \in S_0$, $\overline{s_0} < \overline{f}$ and $S_{\overline{g}}$ has been defined for every $g \in S_0$ with $\overline{g} < \overline{f}$. Define

$$S_{\overline{f}} := \begin{cases} S_{<\overline{f}} & \text{if } \overline{f} \notin Irr(S_{<\overline{f}}), \\ S_{<\overline{f}} \cup \{f\} & \text{if } \overline{f} \in Irr(S_{<\overline{f}}), \end{cases} \qquad \text{where } S_{<\overline{f}} := \bigcup_{\overline{g}<\overline{f},\, g\in S_0} S_{\overline{g}}.$$

Let

$$S_1 := \bigcup_{f\in S_0} S_{\overline{f}}.$$

Then for every $f \in S_0$, we have $f \in S_1 \Leftrightarrow \overline{f} \in Irr(S_{<\overline{f}}) \Leftrightarrow f \in S_{\overline{f}}$.

We first claim that $Irr(S_1) = Irr(S_0)$. Since $S_1 \subseteq S_0$, it suffices to show $Irr(S_1) \subseteq Irr(S_0)$. Assume that there exists a normal triword $[w]_W \in [X^+]_{\mathcal{P}(\mathbb{N})}$ such that $[w]_W \in Irr(S_1)$ and $[w]_W \notin Irr(S_0)$. Since $\overline{S_0} = \overline{I}$, it follows that $[w]_W = \overline{f}$ for some $f \in S_0 \backslash S_1$. If $\overline{f} \in Irr(S_{<\overline{f}})$ then $f \in S_{\overline{f}} \subseteq S_1$, a contradiction. If $\overline{f} \notin Irr(S_{<\overline{f}})$ then $\overline{f} = \overline{[asb]_C}$ for some $s \in S_{<\overline{f}} \subseteq S_1, a, b \in X^*$. This implies that $\overline{f} \notin Irr(S_1)$, a contradiction. Therefore, $Irr(S_1) = Irr(S_0)$. By Lemma 3.20, S_1 is a Gröbner-Shirshov basis for I.

If $f, g \in S_1$, $f \neq g$, $\overline{f} = \overline{[agb]_C}$, then we have $\overline{g} < \overline{f}$, $g \in S_{\overline{g}} \subseteq S_{<\overline{f}}$. So we deduce $\overline{f} \notin Irr(S_{<\overline{f}})$ and $f \notin S_1$, a contradiction. So S_1 is a minimal Gröbner-Shirshov basis for I. By (5), for every $s \in S_1$, we have $supp(s) \subseteq Irr(S_1 \backslash \{s\})$, so S_1 is a reduced Gröbner-Shirshov basis for I.

Now we prove the uniqueness. Suppose that T is an arbitrary reduced Gröbner-Shirshov basis for I. Let $\overline{s_0} = \min \overline{S_1}$ and $\overline{r_0} = \min \overline{T}$, where $s_0 \in S_1, r_0 \in T$. By Theorem 3.15, we have $\overline{s_0} = \overline{[a'r'b']_{C'}} \geq \overline{r'} \geq \overline{r_0}$ for some $r' \in T, a', b' \in X^*$. Similarly, $\overline{r_0} \geq \overline{s_0}$. So we deduce $\overline{r_0} = \overline{s_0}$. We claim that $r_0 = s_0$. Otherwise, we have $0 \neq r_0 - s_0 \in I$. By the above argument again, we obtain that $\overline{r_0} > \overline{r_0 - s_0} \geq \overline{r''} \geq \overline{r_0}$ for some $r'' \in T$, a contradiction. So we have

$$S_1^{\overline{s_0}} = \{s_0\} = \{r_0\} = T^{\overline{r_0}}.$$

For every $[w]_W \in \overline{S_1} \cup \overline{T}$ with $[w]_W > \overline{r_0}$, assume that $S_1^{<[w]_W} = T^{<[w]_W}$. To prove $T = S_1$, it suffices to show $S_1^{[w]_W} \subseteq T^{[w]_W}$. For every $s \in S_1^{[w]_W}$, we have $\overline{s} = \overline{[c'rd']_{L'}} \geq \overline{r}$ for some $r \in T, c', d' \in X^*$. Now we claim that $[w]_W = \overline{s} = \overline{r}$. Otherwise, we have $[w]_W = \overline{s} > \overline{r}$. Then $r \in T^{<[w]_W} = S_1^{<[w]_W}$ and $r \in S_1 \backslash \{s\}$. But $\overline{s} = \overline{[c'rd']_{L'}}$, which contradicts with the fact that S_1 is a reduced Gröbner-Shirshov basis. Now we show $s = r \in T^{[w]_W}$. If $s \neq r$, then $0 \neq s - r \in I$. By Theorem 3.15, $\overline{s-r} = \overline{[ar_1b]_{L_1}} = \overline{[cs_1d]_{L_1}}$ for some $r_1 \in T, s_1 \in S_1, a, b, c, d \in X^*$ with $\overline{r_1}, \overline{s_1} \leq \overline{s-r} < \overline{s} = \overline{r}$. So we deduce $s_1 \in S_1 \backslash \{s\}$ and $r_1 \in T \backslash \{r\}$. Noting that $\overline{s-r} \in supp(s) \cup supp(r)$, we may assume $\overline{s-r} \in supp(s)$. As S_1 is a reduced Gröbner-Shirshov basis, we have $\overline{s-r} \in Irr(S_1 \backslash \{s\})$, which contradicts with the fact that $\overline{s-r} = \overline{[cs_1d]_{L_1}}$, where $s_1 \in S_1 \backslash \{s\}$. Thus $s = r$. Therefore we obtain $S_1^{[w]_W} \subseteq T^{[w]_W}$. It follows that we have $S \subseteq T$. Similarly, we have $T \subseteq S$, which proves the uniqueness. □

Remark 3.22. For associative algebras and polynomial algebras, it is known that every Gröbner-Shirshov basis for an ideal can be reduced to a reduced Gröbner-Shirshov basis. But this is neither the case for dialgebras [8, Example 3.24], nor the case for trialgebras. It suffices to consider the trialgebra defined by the same generators and relations as those in [8, Example 3.24], because the relations form a Gröbner–Shirshov basis for the considered trialgebra.

By using Theorem 3.21, we have the following theorem.

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Theorem 3.23. *Let I_1, I_2 be two ideals of $Tri\langle X\rangle$. Then $I_1 = I_2$ if and only if I_1 and I_2 have the same reduced Gröbner-Shirshov basis.*

4. Applications

In this section, we apply Theorem 3.15 to give a method to find normal forms of elements of an arbitrary trisemigroup. As applications, we reconstruct normal forms of elements of a free commutative trisemigroup which is obtained in [32] and construct normal forms of elements of a free abelian trisemigroup. We also give some characterizations of the Gelfand-Kirillov dimensions of some trialgebras.

Denote by

$$Trisgp\langle X\rangle := ([X^+]_{\mathcal{P}(\mathbb{N})}, \dashv, \vdash, \perp)$$

the free trisemigroup generated by X [24,31]. Clearly, every trisemigroup T is a quotient of some free trisemigroup, say

$$T = Trisgp\langle X|S\rangle := [X^+]_{\mathcal{P}(\mathbb{N})}/\rho(S)$$

for some set X and $S \subseteq [X^+]_{\mathcal{P}(\mathbb{N})} \times [X^+]_{\mathcal{P}(\mathbb{N})}$, where $\rho(S)$ is the congruence on $([X^+]_{\mathcal{P}(\mathbb{N})}, \dashv, \vdash, \perp)$ generated by S. So it is natural to ask the problem: how to find normal forms of elements of an arbitrary quotient trisemigroup of the form $Trisgp\langle X|S\rangle$?

Let $>$ be a monomial-centers ordering on $[X^+]_{\mathcal{P}(\mathbb{N})}$ and

$$S = \{([u_i]_{U_i}, [v_i]_{V_i}) \mid [u_i]_{U_i} > [v_i]_{V_i},\ i \in I\}.$$

Consider the trialgebra $Tri\langle X|S\rangle$, where we identify the set S with the set $\{[u_i]_{U_i} - [v_i]_{V_i} | i \in I\}$. By Shirshov algorithm, we have a Gröbner-Shirshov basis S^{comp} in $Tri\langle X\rangle$ and $Id(S^{comp}) = Id(S)$. It is clear that each element in S^{comp} is of the form $[u]_U - [v]_V$, $[u]_U, [v]_V \in [X^+]_{\mathcal{P}(\mathbb{N})}$. Let

$$\begin{aligned} \sigma \quad : \quad & Tri\langle X|S\rangle \to Tri_{\mathbf{k}}([X^+]_{\mathcal{P}(\mathbb{N})}/\rho(S)), \\ & \sum \alpha_i [u_i]_{U_i} + Id(S) \mapsto \sum \alpha_i [u_i]_{U_i}\rho(S), \quad \alpha_i \in \mathbf{k},\ [u_i]_{U_i} \in [X^+]_{\mathcal{P}(\mathbb{N})}. \end{aligned}$$

Then σ is obviously a trialgebra isomorphism. Noting that by Theorem 3.15 $Irr(S^{comp})$ is a linear basis of $Tri\langle X|S\rangle$, we have $\sigma(Irr(S^{comp}))$ is a linear basis of $Tri_{\mathbf{k}}([X^+]_{\mathcal{P}(\mathbb{N})}/\rho(S))$. It follows that $Irr(S^{comp})$ is exactly a set of normal forms of elements of the trisemigroup $Trisgp\langle X|S\rangle$.

Therefore, we have the following theorem.

Theorem 4.1. *Let $>$ be a monomial-centers ordering on $[X^+]_{\mathcal{P}(\mathbb{N})}$ and $T = Trisgp\langle X|S\rangle$, where $S = \{([u_i]_{U_i}, [v_i]_{V_i}) \mid [u_i]_{U_i} > [v_i]_{V_i},\ i \in I\}$ is a subset of $[X^+]_{\mathcal{P}(\mathbb{N})} \times [X^+]_{\mathcal{P}(\mathbb{N})}$. Then $Irr(S^{comp})$ is a set of normal forms of elements of the trisemigroup $Trisgp\langle X|S\rangle$.*

If we can construct a set of normal forms of certain trialgebra, then we can know how fast the trialgebra grows by the tool of Gelfand-Kirillov dimension. The Gelfand-Kirillov dimension measures the asymptotic growth rate of algebras. Since it provides important structural information, this invariant has become one of the important tools in the study of algebras. In this section, we shall calculate some interesting examples and show how we can apply Gröbner-Shirshov bases in the calculation of Gelfand-Kirillov dimensions of certain trialgebras.

Before introducing a general definition of the Gelfand-Kirillov dimension of a trialgebra T over $\mathbf{k}$, we have to introduce some notations. Let $\mathcal{V}, \mathcal{V}_1$ and $\mathcal{V}_2$ be vector subspaces of T. We first define

$$\mathcal{V}_1 \dashv \mathcal{V}_2 = \mathsf{Span}_{\mathbf{k}}\{a \dashv b \mid a \in \mathcal{V}_1, b \in \mathcal{V}_2\},\ \mathcal{V}_1 \vdash \mathcal{V}_2 = \mathsf{Span}_{\mathbf{k}}\{a \vdash b \mid a \in \mathcal{V}_1, b \in \mathcal{V}_2\},$$

$$\text{and } \mathcal{V}_1 \perp \mathcal{V}_2 = \mathsf{Span}_{\mathsf{k}}\{a \perp b \mid a \in \mathcal{V}_1, b \in \mathcal{V}_2\}.$$

Then we define $\mathcal{V}^1 = \mathcal{V}$ and $\mathcal{V}^n = \sum_{1\leq i\leq n-1}(\mathcal{V}^i \dashv \mathcal{V}^{n-i} + \mathcal{V}^i \vdash \mathcal{V}^{n-i} + \mathcal{V}^i \perp \mathcal{V}^{n-i})$ for every integer number $n \geq 2$. Finally, we define

$$\mathcal{V}^{\leq n} := \mathcal{V}^1 + \mathcal{V}^2 + ... + \mathcal{V}^n.$$

Obviously, we have

$$\mathcal{V}^n = \mathsf{Span}_{\mathbf{k}}\{[a_1...a_n]_U \mid \emptyset \neq U \subseteq \{1, ..., n\}, a_1, ..., a_n \in \mathcal{V}\}$$

and

$$\mathcal{V}^{\leq n} = \mathsf{Span}_{\mathbf{k}}\{[a_1...a_m]_U \mid \emptyset \neq U \subseteq \{1, ..., m\}, m \in \mathbb{N}, m \leq n, a_1, ..., a_m \in \mathcal{V}\}.$$

Now we are ready to introduce the Gelfand-Kirillov dimension of a trialgebra.

Definition 4.2. Let T be a trialgebra over $\mathbf{k}$. Then the Gelfand-Kirillov dimension of a trialgebra T is defined to be

$$\mathsf{GKdim}(T) = \sup_{\mathcal{V}} \overline{\lim_{n\to\infty}} \log_n \mathsf{dim}(\mathcal{V}^{\leq n}),$$

where the supremum is taken over all finite dimensional subspaces $\mathcal{V}$ of T.

We have the following obvious observation, which is well-known in the context [22], for example.

Lemma 4.3. *Let T be a trialgebra generated by a finite set X and $\mathbf{k}X$ the subspace of T spanned by X. Then we have*

$$\mathsf{GKdim}(T) = \overline{\lim_{n\to\infty}} \log_n \mathsf{dim}((\mathbf{k}X)^{\leq n}).$$

Let $X = \{x\}$. It is well known that $\mathsf{GKdim}(\mathbf{k}\langle X\rangle) = 1$ and $\mathsf{GKdim}(Di\langle X\rangle) = 2$, where $\mathbf{k}\langle X\rangle$ (resp. $Di\langle X\rangle$) is the free associative algebra (resp. dialgebra) generated by X. Note that a normal triword of length n in $Tri\langle X\rangle$ is of the form $[x...x]_U$, where U is a nonempty subset of $\{1, ..., n\}$. So by a direct calculation, we have $\mathsf{GKdim}(Tri\langle X\rangle) = +\infty$.

We shall show in subsections 4.1 and 4.2 that the Gelfand-Kirillov dimensions of finitely generated free commutative trialgebras and those of finitely generated free abelian trialgebras are positive integers.

From now on, let $>$ be the deg-lex-centers ordering on $[X^+]_{\mathcal{P}(\mathbb{N})}$, where X is a well-ordered set.

4.1. Normal forms of free commutative trisemigroups

The commutative trisemigroups are introduced and the free commutative trisemigroup generated by a set is constructed by [32]. In this subsection, we give another approach to normal forms of elements of a free commutative trisemigroup.

Definition 4.4. [32] A trisemigroup (trialgebra) $(T, \dashv, \vdash, \perp)$ is *commutative* if $\dashv$, $\vdash$ and $\perp$ are commutative.

Let T_c be the subset of $Tri\langle X\rangle$ consisting of the following polynomials:

$$[u]_U \vdash [v]_V - [v]_V \vdash [u]_U, \quad [u]_U \dashv [v]_V - [v]_V \dashv [u]_U, \quad [u]_U \perp [v]_V - [v]_V \perp [u]_U, \tag{6}$$

where $[u]_U, [v]_V \in [X^+]_{\mathcal{P}(\mathbb{N})}$. Then

$$Tri[X] := Tri\langle X|T_c\rangle$$

is clearly the free commutative trialgebra generated by X. In particular, a linear basis of $Tri[X]$ consisting of normal triwords over X is exactly a set of normal forms of elements of the free commutative trisemigroup generated by X.

Let $X = \{x_i \mid i \in I\}$ be a well-ordered set. For every $u = x_{j_1}x_{j_2}...x_{j_n} \in X^+$, $x_{j_k} \in X$, we define

$$\lfloor u \rfloor = \lfloor x_{j_1}x_{j_2}...x_{j_n} \rfloor := x_{i_1}x_{i_2}...x_{i_n},$$

where $x_{i_1}, x_{i_2}, ..., x_{i_n}$ is a reordering of $x_{j_1}, x_{j_2}, ..., x_{j_n}$ satisfying $x_{i_1} \le x_{i_2} \le \cdots \le x_{i_n}$.

We define

$$\lfloor X^+ \rfloor := \{\lfloor u \rfloor \mid u \in X^+\}; \ \lfloor u \rfloor_U := [\lfloor u \rfloor]_U, \ \emptyset \ne U \subseteq \{1, ..., \ell(u)\};$$

$$\lfloor X^+ \rfloor_{\mathcal{P}(\mathbb{N})} := \{\lfloor u \rfloor_U \mid u \in X^+, \ \emptyset \ne U \subseteq \{1, ..., \ell(u)\}\}.$$

For $u \in X^+$, $[u]_U$ is a normal triword, while $\lfloor u \rfloor_U$ is called a *commutative normal triword*. For instance, assume $u = x_2x_1x_2x_1x_2x_1 \in X^+$ and assume $x_1 < x_2$, where $x_1, x_2 \in X$. Then we have $\lfloor u \rfloor = x_1x_1x_1x_2x_2x_2$, $\lfloor u \rfloor_{\{3,5\}} = [x_1x_1x_1x_2x_2x_2]_{\{3,5\}}$.

Proposition 4.5. *Let $X = \{x_i \mid i \in I\}$ be a well-ordered set. Then we have the followings:*

(i) *$Tri[X] = Tri\langle X|S_c\rangle$, where S_c consists of the following polynomials:*

$$[u]_U - \lfloor u \rfloor_U \quad ([u]_U \in [X^+]_{\mathcal{P}(\mathbb{N})}, \ \ell(u) = 2 \text{ or } |U| = \ell(u) \ge 3),$$

$$[v]_V - \lfloor v \rfloor_1 \quad ([v]_V \in [X^+]_{\mathcal{P}(\mathbb{N})}, \ \ell(v) \ge 3 \text{ and } |V| < \ell(v)).$$

(ii) *S_c is a Gröbner-Shirshov basis in $Tri\langle X\rangle$.*

(iii) *The set*

$$\begin{aligned}\lfloor X^+ \rfloor_c := \{\lfloor v \rfloor_1 \mid \lfloor v \rfloor \in \lfloor X^+ \rfloor\} \quad &\cup \quad \{\lfloor u \rfloor_2 \mid \lfloor u \rfloor \in \lfloor X^+ \rfloor, \ell(u) = 2\} \\ &\cup \quad \{\lfloor u \rfloor_{\{1,2,...,\ell(u)\}} \mid \lfloor u \rfloor \in \lfloor X^+ \rfloor\}.\end{aligned}$$

forms a **k***-basis of the free commutative trialgebra $Tri[X]$.*

Proof. (i) It suffices to show $S_c \subseteq Id(T_c)$ and $T_c \subseteq Id(S_c)$, where T_c consists of the elements described in (6). We first show $S_c \subseteq Id(T_c)$. Since $\dashv$, $\vdash$ and $\perp$ are commutative, we have

$$[x_ix_j]_2 - \lfloor x_ix_j \rfloor_2 \in Id(T_c), \ [u]_{\{1,...,\ell(u)\}} - \lfloor u \rfloor_{\{1,...,\ell(u)\}} \in Id(T_c) \text{ and } [v]_1 - \lfloor v \rfloor_1 \in Id(T_c),$$

where $x_i, x_j \in X$, $u, v \in X^+$, $|u|, |v| \ge 2$. It remains to prove that

$$[v]_V - \lfloor v \rfloor_1 \in Id(T_c), \text{ where } [v]_V \in [X^+]_{\mathcal{P}(\mathbb{N})}, \ \ell(v) \ge 3, \ |V| < \ell(v) \text{ and } V \ne \{1\}.$$

There are two cases to consider:

Case 1. If $1 \notin V$, we assume $[v]_V = [v_0]_{\ell(v_0)} \vdash [v_1]_{V_1}$ for some $v_0, v_1 \in X^+$. Then in $Tri\langle X|T_c\rangle$, we have

$$\begin{aligned}[v]_V - \lfloor v \rfloor_{\ell(v)} &= [v_1]_{V_1} \vdash [v_0]_{\ell(v_0)} - \lfloor v \rfloor_{\ell(v)} \\ &= [v_1v_0]_{\ell(v)} - \lfloor v \rfloor_{\ell(v)} \\ &= 0.\end{aligned}$$

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Assume $\lfloor v \rfloor_{\ell(v)} = (\lfloor v' \rfloor_1 \vdash x) \vdash y$ with $x, y \in X$ and $\lfloor v' \rfloor \in \lfloor X^+ \rfloor$. Then in $Tri\langle X|T_c\rangle$, we obtain

$$\begin{aligned}\lfloor v \rfloor_{\ell(v)} - \lfloor v \rfloor_1 &= ([v']_{(|v'|)} \vdash x) \vdash y - \lfloor v \rfloor_1 \\ &= (\lfloor v' \rfloor_1 \dashv x) \vdash y - \lfloor v \rfloor_1 \\ &= y \vdash (\lfloor v' \rfloor_1 \dashv x) - \lfloor v \rfloor_1 \\ &= x \dashv (y \vdash \lfloor v' \rfloor_1) - \lfloor v \rfloor_1 \\ &= 0.\end{aligned}$$

It follows that $[v]_V - \lfloor v \rfloor_1 \in Id(T_c)$.

Case 2. If $1 \in V$, then by Lemma 2.4, we may assume $[v]_V = ([v_0']_{V_0'} \perp [za]_1)\delta[v_1']_{V_1'}$ with $z \in X$, $a \in X^+$, $v_0', v_1' \in X^*$ and $\delta \in \{\dashv, \perp\}$. Then in $Tri\langle X|T_c\rangle$, we have

$$\begin{aligned}[v]_V - \lfloor v \rfloor_1 &= [v_0']_{V_0'} \perp ([za]_1 \delta [v_1']_{V_1'}) - \lfloor v \rfloor_1 \\ &= [v_0']_{V_0'} \perp ([v_1']_{V_1'} \delta [za]_1) - \lfloor v \rfloor_1 \\ &= ([v_0']_{V_0'} \perp [v_1']_{V_1'})\delta(z \dashv [a]_1) - \lfloor v \rfloor_1 \\ &= (([v_0']_{V_0'} \perp [v_1']_{V_1'})\delta z) \dashv [a]_1 - \lfloor v \rfloor_1 \\ &= [a v_0' v_1' z]_1 - \lfloor v \rfloor_1 \\ &= 0.\end{aligned}$$

It follows that $[v]_V - \lfloor v \rfloor_1 \in Id(T_c)$.

Now we show $T_c \subseteq Id(S_c)$. Clearly we have

$$x \vdash y - y \vdash x \in Id(S_c),\ x \dashv y - y \dashv x \in Id(S_c),\ x \perp y - y \perp x \in Id(S_c),$$

where $x, y \in X$. Suppose that $[u]_U, [v]_V \in [X^+]_{\mathcal{P}(\mathbb{N})}$ with $\ell(uv) > 2$. Then in $Tri\langle X|S_c\rangle$, we have

$$\begin{aligned}&[u]_U \vdash [v]_V - [v]_V \vdash [u]_U = [uv]_{\ell(u)+V} - [vu]_{\ell(v)+U} = \lfloor uv \rfloor_1 - \lfloor vu \rfloor_1 = 0, \\ &[u]_U \dashv [v]_V - [v]_V \dashv [u]_U = [uv]_U - [vu]_V = \lfloor uv \rfloor_1 - \lfloor vu \rfloor_1 = 0,\end{aligned}$$

and $[u]_U \perp [v]_V - [v]_V \perp [u]_U = [uv]_{U\cup(\ell(u)+V)} - [vu]_{V\cup(\ell(v)+U)}$

$$= \begin{cases} \lfloor uv \rfloor_{\{1,2,\dots,|uv|\}} - \lfloor vu \rfloor_{\{1,2,\dots,|uv|\}} = 0, & \text{if } |U| = \ell(u) \text{ and } |V| = \ell(v), \\ \lfloor uv \rfloor_1 - \lfloor vu \rfloor_1 = 0, & \text{otherwise.} \end{cases}$$

This shows that $Id(T_c) = Id(S_c)$ and (i) holds.

(ii) It is easy to check that all possible left (right) multiplication compositions in S_c are equal to zero. For an arbitrary composition $(f,g)_{[w]_W}$ in S_c, we have $-r_f, -r_g \in [X^+]_{\mathcal{P}(\mathbb{N})}$, $\ell(w) \geq 3$, $[w]_W = \overline{[afb]_W} = \overline{[cgd]_W}$ and $\lfloor w \rfloor = \lfloor a\widetilde{r_f}b \rfloor = \lfloor c\widetilde{r_g}d \rfloor$, where $f = \overline{f} + r_f, g = \overline{g} + r_g$, $a, b, c, d \in X^*$. Assume that $[afb]_W = [a\widetilde{f}b]_W - [a\widetilde{r_f}b]_{W_1}$ and $[cgd]_W = [c\widetilde{g}d]_W - [c\widetilde{r_g}d]_{W_2}$. Then we deduce $|W| = \ell(w)$ if and only if $|W_1| = \ell(w) = |W_2|$. It follows that

$$(f,g)_{[w]_W} \quad = \quad [afb]_W - [cgd]_W = -[a\widetilde{r_f}b]_{W_1} + [c\widetilde{r_g}d]_{W_2}$$

$$\equiv \begin{cases} -\lfloor a\widetilde{r_f}b \rfloor_1 + \lfloor c\widetilde{r_g}d \rfloor_1 \equiv 0 \mod(S_c) & \text{if } |W| < \ell(w), \\ -\lfloor a\widetilde{r_f}b \rfloor_{\{1,\dots,\ell(w)\}} + \lfloor c\widetilde{r_g}d \rfloor_{\{1,\dots,\ell(w)\}} \equiv 0 \mod(S_c) & \text{if } |W| = \ell(w). \end{cases}$$

Then all the compositions in S_c are trivial. So S_c is a Gröbner-Shirshov basis in $Tri\langle X\rangle$.

(iii) The claim follows immediately from Theorem 3.15. □

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From Theorem 3.15, Lemma 3.20 and Proposition 4.5, it follows that

Corollary 4.6. *Let $X = \{x_i \mid i \in I\}$ be a well-ordered set and $S'_c \subset Tri\langle X\rangle$ be a set consisting of the following polynomials:*

$$[x_ix_j]_2 - [x_jx_i]_2,\ [x_ix_j]_1 - [x_jx_i]_1,\ [x_ix_j]_{\{1,2\}} - [x_jx_i]_{\{1,2\}}\ (i, j \in I,\ i > j),$$
$$[x_ix_jx_k]_2 - [x_ix_jx_k]_1,\ [x_ix_jx_k]_3 - [x_ix_jx_k]_1,\ [x_ix_jx_k]_{\{1,2\}} - [x_ix_jx_k]_1,$$
$$[x_ix_jx_k]_{\{1,3\}} - [x_ix_jx_k]_1,\ [x_ix_jx_k]_{\{2,3\}} - [x_ix_jx_k]_1\ (i, j, k \in I,\ i \le j \le k).$$

Then S'_c is the reduced Gröbner-Shirshov basis for the free commutative trialgebra $Tri[X]$.

Now by using Theorem 4.1 and Proposition 4.5, we have the following corollary.

Corollary 4.7. [32] *Let $Trisgp[X] := (\lfloor X^+\rfloor_c, \dashv, \vdash, \perp)$, where $\lfloor X^+\rfloor_c$ is defined as in Proposition 4.5. Then $Trisgp[X]$ is the free commutative trisemigroup generated by X, where the operations $\vdash, \dashv$ and $\perp$ are as follows: for any $x, x' \in X$, $\lfloor u\rfloor_U, \lfloor v\rfloor_V \in \lfloor X^+\rfloor_c$ with $\ell(u)\ell(v) > 1$,*

$$\lfloor v\rfloor_V \vdash \lfloor u\rfloor_U = \lfloor u\rfloor_U \vdash \lfloor v\rfloor_V = \lfloor u\rfloor_U \dashv \lfloor v\rfloor_V = \lfloor v\rfloor_V \dashv \lfloor u\rfloor_U = \lfloor uv\rfloor_1;$$
$$\lfloor v\rfloor_V \perp \lfloor u\rfloor_U = \lfloor u\rfloor_U \perp \lfloor v\rfloor_V = \lfloor uv\rfloor_{\{1,2,\dots,|uv|\}}\quad \text{if } |U| = \ell(u) \text{ and } |V| = \ell(v);$$
$$\lfloor v\rfloor_V \perp \lfloor u\rfloor_U = \lfloor u\rfloor_U \perp \lfloor v\rfloor_V = \lfloor uv\rfloor_1\quad \text{if } |U| < \ell(u) \text{ or } |V| < \ell(v);$$
$$x \dashv x' = x' \dashv x = \lfloor xx'\rfloor_1,\ x \vdash x' = x' \vdash x = \lfloor xx'\rfloor_2,\ x \perp x' = x' \perp x = \lfloor xx'\rfloor_{\{1,2\}}.$$

By Lemma 4.3 and Proposition 4.5, we can easily obtain the Gelfand-Kirillov dimension of $Tri[X]$ for every finite set X.

Corollary 4.8. *Let $X = \{x_1, ..., x_r\}$ and $Tri[X]$ be the free commutative trialgebra generated by X. Then we have $\mathsf{GKdim}(Tri[X]) = r$.*

4.2. Normal forms of free abelian trisemigroups

In this subsection, we first introduce a notion of abelian trisemigroups which is an analogy of abelian disemigroups introduced in [33]. Then we construct a set of normal forms of elements of the free abelian trisemigroups.

Definition 4.9. A trisemigroup (trialgebra) $(T, \dashv, \vdash, \perp)$ is *abelian* if $a \vdash b = b \dashv a$ and $a \perp b = b \perp a$ for all $a, b \in T$.

Let X be an arbitrary set and T'_{ab} the subset of $[X^+]_{\mathcal{P}(\mathbb{N})} \times [X^+]_{\mathcal{P}(\mathbb{N})}$ consisting of the following:

$$([u]_U \vdash [v]_V, [v]_V \dashv [u]_U),\ \ ([u]_U \perp [v]_V, [v]_V \perp [u]_U),$$

where $[u]_U, [v]_V \in [X^+]_{\mathcal{P}(\mathbb{N})}$. Let T_{ab} be the set consisting of elements of the form

$$[u]_U \vdash [v]_V - [v]_V \dashv [u]_U,\ \ [u]_U \perp [v]_V - [v]_V \perp [u]_U, \tag{7}$$

where $[u]_U, [v]_V \in [X^+]_{\mathcal{P}(\mathbb{N})}$. Then $Trisgp\langle X|T'_{ab}\rangle$ is clearly the free abelian trisemigroup generated by X, and $Tri\langle X|T_{ab}\rangle$ is the free abelian trialgebra generated by X. By Theorem 4.1, a linear basis of $Tri\langle X|T_{ab}\rangle$ consisting of normal triwords is a set of normal forms of elements of $Trisgp\langle X|T'_{ab}\rangle$.

Now we shall try to construct a linear basis of $Tri\langle X|T_{ab}\rangle$ by the method of Gröbner–Shirshov bases. We introduce a method of writing down a new normal triword from a given one. Let $X = \{x_i \mid i \in I\}$ be a well-ordered set, and let $\dot{X} = \{\dot{x} \mid x \in X\}$ be a copy of X, where by $\dot{x}$ we mean a new symbol. We extend the ordering on X to a well-ordering on $X \cup \dot{X}$ on the following way: (i) $\dot{x}_i < x_i$, (ii) $x_i < x_j$ implies $\dot{x}_i < \dot{x}_j$, $\dot{x}_i < x_j$, $x_i < \dot{x}_j$.

We note that $[X^+]_{\mathcal{P}(\mathbb{N})}$ has a one-to-one correspondence with $(X \cup \dot{X})^+$, and we denote this correspondence by φ. More precisely, φ maps an arbitrary normal triword $[x_{i_1}...x_{i_m}]_U$ to

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a word in $y_1...y_m$ in $(X \cup \dot{X})^+$, such that if $i_t \in U$, then $y_t = \dot{x}_{i_t}$, and if $i_t \notin U$, then $y_t = x_{i_t}$ for every $t \leq m$. For instance, $\varphi([x_1x_2x_2x_1x_3]_{\{2,4\}}) = x_1\dot{x}_2x_2\dot{x}_1x_3$. So we can identity elements in $[X^+]_{\mathcal{P}(\mathbb{N})}$ with those in $(X \cup \dot{X})^+$.

Recall that for every $y_1y_2...y_t \in (X \cup \dot{X})^+$, where each y_i lies in $X \cup \dot{X}$, we have

$$\lfloor y_1y_2...y_t \rfloor = y_{i_1}y_{i_2}...y_{i_t},$$

where $y_{i_1}, y_{i_2}, ..., y_{i_t}$ is a reordering of $y_1, y_2, ..., y_t$ satisfying $y_{i_1} \leq y_{i_2} \leq ... \leq y_{i_t}$. Define

$$\pi : \quad (X \cup \dot{X})^+ \to (X \cup \dot{X})^+, \;\; y_{j_1}y_{j_2}...y_{j_t} \mapsto \lfloor y_{j_1}y_{j_2}...y_{j_t} \rfloor.$$

Finally, define

$$\tau := \varphi^{-1}\pi\varphi : \quad [X^+]_{\mathcal{P}(\mathbb{N})} \to [X^+]_{\mathcal{P}(\mathbb{N})}.$$

For instance,

$$\begin{aligned}\tau([x_1x_2x_2x_1x_3]_{\{2,4\}}) &= \varphi^{-1}\pi\varphi([x_1x_2x_2x_1x_3]_{\{2,4\}}) = \varphi^{-1}\pi(x_1\dot{x}_2x_2\dot{x}_1x_3)\\ &= \varphi^{-1}(\dot{x}_1x_1\dot{x}_2x_2x_3) = [x_1x_1x_2x_2x_3]_{\{1,3\}}.\end{aligned}$$

Roughly speaking, τ reorders the letters in $[u]_U$ such that the middle entries are preserved. Therefore, we immediately deduce that such a map τ satisfies some useful properties, the proof of which is quite easy and so is omitted.

Lemma 4.10. *For all $[u]_U$, $[v]_V \in [X^+]_{\mathcal{P}(\mathbb{N})}$, we have $\tau([uv]_{U\cup(\ell(u)+V)}) = \tau([vu]_{V\cup(\ell(v)+U)})$, $\tau([uv]_{\ell(u)+V}) = \tau([vu]_V)$ and $\tau(\tau([u]_U)) = \tau([u]_U)$.*

Proposition 4.11. *Let $X = \{x_i \mid i \in I\}$ be a well-ordered set, T_{ab} the subset of $Tri\langle X\rangle$ consist of the elements described in (7). Then we have*

(i) *$Tri\langle X|T_{ab}\rangle = Tri\langle X|S_{ab}\rangle$, where $S_{ab} = \{[u]_U - \tau([u]_U) \mid [u]_U \in [X^+]_{\mathcal{P}(\mathbb{N})}, \ell(u) \geq 2\}$;*

(ii) *S_{ab} is a Gröbner-Shirshov basis in $Tri\langle X\rangle$;*

(iii) *The set $\{\tau([u]_U) \mid [u]_U \in [X^+]_{\mathcal{P}(\mathbb{N})}\}$ is a **k**-basis of the free abelian trialgebra $Tri\langle X|T_{ab}\rangle$.*

Proof. (i) It suffices to show $T_{ab} \subseteq Id(S_{ab})$ and $S_{ab} \subseteq Id(T_{ab})$. We first show $T_{ab} \subseteq Id(S_{ab})$. In $Tri\langle X|S_{ab}\rangle$, for all $[u]_U$, $[v]_V \in [X^+]_{\mathcal{P}(\mathbb{N})}$, clearly we have

$$\begin{aligned}[u]_U \vdash [v]_V - [v]_V \dashv [u]_U &= [uv]_{\ell(u)+V} - [vu]_V\\ &= \tau([uv]_{\ell(u)+V}) - \tau([vu]_V),\\ &= 0,\\ [u]_U \perp [v]_V - [v]_V \perp [u]_U &= [uv]_{U\cup(\ell(u)+V)} - [vu]_{V\cup(\ell(v)+U)}\\ &= \tau([uv]_{U\cup(\ell(u)+V)}) - \tau([vu]_{V\cup(\ell(v)+U)})\\ &= 0.\end{aligned}$$

Now we show $S_{ab} \subseteq Id(T_{ab})$. Note that for an arbitrary normal triword, say $[u]_U = [x_{i_1}...x_{i_n}]_U$ for some letters $x_{i_1}, ..., x_{i_n} \in X$ such that $n \geq 2$, the normal triword $\tau([u]_U)$ contains the same letters (with repetitions) as those of $[u]_m$, moreover, the middle entries are preserved. So it suffices to show that, we can reorder x_{i_t} and $x_{i_{t+1}}$ with middle entries preserved. By Lemma 2.4, we may assume

$$[u]_U = ([v]_V\delta_1(x_{i_t}\delta_2x_{i_{t+1}}))\delta_3[w]_W,$$

where if $[v]_V = [\varepsilon]_\emptyset$, then $\delta_1 \in \{\vdash, \perp\}$, and if $[w]_W = [\varepsilon]_\emptyset$, then $\delta_3 \in \{\dashv, \perp\}$. Then by the relations in T_{ab}, we clearly can reorder x_{i_t} and $x_{i_{t+1}}$ with middle entries preserved. It follows that $S_{ab} \subseteq Id(T_{ab})$.

(ii) Clearly all possible left and right multiplication compositions in S_{ab} are equal to zero. Assume for every composition $(f,g)_{[w]_W}$ in S_{ab}, where $f = [u]_U - \tau([u]_U)$ and $g = [v]_V - \tau([v]_V)$. We may assume $[afb]_W = ([a]_A\delta_1 f)\delta_2[b]_B$, $[cgd]_W = ([c]_C\delta_3 g)\delta_4[d]_D$. Then we have

$$\overline{[afb]_W} = \overline{[cgd]_W} = ([a]_A\delta_1[u]_U)\delta_2[b]_B = ([c]_C\delta_3[v]_V)\delta_4[d]_D.$$

It follows that

$$\tau(([a]_A\delta_1\tau([u]_U))\delta_2[b]_B) = \tau(([c]_C\delta_3\tau([v]_V))\delta_4[d]_D).$$

So we obtain

$$\begin{aligned}(f,g)_{[w]_W} &= [afb]_W - [cgd]_W = ([a]_A\delta_1\tau([u]_U))\delta_2[b]_B - ([c]_C\delta_3\tau([v]_V))\delta_4[d]_D \\ &\equiv \tau(([a]_A\delta_1\tau([u]_U))\delta_2[b]_B) - \tau(([c]_C\delta_3\tau([v]_V))\delta_4[d]_D) \\ &\equiv 0 \mod(S_{ab}).\end{aligned}$$

So all the compositions in S_{ab} are trivial, and thus S_{ab} is a Gröbner-Shirshov basis in $Tri\langle X\rangle$.

(iii) This part follows from Theorem 3.15. □

From Theorem 3.15, Lemma 3.20 and Proposition 4.11, it follows that

Corollary 4.12. *Let $X = \{x_i \mid i \in I\}$ be a well-ordered set and $W_{ab} \subset Tri\langle X\rangle$ be a set consisting of the following polynomials:*

$$[x_ix_j]_2 - [x_jx_i]_1,\ [x_ix_j]_1 - [x_jx_i]_2,\ [x_ix_i]_2 - [x_ix_i]_1,\ [x_ix_j]_{\{1,2\}} - [x_jx_i]_{\{1,2\}},\quad (i,j \in I,\ i > j).$$

Then W_{ab} is the reduced Gröbner-Shirshov basis for the free abelian trialgebra $Tri\langle X|T_{ab}\rangle$.

From Lemma 4.3 and Proposition 4.11, it follows that

Corollary 4.13. *Let $X = \{x_1, ..., x_r\}$ and let $Tri\langle X|T_{ab}\rangle$ be the free abelian trialgebra generated by X. Then we have*

$$\mathsf{GKdim}(Tri\langle X|T_{ab}\rangle) = 2r.$$

Proof. Let $\dot{X} = \{\dot{x} \mid x \in X\}$ be a copy of X, and let $\mathbf{k}[X \cup \dot{X}]$ be the commutative ploynomial algebra generated by $X \cup \dot{X}$. It is obvious that $\mathbf{k}[X \cup \dot{X}]$ is isomorphism to $Tri\langle X|T_{ab}\rangle$ as a vector space. So we obtain

$$\mathsf{GKdim}(Tri\langle X|T_{ab}\rangle) = \mathsf{GKdim}(\mathbf{k}[X \cup \dot{X}]) = 2r.$$

The proof is completed. □

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