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Article

From Deterministic Counterspace to Stochastic Shadow: Deriving the Born Rule as a Projection Invariant in TCGS-SEQUENTION

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Abstract

The TCGS-SEQUENTION framework posits that observable 3D reality is a projection of a deterministic, non-linear 4D counterspace. A companion paper demonstrated that source-level nonlinearity is compatible with shadow-level no-signaling through the quotient map structure. This paper addresses the deeper challenge: why does a deterministic source generate probabilistic outcomes at the shadow level, and specifically, why the Born rule $P = |\psi|^2$? We show that the Born measure is the *unique* probability assignment on the shadow state space $\mathcal{S}$ that satisfies three geometric requirements: (i) fiber-independence (well-definedness on equivalence classes), (ii) foliation-invariance (independence of parameterization choices), and (iii) non-contextuality (Gleason's constraint). The projection geometry does not merely *permit* the Born rule—it *forces* it as the only consistent bridge between deterministic source dynamics and operational shadow statistics. Probability in TCGS is neither ontic randomness nor subjective ignorance, but a *projection invariant*: the unique measure that renders the quotient map mathematically coherent.

Keywords: TCGS-SEQUENTION; born rule; quantum probability; projection geometry; Gleason's theorem; foliation invariance; determinism; quotient maps; measurement problem

1. Introduction

1.1. The Challenge

The Timeless Counterspace & Shadow Gravity (TCGS) framework[1,2] proposes that:

1. The 4D counterspace $(\mathcal{C}, G, \Psi)$ is **deterministic**: the source field Ψ evolves (in a foliation-relative sense) according to a well-posed constitutive law.
2. The constitutive law is **non-linear**: $\nabla_{\mathcal{C}} \cdot [\mu(|\nabla_{\mathcal{C}} \Psi|) \nabla_{\mathcal{C}} \Psi] = \text{sources}$.
3. The 3D shadow Σ is a **projection**: observables are pullbacks $\psi = X^* \Psi$.

A companion paper[3] demonstrated that this structure is compatible with operational no-signaling: the quotient map $\pi : \mathcal{C} \rightarrow \mathcal{S}$ ensures that shadow dynamics is affine-linear on density matrices even when source dynamics is nonlinear on configurations.

However, a deeper question remains: **Why does the deterministic source produce probabilistic outcomes at the shadow level, and why specifically the Born rule $P = |\psi|^2$?**

This is not a minor technicality. The Born rule is the *only* axiom of quantum mechanics that introduces probability. If TCGS cannot derive it from projection geometry, the framework remains incomplete.

1.2. The Strategy

We will show that the Born measure is **uniquely forced** by three geometric requirements:

1. **Fiber-independence**: Probabilities must be well-defined on equivalence classes $s \in \mathcal{S}$, not on individual source configurations $c \in \mathcal{C}$.

2. **Foliation-invariance:** Probabilities must be independent of the parameterization (time-slicing) used to describe the projection.
3. **Non-contextuality:** Probabilities for compatible observables must be additive (Gleason's constraint).

The argument proceeds as follows:

- Section 2 reviews the projection framework and quotient map structure.
- Section 3 establishes fiber-independence as the source of effective indeterminism.
- Section 4 shows how foliation-invariance constrains the measure.
- Section 5 invokes Gleason's theorem to force the Born form.
- Section 6 addresses the measurement process as projection refinement.
- Section 7 discusses decoherence as a projection phenomenon.
- Section 8 compares with other derivations of the Born rule.
- Section 9 concludes.

1.3. Ontological Clarification: Probability as Cartographic Incompleteness

It is imperative to rigorously distinguish the *ontic determinism* of the Counterspace from the *effective stochasticity* of the Shadow. The derivation of the Born rule presented in this work does not imply that the fundamental 4D layer of reality is random, nor does it relegate the framework to the class of standard "Local Hidden Variable" theories ruled out by Bell's theorem (as the Source geometry is fundamentally non-local via Axiom A2).

Instead, TCGS-SEQUENTION identifies quantum probability as a manifestation of **Cartographic Incompleteness**. Since the projection map $X : \Sigma \rightarrow \mathcal{C}$ functions as a quotient map (a many-to-one surjection), multiple deterministic source configurations project to the exact same operationally indistinguishable shadow state s . The Born measure μ , therefore, is not an intrinsic law of chance, but the unique geometric measure of the *Fiber Volume* $\pi^{-1}(s)$ —it quantifies the information necessarily lost during dimensional reduction.

This establishes a precise homology across the framework's three domains, identifying all apparent indeterminacies as projection artifacts:

- **In Gravity:** Information loss manifests as *Dark Matter* (the artifact of projecting 4D extrinsic curvature onto a 3D metric).
- **In Biology:** Information loss manifests as *Darwinian Chance* (the artifact of projecting 4D invariant structure onto a 3D lineage).
- **In Quantum Mechanics:** Information loss manifests as the *Born Rule* (the artifact of projecting 4D deterministic content onto a 3D observation).

Thus, the probabilistic nature of the shadow is the rigorous consequence of the determinism of the source. To demand that the 3D shadow display the full determinism of the 4D source is to demand that a map contain the full resolution of the territory—a metamathematical impossibility akin to the Gödelian gap between truth and provability.

2. The Projection Framework

2.1. Source and Shadow Spaces

Let $\mathcal{C}$ denote the **counterspace**: the space of all source configurations. In the full TCGS framework, $\mathcal{C}$ carries a 4D geometry $(\mathcal{C}, G, \Psi)$; for the present purpose, we treat $\mathcal{C}$ as an abstract configuration space.

Let $\mathcal{S}$ denote the **shadow state space**: the space of operationally distinguishable states. In standard quantum mechanics, $\mathcal{S}$ is the space of density matrices on a Hilbert space $\mathcal{H}$:

$$\mathcal{S} = \{\rho \in \mathcal{L}(\mathcal{H}) : \rho \geq 0, \text{Tr}(\rho) = 1\}. \quad (1)$$

2.2. The Quotient Map

The foundational structure is the **quotient map**:

$$\pi : \mathcal{C} \rightarrow \mathcal{S}, \quad s = \pi(c). \quad (2)$$

This map is surjective: every shadow state has at least one source representative. The **fiber** over s is:

$$\pi^{-1}(s) := \{c \in \mathcal{C} : \pi(c) = s\}. \quad (3)$$

Two source configurations c_1, c_2 belong to the same fiber if and only if they are **operationally indistinguishable**: all shadow observables yield identical statistics.

2.3. The Interpretive Principle

Axiom 1 (Operational Equivalence). *Source configurations in the same fiber are physically equivalent at the shadow level. No shadow-accessible experiment can distinguish c_1 from c_2 if $\pi(c_1) = \pi(c_2)$.*

This axiom is not optional; it is the *definition* of the shadow state space. The shadow state s is the equivalence class $\pi^{-1}(s)$.

3. Fiber-Independence and Effective Indeterminism

3.1. The Source of Apparent Randomness

In TCGS, the source dynamics is deterministic:

$$F_t : \mathcal{C} \rightarrow \mathcal{C}, \quad c(t) = F_t(c_0). \quad (4)$$

Given initial conditions c_0 , the future configuration $c(t)$ is uniquely determined.

However, the shadow observer does not have access to c_0 ; they have access only to $s_0 = \pi(c_0)$. Different representatives $c_0, c'_0 \in \pi^{-1}(s_0)$ may evolve to configurations $c(t), c'(t)$ that project to *different* shadow states:

$$\pi(F_t(c_0)) \neq \pi(F_t(c'_0)) \quad \text{even though} \quad \pi(c_0) = \pi(c'_0). \quad (5)$$

This is the geometric origin of **effective indeterminism**: the shadow dynamics is one-to-many because the fiber structure is not preserved under source evolution.

3.2. The Measure Problem

If the shadow observer cannot determine which representative $c_0 \in \pi^{-1}(s_0)$ obtains, they must assign probabilities to the possible shadow outcomes. This requires a **measure** μ on the fiber:

$$\mu : \pi^{-1}(s) \rightarrow [0, 1], \quad \int_{\pi^{-1}(s)} d\mu(c) = 1. \quad (6)$$

The probability of transitioning to shadow state s' is then:

$$P(s'|s) = \mu\left(\{c \in \pi^{-1}(s) : \pi(F_t(c)) = s'\}\right). \quad (7)$$

The central question: What determines μ ?

3.3. Fiber-Independence as a Constraint

Whatever measure μ is chosen, it must satisfy a fundamental consistency requirement:

Definition 1 (Fiber-Independence). *A probability assignment P is **fiber-independent** if it depends only on the equivalence class $s = \pi(c)$, not on the choice of representative $c \in \pi^{-1}(s)$.*

This is not a physical assumption; it is a *logical* requirement. If P depended on which representative c obtained, then the shadow state s would not be a complete description—contradicting the definition of S as the space of operationally complete states.

Proposition 1. *Fiber-independence forces the measure μ to be **canonical**: there exists a unique (up to normalization) measure on each fiber that is invariant under the symmetries that define the fiber.*

Proof sketch. The fiber $\pi^{-1}(s)$ is the orbit of any representative c under the symmetry group $\mathcal{G}_s$ that leaves the shadow state invariant. By standard results on invariant measures, there is a unique (up to normalization) $\mathcal{G}_s$ -invariant measure on the orbit. $\square$

4. Foliation-Invariance

4.1. The Parameterization Freedom

In TCGS, time is not ontic; it is a foliation parameter. Different observers may use different parameterizations (foliations) $\mathcal{F}, \mathcal{F}'$ to describe the same physical content.

A probability assignment is **foliation-invariant** if it is unchanged under reparameterization:

$$P_{\mathcal{F}}(s'|s) = P_{\mathcal{F}'}(s'|s) \quad \text{for all admissible } \mathcal{F}, \mathcal{F}'. \quad (8)$$

4.2. Constraint on the Measure

Foliation-invariance imposes strong constraints on the allowed measures.

Theorem 1 (Foliation-Invariant Measures). *Let S be the space of density matrices on $\mathcal{H}$. The only probability measures on S that are:*

1. *Fiber-independent*
2. *Foliation-invariant*
3. *Continuous in the trace-norm topology*

are of the form:

$$P(\rho \rightarrow \rho') = \text{Tr}(E_{\rho'} \rho) \quad (9)$$

for some positive operator $E_{\rho'}$ with $\sum_{\rho'} E_{\rho'} = I$.

Proof sketch. Foliation-invariance requires that the transition probabilities depend only on the intrinsic geometric relationship between ρ and ρ' , not on any external parameter. Combined with continuity and fiber-independence, this forces the linear form $\text{Tr}(E\rho)$. The positivity of probabilities requires $E \geq 0$, and normalization requires $\sum E = I$. $\square$

This theorem establishes that shadow probabilities must be **linear functionals** of the density matrix. But it does not yet fix the specific form of $E_{\rho'}$. For that, we need Gleason's theorem.

5. Gleason's Theorem and the Born Rule

5.1. The Non-Contextuality Constraint

Consider a measurement with possible outcomes $\{a_i\}$ corresponding to orthogonal projectors $\{P_i\}$ with $\sum_i P_i = I$. The probabilities $p_i = P(a_i|\rho)$ must satisfy:

$$\sum_i p_i = 1, \quad p_i \geq 0. \quad (10)$$

A probability assignment is **non-contextual** if the probability p_i for outcome a_i depends only on P_i and ρ , not on which other projectors $\{P_j\}_{j \neq i}$ appear in the measurement.

In TCGS terms: the probability for a shadow outcome depends only on the geometric relationship between the source projection and the outcome, not on what other outcomes were “possible.”

5.2. Gleason's Theorem

Theorem 2 (Gleason, 1957[4]). *Let $\mathcal{H}$ be a Hilbert space of dimension ≥ 3 . Every non-contextual probability measure on the projection lattice of $\mathcal{H}$ has the form:*

$$P(P|\rho) = \text{Tr}(P\rho) \quad (11)$$

for some density matrix ρ .

5.3. Application to TCGS

In the TCGS framework, non-contextuality is not an assumption; it is a *consequence* of the projection geometry:

Proposition 2. *The pullback operation X^* is non-contextual: the value of $X^*\Psi$ on a shadow region depends only on the local geometry of X and Ψ , not on what happens in spacelike-separated regions.*

Proof sketch. The pullback $X^*\Psi$ is defined pointwise: at each point $p \in \Sigma$, the pulled-back field depends only on the value of Ψ at $X(p) \in \mathcal{C}$ and the differential dX_p . This is manifestly local and non-contextual. $\square$

Combining with Gleason's theorem:

Corollary 1 (Born Rule from Projection Geometry). *In TCGS, the probability for a shadow outcome corresponding to projector P is:*

$$\mathbb{P}_{\text{Born}}(P|s) = \text{Tr}(P\rho_s) \quad (12)$$

where ρ_s is the density matrix representing shadow state s .

For a pure state $s = |\psi\rangle$, this reduces to:

$$\mathbb{P}_{\text{Born}}(P|\psi) = \langle \psi | P | \psi \rangle = |\langle \phi | \psi \rangle|^2 \quad (13)$$

if $P = |\phi\rangle\langle\phi|$.

This is the **Born rule**, derived from projection geometry rather than postulated.

6. Measurement as Projection Refinement

6.1. The Measurement Problem Restated

The standard measurement problem asks: why does a superposition $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ yield a definite outcome, and why with probabilities $|\alpha|^2, |\beta|^2$?

In TCGS, this question is reframed:

A measurement is a **refinement of the projection**: the apparatus selects a finer equivalence relation on $\mathcal{C}$, splitting the original fiber into sub-fibers corresponding to distinct outcomes.

6.2. Formal Description

Before measurement, the shadow state is $s = \pi(c)$ for some (unknown) $c \in \pi^{-1}(s)$.

The measurement apparatus defines a **refined quotient map**:

$$\pi_M : \mathcal{C} \rightarrow \mathcal{S}_M \quad (14)$$

where $\mathcal{S}_M$ is a finer state space (more equivalence classes). The original fiber $\pi^{-1}(s)$ is partitioned:

$$\pi^{-1}(s) = \bigsqcup_i \pi_M^{-1}(s_i) \quad (15)$$

where $\{s_i\}$ are the possible outcomes.

6.3. Probabilities from Fiber Volumes

The probability of outcome s_i is the **canonical measure** of the corresponding sub-fiber:

$$P(s_i|s) = \frac{\mu(\pi_M^{-1}(s_i) \cap \pi^{-1}(s))}{\mu(\pi^{-1}(s))}. \quad (16)$$

By the results of Sections 3–5, this canonical measure is the Born measure:

$$P(s_i|s) = \text{Tr}(P_i \rho_s). \quad (17)$$

6.4. “Collapse” as Conditioning

What standard quantum mechanics calls “collapse” is, in TCGS, simply **Bayesian conditioning**:

Learning the outcome s_i updates the shadow state from s to s_i . This is not a physical process; it is an epistemic update reflecting the refinement of the projection.

The source configuration c does not change; what changes is the observer’s knowledge of which sub-fiber c belongs to.

7. Decoherence from Projection Geometry

7.1. The Pointer Basis Problem

Why do macroscopic systems appear in definite states (position, not superposition)? Standard decoherence theory[5] answers: interaction with the environment selects a preferred “pointer basis.”

In TCGS, this is reinterpreted geometrically:

The pointer basis is the **natural basis for the projection**: it is determined by the structure of the quotient map π , not by environmental interaction.

7.2. Projection-Induced Decoherence

Consider a system S coupled to an apparatus A . The combined shadow state space $\mathcal{S}_{SA}$ has a quotient map:

$$\pi_{SA} : \mathcal{C} \rightarrow \mathcal{S}_{SA}. \quad (18)$$

For macroscopic systems, the fiber structure is highly constrained: only certain combinations of S and A configurations belong to the same fiber.

Proposition 3. *For macroscopic systems, the fibers are approximately factorized:*

$$\pi^{-1}(s_{SA}) \approx \pi^{-1}(s_S) \times \pi^{-1}(s_A) \quad (19)$$

only for states s_S in the pointer basis.

This means: superpositions of macroscopically distinct states correspond to fibers that are *not* productizable, hence are operationally inaccessible at the shadow level. This is projection-induced decoherence.

7.3. No Ontic Collapse Required

In TCGS, there is no “collapse of the wave function” as a physical process. The source evolves deterministically; decoherence is the shadow-level manifestation of fiber structure. The appearance of definite outcomes is a projection artifact, not an ontic event.

8. Comparison with Other Derivations

Several approaches have attempted to derive the Born rule from more fundamental principles. We compare the TCGS derivation with the main alternatives.

8.1. Everettian/Decision-Theoretic Approaches

Deutsch[7] and Wallace[8] derive the Born rule from decision-theoretic rationality in the many-worlds interpretation.

TCGS comparison: Both approaches treat probability as emerging from a deterministic substrate. However, Everett requires “branch counting” or “caring measure” arguments that remain controversial. TCGS derives the measure from geometric uniqueness (fiber-invariance, foliation-invariance), avoiding the need for decision theory.

8.2. Zurek’s Enviance

Zurek[6] derives the Born rule from “environment-assisted invariance” (enviance): the requirement that probabilities be invariant under certain symmetry operations involving the environment.

TCGS comparison: Enviance is structurally similar to our foliation-invariance argument. In TCGS, the “environment” is the fiber—the degrees of freedom integrated out by the quotient map. The symmetry requirement is the same; the physical interpretation differs.

8.3. Gleason-Based Approaches

Several authors[9,10] use Gleason’s theorem directly, treating non-contextuality as an axiom.

TCGS comparison: We also invoke Gleason, but derive non-contextuality from the local nature of the pullback operation, rather than postulating it.

8.4. Typicality Arguments

Dürr, Goldstein, and Zanghi[11] derive Born-rule statistics in Bohmian mechanics from typicality: most initial conditions (with respect to a natural measure) yield Born-distributed outcomes.

TCGS comparison: Our fiber-measure argument is analogous to typicality. The difference is that in TCGS, the “natural measure” is not postulated but *forced* by the requirement of fiber-independence and foliation-invariance.

Table 1. Comparison of Born rule derivations.

Approach	Key Assumption	TCGS Analog
Everett/Decision	Rational betting behavior	Not needed
Zurek (Enviance)	Environment-assisted invariance	Foliation-invariance
Gleason direct	Non-contextuality postulate	Non-contextuality derived from pullback locality
Bohmian typicality	Natural measure on initial conditions	Canonical fiber measure (derived, not postulated)

9. Conclusion

We have shown that the Born rule is not an independent axiom in TCGS-SEQUENTION; it is a **theorem** that follows from the projection geometry.

The argument proceeds in three steps:

1. **Fiber-independence:** Probabilities must be well-defined on equivalence classes. This forces the measure to be canonical on fibers.
2. **Foliation-invariance:** Probabilities must be independent of parameterization. This forces the measure to be a linear functional of the density matrix.

3. **Non-contextuality:** The pullback operation is local, hence non-contextual. By Gleason's theorem, the only non-contextual measure is the Born measure.

The result:

$$P(P|\psi) = |\langle \phi|\psi \rangle|^2 = \text{Tr}(P\rho) \quad (20)$$

is the unique probability assignment consistent with TCGS projection geometry.

9.1. What This Achieves

- **Probability is neither ontic nor subjective:** It is a *projection invariant*—the unique measure that makes the quotient map mathematically coherent.
- **Determinism and probability coexist:** The source is deterministic; probability arises from fiber-averaging, not from ontic randomness.
- **The measurement problem is dissolved:** “Collapse” is Bayesian conditioning on a projection refinement, not a physical process.
- **Decoherence is geometric:** The pointer basis is determined by fiber structure, not by environmental interaction.

9.2. Open Questions

Several questions remain for future work:

1. **Explicit fiber construction:** Can the fibers $\pi^{-1}(s)$ be explicitly constructed for physically realistic systems?
2. **Hilbert space emergence:** Why does the shadow state space have Hilbert space structure? This paper assumed $\mathcal{S}$ is the space of density matrices; deriving this from more primitive TCGS axioms remains open.
3. **Relativistic extension:** How does the fiber/foliation structure generalize to quantum field theory?
4. **Gravity-matter coupling:** In full TCGS, the projection $X : \Sigma \rightarrow \mathcal{C}$ is dynamical. How does this affect the Born rule derivation?

These are directions for the ongoing cartographic program of TCGS-SEQUENTION.

Appendix A. Gleason's Theorem: Technical Details

Appendix A.1. Statement

Theorem A1 (Gleason, 1957). *Let $\mathcal{H}$ be a real or complex Hilbert space of dimension $d \geq 3$. Let $\nu : \mathcal{P}(\mathcal{H}) \rightarrow [0, 1]$ be a function on the projection lattice satisfying:*

1. $\nu(I) = 1$
2. $\nu(P_1 + P_2) = \nu(P_1) + \nu(P_2)$ for orthogonal projections $P_1 \perp P_2$

Then there exists a unique density matrix ρ such that:

$$\nu(P) = \text{Tr}(P\rho) \quad (\text{A1})$$

for all projections P .

Appendix A.2. Significance for TCGS

Condition (2) is additivity for orthogonal projections. In TCGS terms, this means: if two shadow outcomes are mutually exclusive (corresponding to orthogonal subspaces of the fiber), their probabilities add.

This is a geometric property of the fiber partition: mutually exclusive outcomes correspond to disjoint sub-fibers, and the canonical measure on a disjoint union is the sum of the measures on the parts.

Thus, Gleason's additivity condition is not an assumption; it is a consequence of the fiber geometry.

Appendix B. Uniqueness of the Canonical Fiber Measure

Appendix B.1. Fiber as Orbit

Let $\mathcal{G}_s$ be the group of source transformations that leave the shadow state s invariant:

$$\mathcal{G}_s := \{g \in \text{Aut}(\mathcal{C}) : \pi(g \cdot c) = \pi(c) \text{ for all } c \in \pi^{-1}(s)\}. \quad (\text{A2})$$

The fiber $\pi^{-1}(s)$ is an orbit of $\mathcal{G}_s$.

Appendix B.2. Haar Measure

By standard results on locally compact groups, there exists a unique (up to normalization) left-invariant measure $\mu_{\mathcal{G}_s}$ on $\mathcal{G}_s$ —the **Haar measure**.

This induces a unique measure on the orbit $\pi^{-1}(s)$:

$$\mu_{\pi^{-1}(s)}(A) := \mu_{\mathcal{G}_s}(\{g : g \cdot c_0 \in A\}) \quad (\text{A3})$$

for any reference point $c_0 \in \pi^{-1}(s)$.

Appendix B.3. Connection to Born Measure

For the quantum mechanical case, the relevant symmetry group is $U(n)$ (unitaries that leave the density matrix invariant), and the Haar measure on $U(n)$ induces precisely the Born measure on the space of pure states (the projective Hilbert space $\mathbb{CP}^{n-1}$).

This is the technical underpinning of the claim that fiber-independence forces the Born measure.

Appendix C. Foliation-Invariance and the Trace Form

Appendix C.1. Setup

Let $\rho, \sigma \in \mathcal{S}$ be density matrices. A transition probability $P(\rho \rightarrow \sigma)$ is a function:

$$P : \mathcal{S} \times \mathcal{S} \rightarrow [0, 1]. \quad (\text{A4})$$

Appendix C.2. Foliation-Invariance

A foliation is a one-parameter family of shadow slices. Foliation-invariance requires:

$$P(\rho \rightarrow \sigma) = P(U_t \rho U_t^\dagger \rightarrow U_t \sigma U_t^\dagger) \quad (\text{A5})$$

for all unitaries U_t generated by admissible Hamiltonians.

Appendix C.3. Consequence

Proposition A1. If $P(\rho \rightarrow \sigma)$ is:

1. Foliation-invariant
2. Continuous in both arguments
3. Normalized: $\sum_{\sigma} P(\rho \rightarrow \sigma) = 1$

then P has the form:

$$P(\rho \rightarrow \sigma) = \text{Tr}(E_{\sigma} \rho) \quad (\text{A6})$$

for some POVM $\{E_{\sigma}\}$.

Proof sketch. Foliation-invariance means P depends only on unitary-invariant quantities. For density matrices, the only such quantities are traces of products. Continuity and normalization then force the POVM form. $\square$

Combined with Gleason's theorem (which fixes $E_\sigma = P_\sigma$ for projective measurements), this yields the Born rule.

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