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New fractional generalized Gronwall inequalities and Lyapunov theorems with applications

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New Fractional Generalized Gronwall Inequalities and Lyapunov Theorems with Applications

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Abstract

This paper deals with some expressions of the fractional generalized Gronwall inequality when associated with both non-negative and non-positive singular kernels and establishes sharp Mittag-Leffler bounds containing different ingredients. The long-term behavior of non-autonomous fractional order systems by means of modified fractional Lyapunov theorems is analyzed. As an application, we give a few examples that use quadratic Lyapunov functions for typical fractional order systems to predict trajectories that ultimately aim to reach vector 0 as $t \to \infty$.

Keywords: fractional generalized Gronwall inequality; non-autonomous fractional order systems; Lyapunov function; fractional Lyapunov theorems; global asymptotic stability

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1. Introduction

In the studies of non-autonomous fractional order systems, one often concerns generalized Gronwall inequality to understand some basic properties of solutions to many such systems. This paper examines a typical mathematical tool that concerns the fractional generalized Gronwall inequality associated with both non-positive and non-negative singular kernels and develops new insights into fractional Lyapunov theory.

The generalized Gronwall inequality with a non-negative singular kernel describes a fractional integral inequality problem that deals with finding some reasonable bound to its solutions (see, e.g., [3,4]). This inequality and its many different forms are widely known to be useful in the understanding of basic stability and bounds of solutions for the theory of fractional order systems [26–30]. The stability theory of fractional order systems has made some progress in the last two decades, and many interesting results are still expected. The readers can learn some interesting results and new insights that concern fractional Lyapunov theory for many Caputo-type fractional order systems devoted to a survey paper [5].

The non-autonomous fractional order systems are diverse in sciences and engineering, and a lot of scientific works are devoted to holding their precise construction using the notions of fractional integrals and fractional derivatives [1,2]. These systems seem harder owing to time-dependent structures that may excite or force the evolution of system dynamics and reasonable mathematical insights that are very less focused on for rigorous analysis of their behaviors.

In many applications of interest, many of these governing systems usually carry time-dependent terms or coefficients when the state operates from one position to reach another position operated by reasonable fractional derivative operators, and control methods were demonstrated for achieving effective system control goal dynamics [18–24]. But how these systems' states evolve when time progresses forward toward ∞ remains unknown and quite puzzling even if one considers a sufficiently large number of initial positions. The qualitative properties of solutions of many such systems, such as asymptotic stability and bounds, often demand new mathematical tools and are much less understood. In this context, the ideas of the fractional Lyapunov direct method (see, e.g., [8–16,25]) were published to give new insights to reasonably well predict some basic understanding of many such systems. But the fractional Lyapunov theory still lacks many, many different new mathematical ingredients in its forms and seems crucial for some fundamental classes of such non-autonomous systems. One major key aspect of these could be the right formulations of advanced tools are missing and not known in the current state of the art of knowledge.

Some forms of generalized Gronwall inequality with singular kernels are known to give some breakthrough results to many different versions of fractional Lyapunov stability theorems that concern fractional-order systems. Many results in this issue are broadly not known, and they seem to be less focused in the literature. First, the applications of well-known generalized Gronwall inequalities suggested by Henry [4] and Ye et al. [3] do not give new pathways to the understanding of some stability problems of fractional order systems. This is indeed true in the sense that these works consider only non-negative singular kernels and do not develop good directions to basic fractional Lyapunov theory. Second, the right mathematical insights to reasonable formulations of non-positive singular kernel generalized Gronwall inequality remain mysterious. Recently, Lenka [7] formulated some intuitive generalized Gronwall inequalities that contain a non-positive kernel and used them to prove some elementary Lyapunov theorems to predict at least the asymptotic stability of non-autonomous fractional order systems. The mentioned author has also introduced some stronger versions of generalized Gronwall inequalities with non-positive singular kernel functions in a work [17] and used them masterfully in predicting global asymptotic stability and bounds to solutions of many such fractional systems.

It has been learned that there still exists some difficulty in the progress of generalized Gronwall inequality with a non-positive kernel, which remains less understood. A typical question that remained open is as follows: Is it possible to classify the generalized Gronwall inequality that contains both non-positive and non-negative kernels that can give a definite reasonable Mittag-Leffler bound? To the best of the author's knowledge, no adequate new mathematical knowledge has yet been known in the literature for the existence of such an inequality. We are curious about such an unknown problem and wish to address some rigorous formulation of the mentioned inequality.

One goal of this paper concerns providing a reasonable answer to the question by means of new modifications of previously partially known generalized Gronwall inequalities. We also study some applications of these inequalities in the direction of developing modifications to elementary fractional Lyapunov theorems that can give conclusive notions to asymptotic stability and boundedness for fractional order systems. In this context, we discuss some reasonable theoretical bounds for equations that may carry time-varying phenomena in some system configurations that were established. Finally, some examples are presented that deal with equations of fractional order, and our new results demonstrate the effectiveness and nature of long-term solutions and predicting sharp bounds.

Hereafter in Section 2, we first give some notations and basic ingredients that provide some useful tools for understanding fractional order systems. In section 3 we give some new formulations to the so-called fractional generalized Gronwall inequality, which is the main topic of this paper. In Section 4, some new Lyapunov theorems for non-autonomous fractional order systems were addressed that build new knowledge by using generalized Gronwall inequalities. In Section 5 we discuss some examples that describe equations that involve Caputo fractional derivatives, and some reasonable results were utilized to predict asymptotic dynamics and bounds of solutions. Conclusions roughly close to this paper are summarized in Section 6.

2. Preliminaries

We shall use the following notation throughout this paper. We fix $\mathbb{N}$ as the set of natural numbers, $\mathbb{R}_+$ as the set of positive real numbers, $\mathbb{R}_{\geq 0}$ as the set of non-negative real numbers, $\mathbb{R}$ as the set of real numbers, and U^T as the transpose of $U \in \mathbb{R}^{m\times n}$. $\mathbb{R}^n$ is the Euclidean space, $\|\cdot\|$ is the standard Euclidean norm, $\mathcal{C}^0$ is the space of continuous functions, and $\mathcal{C}^n$ is the space of n-times continuously differentiable functions.

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Definition 1. *The Riemann-Liouville fractional integral of an integrable function* $u : [b, \infty) \to \mathbb{R}$ *is defined by* [1,2]:

$$ {}^{RL}\mathcal{I}^{\alpha}_{b,t} u(t) = \frac{1}{\Gamma(\alpha)} \int_b^t (t-\zeta)^{\alpha-1} u(\zeta) d\zeta, \ t > b, \tag{1} $$

where $b \in \mathbb{R}$, $\alpha \in \mathbb{R}_+$ *and Gamma function* $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt$.

Definition 2. *The Caputo fractional derivative of real valued function u which is* C^0 *on* $[b, \infty)$ *and* C^n *on* (b, ∞) *is defined by* [1,2]:

$$ {}^{C}\mathcal{D}^{\alpha}_{b,t} u(t) = \frac{1}{\Gamma(n-\alpha)} \int_b^t (t-\zeta)^{n-\alpha-1} \frac{\mathrm{d}^n}{\mathrm{d}\zeta^n} u(\zeta) d\zeta, \ t > b, \tag{2} $$

whenever $n - 1 < \alpha < n$ *and* ${}^{C}\mathcal{D}^{\alpha}_{b,t} u(t) = \frac{\mathrm{d}^n}{\mathrm{d}t^n} u(t)$ *whenever* $\alpha = n$, *where* $b \in \mathbb{R}$, $\alpha \in \mathbb{R}_+$ *and* $n \in \mathbb{N}$.

Definition 3. *The two-parameter Mittag-Leffler function is defined by an expression* [6]:

$$ E_{\delta,\gamma}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\delta k + \gamma)}, \ \delta, \gamma > 0, \ t \in \mathbb{R}. \tag{3} $$

In many theoretical and applied areas of interest, one is often concerned with a governing equation that can be described as a non-autonomous Caputo-type fractional-order system [5,7,17]:

$$ \begin{aligned} {}^{C}\mathcal{D}^{\alpha}_{b,t} u(t) &= \tilde{f}(t, u(t)), \\ u(b) &= u_b, \end{aligned} \tag{4} $$

where $u(t) \in \mathbb{R}^n$, ${}^{C}\mathcal{D}^{\alpha}_{b,t} u(t) = ({}^{C}\mathcal{D}^{\alpha}_{b,t} u_1(t), \ldots, {}^{C}\mathcal{D}^{\alpha}_{b,t} u_n(t))^T \in \mathbb{R}^n$, $b \in \mathbb{R}$, $\alpha \in (0, 1]$, and function $\tilde{f} : [b, \infty) \times \mathbb{D} \subset \mathbb{R}^n \to \mathbb{R}^n$ is continuous with its arguments.

Definition 4. [17] *To say a function* $\xi : [0, \infty) \to [0, \infty)$ *belongs to* ***class-***$\mathcal{K}_\infty$ *if*

i) *it is continuous and strictly increasing,*
ii) *it satisfies* $\xi(0) = 0$ *and* $\xi(r) \to \infty$ *as* $r \to \infty$.

Definition 5. [17] *To say the system* (4) *is globally asymptotically stable if the non-trivial solution* $u(t) \to 0$ *as* $t \to \infty$, *for all* $u(b) \in \mathbb{R}^n$.

3. Fractional Generalized Gronwall Inequality

In this section our goal is to establish new modifications to generalized Gronwall inequalities with non-positive singular kernels that appeared recently in some works [7,17]. Our novel insights have come to light in understanding when an integral equation that involves a Riemann-Liouville fractional integral with kernels can be non-positive and non-negative.

We begin with the following:

Theorem 1. *Let* $u \in \mathbb{R}_{\geq 0}$ *be continuous on the interval* $[b, T)$, *where* $b \in \mathbb{R}$ *and* $b < T \leq \infty$. *Let* $w \in \mathbb{R}_{\geq 0}$ *be continuous on the interval* $[b, T)$. *Let* $v \in \mathbb{R}_{\geq 0}$ *be continuous, bounded, and non-decreasing on* $[b, T)$, *and suppose that* $g_1, g_2 \in \mathbb{R}_{\geq 0}$ *are continuous on* $[b, T)$ *that obey the integral inequality*

$$ u(t) \leq v(t) + g_1(t) \int_b^t (t-\tau)^{\alpha-1} u(\tau) d\tau - g_2(t) \int_b^t (t-\tau)^{\alpha-1} u(\tau) d\tau + \int_b^t (t-\tau)^{\alpha-1} w(\tau) d\tau, \tag{5} $$

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for all $t \in [b, T)$, where constant $\alpha \in (0, 1]$. Then, we have

$$u(t) \leq E_{\alpha,1}\left(\Gamma(\alpha)(g_1(t) - g_2(t))(t-b)^{\alpha}\right)\left[v(t) + \left(E_{\alpha,1}\left(\alpha\Gamma(\alpha)\int_b^t (t-\tau)^{\alpha-1}w(\tau)d\tau\right) - 1\right)\right], \quad (6)$$

for all $t \in [b, T)$.

Proof. The proof is immediate when

$$u(t) = E_{\alpha,1}\left(\Gamma(\alpha)(g_1(t) - g_2(t))(t-b)^{\alpha}\right)\left[v(t) + \left(E_{\alpha,1}\left(\alpha\Gamma(\alpha)\int_b^t (t-\tau)^{\alpha-1}w(\tau)d\tau\right) - 1\right)\right]. \quad (7)$$

Thus, on contrary, we may assume that

$$u(t) > E_{\alpha,1}\left(\Gamma(\alpha)(g_1(t) - g_2(t))(t-b)^{\alpha}\right)\left[v(t) + \left(E_{\alpha,1}\left(\alpha\Gamma(\alpha)\int_b^t (t-\tau)^{\alpha-1}w(\tau)d\tau\right) - 1\right)\right], \quad (8)$$

for all $t \in [b, T)$. Next, it is possible to define

$$z(t) = u(t) - E_{\alpha,1}\left(\Gamma(\alpha)(g_1(t) - g_2(t))(t-b)^{\alpha}\right)[v(t) + h(t)], \ \forall t \geq b, \quad (9)$$

where $h(t) = \left(E_{\alpha,1}\left(\alpha\Gamma(\alpha)\int_b^t (t-\tau)^{\alpha-1}w(\tau)d\tau\right) - 1\right)$. Then, one obtains from (5) that

$$\begin{aligned} u(t) \leq\ & v(t) + (g_1(t) - g_2(t))\int_b^t (t-\tau)^{\alpha-1}z(\tau)d\tau \\ & + (g_1(t) - g_2(t))\int_b^t (t-\tau)^{\alpha-1}[v(\tau) + h(\tau)]E_{\alpha,1}\left(\Gamma(\alpha)(g_1(\tau) - g_2(\tau))(\tau-b)^{\alpha}\right)d\tau \\ & + \int_b^t (t-\tau)^{\alpha-1}w(\tau)d\tau, \ \forall t \geq b. \end{aligned} \quad (10)$$

By noting v is continuous, bounded and non-decreasing, and h is non-negative, one has from (10) that

$$\begin{aligned} u(t) \leq\ & v(t) + |(g_1(t) - g_2(t))|\int_b^t (t-\tau)^{\alpha-1}|z(\tau)|d\tau \\ & + |(g_1(t) - g_2(t))|v_c\int_b^t (t-\tau)^{\alpha-1}E_{\alpha,1}\left(\Gamma(\alpha)(g_1(\tau) - g_2(\tau))(\tau-b)^{\alpha}\right)d\tau \\ & + |(g_1(t) - g_2(t))|\int_b^t (t-\tau)^{\alpha-1}|h(\tau)|E_{\alpha,1}\left(\Gamma(\alpha)(g_1(\tau) - g_2(\tau))(\tau-b)^{\alpha}\right)d\tau \\ & + \left(E_{\alpha,1}\left(\alpha\Gamma(\alpha)\int_b^t (t-\tau)^{\alpha-1}w(\tau)d\tau\right) - 1\right), \ \forall t \geq b, \end{aligned} \quad (11)$$

where $v(t) \leq v_c$, for some constant $0 < v_c < \infty$. Since u, v, w, g_1, and g_2 were continuous, by letting $t \to b$ in (11), one gets $u(b) \leq v(b)$. This contradicts to our assumption that $u(b) > v(b)$. Therefore, one expects intuitively the inequality (6). This finished the proof. □

It may be noted that Theorem 1 provides a case study justification for the existence of both non-positive and non-negative singular kernels associated with the fractional integral inequality (5). This Theorem 1 has a special case whenever the function $g_1 = 0$ and can be found in the author's work [17].

Two simpler versions of such a theorem can be treated as corollaries. Here we give two corollaries that are immediate from Theorem 1.

Corollary 1. *Let $u \in \mathbb{R}_{\geq 0}$ be continuous on the interval $[b, T)$, where $b \in \mathbb{R}$ and $b < T \leq \infty$. Let $w \in \mathbb{R}_{\geq 0}$ be continuous on the interval $[b, T)$. Let $v \in \mathbb{R}_{\geq 0}$ be continuous, bounded, and non-decreasing on $[b, T)$, and suppose it obeys the integral inequality*

$$u(t) \leq v(t) + \lambda_1 \int_b^t (t-\tau)^{\alpha-1} u(\tau)d\tau - \lambda_2 \int_b^t (t-\tau)^{\alpha-1} u(\tau)d\tau + \int_b^t (t-\tau)^{\alpha-1} w(\tau)d\tau, \tag{12}$$

for all $t \in [b, T)$, where constants $\lambda_1 \geq 0$, $\lambda_2 \geq 0$, and $\alpha \in (0, 1]$. Then, we have

$$u(t) \leq E_{\alpha,1}\left(\Gamma(\alpha)(\lambda_1 - \lambda_2)(t-b)^\alpha\right)\left[v(t) + \left(E_{\alpha,1}\left(\alpha\Gamma(\alpha)\int_b^t (t-\tau)^{\alpha-1} w(\tau)d\tau\right) - 1\right)\right], \tag{13}$$

for all $t \in [b, T)$.

Proof. Take $g_i(t) = \lambda_i \geq 0$, $i = 1, 2$, in Theorem 1. □

Corollary 2. *Let $u \in \mathbb{R}_{\geq 0}$ be continuous on the interval $[b, T)$ and let $w \in \mathbb{R}_{\geq 0}$ be continuous on the interval $[b, T)$, where $b \in \mathbb{R}$ and $b < T \leq \infty$, and let it obey the integral inequality*

$$u(t) \leq A + \lambda_1 \int_b^t (t-\tau)^{\alpha-1} u(\tau)d\tau - \lambda_2 \int_b^t (t-\tau)^{\alpha-1} u(\tau)d\tau + \int_b^t (t-\tau)^{\alpha-1} w(\tau)d\tau, \tag{14}$$

for all $t \in [b, T)$, where constants $A > 0$, $\lambda_1 \geq 0$, $\lambda_2 \geq 0$, and $\alpha \in (0, 1]$. Then, we have

$$u(t) \leq E_{\alpha,1}\left(\Gamma(\alpha)(\lambda_1 - \lambda_2)(t-b)^\alpha\right)\left[A + \left(E_{\alpha,1}\left(\alpha\Gamma(\alpha)\int_b^t (t-\tau)^{\alpha-1} w(\tau)d\tau\right) - 1\right)\right], \tag{15}$$

for all $t \in [b, T)$.

Proof. Take $v(t) = A > 0$ in Corollary 1. □

The new result below develops a mathematical insight whenever a kernel function involves a time-dependent, non-constant, continuous, and increasing function. It gives a new analogue of the result as compared to the previous Theorem 1, especially whenever the Mittag-Leffler function holds the integral of the mentioned time-dependent function.

Theorem 2. *Let $u \in \mathbb{R}_{\geq 0}$ be continuous on the interval $[b, T)$, where $b \in \mathbb{R}$ and $b < T \leq \infty$. Let $w \in \mathbb{R}_{\geq 0}$ be continuous on the interval $[b, T)$. Let $v \in \mathbb{R}_{\geq 0}$ be continuous, bounded, and non-decreasing on $[b, T)$, and suppose that $g_1, g_2 \in \mathbb{R}_{\geq 0}$ are non-constant, continuous, and increasing on $[b, T)$ that obey the integral inequality*

$$\begin{aligned} u(t) \leq v(t) &+ \lambda_1 \int_b^t (t-\tau)^{\alpha-1} g_1(\tau) u(\tau)d\tau - \lambda_2 \int_b^t (t-\tau)^{\alpha-1} g_2(\tau) u(\tau)d\tau \\ &+ \int_b^t (t-\tau)^{\alpha-1} w(\tau)d\tau, \end{aligned} \tag{16}$$

for all $t \in [b, T)$, where constants $\lambda_1 \geq 0$, $\lambda_2 \geq 0$, and $\alpha \in (0, 1]$. Then, we have

$$\begin{aligned} u(t) \leq\ & E_{\alpha,1}\left(\lambda_1\Gamma(\alpha)\left(\int_b^t g_1(\tau)d\tau\right)^\alpha - \lambda_2\Gamma(\alpha)\left(\int_b^t g_2(\tau)d\tau\right)^\alpha\right) \\ & \times \left[v(t) + \left(E_{\alpha,1}\left(\alpha\Gamma(\alpha)\int_b^t (t-\tau)^{\alpha-1} w(\tau)d\tau\right) - 1\right)\right], \end{aligned} \tag{17}$$

for all $t \in [b, T)$.

Proof. The conclusion in (17) is obvious when

$$u(t) = E_{\alpha,1}\left(\lambda_1\Gamma(\alpha)\left(\int_b^t g_1(\tau)d\tau\right)^{\alpha} - \lambda_2\Gamma(\alpha)\left(\int_b^t g_2(\tau)d\tau\right)^{\alpha}\right) \times \left[v(t) + \left(E_{\alpha,1}\left(\alpha\Gamma(\alpha)\int_b^t (t-\tau)^{\alpha-1}w(\tau)d\tau\right) - 1\right)\right], \tag{18}$$

for all $t \in [b, T)$. On contrary, it is assumed that

$$u(t) > E_{\alpha,1}\left(\lambda_1\Gamma(\alpha)\left(\int_b^t g_1(\tau)d\tau\right)^{\alpha} - \lambda_2\Gamma(\alpha)\left(\int_b^t g_2(\tau)d\tau\right)^{\alpha}\right) \times \left[v(t) + \left(E_{\alpha,1}\left(\alpha\Gamma(\alpha)\int_b^t (t-\tau)^{\alpha-1}w(\tau)d\tau\right) - 1\right)\right], \tag{19}$$

for all $t \in [b, T)$. Then, we define

$$F(t) = u(t) - E(t)[v(t) + h(t)], \tag{20}$$

for all $t \in [b, T)$, where the notations $E(t) = E_{\alpha,1}\left(\lambda_1\Gamma(\alpha)\left(\int_b^t g_1(\tau)d\tau\right)^{\alpha} - \lambda_2\Gamma(\alpha)\left(\int_b^t g_2(\tau)d\tau\right)^{\alpha}\right)$ and $h(t) = \left(E_{\alpha,1}\left(\alpha\Gamma(\alpha)\int_b^t (t-\tau)^{\alpha-1}w(\tau)d\tau\right) - 1\right)$ were used. In spirit of (16), one has

$$\begin{aligned} u(t) \le v(t) &+ \int_b^t (t-\tau)^{\alpha-1}(\lambda_1 g_1(\tau) - \lambda_2 g_2(\tau))F(\tau)d\tau \\ &+ \int_b^t (t-\tau)^{\alpha-1}(\lambda_1 g_1(\tau) - \lambda_2 g_2(\tau))[v(\tau) + h(\tau)]E(\tau)d\tau \\ &+ \int_b^t (t-\tau)^{\alpha-1}w(\tau)d\tau, \ \forall t \ge b. \end{aligned} \tag{21}$$

Since v is continuous, bounded and non-decreasing, and g_1, g_2 are increasing, the inequality (21) can be majorized by

$$\begin{aligned} u(t) &\le v(t) + (\lambda_1|g_1(t)| + \lambda_2|g_2(t)|)\int_b^t (t-\tau)^{\alpha-1}|F(\tau)|d\tau \\ &\quad + (\lambda_1|g_1(t)| + \lambda_2|g_2(t)|)\int_b^t (t-\tau)^{\alpha-1}[v_c + |h(\tau)|]|E(\tau)|d\tau + \int_b^t (t-\tau)^{\alpha-1}w(\tau)d\tau \\ &\le v(t) + (\lambda_1|g_1(t)| + \lambda_2|g_2(t)|)\int_b^t (t-\tau)^{\alpha-1}|F(\tau)|d\tau \\ &\quad + (\lambda_1|g_1(t)| + \lambda_2|g_2(t)|)\int_b^t (t-\tau)^{\alpha-1}|h(\tau)||E(\tau)|d\tau + (\lambda_1|g_1(t)| + \lambda_2|g_2(t)|)v_c\frac{(t-b)^{\alpha}}{\alpha} \\ &\quad + \left(E_{\alpha,1}\left(\alpha\Gamma(\alpha)\int_b^t (t-\tau)^{\alpha-1}w(\tau)d\tau\right) - 1\right), \end{aligned} \tag{22}$$

for all $t \ge b$, where $v(t) \le v_c < \infty$ with constant $v_c > 0$. Since u, v, w, g_1, and g_2 were continuous, by letting $t \to b$ in (22), one obtains $u(b) \le v(b)$. This contradicts to priori assumption $u(b) > v(b)$. Therefore, the inequality (17) should be true. This finished the proof. □

A complementary result to Theorem 2 is stated below. It introduces a time-dependent continuous and decreasing function in the kernel arguments of inequality (23).

Theorem 3. *Let $u \in \mathbb{R}_{\ge 0}$ be continuous on the interval $[b, T)$, where $b \in \mathbb{R}$ and $b < T \le \infty$. Let $w \in \mathbb{R}_{\ge 0}$ be continuous on the interval $[b, T)$. Let $v \in \mathbb{R}_{\ge 0}$ be continuous, bounded, and non-decreasing on $[b, T)$,*

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and suppose that $g_1, g_2 \in \mathbb{R}_{\geq 0}$ are non-constant, continuous, and decreasing on $[b, T)$ that obey the integral inequality

$$u(t) \leq v(t) + \lambda_1 \int_b^t (t-\tau)^{\alpha-1} g_1(\tau)u(\tau)d\tau - \lambda_2 \int_b^t (t-\tau)^{\alpha-1} g_2(\tau)u(\tau)d\tau + \int_b^t (t-\tau)^{\alpha-1} w(\tau)d\tau, \tag{23}$$

for all $t \in [b, T)$, where constants $\lambda_1 \geq 0$, $\lambda_2 \geq 0$, and $\alpha \in (0, 1]$. Then, we have

$$u(t) \leq E_{\alpha,1}\left(\lambda_1 \Gamma(\alpha)(t-b)^{\alpha-1}\left(\int_b^t g_1(\tau)d\tau\right) - \lambda_2 \Gamma(\alpha)(t-b)^{\alpha-1}\left(\int_b^t g_2(\tau)d\tau\right)\right) \times \left[v(t) + \left(E_{\alpha,1}\left(\alpha\Gamma(\alpha)\int_b^t (t-\tau)^{\alpha-1} w(\tau)d\tau\right) - 1\right)\right], \tag{24}$$

for all $t \in [b, T)$.

Proof. Refer to the proof of Theorem 2. □

Our next Theorem 4 below develops a complementary result to Theorem 2 and establishes a new sharp bound in (26) in contrast to the previous bound (17).

Theorem 4. *Let $u \in \mathbb{R}_{\geq 0}$ be continuous on the interval $[b, T)$, where $b \in \mathbb{R}$ and $b < T \leq \infty$. Let $w \in \mathbb{R}_{\geq 0}$ be continuous on the interval $[b, T)$. Let $v \in \mathbb{R}_{\geq 0}$ be continuous, bounded, and non-decreasing on $[b, T)$, and suppose that $g_1, g_2 \in \mathbb{R}_{\geq 0}$ are continuous and non-decreasing on $[b, T)$ that obey the integral inequality*

$$u(t) \leq v(t) + \int_b^t (t-\tau)^{\alpha-1} g_1(\tau)u(\tau)d\tau - \int_b^t (t-\tau)^{\alpha-1} g_2(\tau)u(\tau)d\tau + \int_b^t (t-\tau)^{\alpha-1} w(\tau)d\tau, \tag{25}$$

for all $t \in [b, T)$, where constant $\alpha \in (0, 1]$. Then, we have

$$u(t) \leq E_{\alpha,1}\left(\alpha\Gamma(\alpha)\int_b^t (t-\tau)^{\alpha-1}[g_1(\tau) - g_2(\tau)]d\tau\right) \times \left[v(t) + \left(E_{\alpha,1}\left(\alpha\Gamma(\alpha)\int_b^t (t-\tau)^{\alpha-1} w(\tau)d\tau\right) - 1\right)\right], \tag{26}$$

for all $t \in [b, T)$.

Proof. The proof can be repeated in a similar manner as in Theorem 2. □

However, the previously formulated Theorems 2 and 4 have some shortcomings that can be imagined whenever the functions g_1 and g_2 are not only continuous as in Theorem 1. We give some reasonable novel insights in this direction for a rigorous formulation to estimate probable theoretical bounds of appropriate inequalities.

We first introduce two basic results that only consider either a non-positive or non-negative singular kernel in the generalized Gronwall inequality.

Theorem 5. *Let $u \in \mathbb{R}_{\geq 0}$ be continuous on the interval $[b, T)$, where $b \in \mathbb{R}$ and $b < T \leq \infty$. Let $w \in \mathbb{R}_{\geq 0}$ be continuous on the interval $[b, T)$. Let $v \in \mathbb{R}_{\geq 0}$ be continuous, bounded, and non-decreasing on $[b, T)$, and suppose that $g \in \mathbb{R}_{\geq 0}$ is continuous on $[b, T)$ that obeys the integral inequality*

$$u(t) \leq v(t) + \int_b^t (t-\tau)^{\alpha-1} g(\tau)u(\tau)d\tau + \int_b^t (t-\tau)^{\alpha-1} w(\tau)d\tau, \tag{27}$$

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for all $t \in [b, T)$, *where constant* $\alpha \in (0, 1]$. *Then, we have*

$$u(t) \leq E_{\alpha,1}\left(\alpha\Gamma(\alpha)\int_b^t (t-\tau)^{\alpha-1} g(\tau)d\tau\right) \times \left[v(t) + \left(E_{\alpha,1}\left(\alpha\Gamma(\alpha)\int_b^t (t-\tau)^{\alpha-1} w(\tau)d\tau\right) - 1\right)\right], \tag{28}$$

for all $t \in [b, T)$.

Proof. See the proof of Theorem 2. □

Theorem 6. *Let* $u \in \mathbb{R}_{\geq 0}$ *be continuous on the interval* $[b, T)$, *where* $b \in \mathbb{R}$ *and* $b < T \leq \infty$. *Let* $w \in \mathbb{R}_{\geq 0}$ *be continuous on the interval* $[b, T)$. *Let* $v \in \mathbb{R}_{\geq 0}$ *be continuous, bounded, and non-decreasing on* $[b, T)$, *and suppose that* $g \in \mathbb{R}_{\geq 0}$ *is continuous on* $[b, T)$ *that obeys the integral inequality*

$$u(t) \leq v(t) - \int_b^t (t-\tau)^{\alpha-1} g(\tau)u(\tau)d\tau + \int_b^t (t-\tau)^{\alpha-1} w(\tau)d\tau, \tag{29}$$

for all $t \in [b, T)$, *where constant* $\alpha \in (0, 1]$. *Then, we have*

$$u(t) \leq E_{\alpha,1}\left(-\alpha\Gamma(\alpha)\int_b^t (t-\tau)^{\alpha-1} g(\tau)d\tau\right) \times \left[v(t) + \left(E_{\alpha,1}\left(\alpha\Gamma(\alpha)\int_b^t (t-\tau)^{\alpha-1} w(\tau)d\tau\right) - 1\right)\right], \tag{30}$$

for all $t \in [b, T)$.

Proof. See the proof of Theorem 2. □

Now we are in a position to formalize a generalized result that concerns the generalized Gronwall inequality stated below in Theorem 7. This theorem can be viewed as an extension of Theorem 5, Theorem 6, or Theorem 4.

Theorem 7. *Let* $u \in \mathbb{R}_{\geq 0}$ *be continuous on the interval* $[b, T)$, *where* $b \in \mathbb{R}$ *and* $b < T \leq \infty$. *Let* $w \in \mathbb{R}_{\geq 0}$ *be continuous on the interval* $[b, T)$. *Let* $v \in \mathbb{R}_{\geq 0}$ *be continuous, bounded, and non-decreasing on* $[b, T)$, *and suppose that* $g_1, g_2 \in \mathbb{R}_{\geq 0}$ *are continuous on* $[b, T)$ *that obey the integral inequality*

$$u(t) \leq v(t) + \int_b^t (t-\tau)^{\alpha-1} g_1(\tau)u(\tau)d\tau - \int_b^t (t-\tau)^{\alpha-1} g_2(\tau)u(\tau)d\tau + \int_b^t (t-\tau)^{\alpha-1} w(\tau)d\tau, \tag{31}$$

for all $t \in [b, T)$, *where constant* $\alpha \in (0, 1]$. *Then, we have*

$$u(t) \leq E_{\alpha,1}\left(\alpha\Gamma(\alpha)\int_b^t (t-\tau)^{\alpha-1} [g_1(\tau) - g_2(\tau)]d\tau\right) \times \left[v(t) + \left(E_{\alpha,1}\left(\alpha\Gamma(\alpha)\int_b^t (t-\tau)^{\alpha-1} w(\tau)d\tau\right) - 1\right)\right], \tag{32}$$

for all $t \in [b, T)$.

Proof. The proof strategy is as in Theorem 2. □

Owing to the aforementioned results, here we state two propositions that seem quite adequate in their form and can be proved in the preceding strategies.

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Proposition 1. *Let $u \in \mathbb{R}_{\geq 0}$ be continuous on the interval $[b, T)$, where $b \in \mathbb{R}$ and $b < T \leq \infty$. Let $w \in \mathbb{R}_{\geq 0}$ be continuous on the interval $[b, T)$. Let $v \in \mathbb{R}_{\geq 0}$ be continuous, bounded, and non-decreasing on $[b, T)$, and suppose that $g \in \mathbb{R}$ is continuous on $[b, T)$ that obeys the integral inequality*

$$u(t) \leq v(t) + \lambda \int_b^t (t-\tau)^{\alpha-1} g(\tau) u(\tau) d\tau + \int_b^t (t-\tau)^{\alpha-1} w(\tau) d\tau, \tag{33}$$

for all $t \in [b, T)$, where constants $\lambda \in \mathbb{R}$, and $\alpha \in (0, 1]$. Then, we have

$$\begin{aligned} u(t) \leq\ & E_{\alpha,1}\left(\alpha\lambda\Gamma(\alpha) \int_b^t (\tau-b)^{\alpha-1} g(\tau) d\tau\right) \\ & \times \left[v(t) + \left(E_{\alpha,1}\left(\alpha\Gamma(\alpha) \int_b^t (t-\tau)^{\alpha-1} w(\tau) d\tau\right) - 1\right)\right], \end{aligned} \tag{34}$$

for all $t \in [b, T)$.

Proposition 2. *Let $u \in \mathbb{R}_{\geq 0}$ be continuous on the interval $[b, T)$, where $b \in \mathbb{R}$ and $b < T \leq \infty$. Let $w \in \mathbb{R}_{\geq 0}$ be continuous on the interval $[b, T)$. Let $v \in \mathbb{R}_{\geq 0}$ be continuous, bounded, and non-decreasing on $[b, T)$, and suppose that $g \in \mathbb{R}$ is continuous on $[b, T)$ that obeys the integral inequality*

$$u(t) \leq v(t) + \lambda \int_b^t (t-\tau)^{\alpha-1} g(\tau) u(\tau) d\tau + \int_b^t (t-\tau)^{\alpha-1} w(\tau) d\tau, \tag{35}$$

for all $t \in [b, T)$, where constants $\lambda > 0$, and $\alpha \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1]$. Then, we have

$$\begin{aligned} u(t) \leq\ & E_{\alpha,1}\left(\frac{1}{2}\lambda\Gamma(2\alpha-1)(t-b)^{2\alpha-1} + \frac{1}{2}\lambda \int_b^t g^2(\tau) d\tau\right) \\ & \times \left[v(t) + \left(E_{\alpha,1}\left(\alpha\Gamma(\alpha) \int_b^t (t-\tau)^{\alpha-1} w(\tau) d\tau\right) - 1\right)\right], \end{aligned} \tag{36}$$

for all $t \in [b, T)$.

Remark 1. *The new Theorems 1- 7 develop some meaningful bounds to the unknown solution $u(t)$ associated with them. The proofs to these theorems are somewhat analogous to previous works detailed in [7,17]. However, proving these theorems by means of any new different methods might bring good understanding; it remains an open exercise.*

4. Global Dynamics of Fractional Order Systems

This section's goal is to give some promising applications of the generalized Gronwall inequality that have been established in the previous section. First of all, some typical special inequalities enable new ways to understand basic Lyapunov theorems for fractional order systems. Secondly, they can be useful to predict sharp bounds to solutions of fractional order systems.

The Subsection 4.1 develops some novel fractional Lyapunov theorems that use the knowledge of Lyapunov functions and associated fractional Lyapunov differential inequalities for the system (4). The Subsection 4.2 establishes results where the solutions of (4) can at most be majorized by some reasonable theoretical bounds.

4.1. Some Modified Elementary Lyapunov Theorems

In this subsection we give some new global asymptotic stability theorems for the non-autonomous fractional order system (4). We follow the ideas of basic fractional Lyapunov theory recently modernized in some works [7,17].

The main ingredients to our results are Lyapunov functions, fractional Lyapunov differential inequalities, and some reasonable conditions attached to them.

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Theorem 8. *Suppose that $\mathcal{V} : [b,\infty) \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is a C^1 function and satisfies*

A_1. *$k_1\|u\|^p \leq \mathcal{V}(t,u) \leq k_2(t)\|u\|^q$, where k_1, p, q are some positive constants and $k_2 : [b,\infty) \to \mathbb{R}_+$ is a continuous function,*

A_2. *along system (4) non-trivial solution $u(t)$:*

$$ {}^{C}\mathcal{D}^{\beta}_{b,t}\mathcal{V}(t,u(t)) \leq \mu\mathcal{V}(t,u(t)) - g(t)\mathcal{V}(t,u(t)) + w(t),\ \forall t > b,\ \forall u \in \mathbb{R}^n - \{0\}, \tag{37} $$

where

- *i) fractional order $\beta \in (0,1]$, and some constant $\mu > 0$,*
- *ii) the function $g \in \mathbb{R}_{\geq 0}$ is continuous on $[b,\infty)$ and satisfies $g(t) \geq \lambda > 0$ for all $t \in [b,\infty)$ with some constant λ,*
- *iii) the function $w \in \mathbb{R}_{\geq 0}$ is continuous on $[b,\infty)$ and obeys $\int_b^t (t-\tau)^{\beta-1}w(\tau)d\tau \to 0$ as $t \to \infty$.*

Then, the system (4) is globally asymptotically stable if the condition $\mu < \lambda$ is satisfied.

Proof. We modify the inequality (37) in A_2 and consider an equation

$$ \begin{aligned} {}^{C}\mathcal{D}^{\beta}_{b,t}\mathcal{V}(t,u(t)) &= \mu\mathcal{V}(t,u(t)) - g(t)\mathcal{V}(t,u(t)) + w(t) - N(t) \\ \mathcal{V}(b,u(b)) &= \mathcal{V}_b \end{aligned} \tag{38} $$

where N is non-negative continuous function on $[b,\infty)$. We use Riemann-Liouville integral on (38), and obtain

$$ \begin{aligned} \mathcal{V}(t,u(t)) = \mathcal{V}(b,u(b)) &+ \frac{\mu}{\Gamma(\beta)}\int_b^t (t-\tau)^{\beta-1}\mathcal{V}(\tau,u(\tau))d\tau \\ &- \frac{1}{\Gamma(\beta)}\int_b^t (t-\tau)^{\beta-1}g(\tau)\mathcal{V}(\tau,u(\tau))d\tau \\ &+ \frac{1}{\Gamma(\beta)}\int_b^t (t-\tau)^{\beta-1}w(\tau)d\tau - \frac{1}{\Gamma(\beta)}\int_b^t (t-\tau)^{\beta-1}N(\tau)d\tau. \end{aligned} \tag{39} $$

Since N is non-negative and $g(t) \geq \lambda > 0$ for all $t \in [b,\infty)$, it follows from (39) that

$$ \begin{aligned} \mathcal{V}(t,u(t)) \leq \mathcal{V}(b,u(b)) &+ \frac{\mu}{\Gamma(\beta)}\int_b^t (t-\tau)^{\beta-1}\mathcal{V}(\tau,u(\tau))d\tau \\ &- \frac{\lambda}{\Gamma(\beta)}\int_b^t (t-\tau)^{\beta-1}\mathcal{V}(\tau,u(\tau))d\tau + \frac{1}{\Gamma(\beta)}\int_b^t (t-\tau)^{\beta-1}w(\tau)d\tau. \end{aligned} \tag{40} $$

By using Corollary 2 in (40), one gets

$$ \mathcal{V}(t,u(t)) \leq E_{\beta,1}\Big((\mu-\lambda)(t-b)^\beta\Big)\left[\mathcal{V}(b,u(b)) + \left(E_{\beta,1}\left(\beta\int_b^t (t-\tau)^{\beta-1}w(\tau)d\tau\right) - 1\right)\right]. \tag{41} $$

Then from A_1 and (41), we estimate

$$ \|u(t)\| \leq C^{\#}\left[\frac{1}{k_1}E_{\beta,1}\Big(-(\lambda-\mu)(t-b)^\beta\Big)\left[k_2(b)\|u(b)\|^q + \left(E_{\beta,1}\left(\beta\int_b^t (t-\tau)^{\beta-1}w(\tau)d\tau\right) - 1\right)\right]\right]^{\frac{1}{p}}, \tag{42} $$

for all $t \geq b$, where constant $C^{\#} \geq 1$. Whenever $\lambda > \mu$, $E_{\beta,1}\Big(-(\lambda-\mu)(t-b)^\beta\Big) \to 0$ as $t \to \infty$ (see, e.g., Theorem 1.6 of [1]), and by noticing $\int_b^t (t-\tau)^{\beta-1}w(\tau)d\tau \to 0$ as $t \to \infty$, it follows from (42) that

$$ \lim_{t\to\infty} u(t) = 0. \tag{43} $$

Thus, the system (4) should be globally asymptotically stable provided the condition $\lambda > \mu$ must hold. This completes the proof. □

Theorem 9. *Suppose that $\mathcal{V} : [b, \infty) \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is a $\mathcal{C}^1$ function and satisfies*

A_1. *$\alpha_1(\|u\|) \leq \mathcal{V}(t, u) \leq k_2(t)\alpha_2(\|u\|)$, where $\alpha_1(\cdot), \alpha_2(\cdot)$ belongs to **class-**$\mathcal{K}_\infty$ and $k_2 : [b, \infty) \to \mathbb{R}_+$ is a continuous,*

A_2. *along system (4) non-trivial solution $u(t)$:*

$$ {}^{C}\mathcal{D}^{\beta}_{b,t}\mathcal{V}(t, u(t)) \leq -g(t)\mathcal{V}(t, u(t)) + w(t), \ \forall t > b, \ \forall u \in \mathbb{R}^n - \{0\}, \tag{44} $$

where

i) fractional order $\beta \in (0, 1]$,

ii) the function $g \in \mathbb{R}_{\geq 0}$ is continuous on $[b, \infty)$ and satisfies $\int_b^t (t-\tau)^{\beta-1} g(s) ds \to \infty$ as $t \to \infty$,

iii) the function $w \in \mathbb{R}_{\geq 0}$ is continuous on $[b, \infty)$ and satisfies $\int_b^t (t-\tau)^{\beta-1} w(\tau) d\tau \to 0$ as $t \to \infty$.

Then, the system (4) is globally asymptotically stable.

Proof. In view of A_2, we define an initial value problem

$$ \begin{aligned} {}^{C}\mathcal{D}^{\beta}_{b,t}\mathcal{V}(t, u(t)) &= -g(t)\mathcal{V}(t, u(t)) + w(t) - N(t) \\ \mathcal{V}(b, u(b)) &= \mathcal{V}_b \end{aligned} \tag{45} $$

where N is non-negative continuous function on $[b, \infty)$. Taking Riemann-Liouville integral on (45), we obtain

$$ \begin{aligned} \mathcal{V}(t, u(t)) = \mathcal{V}(b, u(b)) &- \frac{1}{\Gamma(\beta)} \int_b^t (t-\tau)^{\beta-1} g(\tau)\mathcal{V}(\tau, u(\tau)) d\tau \\ &+ \frac{1}{\Gamma(\beta)} \int_b^t (t-\tau)^{\beta-1} w(\tau) d\tau - \frac{1}{\Gamma(\beta)} \int_b^t (t-\tau)^{\beta-1} N(\tau) d\tau. \end{aligned} \tag{46} $$

Since N is non-negative, we have from (46) that

$$ \begin{aligned} \mathcal{V}(t, u(t)) \leq \mathcal{V}(b, u(b)) &- \frac{1}{\Gamma(\beta)} \int_b^t (t-\tau)^{\beta-1} g(\tau)\mathcal{V}(\tau, u(\tau)) d\tau \\ &+ \frac{1}{\Gamma(\beta)} \int_b^t (t-\tau)^{\beta-1} w(\tau) d\tau. \end{aligned} \tag{47} $$

Now we employ Theorem 6 to inequality (47), and establish

$$ \mathcal{V}(t, u(t)) \leq E_{\beta,1}\left(-\beta \int_b^t (t-\tau)^{\beta-1} g(\tau) d\tau\right)\left[\mathcal{V}(b, u(b)) + \left(E_{\beta,1}\left(\beta \int_b^t (t-\tau)^{\beta-1} w(\tau) d\tau\right) - 1\right)\right]. \tag{48} $$

As a result, from A_1 and (48), we get

$$ \begin{aligned} \alpha_1(\|u(t)\|) \leq C^{\#}&\left[E_{\beta,1}\left(-\beta \int_b^t (t-\tau)^{\beta-1} g(\tau) d\tau\right) k_2(b)\alpha_2(\|u(b)\|)\right] \\ &+ C^{\#} E_{\beta,1}\left(-\beta \int_b^t (t-\tau)^{\beta-1} g(\tau) d\tau\right)\left[\left(E_{\beta,1}\left(\beta \int_b^t (t-\tau)^{\beta-1} w(\tau) d\tau\right) - 1\right)\right], \end{aligned} \tag{49} $$

for all $t \geq b$, where constant $C^{\#} \geq 1$. Letting $t \to \infty$, and using $\int_b^t (t-\tau)^{\beta-1} g(\tau) d\tau \to \infty$ as $t \to \infty$, and $\int_b^t (t-\tau)^{\beta-1} w(\tau) d\tau \to 0$ as $t \to \infty$, it follows from (49) that

$$ \lim_{t \to \infty} u(t) = 0. \tag{50} $$

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Thus, the system (4) should be globally asymptotically stable. This completes the proof. □

Theorem 10. *Suppose that $\mathcal{V} : [b, \infty) \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is a C^1 function and satisfies*

A_1. *$\alpha_1(\|u\|) \leq \mathcal{V}(t, u) \leq k_2(t)\alpha_2(\|u\|)$, where $\alpha_1(\cdot), \alpha_2(\cdot)$ belongs to **class-$\mathcal{K}_\infty$** and $k_2 : [b, \infty) \to \mathbb{R}_+$ is a continuous,*

A_2. *along system (4) non-trivial solution $u(t)$:*

$$^{C}\mathcal{D}^{\beta}_{b,t}\mathcal{V}(t, u(t)) \leq -\lambda^+ g(t)\mathcal{V}(t, u(t)) + w(t),\ \forall t > b,\ \forall u \in \mathbb{R}^n - \{0\}, \tag{51}$$

where

i) fractional order $\beta \in (0, 1]$, and some constant $\lambda^+ > 0$,

ii) the function $g \in \mathbb{R}_{\geq 0}$ is non-constant, continuous, and decreasing on $[b, \infty)$, and satisfies $\int_b^t g(s)ds \to \infty$ as $t \to \infty$,

iii) the function $w \in \mathbb{R}_{\geq 0}$ is continuous on $[b, \infty)$ and satisfies $\int_b^t (t - \tau)^{\beta-1} w(\tau)d\tau \to 0$ as $t \to \infty$.

Then, the system (4) is globally asymptotically stable.

Proof. By following the proof of Theorem 9 and using Theorem 3, one obtains the result. □

Theorem 11. *Suppose that $\mathcal{V} : [b, \infty) \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is a C^1 function and satisfies*

A_1. *$\alpha_1(\|u\|) \leq \mathcal{V}(t, u) \leq k_2(t)\alpha_2(\|u\|)$, where $\alpha_1(\cdot), \alpha_2(\cdot)$ belongs to **class-$\mathcal{K}_\infty$** and $k_2 : [b, \infty) \to \mathbb{R}_+$ is a continuous,*

A_2. *along system (4) non-trivial solution $u(t)$:*

$$^{C}\mathcal{D}^{\beta}_{b,t}\mathcal{V}(t, u(t)) \leq g_1(t)\mathcal{V}(t, u(t)) - g_2(t)\mathcal{V}(t, u(t)) + w(t),\ \forall t > b,\ \forall u \in \mathbb{R}^n - \{0\}, \tag{52}$$

where

i) fractional order $\beta \in (0, 1]$,

ii) the functions $g_1, g_2 \in \mathbb{R}_{\geq 0}$ are continuous on $[b, \infty)$ and $\int_b^t (t - s)^{\beta-1}(g_1(s) - g_2(s))ds \to -\infty$ as $t \to \infty$,

iii) the function $w \in \mathbb{R}_{\geq 0}$ is continuous on $[b, \infty)$ and satisfies $\int_b^t (t - \tau)^{\beta-1} w(\tau)d\tau \to 0$ as $t \to \infty$.

Then, the system (4) is globally asymptotically stable.

Proof. By using Theorem 7, one obtains the result. □

Remark 2. *The proofs of Theorem 8, Theorem 9, and Theorem 11 are new in the sense of novel applications of some of our introduced generalized Gronwall inequalities. Proving these theorems by means of any different method remains a challenging exercise problem.*

4.2. Some growth and decay bounds

In this subsection we establish new growth and decay bounds of solutions of system (4). Our results improve recent results of growth-decay estimates found in work [17].

Theorem 12. *Suppose that $\mathcal{V} : [b, \infty) \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is a C^1 function and satisfies*

A_1. *$\alpha_1(\|u\|) \leq \mathcal{V}(t, u) \leq k_2(t)\alpha_2(\|u\|)$, where $\alpha_1(\cdot), \alpha_2(\cdot)$ belongs to **class-$\mathcal{K}_\infty$** and $k_2 : [b, \infty) \to \mathbb{R}_+$ is a continuous,*

A_2. *along system (4) non-trivial solution $u(t)$:*

$$^{C}\mathcal{D}^{\beta}_{b,t}\mathcal{V}(t, u(t)) \leq -g(t)\mathcal{V}(t, u(t)) + w(t),\ \forall t > b,\ \forall u \in \mathbb{R}^n - \{0\}, \tag{53}$$

where $\beta \in (0, 1]$, $g \in \mathbb{R}_{\geq 0}$ is continuous on $[b, \infty)$, and $w \in \mathbb{R}_{\geq 0}$ is continuous on $[b, \infty)$.

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Then, the solution $u(t)$ of system (4) obey the bound

$$\begin{aligned}\alpha_1(\|u(t)\|) \leq C^{\#}\Big[E_{\beta,1}\Big(-\beta\int_b^t (t-\tau)^{\beta-1}g(\tau)d\tau\Big)k_2(b)\alpha_2(\|u(b)\|)\Big] \\ + C^{\#}E_{\beta,1}\Big(-\beta\int_b^t (t-\tau)^{\beta-1}g(\tau)d\tau\Big)\Big[\Big(E_{\beta,1}\Big(\beta\int_b^t (t-\tau)^{\beta-1}w(\tau)d\tau\Big)-1\Big)\Big],\end{aligned} \tag{54}$$

for all $t \geq b$, where constant $C^{\#} \geq 1$.

Proof. First, we modify the inequality in A_2 and construct an equation

$$\begin{aligned}{}^{C}\mathcal{D}^{\beta}_{b,t}\mathcal{V}(t,u(t)) &= -g(t)\mathcal{V}(t,u(t)) + w(t) - N(t) \\ \mathcal{V}(b,u(b)) &= \mathcal{V}_b\end{aligned} \tag{55}$$

where N is non-negative continuous function on $[b,\infty)$. Then, we apply Riemann-Liouville integral on (55) and obtain

$$\begin{aligned}\mathcal{V}(t,u(t)) = \mathcal{V}(b,u(b)) - \frac{1}{\Gamma(\beta)}\int_b^t (t-\tau)^{\beta-1}g(\tau)\mathcal{V}(\tau,u(\tau))d\tau \\ + \frac{1}{\Gamma(\beta)}\int_b^t (t-\tau)^{\beta-1}w(\tau)d\tau - \frac{1}{\Gamma(\beta)}\int_b^t (t-\tau)^{\beta-1}N(\tau)d\tau.\end{aligned} \tag{56}$$

Since N is non-negative, the expression (56) gives

$$\begin{aligned}\mathcal{V}(t,u(t)) \leq \mathcal{V}(b,u(b)) - \frac{1}{\Gamma(\beta)}\int_b^t (t-\tau)^{\beta-1}g(\tau)\mathcal{V}(\tau,u(\tau))d\tau \\ + \frac{1}{\Gamma(\beta)}\int_b^t (t-\tau)^{\beta-1}w(\tau)d\tau.\end{aligned} \tag{57}$$

Subsequently, by invoking Theorem 6 to inequality (57), one obtains

$$\mathcal{V}(t,u(t)) \leq E_{\beta,1}\Big(-\beta\int_b^t (t-\tau)^{\beta-1}g(\tau)d\tau\Big)\Big[\mathcal{V}(b,u(b)) + \Big(E_{\beta,1}\Big(\beta\int_b^t (t-\tau)^{\beta-1}w(\tau)d\tau\Big)-1\Big)\Big]. \tag{58}$$

As a result, we can obtain from A_1 and (58) that

$$\begin{aligned}\alpha_1(\|u(t)\|) \leq C^{\#}\Big[E_{\beta,1}\Big(-\beta\int_b^t (t-\tau)^{\beta-1}g(\tau)d\tau\Big)k_2(b)\alpha_2(\|u(b)\|)\Big] \\ + C^{\#}E_{\beta,1}\Big(-\beta\int_b^t (t-\tau)^{\beta-1}g(\tau)d\tau\Big)\Big[\Big(E_{\beta,1}\Big(\beta\int_b^t (t-\tau)^{\beta-1}w(\tau)d\tau\Big)-1\Big)\Big],\end{aligned} \tag{59}$$

for all $t \geq b$, where constant $C^{\#} \geq 1$. This closes the proof. □

Theorem 13. *Suppose that $\mathcal{V} : [b,\infty) \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is a C^1 function and satisfies*

A_1. *$\alpha_1(\|u\|) \leq \mathcal{V}(t,u) \leq k_2(t)\alpha_2(\|u\|)$, where $\alpha_1(\cdot), \alpha_2(\cdot)$ belongs to **class-$\mathcal{K}_\infty$** and $k_2 : [b,\infty) \to \mathbb{R}_+$ is a continuous,*

A_2. *along system (4) non-trivial solution $u(t)$:*

$$ {}^{C}\mathcal{D}^{\beta}_{b,t}\mathcal{V}(t,u(t)) \leq g(t)\mathcal{V}(t,u(t)) + w(t),\ \forall t > b,\ \forall u \in \mathbb{R}^n - \{0\}, \tag{60}$$

where $\beta \in (0,1]$, $g \in \mathbb{R}_{\geq 0}$ is continuous on $[b,\infty)$, and $w \in \mathbb{R}_{\geq 0}$ is continuous on $[b,\infty)$.

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Then, the solution $u(t)$ of system (4) obey the bound

$$\alpha_1(\|u(t)\|) \leq C^{\#}\left[E_{\beta,1}\left(\beta\int_b^t (t-\tau)^{\beta-1}g(\tau)d\tau\right)k_2(b)\alpha_2(\|u(b)\|)\right] + C^{\#}E_{\beta,1}\left(\beta\int_b^t (t-\tau)^{\beta-1}g(\tau)d\tau\right)\left[\left(E_{\beta,1}\left(\beta\int_b^t (t-\tau)^{\beta-1}w(\tau)d\tau\right)-1\right)\right], \tag{61}$$

for all $t \geq b$, where constant $C^{\#} \geq 1$.

Proof. By using Theorem 5, one obtains the result. □

Theorem 14. *Suppose that $\mathcal{V} : [b,\infty) \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is a C^1 function and satisfies*

A_1. *$\alpha_1(\|u\|) \leq \mathcal{V}(t,u) \leq k_2(t)\alpha_2(\|u\|)$, where $\alpha_1(\cdot), \alpha_2(\cdot)$ belongs to **class-$\mathcal{K}_\infty$** and $k_2 : [b,\infty) \to \mathbb{R}_+$ is a continuous,*

A_2. *along system (4) non-trivial solution $u(t)$:*

$$^{C}\mathcal{D}^{\beta}_{b,t}\mathcal{V}(t,u(t)) \leq g_1(t)\mathcal{V}(t,u(t)) - g_2(t)\mathcal{V}(t,u(t)) + w(t),\ \forall t > b,\ \forall u \in \mathbb{R}^n - \{0\}, \tag{62}$$

where $\beta \in (0,1]$, $g_1, g_2 \in \mathbb{R}_{\geq 0}$ are continuous on $[b,\infty)$, and $w \in \mathbb{R}_{\geq 0}$ is continuous on $[b,\infty)$.

Then, the solution $u(t)$ of system (4) obey the bound

$$\alpha_1(\|u(t)\|) \leq C^{\#}\left[E_{\beta,1}\left(\beta\int_b^t (t-\tau)^{\beta-1}[g_1(\tau)-g_2(\tau)]d\tau\right)k_2(b)\alpha_2(\|u(b)\|)\right] + C^{\#}E_{\beta,1}\left(\beta\int_b^t (t-\tau)^{\beta-1}[g_1(\tau)-g_2(\tau)]d\tau\right)\left[\left(E_{\beta,1}\left(\beta\int_b^t (t-\tau)^{\beta-1}w(\tau)d\tau\right)-1\right)\right], \tag{63}$$

for all $t \geq b$, where constant $C^{\#} \geq 1$.

Proof. By using Theorem 7, one obtains the result. □

Remark 3. *It may be noted that proving the obtained sharp bounds (53) of Theorem 12, (60) of Theorem 13, and (62) of Theorem 14 by means of any different methods remains an open problem.*

5. Applications

This section provides some peculiar examples of fractional order systems to illustrate a few applicable results.

Example 1. *Let us consider the non-autonomous linear fractional order system*

$$^{C}\mathcal{D}^{\alpha}_{0,t}u(t) = -E_{\alpha,1}(t^{\alpha})u(t) \tag{64}$$

where $\alpha \in (\frac{1}{2},1)$, and $u(0) = u_0 \in \mathbb{R}$.

Here we use the Lyapunov function $\mathcal{V}(t,u) = u^2$. Set $\beta = \alpha$. Then, by using Lemma 1 of [10] along equation (64), we obtain

$$\begin{aligned} ^{C}\mathcal{D}^{\beta}_{0,t}\mathcal{V}(t,u(t)) &\leq 2u(t)\,^{C}\mathcal{D}^{\beta}_{0,t}u(t) \\ &= -2E_{\beta,1}\left(t^{\beta}\right)\mathcal{V}(t,u(t)),\ \forall t > 0,\ u \neq 0. \end{aligned} \tag{65}$$

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Denote by $g(t) = 2E_{\beta,1}(t^{\beta})$ and $w(t) = 0$. Observe that

$$\begin{aligned}\int_0^t (t-\tau)^{\beta-1} g(\tau) d\tau &= 2\int_0^t (t-\tau)^{\beta-1} E_{\beta,1}\left(\tau^{\beta}\right) d\tau \\ &= 2\Gamma(\beta) t^{2\beta-1} E_{\beta,2\beta}\left(t^{\beta}\right).\end{aligned} \tag{66}$$

Since $\int_0^t (t-\tau)^{\beta-1} g(\tau) d\tau \to \infty$ as $t \to \infty$, and $\int_0^t (t-\tau)^{\beta-1} w(\tau) d\tau \to 0$ as $t \to \infty$, by Theorem 9 the system (64) should be globally asymptotically stable.

Remark 4. *It may be noted that using Theorem* 1 *of* [9], *one can predict global asymptotic stability of the zero solution of equation* (64). *In contrast to Theorem* 1 *of* [9], *our new Theorem 9 gives a more general form to predict global asymptotic stability of* (64).

Example 2. *Let us consider the non-autonomous linear fractional order system*

$${}^{C}\mathcal{D}_{b,t}^{\alpha} u(t) = -\frac{1}{1+t-b} u(t) \tag{67}$$

where $b \in \mathbb{R}$, $\alpha \in (0,1)$, *and* $u(b) = u_b \in \mathbb{R}$.

Take the Lyapunov function $\mathcal{V}(t,u) = u^2$. Set $\beta = \alpha$. Then, by using Lemma 1 of [10] along equation (67), one has

$$\begin{aligned}{}^{C}\mathcal{D}_{b,t}^{\beta} \mathcal{V}(t,u(t)) &\le 2u(t)\,{}^{C}\mathcal{D}_{b,t}^{\beta} u(t) \\ &= -\frac{2}{1+t-b} \mathcal{V}(t,u(t)), \ \forall t > b, \ u \neq 0.\end{aligned} \tag{68}$$

Denote by $g(t) = \frac{1}{1+t-b}$, $\lambda^{+} = 2$ and $w(t) = 0$. Since $\int_b^t g(\tau) d\tau = \log(1+t-b) \to \infty$ as $t \to \infty$, and $\int_b^t (t-\tau)^{\beta-1} w(\tau) d\tau \to 0$ as $t \to \infty$, by Theorem 10 the system (67) should be globally asymptotically stable.

Remark 5. *It may be noted that the literature results* [8,9,11–14] *seem not effective for equation* (67) *to predict global asymptotic stability of the zero solution as* $t \to \infty$. *In contrast, our new Theorem 10 gives a new mathematical tool to predict global asymptotic stability of* (67).

In Example 3, we have considered an autonomous nonlinear fractional order system (69), and it is demonstrated that by using our new result, Theorem 9 is effective to predict global asymptotic stability of such a system.

Example 3. *Let* $u = (u_1, u_2)^T \in \mathbb{R}^2$ *and consider the autonomous fractional order system*

$$\begin{aligned}{}^{C}\mathcal{D}_{b,t}^{\alpha} u_1(t) &= -4u_1(t) + u_2(t) + \sin(u_1(t)) \\ {}^{C}\mathcal{D}_{b,t}^{\alpha} u_2(t) &= u_1(t) - 4u_2(t) + \sin(u_2(t))\end{aligned} \tag{69}$$

where $b \in \mathbb{R}$, $\alpha \in (0,1)$ *and* $u(b) = (u_1(b), u_2(b))^T \in \mathbb{R}^2$.

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First, we let the Lyapunov function $\mathcal{V}(t,u) = u_1^2 + u_2^2$. Set $\beta = \alpha$. Then, by using Lemma 1 of [10] along equation (69), one gets

$$\begin{aligned} {}^{C}\mathcal{D}^{\beta}_{b,t}\mathcal{V}(t,u(t)) &\leq 2u_1(t)\,{}^{C}\mathcal{D}^{\beta}_{b,t}u_1(t) + 2u_2(t)\,{}^{C}\mathcal{D}^{\beta}_{b,t}u_2(t) \\ &= -8u_1^2(t) + 2u_1(t)u_2(t) + 2u_1(t)\sin(u_1(t)) \\ &\quad + 2u_2(t)u_1(t) - 8u_2^2(t) + 2u_2(t)\sin(u_2(t)) \\ &\leq -8u_1^2(t) + 4u_1(t)u_2(t) - 8u_2^2(t) + 2u_1^2(t) + 2u_2^2(t) \\ &\leq -4\mathcal{V}(t,u(t)),\ \forall t > b,\ \forall u \in \mathbb{R}^2 - \{(0,0)^T\}. \end{aligned} \tag{70}$$

Take the functions $g(t) = 4$ and $w(t) = 0$. Since $\int_b^t (t-\tau)^{\beta-1} g(t) = \frac{4}{\beta}(t-b)^{\beta} \to \infty$ as $t \to \infty$, and $\int_b^t (t-\tau)^{\beta-1} w(\tau)d\tau \to 0$ as $t \to \infty$, clearly, the assumptions A_1 and A_2 of Theorem 9 are satisfied. Thus, we conclude that the system (69) should be globally asymptotically stable.

In Example 4, one of our new results has been utilized to establish sharp bounds to solutions of equation (71).

Example 4. *Let $\alpha \in (0,1)$, let $b \in \mathbb{R}$, and let $\gamma \in \mathbb{R}_+$. Consider the non-autonomous fractional order system*

$$\begin{aligned} {}^{C}\mathcal{D}^{\alpha}_{b,t}u(t) &= 2u(t) - \Big(3 + (t-b)^5\Big)u(t) + (t-b)^{\gamma} \\ u(b) &= u_b. \end{aligned} \tag{71}$$

Take the Lyapunov function $\mathcal{V}(t,u) = u^2$. Observe that $\frac{1}{2}u^2 \leq \mathcal{V}(t,u) \leq 2u^2$. By using Lemma 2.1 of [9] along system (71) solution $u(t)$, we obtain

$$\begin{aligned} {}^{C}\mathcal{D}^{\alpha}_{b,t}\mathcal{V}(t,u(t)) &\leq 2u(t)\,{}^{C}\mathcal{D}^{\alpha}_{b,t}u(t) \\ &= 4u^2(t) - 2\Big(3 + (t-b)^5\Big)u^2(t) + 2u(t)(t-b)^{\gamma} \\ &\leq 4u^2(t) - 2\Big(3 + (t-b)^5\Big)u^2(t) + u^2(t) + (t-b)^{2\gamma} \\ &= 5\mathcal{V}(t,u(t)) - \Big[6 + 2(t-b)^5\Big]\mathcal{V}(t,u(t)) + (t-b)^{2\gamma},\ \forall t > b,\ \forall u \in \mathbb{R} - \{0\}. \end{aligned} \tag{72}$$

Set $\beta = \alpha$, $g_1(t) = 5$, $g_2(t) = 6 + 2(t-b)^5$ and $w(t) = (t-b)^{2\gamma}$. Take $\alpha_1(\|u\|) = \frac{1}{2}u^2$, $\alpha_2(\|u\|) = 2u^2$ and $k_2(t) = 1$. Observe that A_1 and A_2 of Theorem 14 holds, and one has

$$\begin{aligned} \frac{1}{2}u^2(t) &\leq C^{\#}\Big[E_{\alpha,1}\Big(\alpha\int_b^t (t-\tau)^{\alpha-1}\Big[5 - 6 - 2(\tau-b)^5\Big]d\tau\Big)2u^2(b)\Big] \\ &\quad + C^{\#}E_{\alpha,1}\Big(\alpha\int_b^t (t-\tau)^{\alpha-1}\Big[5 - 6 - 2(\tau-b)^5\Big]d\tau\Big) \\ &\quad \times \Big[\Big(E_{\alpha,1}\Big(\alpha\int_b^t (t-\tau)^{\alpha-1}(\tau-b)^{2\gamma}d\tau\Big) - 1\Big)\Big], \end{aligned} \tag{73}$$

for all $t \geq b$, where constant $C^{\#} \geq 1$. Thus, we have the estimate for the solution of (71):

$$\begin{aligned} u^2(t) &\leq 4C^{\#}\Big[E_{\alpha,1}\Big(-(t-b)^{\alpha} - \frac{2\alpha\Gamma(\alpha)\Gamma(6)}{\Gamma(\alpha+6)}(t-b)^{5+\alpha}\Big)u^2(b)\Big] \\ &\quad + 2C^{\#}E_{\alpha,1}\Big(-(t-b)^{\alpha} - \frac{2\alpha\Gamma(\alpha)\Gamma(6)}{\Gamma(\alpha+6)}(t-b)^{5+\alpha}\Big) \\ &\quad \times \Big[E_{\alpha,1}\Big(\frac{\alpha\Gamma(\alpha)\Gamma(2\gamma+1)}{\Gamma(\alpha+2\gamma+1)}(t-b)^{2\gamma+\alpha}\Big) - 1\Big], \end{aligned} \tag{74}$$

for all $t \geq b$, where constant $C^{\#} \geq 1$.

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6. Conclusions

Many different versions of fractional generalized Gronwall inequalities are widely not known in the literature. Is it possible to formulate the right kinds of many such inequalities? This paper discusses some new formulations of fractional generalized Gronwall inequalities containing both non-positive and non-negative singular kernels that were not available in the literature previously. Some typical inequalities have been used in the rigorous developments of new fractional Lyapunov theorems that concern the global asymptotic stability of non-autonomous fractional order systems. The new Lyapunov theorems give novel sufficient conditions that are linked with the integral coefficient of the fractional Lyapunov differential inequality attached to them. The ideas of Lyapunov functions and some inequalities have further been used to establish bounds of non-autonomous fractional order systems. To close this discussion, we have provided some examples for the novel importance of special results.

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