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Alena-Rainich-Codazzi Branch Geometry and Integral Twistor-Link Zero Modes: Structural Reconstruction of Standard Model Multiplets

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Abstract

This article develops an Alena-Rainich-Codazzi (ARC) branch geometry for traceless gauge-side stress tensors of non-null Rainich type. In the admissible sector, the stress-balance condition is represented by a Codazzi-closed anisotropic Lorentzian branch metric whose residual scalar acts as a Codazzi multiplier. The resulting branch has shear-free geodesic Rainich principal null directions and admits explicit warped examples. Compact-leaf obstructions are treated through regular integral link data. For a branch worldline, the link sphere is identified with the projective spinor sphere, and the associated spin^c zero modes give the finite link spaces used in the reconstruction. A gap-current Fredholm functional then selects the minimal active support $E_3 \oplus E_2$, yielding the compact zero-mode basis group $S(U(3) \times U(2))$. Applying the exterior-algebra package to this reconstructed module gives one Standard Model generation, a gap-selected chirality locking, and the invariant one-Higgs channels; a projective color lift gives three central sectors when the stated non-degeneracy condition holds. The result is structural: Yukawa matrices, fermion masses, mixing data, confinement scales, and full spectra remain spectral data of the global ARC branch problem.

Keywords: Rainich geometry; Codazzi tensors; optical structures; anisotropic Lorentzian metrics; gauge stress tensors; spinor geometry; holonomy; geometric field theory; Alena Tensor

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1. Introduction

The geometrization of field interactions usually proceeds by enlarging the geometric structure. In Kaluza-Klein-type models, gauge variables are represented by a higher-dimensional or bundle metric structure [1]. In Eisenhart-Duval-type constructions, forced dynamics is rewritten as geodesic dynamics on an enlarged space [2]. Randers and Finsler descriptions encode charged motion by an effective geometry of trajectories [3]. The question considered here is narrower: whether a gauge-side stress tensor of the appropriate algebraic type can be represented by a four-dimensional anisotropic Lorentzian branch, without adding a spacetime dimension and without introducing an independent Finsler norm.

The starting point is the Alena Tensor identification (14) [4]. The Alena Tensor setting supplies the residual density, the translational-current coefficient, and the vortex terms used below [5]. The continuum, variational, and Higgs-like branch-potential parts of this background are the Alena Tensor inputs used here [6,7], and [8]. In the present construction these ingredients are constrained by a Codazzi closure and by finite link data. The finite internal labels are therefore not introduced as local tensor indices; they are recovered from the regular projective link sector of the closed branch.

The physically ARC-admissible sector consists of stationary finite-action non-null Rainich branches for which the Hilbert stress balance is absorbed by a Codazzi-closed branch geometry, the anisotropy has a single gauge-vorticity reading, compact-leaf obstructions carry regular integral low-trace links, and the observed fermion space is the lowest gap-locked spectral cluster. The Codazzi multiplier supplies the differential closure which complements the algebraic Rainich conditions [9], as in the classical reconstruction problem [10]. Physical ARC admissibility is a selection condition on Alena-Rainich branch configurations.

The local branch is built from a tetrad (U, N, W, S). The Rainich plane is span(U, N), while (W, S) spans the transverse two-plane. The branch metric leaves the Rainich plane unscaled and rescales the transverse plane by χ. The same anisotropy is read from the algebraic gauge-to-branch relation and from the normalized vorticity-flux closure. The trace-adjusted Codazzi tensor gives the Levi-Civita divergence identity for the weighted branch stress, so the flat-reference stress response is represented by the branch geometry.

The Codazzi condition also supplies the optical input used by the link construction. On the non-degenerate Codazzi set, the Rainich principal null directions are geodesic and shear-free. The optical conclusion is obtained at the Codazzi level, without imposing the vacuum Einstein equations or Petrov-speciality assumptions. The resulting local twistor-line reading is the standard projective-spinor one [11,12]; related projective-spinor ideas also occur in twistor approaches to Standard Model geometry [13].

The local branch sector is non-empty. In a flat-reference warped class the Codazzi equations reduce to two first integrals, and a smooth noncompact hyperbolic representative is obtained. Compact transverse leaves have a mean-value obstruction in the null-normalization equation. A defect traced along the Rainich plane gives a branch worldline Γ, and a small sphere linking Γ becomes the link sphere. When the defect is resolved by patching the oriented transverse frame and the regular phase-current sector, the corresponding link connection has integral period and supplies the spin^c determinant line used below.

Closed local branch observables are invariant under finite non-singlet factors. Finite non-singlet data are therefore assigned to link, boundary, or sector degrees of freedom. For a branch worldline, the optical structure identifies the small link sphere with $S^2_\Gamma \simeq \mathbb{CP}^1$. The corresponding finite spaces are the positive spin^c zero-mode spaces $E_q = \ker D_q^+ \simeq H^0(\mathbb{CP}^1, \mathcal{O}(q-1))$. This is the standard monopole zero-mode calculation on S^2 [14], also appearing in the Taub-NUT Dirac setting [15]; equivalently, it is the $\mathbb{CP}^1$ case of Borel-Weil [16,17].

The Toeplitz cutoff on E_q makes E_2 the first dipole block and E_3 the first quadrupole-capable block whose low modes generically generate $\mathfrak{su}(3)$. The active low sources have two branch origins: the phase-current dipole source and the normal Codazzi-gap source. The gap-current Fredholm energy combines the positive Toeplitz-Casimir boundary Hessian with a Codazzi-gap support tension. Its minimum selects the two-channel support $E_3 \oplus E_2$. Standard Toeplitz quantization gives the finite-mode comparison class [18]; the same cutoff has the fuzzy-sphere form [19], with brane/fuzzy-sphere models providing the broader finite-matrix setting [20].

Once $\mathcal{V} = C \oplus W$ has been reconstructed from the link, the compact zero-mode basis group is reduced by the total top form to $S(U(3) \times U(2))$. The remaining one-generation representation theory is the standard $3 + 2$ exterior-algebra package familiar from $SU(5)/\mathrm{Spin}(10)$ [21] and from Clifford-ideal descriptions [22,23]. In the present construction this package is applied after $\mathcal{V}$ has been obtained from the ARC branch and its projective link.

The comparison with almost-commutative geometry is structural. In the noncommutative-geometric Standard Model the finite internal algebra is part of the input [24], and the spectral action supplies the dynamical principle [25]. Here the finite internal role is played by link zero modes of a closed four-dimensional branch. The construction is closer in this respect to geometric matter models, in which geometry and topology recover particle labels without introducing them as local tensor indices [26].

The fermionic sector is obtained by the gap-selected diagonal locking of branch chirality with total exterior parity. This is the primitive finite grading compatible with the branch Dirac term, the weak Clifford-odd map, and charge conjugation. When the chiral Fredholm gap condition holds, the mirror exterior-parity sector is separated from the isolated low cluster. The positive operator $K_F = Q_F^\dagger Q_F$ contains the branch Dirac term, the weak Clifford-odd map, and the holonomy-Codazzi response terms. The exterior-algebra identity fixes the representation structure, while the numerical fermion masses belong to the spectral data of the closed branch operator. When the locked low cluster is isolated, the same first-order structure gives the effective Dirac interface with fermionic QFT. The use of fixed exterior parity is compatible with differential-form descriptions of fermions [27].

The family refinement is global. Since closed local color observables are singlet-valued, the closed color holonomy is naturally projective. The lift from $PSU(3)$ to $SU(3)$ over the link sphere has three central classes. Boundary traces of the linearized Codazzi response may act as central-degree intertwiners; when the adjacent components are nonzero, the central phase and the shift generate $M_3(\mathbb{C})$. Flavor relations require additional spectral input. In the present setting, the relevant restricted source is the lowest central clock-shift sector.

The result is structural. The Codazzi part gives the closed branch geometry and stress-balance absorption. The integral projective link gives the spin^c zero-mode spaces and the Toeplitz visibility cutoff. The gap-current Fredholm problem gives the minimal $3+2$ link module. The reconstructed module gives the unimodular zero-mode basis group, the one-generation exterior module, the gap-selected diagonal locking, and the three-sector family algebra under the stated central-degree condition. The article follows this order: branch closure, link reconstruction, finite representation theory, locked fermions and families, and the structural reconstruction theorem. A reduced one-loop branch-level bosonic normalization check is recorded in Appendix F. Numerical Yukawa matrices, confinement energies, threshold corrections, and full spectra remain spectral and matching problems for the full Alena-Rainich-Codazzi branch dynamics.

2. ARC Branch Closure, Translational Phase, and Optical Sector

The ARC branch used below is fixed by a non-null Rainich stress type, a Codazzi-density closure, a regular translational-current coefficient, and the compact-leaf obstruction of the resulting optical branch. The first part of the section records the local branch tensor, the residual scalar, and the translational-current closure. The second part records the optical splitting, a non-empty warped class, and the obstruction which supplies the link data used in Section 3.

The construction is local. A flat Lorentzian metric $\eta_{\mu\nu}$ is used as the reference metric for the gauge-side description, while $k_{\mu\nu}$ denotes the anisotropic branch metric. The flow field U^μ is normalized in both descriptions, $\eta_{\mu\nu}U^\mu U^\nu = 1$ and $k_{\mu\nu}U^\mu U^\nu = 1$. Gauge-side contractions are made with $\eta_{\mu\nu}$ unless the branch metric is written explicitly. Standard gauge and spinor conventions are as in [28] and [29]; the branch connection is the Levi-Civita connection of $k_{\mu\nu}$ [30].

2.1. Non-Null Branch Tensor and Translational-Current Codazzi Closure

The branch is built from a local orthonormal tetrad (U, N, W, S). The vector U is timelike, N is selected by the normalized vorticity flux, and (W, S) spans the transverse two-plane. With tetrad covectors lowered by $\eta_{\mu\nu}$,

$$\eta_{\mu\nu} = U_\mu U_\nu - N_\mu N_\nu - W_\mu W_\nu - S_\mu S_\nu. \tag{1}$$

The anisotropic metric branch is

$$k_{\mu\nu} = U_\mu U_\nu - N_\mu N_\nu - e^{2\chi}\left(W_\mu W_\nu + S_\mu S_\nu\right), \tag{2}$$

so the Rainich plane span(U, N) is unchanged and the transverse plane is rescaled by χ. The branch trace and branch density are

$$k = \eta^{\mu\nu}k_{\mu\nu} = 2 + 2e^{2\chi}, \tag{3}$$

$$p_\Lambda = p_0 k^2. \tag{4}$$

The construction is used on the non-degenerate Lorentzian region of k.

The vorticity tensor is $\omega_{\mu\nu} = \Delta_\mu{}^\alpha \Delta_\nu{}^\beta \nabla^{(k)}_{[\alpha} U_{\beta]}$, with $\Delta^\mu{}_\nu = \delta^\mu{}_\nu - U^\mu U_\nu$. The vorticity flux vector is

$$D^\mu_\omega = a_\nu \omega^{\nu\mu} + \frac{c^2}{2} \Delta^\mu{}_\nu \nabla^{(k)}_\lambda \omega^{\lambda\nu}, \qquad a^\mu = U^\alpha \nabla^{(k)}_\alpha U^\mu. \tag{5}$$

The normalized flux branch is the branch for which $q^\mu = p_\Lambda N^\mu$. If $D^\mu_\omega \neq 0$, then

$$N^\mu = -\frac{D^\mu_\omega}{|D_\omega|}, \qquad |D_\omega| = \sqrt{-D_{\omega\alpha} D^\alpha_\omega}, \qquad \alpha = \frac{1}{|D_\omega|}. \tag{6}$$

The remaining transverse frame may be fixed locally from $\omega^\mu{}_\nu N^\nu$. Points at which this construction degenerates belong to a different branch sector or to the boundary of the normalized sector.

The branch tensor is

$$Y_{\mu\nu} = p_0 k^2 \left(\frac{4}{k} k_{\mu\nu} - \eta_{\mu\nu} \right). \tag{7}$$

In the tetrad above its nonzero diagonal components are

$$Y_{UU} = 4p_0(1 - e^{4\chi}), \qquad Y_{NN} = -Y_{UU}, \qquad Y_{WW} = Y_{SS} = Y_{UU}. \tag{8}$$

Equivalently,

$$\frac{Y_{UU}}{p_\Lambda} = -\tanh\chi. \tag{9}$$

The mixed tensor $Y^\mu{}_\nu$ has non-null Rainich type $(y, y, -y, -y)$,

$$\eta^{\mu\nu} Y_{\mu\nu} = 0, \qquad Y^\mu{}_\alpha Y^\alpha{}_\nu = \frac{1}{4} Y_{\alpha\beta} Y^{\alpha\beta} \delta^\mu{}_\nu. \tag{10}$$

This is the algebraic type used in the classical Rainich reconstruction problem [9]. Infinitesimal branch variations are constrained by preservation of this stratum:

$$\delta(\eta^{\mu\nu} Y_{\mu\nu}) = 0, \tag{11}$$

$$\delta\left(Y^\mu{}_\alpha Y^\alpha{}_\nu - \frac{1}{4} Y_{\alpha\beta} Y^{\alpha\beta} \delta^\mu{}_\nu \right) = 0. \tag{12}$$

Equivalently, if $\mathcal{R}_\eta$ is the non-null Rainich stratum and $Y = \rho J$ with $J^2 = 1$ and $\operatorname{tr} J = 0$, then admissible tangent vectors have the form $h = aJ + \rho K$, where $K^\dagger_\eta = K$, $\operatorname{tr} K = \operatorname{tr}(JK) = 0$, and $JK + KJ = 0$. The integrating curve is

$$J(t) = e^{t\mathcal{B}} J e^{-t\mathcal{B}}, \qquad \rho(t) = \rho + ta, \tag{13}$$

with $\mathcal{B} = -JK/2$. Generic stress perturbations leave this stratum [31]. Self-dual and two-form descriptions of this tensor class are standard in chiral and gauge-gravity formulations [32]; Urbantke reconstruction gives the corresponding conformal-geometric viewpoint [33].

The convention used below is

$$Y_{\mu\nu} = -\mathrm{Y}_{\mu\nu}. \tag{14}$$

Thus the gauge-side tensor is represented only in a tetrad in which

$$\mathrm{Y}_{NN} = -\mathrm{Y}_{UU}, \qquad \mathrm{Y}_{WW} = \mathrm{Y}_{SS} = \mathrm{Y}_{UU}, \tag{15}$$

and the anisotropy is fixed by

$$\tanh\chi = \frac{\mathrm{Y}_{UU}}{p_\Lambda}. \tag{16}$$

Equivalently,

$$e^{4\chi} = 1 + \frac{Y_{UU}}{4p_0}. \tag{17}$$

The real branch condition is

$$1 + \frac{Y_{UU}}{4p_0} > 0. \tag{18}$$

This completes the algebraic gauge-to-branch map.

Put

$$R_\omega = \frac{\omega^2}{|D_\omega|}, \qquad \omega^2 = \omega^{\mu\nu}\omega_{\mu\nu}. \tag{19}$$

The translational-current coefficient is regularized by a phase-amplitude pair

$$\Psi_\zeta = \rho_\zeta e^{if_\zeta}, \qquad \mu_\zeta = e^{-\vartheta(\rho_\zeta)}, \qquad \mu_\zeta > 0. \tag{20}$$

The vacuum normalization is $\mu_\zeta = 1$ on the unexcited link background. The scalar branch factor is

$$\phi = 1 - \zeta^2 - \mu_\zeta R_\omega. \tag{21}$$

The coefficient ζ is the translational-current coefficient of the flat-reference Alena Tensor description [5]. Thus $p_\Lambda\zeta^2$ is the translational part of the available density, while $p_\Lambda\mu_\zeta R_\omega$ is the stiffness-weighted rotational part. The scalar ϕ is the branch residual used below as the Codazzi multiplier.

The translational current is

$$J^\mu_{\rm tr} = p_\Lambda\zeta^2 U^\mu. \tag{22}$$

Its conservation is imposed variationally by the phase term

$$S_{\rm cur} = \int_X f_\zeta \nabla^{(k)}_\mu\big(p_\Lambda\zeta^2 U^\mu\big)\,d\mathrm{vol}_k. \tag{23}$$

Variation with respect to f_ζ gives

$$\nabla^{(k)}_\mu J^\mu_{\rm tr} = 0. \tag{24}$$

The amplitude part of the same effective sector is taken in the form

$$S_{\rho_\zeta} = -\int_X p_\Lambda\left(\frac{\ell_\zeta^2}{2}k^{\mu\nu}\partial_\mu\rho_\zeta\partial_\nu\rho_\zeta + V_\zeta(\rho_\zeta)\right)d\mathrm{vol}_k. \tag{25}$$

In the branch-sign convention used below, and after freezing the density ratio between $d\mathrm{vol}_\eta$ and $d\mathrm{vol}_k$ in the principal link problem, its local Euler equation is

$$\nabla^{(k)}_\mu\big(p_\Lambda\ell_\zeta^2\nabla^{(k)\mu}\rho_\zeta\big) = p_\Lambda\big(V_\zeta'(\rho_\zeta) + R_\omega\mu_\zeta'(\rho_\zeta)\big). \tag{26}$$

In the frozen link sector this reduces to

$$V_\zeta'(\rho_\zeta) + R_\omega\mu_\zeta'(\rho_\zeta) = 0. \tag{27}$$

The local stiffness is positive when

$$V_\zeta''(\rho_\zeta) + R_\omega\mu_\zeta''(\rho_\zeta) > 0. \tag{28}$$

The branch action used for the Hilbert response is correspondingly

$$S^{(\zeta)}_{\rm ARC} = \pm\int_X \phi p_\Lambda\,d\mathrm{vol}_\eta + S_{\rm cur} + S_{\rho_\zeta}. \tag{29}$$

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The coefficient ζ is therefore not an independent scalar multiplier in (29); it is the coefficient of the conserved translational current (22).

The rotational energy density is

$$E_{\rm rot} = \frac{p_\Lambda}{2}\mu_\zeta R_\omega. \tag{30}$$

In the canonical vorticity plane $\omega_{NS} = \varpi$, so that $\omega^2 = 2\varpi^2$. The normalized flux sector gives

$$T^{\mu\nu}_{\rm matter} = \varrho U^\mu U^\nu + \frac{p_\Lambda}{c^2}(U^\mu N^\nu + U^\nu N^\mu) - \tau^{\mu\nu}_{\rm new} + T^{\mu\nu}_{\rm grad}, \tag{31}$$

where, in the frozen stiffness sector,

$$\tau^{\mu\nu}_{\rm new} = \mu_\zeta \frac{p_\Lambda}{2|D_\omega|}\left(\sigma^\mu{}_\lambda\omega^{\nu\lambda} + \sigma^\nu{}_\lambda\omega^{\mu\lambda}\right) + \frac{p_\Lambda}{2}\mu_\zeta R_\omega\left(N^\mu N^\nu + \frac{1}{3}\Delta^{\mu\nu}\right), \tag{32}$$

$$T^{\mu\nu}_{\rm grad} = \nabla^{(k)}_\lambda\left(c^2\frac{E_{\rm rot}}{|D_\omega|}N^{(\mu}\omega^{\nu)\lambda}\right) + T^{\mu\nu}_{\rho_\zeta}. \tag{33}$$

Here $T^{\mu\nu}_{\rho_\zeta}$ denotes the Hilbert response of (25). In the link-frozen sector it is lower order in the normal boundary problem. The full Alena stress in the branch variables is $T^{\mu\nu}_{\rm matter} \pm \phi Y^{\mu\nu}$, with the sign fixed by the convention in (29). In the closed branch sector the matter part and the residual branch part are taken to be separately k-conserved:

$$\nabla^{(k)}_\mu T^{\mu\nu}_{\rm matter} = 0, \tag{34}$$

$$\nabla^{(k)\mu}(\phi Y_{\mu\nu}) = 0. \tag{35}$$

The first identity is the material Alena balance in the same branch geometry [5]. At a current-supported regular link, the frozen principal boundary problem of (34) is natural on S^2_Γ. Hence its principal part is $SU(2)$-equivariant. The standard spherical-harmonic decomposition and Schur lemma then make the inequivalent low types invariant; for tensor traces the same statement is expressed by tensor spherical harmonics [34,35]. The tangential second-order part is carried by (33); in the normalized non-degenerate sector its principal boundary Hessian is the positive angular Casimir on the low boundary trace.

The gradient term is the hyperstress contribution generated by the variation of $\alpha = |D_\omega|^{-1}$ and by the stiffness response. The trace is

$$k_{\mu\nu}T^{\mu\nu}_{\rm matter} = \varrho + \nabla^{(k)}_\lambda J^\lambda_{\rm grad} + k_{\mu\nu}T^{\mu\nu}_{\rho_\zeta}, \qquad J^\lambda_{\rm grad} = c^2\frac{E_{\rm rot}}{|D_\omega|}\varpi S^\lambda. \tag{36}$$

The gradient-closed sector has $\nabla^{(k)}_\lambda J^\lambda_{\rm grad} = 0$, while the active rotational-gradient sector has $\nabla^{(k)}_\lambda J^\lambda_{\rm grad} = E_{\rm rot}$.

The same branch fixes the material value of χ. If the diagonal shear conditions $\sigma_{NN} = \sigma_{SS}$, $\sigma_{NW} = 0$, and $\sigma_{SW} = 0$ hold, then

$$\tau_{{\rm new},NN} - \tau_{{\rm new},SS} = \mu_\zeta p_\Lambda \frac{2\varpi\sigma_{NS} + \varpi^2}{|D_\omega|}. \tag{37}$$

The branch tensor gives

$$Y_{NN} - Y_{SS} = 2p_\Lambda \tanh\chi. \tag{38}$$

Hence the reduced anisotropic closure gives

$$\tanh\chi = \mu_\zeta\frac{2\varpi\sigma_{NS} + \varpi^2}{2|D_\omega|}. \tag{39}$$

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Together with (16), this is the gauge-matter-vorticity compatibility condition

$$\frac{Y_{UU}}{p_\Lambda} = \mu_\zeta \frac{2\varpi\sigma_{NS} + \varpi^2}{2|D_\omega|}. \quad (40)$$

Thus the same anisotropy is read from the gauge stress and from the stiffness-weighted normalized shear-vorticity branch.

The Hilbert variation of (29) defines the ARC stress response by

$$\delta_\eta S^{(\zeta)}_{\text{ARC}} = \frac{1}{2}\int_X T^{\text{ARC}}_{\mu\nu}\delta\eta^{\mu\nu}\,\mathrm{dvol}_\eta + \text{boundary}. \quad (41)$$

On stationary compactly supported variations this gives the flat-reference balance law for the total ARC stress. The Codazzi-closed ARC sector is the sub-sector in which this balance is represented by the Levi-Civita connection of the branch metric.

Closure of the residual force-density term requires differential data in addition to the algebraic Rainich conditions. The residual scalar ϕ in (21) acts as a Codazzi multiplier, in parallel with the differential closure required in the classical Rainich problem [10]. Put $\tau = k^{\mu\nu}Y_{\mu\nu}$ and

$$B_{\mu\nu} = Y_{\mu\nu} - \frac{1}{3}\tau k_{\mu\nu}. \quad (42)$$

The residual scalar $\phi \neq 0$ is a Codazzi multiplier if $A_{\mu\nu} = \phi B_{\mu\nu}$ satisfies (51). If $C^B_{\alpha\mu\nu} = \nabla^{(k)}_\alpha B_{\mu\nu} - \nabla^{(k)}_\mu B_{\alpha\nu}$ and $\theta = \mathrm{d}\log|\phi|$, this is the scalar multiplier equation

$$C^B_{\alpha\mu\nu} + \theta_\alpha B_{\mu\nu} - \theta_\mu B_{\alpha\nu} = 0. \quad (43)$$

Thus the ARC sector is the branch sub-sector for which the same scalar (21) satisfies (43) with a locally exact θ. On a non-degenerate open set the scalar is fixed by the branch data.

Proposition 1 (Codazzi multiplier criterion)**.** *Let X be a connected non-degenerate open set on which $B_{\mu\nu}$ is non-degenerate, and let $\beta^{\mu\nu}$ be its inverse. If (43) holds, then θ is fixed by B:*

$$\theta_\alpha = -\frac{1}{3}\beta^{\mu\nu}C^B_{\alpha\mu\nu}. \quad (44)$$

Conversely, define θ_B by (44). A nonzero scalar Codazzi multiplier exists on X if and only if (43) holds with $\theta = \theta_B$ and θ_B is exact. Equivalently, $\mathrm{d}\theta_B = 0$ and all periods of θ_B vanish. In that case ϕ is determined on X up to multiplication by a nonzero constant.

Proof. Contracting (43) with $\beta^{\mu\nu}$ gives

$$\beta^{\mu\nu}C^B_{\alpha\mu\nu} + 4\theta_\alpha - \theta_\alpha = 0. \quad (45)$$

This gives (44). Hence the multiplier equation has no independent scalar freedom on the non-degenerate set. If a multiplier exists, then $\theta = \mathrm{d}\log|\phi|$, so θ_B is exact. Conversely, if (43) holds with $\theta = \theta_B$ and $\theta_B = \mathrm{d}f$, then $\phi = e^f$ gives a local Codazzi multiplier, with an arbitrary nonzero constant factor. The global condition is the vanishing of the de Rham class $[\theta_B] \in H^1_{\mathrm{dR}}(X)$, equivalently the vanishing of the periods $\int_\gamma \theta_B$ for closed cycles γ. □

Proposition 2 (Codazzi-current reduction of the translational coefficient)**.** *Let the residual scalar (21) be a Codazzi multiplier on a non-degenerate connected open set. If $\theta_B = \mathrm{d}f_B$, then*

$$\phi = Ce^{f_B}, \qquad \zeta^2 = 1 - \mu_\zeta R_\omega - Ce^{f_B}, \quad (46)$$

for a nonzero constant C. If, in addition, the translational current (22) is conserved, then

$$U(\mu_\zeta R_\omega) + Ce^{f_B}\theta_B(U) = \left(1 - \mu_\zeta R_\omega - Ce^{f_B}\right)\left(\nabla^{(k)}_\mu U^\mu + U(\log p_\Lambda)\right). \tag{47}$$

Proof. The first statement is the scalar part of Proposition 1 applied to (21). This gives $\phi = Ce^{f_B}$ and hence (46). The conservation law (24) gives

$$U(\zeta^2) + \zeta^2\left(\nabla^{(k)}_\mu U^\mu + U(\log p_\Lambda)\right) = 0. \tag{48}$$

Substitution of (46) gives (47). □

For the normalized branch, the trace-adjusted tensor is

$$A_{\mu\nu} = \phi p_\Lambda\left(\frac{1}{3}\left(\bar{k} - \frac{4}{k}\right)k_{\mu\nu} - \eta_{\mu\nu}\right), \qquad \bar{k} := k^{\mu\nu}\eta_{\mu\nu} = 2 + 2e^{-2\chi}. \tag{49}$$

It is chosen so that

$$\phi Y_{\mu\nu} = A_{\mu\nu} - \mathcal{A}k_{\mu\nu}, \qquad \mathcal{A} = A_{\alpha\beta}k^{\alpha\beta}. \tag{50}$$

Here k is the η-trace of the branch metric from (3), while $\bar{k}$ is the k-trace of $\eta_{\mu\nu}$. The latter is the trace used in $\mathcal{A}$.

Proposition 3 (Residual divergence closure)**.** *If*

$$\nabla^{(k)}_\alpha A_{\mu\nu} = \nabla^{(k)}_\mu A_{\alpha\nu}, \tag{51}$$

then (35) holds.

Proof. Contracting (51) gives

$$k^{\alpha\nu}\nabla^{(k)}_\alpha A_{\mu\nu} = k^{\alpha\nu}\nabla^{(k)}_\mu A_{\alpha\nu}. \tag{52}$$

By metric compatibility and symmetry,

$$\nabla^{(k)\nu} A_{\mu\nu} = \partial_\mu \mathcal{A}. \tag{53}$$

Equivalently,

$$\nabla^{(k)\mu} A_{\mu\nu} = \partial_\nu \mathcal{A}. \tag{54}$$

Together with (50), this gives

$$\nabla^{(k)\mu}(\phi Y_{\mu\nu}) = 0. \tag{55}$$

□

The implication is one-way. If $C^A_{\alpha\mu\nu} = \nabla^{(k)}_\alpha A_{\mu\nu} - \nabla^{(k)}_\mu A_{\alpha\nu}$, then

$$\nabla^{(k)\nu}(\phi Y_{\mu\nu}) = k^{\alpha\nu}C^A_{\alpha\mu\nu}. \tag{56}$$

Thus (55) expresses the vanishing of the contracted defect, whereas (51) gives the full branch-integrability condition.

2.2. Optical Splitting, Warped Realization, and Compact-Leaf Obstruction

In the k-orthonormal frame $E_0 = U$, $E_1 = N$, $E_2 = e^{-\chi}W$, $E_3 = e^{-\chi}S$, the endomorphism $A^a{}_b$ has type (M, M, P, P), with

$$M = \phi p_\Lambda \frac{\tanh\chi(\tanh\chi - 3)}{3(1 + \tanh\chi)}, \qquad P = \phi p_\Lambda \frac{\tanh\chi(\tanh\chi + 3)}{3(1 + \tanh\chi)}. \tag{57}$$

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Only the standard two-eigenvalue Codazzi splitting is needed below. The Riemannian background may be compared with [36], while Codazzi spacetime structures are treated in [37]. In the ARC branch this splitting has the following $2+2$ form.

Theorem 1 (Rainich-Codazzi optical decomposition)**.** *Let $A^\sharp$ be the k-self-adjoint endomorphism determined by $A_{\mu\nu}$. On an open set $\mathcal{U}$ on which the two eigenvalues in (57) are distinct, put*

$$T\mathcal{U} = E \oplus F, \qquad E = \mathrm{span}(U, N), \qquad F = \mathrm{span}(W, S). \tag{58}$$

Then (51) holds if and only if

$$d_E M = 0, \qquad\qquad d_F P = 0, \tag{59}$$

$$(\nabla^{(k)}_{X_1} X_2)_F = \frac{k(X_1, X_2)}{M - P}(\nabla^{(k)} M)_F, \qquad X_1, X_2 \in \Gamma(E), \tag{60}$$

$$(\nabla^{(k)}_{V_1} V_2)_E = \frac{k(V_1, V_2)}{P - M}(\nabla^{(k)} P)_E, \qquad V_1, V_2 \in \Gamma(F). \tag{61}$$

In particular, for

$$\ell = U + N, \qquad r = U - N, \tag{62}$$

one has

$$\nabla^{(k)}_\ell \ell \parallel \ell, \qquad \nabla^{(k)}_r r \parallel r, \tag{63}$$

and

$$\sigma_\ell = 0, \qquad \sigma_r = 0. \tag{64}$$

Proof. The component decomposition of (51) with respect to (58) gives the standard two-eigenvalue Codazzi identities. The pure EEE and FFF components give (59), and the mixed components give the umbilicity equations (60) and (61); the converse is obtained by recombining the same components.

For ℓ and r in (62), (60) removes the F-component of the corresponding accelerations. Metric compatibility leaves the E-component orthogonal to the same null vector, hence proportional to it. This gives (63). The trace-free transverse optical parts vanish by (61), which gives (64). □

The EEE and FFF parts of Theorem 1 give, after absorbing the numerical factor in (57) into the constants, the two first integrals

$$\phi p_\Lambda \frac{\tanh\chi(\tanh\chi - 3)}{1 + \tanh\chi} = C_{UN}, \qquad E_0 C_{UN} = E_1 C_{UN} = 0, \tag{65}$$

$$\phi p_\Lambda \frac{\tanh\chi(\tanh\chi + 3)}{1 + \tanh\chi} = C_{WS}, \qquad E_2 C_{WS} = E_3 C_{WS} = 0. \tag{66}$$

Using (21), the first integral may also be written as

$$\zeta^2 = 1 - \mu_\zeta R_\omega - \frac{C_{UN}(1 + \tanh\chi)}{p_\Lambda \tanh\chi(\tanh\chi - 3)}. \tag{67}$$

Since ζ is the translational-current coefficient, (67) is the Codazzi-fixed balance between the translational current, the stiffness-weighted rotational contribution, and the branch residual. With $t = \tanh\chi$, the non-isotropic, non-degenerate sector satisfies

$$t = 3\,\frac{C_{WS} + C_{UN}}{C_{WS} - C_{UN}}. \tag{68}$$

The rotational energy becomes

$$E_{\rm rot} = \frac{p_\Lambda}{2}(1-\zeta^2) - \frac{1+t}{12t}(C_{WS} - C_{UN}). \tag{69}$$

The rotational terms are the Alena Tensor terms used in the continuum and vortex setting of [5]. After Codazzi closure they obey the local equation of state (69).

The Codazzi gap is denoted

$$\Delta_C = M - P. \tag{70}$$

It sets the local scale of the two-eigenvalue splitting. In the link problem below, the same gap supplies the natural scale of the Fredholm-domain barrier.

Theorem 1 separates the full Codazzi condition from the optical input used by the link construction. The divergence closure (55) depends only on the contracted defect (56), while the link construction uses (63)-(64). This is a Goldberg-Sachs type optical structure at the Codazzi level. The same $2+2$ geometry appears in type $2+2$ conformal Killing tensors [38] and in umbilical space-time structures with two shear-free geodesic null congruences [39]. The divergence-free non-null Maxwell-Minkowski case gives the Rainich comparison class [10]; Codazzi space-time structures give another comparison class [37].

Definition 1 (Optical Alena-Rainich branch)**.** *A non-null Alena-Rainich branch tensor is called optical on an open set if:*

(i) *it satisfies the Rainich algebraic type (10) and determines a non-degenerate orthogonal splitting $T\mathcal{U} = E \oplus F$, where E is Lorentzian, F is spacelike, and the associated trace-adjusted endomorphism has two distinct eigenvalues;*

(ii) *there exists a nonzero scalar ϕ such that the branch divergence closure (55) holds;*

(iii) *the two principal planes are umbilic: for some $H_E \in \Gamma(F)$ and $H_F \in \Gamma(E)$,*

$$(\nabla^{(k)}_{X_1} X_2)_F = k(X_1, X_2) H_E, \qquad X_1, X_2 \in \Gamma(E), \tag{71}$$

$$(\nabla^{(k)}_{V_1} V_2)_E = k(V_1, V_2) H_F, \qquad V_1, V_2 \in \Gamma(F). \tag{72}$$

A Codazzi-closed ARC branch is the calibrated case of this definition, with H_E and H_F fixed by (60) and (61).

Definition 1 isolates the local structure needed by the link construction. The Codazzi-closed sector is the integrable realization used in the explicit branch solution. It supplies the local projective spinor sphere attached to the branch, in the standard sense of spinor and twistor geometry [11] and [40]. In the link construction below this sphere is the Riemann sphere $S^2_\Gamma \simeq \mathbb{CP}^1$. Twistor unification models use related projective-spinor geometry at the level of the full twistor space [13]. Curved twistor constructions give the broader comparison class [41]. The present construction uses the local twistor line associated with the ARC defect.

A solvable class is obtained when the two eigendistributions are written in warped form. Put

$$u = \tanh\chi. \tag{73}$$

On a product patch take

$$\eta = h - \mathcal{R}^2\gamma, \qquad k = h - e^{2\chi}\mathcal{R}^2\gamma. \tag{74}$$

Here h is Lorentzian on E, γ is Riemannian on the transverse leaves, and $\mathcal{R} > 0$. Put $F = \phi p_\Lambda$. In the non-isotropic sector,

$$M = F\frac{u(u-3)}{3(1+u)}, \qquad P = F\frac{u(u+3)}{3(1+u)}. \tag{75}$$

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If u, $\mathcal{R}$, and F depend only on the E-variables, the Codazzi condition is equivalent to

$$d_E M = 0, \qquad d_E P = (M - P) d_E \log(\mathcal{R} e^{\chi}). \tag{76}$$

Solving gives

$$F(u) = F_0 \frac{1+u}{u(3-u)}, \tag{77}$$

and

$$\mathcal{R}^2 \frac{4(1+u)}{(1-u)(3-u)^2} = L^2. \tag{78}$$

Thus

$$\eta = h - b(u)^2 \gamma, \qquad k = h - a(u)^2 \gamma, \tag{79}$$

with

$$a(u) = \frac{L}{2}(3-u), \qquad b(u) = \frac{L}{2}(3-u)\sqrt{\frac{1-u}{1+u}}. \tag{80}$$

On $0 < u < 1$,

$$\frac{\mathrm{d}}{\mathrm{d}u} \log b(u) = -\frac{4 - u - u^2}{(3-u)(1-u^2)} < 0. \tag{81}$$

Hence b maps $(0,1)$ to $(0, 3L/2)$. Its inverse is denoted by $u = u(\tau)$, where

$$0 < \tau < \frac{3L}{2}, \qquad \tau = b(u). \tag{82}$$

The eigenvalue gap and the branch scale are tied by

$$M = -\frac{F_0}{3}, \qquad P = \frac{F_0(3+u)}{3(3-u)}, \qquad M - P = -\frac{2F_0}{3-u}. \tag{83}$$

In particular,

$$a(u)(M-P) = -F_0 L. \tag{84}$$

Moreover,

$$p_\Lambda(u) = \frac{16 p_0}{(1-u)^2}, \tag{85}$$

and

$$\phi(u) = \frac{F_0(1+u)(1-u)^2}{16 p_0 u(3-u)}. \tag{86}$$

The constant contribution to the Einstein trace is

$$\Lambda = 8\kappa p_0. \tag{87}$$

The curvature identities are the standard warped-product formulas applied to (79); the convention agrees with [42] and [43]. In particular,

$$\mathrm{Ric}^{(k)}_{Ai} = 0. \tag{88}$$

Flatness of $\eta = h - b^2\gamma$ is equivalent to

$$\mathrm{Riem}(h) = 0, \qquad \nabla_h^2 b = 0, \qquad K_\gamma + |db|_h^2 = 0. \tag{89}$$

The scalar-curvature and E-block formulas following from (79) are recorded in the appendix; they are not used in the finite-sector reconstruction.

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The normalized vorticity closure is imposed by rescaling the two principal null directions and putting

$$\ell_\alpha = e^\alpha \ell, \qquad r_\alpha = e^{-\alpha} r, \tag{90}$$

$$U_\alpha = \frac{1}{2}\left(e^\alpha \ell + e^{-\alpha} r\right), \qquad N_\alpha = \frac{1}{2}\left(e^\alpha \ell - e^{-\alpha} r\right). \tag{91}$$

The scalar normalization

$$|D_\omega| u = \frac{D_o^2}{2} \tag{92}$$

is solved by

$$\varpi = D_o e^{-\lambda}, \qquad \sigma_{NS} = D_o \sinh \lambda. \tag{93}$$

If the vorticity is generated by transverse variation of α, then

$$\lambda = \log \frac{2a(u) D_o}{|\nabla_\gamma \alpha|}. \tag{94}$$

The condition $D_\omega \parallel N_\alpha$ gives

$$\Delta_\gamma \alpha = \varepsilon \frac{2a(u)^2 D_o^2}{c^2 u}, \qquad \varepsilon = \pm 1. \tag{95}$$

Equivalently, with $\nabla_\gamma \alpha = |\nabla_\gamma \alpha| \widehat{S}$,

$$\operatorname{div}_\gamma \widehat{S} - \widehat{S}(\lambda) = \varepsilon \frac{a(u) D_o}{c^2 u} e^\lambda. \tag{96}$$

Thus the normalized vorticity closure fixes the null normalization inside the closed branch.

The global hyperbolic representative is obtained on

$$M_H = \left(0, \frac{3L}{2}\right) \times \mathbb{R} \times H^2, \tag{97}$$

with $h = d\tau^2 - dz^2$ and

$$\gamma_H = d\rho^2 + e^{2\rho} d\psi^2. \tag{98}$$

Then

$$\eta_H = d\tau^2 - dz^2 - \tau^2 \gamma_H \tag{99}$$

is flat, and

$$k_H = d\tau^2 - dz^2 - a(u(\tau))^2 \gamma_H. \tag{100}$$

The horocyclic coordinate satisfies

$$|\nabla_{\gamma_H} \rho| = 1, \qquad \Delta_{\gamma_H} \rho = 1. \tag{101}$$

Define

$$C(\tau) = \frac{2a(u(\tau))^2 D_o^2}{c^2 u(\tau)}, \qquad \alpha = \varepsilon C(\tau) \rho. \tag{102}$$

Then

$$|\nabla_{\gamma_H} \alpha| = C(\tau), \qquad \lambda = \log \frac{c^2 u(\tau)}{a(u(\tau)) D_o}. \tag{103}$$

Theorem 2 (Global hyperbolic warped branch)**.** *Let $L > 0$, $p_0 > 0$, $F_0 \neq 0$, and $D_o > 0$. On M_H in (97), define $u(\tau)$ by (82), k_H by (100), F by (77), and α by (102). Then η_H is flat, k_H is smooth and Lorentzian, $A_{\mu\nu}$ satisfies (51), the divergence identity (55) holds, $Y_{\mu\nu}$ has the Rainich type (10), and the normalized-vorticity equations (95) and (96) hold.*

Proof. The monotonicity in (81) gives a smooth inverse $u(\tau)$. The identities $K_{\gamma_H} = -1$, $|d\tau|_h^2 = 1$, and $\nabla_h^2\tau = 0$ give (89); hence (99) is flat. Equations (77) and (78) solve (76), so $A_{\mu\nu}$ is Codazzi. The divergence identity and Rainich type follow from (50), (51), and (10). The function ρ satisfies (101); substituting (102) gives (95), (103), and (96). □

The flat-reference warped sector is therefore non-empty. Compact transverse leaves behave differently. Since u is constant on each transverse leaf, integration of (95) over a closed leaf Σ gives

$$0 = \int_\Sigma \Delta_\gamma \alpha \, dA_\gamma = \varepsilon \frac{2a(u)^2 D_o^2}{c^2 u} \mathrm{Area}_\gamma(\Sigma). \tag{104}$$

For $D_o \neq 0$ and $0 < u < 1$ this is impossible for smooth global α. A distributional closure has the form

$$\Delta_\gamma \alpha = \varepsilon \frac{2a(u)^2 D_o^2}{c^2 u} + \sum_j q_j \delta_{p_j}, \qquad \sum_j q_j = -\varepsilon \frac{2a(u)^2 D_o^2}{c^2 u} \mathrm{Area}_\gamma(\Sigma). \tag{105}$$

Thus the compact-leaf obstruction is the place where link, boundary, or defect data enter the closed branch problem. A defect core followed along a branch trajectory in the E-plane gives a branch worldline Γ; a small two-sphere linking it is the twistor link sphere used in Section 3. Compatible local ARC branches patch on $X = M \setminus \Gamma$ by the ordinary tensorial cocycle condition on k, Y, A, ϕ, χ, the translational phase data, and the $2+2$ splitting. At this stage the distributional source in (105) is a real boundary source. The integral spin^c link used below is obtained when the defect resolution includes patching of the oriented transverse frame and the regular phase-current sector. Standard elliptic patching and removable-singularity theory provide the comparison class [44]. The compactness obstruction functions as a boundary-source mechanism for the closed branch problem. The local ARC sector remains governed by the standard elliptic regularity background [45].

3. Integral Projective Link and Gap-Current Fredholm Support

The compact-leaf obstruction of Section 2.2 supplies the place where finite link data enter the closed branch problem. In the sector used below the obstruction is resolved by patching the oriented transverse frame and by the regular phase-current sector of (22). This gives an integral spin^c link, its Borel-Weil zero-mode spaces, and the Toeplitz visibility cutoff. The phase-current source and the normal Codazzi-gap source then give the two active low channels. The finite support is selected by the gap-current Fredholm energy rather than by a separate no-extra-visible postulate.

3.1. Integral Spin^c link and Borel-Weil Zero Modes

On the non-degenerate set $\Delta_C \neq 0$, mixed frame rotations between the Rainich and transverse planes are massive relative to the Codazzi eigenvalue gap (70). The transverse connection on $F = \mathrm{span}(W, S)$ is denoted

$$A_{WS} = k(W, \nabla^{(k)} S). \tag{106}$$

Its curvature is

$$F_{WS} = \mathrm{d}A_{WS}. \tag{107}$$

A compact-leaf defect is called transverse-frame resolved if its resolution on $X = M \setminus \Gamma$ includes patching of the oriented transverse frame (W, S) around the link sphere.

Proposition 4 (Transverse-frame integrality)**.** *If a compact-leaf defect is transverse-frame resolved, then the transverse link curvature has integral period:*

$$\frac{1}{2\pi} \int_{S_\Gamma^2} F_{WS} \in \mathbb{Z}. \tag{108}$$

Consequently the transverse-frame resolution supplies an integral spin^c determinant line on the link.

Proof. Let U_N and U_S be two frame patches on S^2_Γ. On $U_N \cap U_S$ the oriented frames differ by a map to $SO(2)$, written as $g_{NS} = e^{i\beta}$. Hence the local connection forms satisfy $A_S = A_N + \mathrm{d}\beta$. By Stokes' theorem on the two hemispheres,

$$\int_{S^2_\Gamma} F_{WS} = -\int_{S^1} \mathrm{d}\beta. \tag{109}$$

The transition function $e^{i\beta}$ is single-valued on the equator, so the last integral is an integer multiple of 2π. This gives (108). □

This is the topological part of the link condition. The analytic part is the normal spin^c Dirac-linearized-Codazzi problem on the punctured neighbourhood of Γ. Proposition 4 identifies the integral link sector with the compact-leaf sector resolved by transverse-frame patching. General real distributional solutions of (105) may lie outside this sector. After vacuum selection this residual transverse holonomy is the electromagnetic connection. The full electroweak group appears later as the stabilizer of the reconstructed two-dimensional block W_Γ.

Proposition 5 (Canonical projective link)**.** *For an optical ARC branch carrying a transverse-frame resolved compact-leaf defect, the link sphere S^2_Γ carries a canonical projective-spinor reading, up to spin^c isomorphism. In this reading the finite zero-mode spaces are the positive spin^c link modes in* (110).

Proof. The optical branch supplies the oriented $2+2$ splitting and the shear-free principal null pair of Definition 1. The transverse-frame resolution supplies the integral determinant line by Proposition 4. Thus the small oriented link sphere represents the projectivized local spinor space of the branch, with the spin^c twisting fixed by the integral link period. Changing the small sphere or the local frame changes the construction by the corresponding spin^c bundle isomorphism. □

The self-dual three-dimensional spacetime sector has an orthogonal bilinear structure, as in the geometry of self-dual forms [46] and its spinorial descriptions [47]. Color is treated as a multiplicity of equivalent closed branch copies. Chiral gravitational variables are used only for the branch geometry [48,49]; pure-connection and self-dual formulations provide the comparison class [50,51].

Lemma 1 (No open finite index)**.** *Let E be a finite multiplicity space of equivalent closed ARC branch modes, and let G_E be the compact group preserving its Hermitian structure and the imposed orientation data. A closed local branch observable is invariant in the E-factor. If E is a nontrivial irreducible G_E-module, then an open vector in E cannot define a closed local branch tensor.*

Proof. The closure equation (55) is an equation for the spacetime tensor $\phi Y_{\mu\nu}$. The finite label distinguishes equivalent copies of the same local closed data rather than spacetime tensor components. A local closed observable is basis-independent in E, and hence lies in the invariant part of the finite factor. For nontrivial irreducible E, $\mathrm{Hom}_{G_E}(E, \mathbb{C}) = 0$. □

For color this leaves only singlet local closures, for example $\bar{q}_A q^A$ and $\epsilon_{ABC} q^A q^B q^C$. Thus $3 \otimes \bar{3}$ and $3 \otimes 3 \otimes 3$ contain local singlets, while open color indices are excluded from closed local branch tensors. The finite non-singlet data are therefore link, boundary, or sector data. This is the local branch form of the usual separation between closed observables and finite sector labels, as in Doplicher-Roberts reconstruction [52].

Let Γ be a branch worldline produced by the compact-leaf mechanism in (104) and (105). The optical branch structure identifies the small link sphere with the projective spinor sphere of the branch, $S^2_\Gamma \simeq \mathbb{CP}^1$; in the Codazzi sector this structure is supplied by Theorem 1. Thus the finite link module is assigned to the Borel-Weil realization of the branch twistor line, while local branch tensors remain without finite open indices. This is the local analogue of using projective twistor geometry for internal symmetry in twistor unification [13,53].

Near the branch worldline, after separating the tangential direction along Γ, the normal Dirac-linearized-Codazzi problem has the standard spin^c boundary reduction on a collar or conic neighbourhood [54,55]. Let D_q be the resulting spin^c Dirac operator on the link sphere, with determinant degree fixed by the integral period in (108). With the chosen orientation, the positive link modes are the standard monopole zero modes on $S^2_\Gamma \simeq \mathbb{CP}^1$ [14], equivalently the $\mathbb{CP}^1$ Borel-Weil spaces [16]:

$$E_q = \ker D_q^+ \simeq H^0(\mathbb{CP}^1, \mathcal{O}(q-1)). \tag{110}$$

The Aharonov-Casher count gives the same index reading [56]. Hence $\dim_{\mathbb{C}} E_q = q$, with the standard $SL(2)$ convention used below [17]. The same zero-mode realization appears in the Taub-NUT Dirac setting [15].

Let $\mathcal{H}_\ell$ denote the degree-ℓ scalar spherical harmonics on S^2_Γ. The Toeplitz operator on E_q is $T_q(f) = P_q M_f P_q$, where P_q is the orthogonal projection to E_q. Standard $\mathbb{CP}^1$ Toeplitz quantization gives the finite-mode comparison class [18]. The same cutoff has the fuzzy-sphere form [19], with brane/fuzzy-sphere models providing the broader finite-matrix setting [20].

Lemma 2 (Toeplitz visibility cutoff)**.** *The traceless endomorphism space of E_q decomposes as*

$$\mathrm{End}_0(E_q) = \bigoplus_{\ell=1}^{q-1} V_\ell, \qquad V_\ell \simeq \mathcal{H}_\ell. \tag{111}$$

Equivalently,

$$T_q(\mathcal{H}_\ell) = \begin{cases} V_\ell, & 0 \le \ell \le q-1, \\ 0, & \ell \ge q, \end{cases} \tag{112}$$

after projection to the irreducible $SU(2)$ types.

Proof. Equation (110) identifies E_q with the spin $(q-1)/2$ irreducible $SU(2)$ module. The Clebsch-Gordan decomposition of $E_q \otimes E_q^*$ gives the summands V_ℓ for $0 \le \ell \le q-1$. Standard $\mathbb{CP}^1$ Toeplitz quantization gives a nonzero equivariant map on each of these summands and no finite-mode image for higher ℓ. □

The visible non-scalar Toeplitz content of E_q is denoted by $\mathrm{Vis}(E_q)$ and is read from (111): $\mathrm{Vis}(E_q) = \bigoplus_{\ell=1}^{q-1} V_\ell$. Hence E_2 is the first dipole-only block and E_3 is the first block with a degree-two response.

Let L_a be the $SU(2)$ generators on E_q. The fuzzy adjoint Laplacian is

$$\Delta_q X = \sum_{a=1}^{3} [L_a, [L_a, X]]. \tag{113}$$

The Toeplitz link energy on the active response $B \in \mathrm{End}_0(E_q)$ is

$$\mathcal{E}_q(B) = \kappa_q \, \mathrm{Tr}(B^\dagger \Delta_q B), \qquad \kappa_q > 0. \tag{114}$$

On the summand V_ℓ this energy is proportional to $\ell(\ell+1)$.

3.2. Phase-Current and Codazzi-Gap Low Sources

The regular phase current and the Codazzi gap give the two low sources used below. The phase source is first order on the link, while the gap source contains the normal Hessian of $\log|\Delta_C|$. Put

$$S_W = \Pi^{\mathrm{offdiag}}_{\ell=1}(df_\zeta + A_{WS}), \tag{115}$$

$$S_C^{(1)} = \Pi_{\ell=1}(d_\perp \log|\Delta_C|), \tag{116}$$

$$S_C^{(2)} = \Pi_{\ell=2}\Big(\nabla_\perp^2 \log|\Delta_C|\Big)_0. \tag{117}$$

Here the projections are taken after the frozen boundary-trace map from link one-forms and normal tensors to the corresponding Toeplitz source space. The symbol $\Pi^{\mathrm{offdiag}}_{\ell=1}$ denotes the off-diagonal part of the degree-one projected response on the dipole block, $d_\perp$ and $\nabla_\perp$ are normal derivatives to Γ, and the subscript 0 denotes the trace-free part. The color source is $S_C = S_C^{(1)} + S_C^{(2)}$.

The projected link traces of the linearized branch response are denoted by

$$b_C^{(2)} = \Pi_{\ell=2} T_\Gamma(\delta A), \qquad b_W^{\mathrm{offdiag}} = \Pi_{\mathrm{offdiag}}\Pi_{\ell=1} T_\Gamma(\delta A). \tag{118}$$

The expressions (115)-(117) specify the branch origin of the two active low responses: the weak dipole is tied to the phase-current link one-form, while the color quadrupole is tied to the normal Hessian of the Codazzi gap.

Proposition 6 (Low Codazzi link traces). *The link components $\ell = 0, 1, 2$ survive the principal linearized Codazzi equation on a punctured normal neighbourhood of Γ. In the annulus model of Appendix C, the boundary trace map has nonzero image in these three components. This low-mode non-vanishing is stable under sufficiently small smooth perturbations of the normal branch geometry. Consequently, the central, degree-one, and degree-two low responses used in (118) are realized by boundary traces of linearized branch responses whenever the corresponding projected low trace is nonzero.*

Proof. The principal annulus trace calculation and its perturbation stability are recorded in Appendix C. The result is the finite-dimensional Calderon-trace stability of the projected low map. The assignment to the phase-current and Codazzi-gap channels is made by (115)-(117). □

The same low-mode language is compatible with the anisotropy variation. If $\chi = \chi_0 + \varepsilon f$, $\zeta = \zeta_0 + \varepsilon z$, and $\mu_\zeta R_\omega = B_0 + \varepsilon b$, with $T_0 = \tanh\chi_0$ and $\phi_0 = 1 - \zeta_0^2 - B_0$, then the linearized first-integral difference gives

$$\frac{c_{WS}}{A_{WS,0}} - \frac{c_{UN}}{A_{UN,0}} = \phi_0 \frac{6(1-T_0^2)}{9-T_0^2} f. \tag{119}$$

Here c_{UN} and c_{WS} are the variations of the first-integral constants, while $A_{UN,0}$ and $A_{WS,0}$ denote the unperturbed scalar factors in (65) and (66). In the real non-degenerate branch, the scalar multiplier in (119) is nonzero. Thus, at the frozen-coefficient level on a small link sphere, the linearized anisotropy component f is not removed by the Codazzi first-integral difference. Its axisymmetric low components are the Legendre components of the spherical modes used in the annulus trace calculation.

Definition 2 (Non-degenerate low Codazzi-gap source). *A projected low Codazzi-gap source on the link is called non-degenerate if the central projected trace is nonzero, $S_W \neq 0$, and $S_C^{(2)} \neq 0$. It is called color-generating if the degree-one component $S_C^{(1)}$ and the degree-two component $S_C^{(2)}$ generate the traceless algebra on the minimal quadrupole-capable block after Toeplitz projection.*

Corollary 1 (Generic low Codazzi-gap traces). *Let $\mathcal{B}$ be a Banach space of admissible linearized boundary data for which the projected trace maps to the central component, to S_W, and to $S_C^{(2)}$ are continuous and not identically zero. Then the subset of $\mathcal{B}$ on which all three projected traces are nonzero is open and dense. If the*

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degree-one color companion is not symmetry-forbidden, the color-generating condition is also open and dense inside the non-degenerate locus.

Proof. The zero set of each nonzero continuous projected trace map is a closed proper linear subspace of $\mathcal{B}$. Its complement is open and dense. The first statement follows by finite intersection. The non-generating color pairs are cut out by the corresponding commutator minors in the finite-dimensional space $\mathrm{End}_0(E_3)$, hence form a proper closed exceptional set whenever one generating pair exists. □

Definition 3 (Second-jet low-source Fredholm defect)**.** *A regular low-trace Fredholm ARC defect is called second-jet low-source if, in a normal collar of Γ, the principal boundary source of the normal Dirac-linearized-Codazzi problem is determined, modulo higher link-energy terms, by the normal two-jet of the trace-adjusted branch tensor, the regular translational current* (22)*, and the normal Codazzi gap* (70)*. Equivalently, after restriction to a small link sphere, its principal low part lies in $V_0 \oplus V_1 \oplus V_2$. It is called non-degenerate if the V_1 phase-current projection and the V_2 Codazzi-gap projection are both nonzero.*

The condition fixes the highest principal harmonic degree of the source before zero-mode block sizes are assigned. The support selection is made by the gap-current Fredholm energy in (121).

Proposition 7 (Second-jet low-source reduction)**.** *For a second-jet low-source Fredholm ARC defect, the principal non-scalar low source has only the degree-one and degree-two harmonic types. If the defect is non-degenerate, both types are present. At the frozen-coefficient link level these are the irreducible $SU(2)$ types V_1 and V_2.*

Proof. In a normal frame, restriction of a two-jet source to S^2_Γ gives a scalar term, a term linear in the unit normal, and a traceless quadratic term. These are the standard V_0, V_1, and V_2 spherical types; the trace part of the quadratic term belongs to V_0. Higher harmonic degrees do not occur in the principal low source. The non-degenerate condition gives the nonzero V_1 and V_2 projections. The frozen-coefficient link operator is the round $S^2_\Gamma \simeq \mathbb{CP}^1$ model used in (110), so these finite low types are irreducible $SU(2)$ modules. □

Proposition 8 (Natural principal two-channel splitting)**.** *Let a regular Fredholm ARC defect be non-degenerate second-jet low-source and current-supported in the split-conserved matter-Codazzi sector* (34)*-*(35)*. Then the frozen principal boundary problem has two invariant non-scalar low channels: the phase-current degree-one channel and the Codazzi-gap degree-two channel.*

Proof. By Proposition 7, the principal non-scalar source consists of the nonzero V_1 and V_2 types. The frozen principal boundary operator is natural on the canonical link $S^2_\Gamma \simeq \mathbb{CP}^1$; hence it commutes with the $SU(2)$ action of the link. The irreducible spherical types of different degree are inequivalent, and Schur's lemma forbids principal mixing between them. The matter part supplies the positive Casimir boundary Hessian by (34), while the residual part is closed by (35). □

Definition 4 (Primitive two-channel low response)**.** *A non-degenerate second-jet low-source Fredholm defect is called primitive two-channel if S_W has nonzero off-diagonal degree-one projection in the phase-current channel and if S_C has a nonzero degree-two projection in the Codazzi-gap channel, with a degree-one color companion not symmetry-forbidden. The condition concerns boundary response data on the link and leaves the local ARC tensor without open finite indices.*

Proposition 9 (Primitive-visible genericity)**.** *For a non-degenerate second-jet low-source defect in the split-conserved matter-Codazzi sector, the principal non-scalar response has invariant degree-one and degree-two channels. The primitive two-channel condition is open and dense in the corresponding non-degenerate low-trace boundary data, after the phase-current and Codazzi-gap assignments in* (115)*-*(117)*.*

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Proof. The two invariant channels are supplied by Proposition 8. The nonvanishing of the central, degree-one, and degree-two projected traces is open dense by Corollary 1. The failure of the degree-one phase channel to contain the required off-diagonal component is a proper linear condition. On the color channel, the failure of the projected degree-one and degree-two pair to generate $\mathrm{End}_0(E_3)$ is cut out by the corresponding commutator minors. Thus the complement of the primitive-visible locus is contained in a proper closed exceptional set. □

3.3. Gap-Current Fredholm Energy and Minimal Support

Lemma 3 (Casimir compression of the split-conserved link energy)**.** *At a regular link in the normalized split-conserved matter-Codazzi sector, the principal boundary Hessian of the k-conserved matter problem has positive $SU(2)$-Casimir tangential part on the low boundary trace. Its compression to E_q is the Toeplitz link energy (114), up to the same positive normalization.*

Proof. By (34), the tangential second-order part of the normalized matter boundary problem is carried by the hyperstress term (33). On the canonical link its frozen principal symbol is the $SU(2)$ Casimir on the harmonic types of S^2_Γ. On degree ℓ it has eigenvalue $\ell(\ell+1)$. The Toeplitz compression sends the degree-ℓ harmonic type to $V_\ell \subset \mathrm{End}(E_q)$ for $0 \le \ell \le q-1$ and kills the higher degrees, as recorded in (111). Hence the compressed Casimir is the fuzzy adjoint Laplacian appearing in (113). The corresponding quadratic form is (114), with the positive normalization inherited from (34). □

Let P_W project to a dipole support E_{q_W} and let P_C project to a quadrupole-capable support E_{q_C}. The two support tensions are scaled by the Codazzi gap:

$$\lambda_W = \lambda_{W,0}|\Delta_C|_\Gamma, \qquad \lambda_C = \lambda_{C,0}|\Delta_C|_\Gamma, \qquad \lambda_{W,0}, \lambda_{C,0} > 0. \tag{120}$$

The gap-current Fredholm energy is

$$\begin{aligned}\mathcal{E}_\Gamma = {} & \kappa_W \mu_{\zeta,\Gamma} \operatorname{Tr}(B_W^\dagger \Delta_{P_W} B_W) + \kappa_C \operatorname{Tr}(B_C^\dagger \Delta_{P_C} B_C) + \lambda_W \operatorname{Tr} P_W + \lambda_C \operatorname{Tr} P_C \\ & - 2\mathrm{Re}\langle B_W, S_W\rangle - 2\mathrm{Re}\langle B_C, S_C\rangle. \end{aligned}\tag{121}$$

Here Δ_{P_W} and Δ_{P_C} are the fuzzy adjoint Laplacians on the selected supports, while B_W and B_C are the active low-mode responses.

Definition 5 (Hessian-ground-state link support)**.** *For a fixed non-degenerate gap-current low source, a two-channel link support is called Hessian-ground-state if it minimizes (121) among supports carrying the phase-current dipole source and the Codazzi-gap quadrupole source. The support is called no-extra-visible if the minimizing support contains no visible Toeplitz type beyond those required by the active source on each channel.*

Proposition 10 (Gap-current Euler equations)**.** *For fixed P_W and P_C, the stationary active responses of (121) satisfy*

$$\kappa_W \mu_{\zeta,\Gamma} \Delta_{P_W} B_W = S_W, \tag{122}$$
$$\kappa_C \Delta_{P_C} B_C = S_C. \tag{123}$$

Consequently all unforced visible components vanish in the ground state.

Proof. Variation of (121) with respect to $B_W^\dagger$ and $B_C^\dagger$ gives (122) and (123). On every visible summand with zero source, the corresponding equation has only the zero solution because the compressed Casimir is positive on non-scalar modes. □

Definition 6 (Regular low-trace Fredholm ARC defect)**.** *A branch worldline Γ is a regular low-trace Fredholm ARC defect if the compact-leaf obstruction in (104) and (105) is resolved on $X = M \setminus \Gamma$ by a regular integral spin^c link problem with the following properties:*

(i) *the finite non-singlet sectors are boundary zero-mode sectors of the normal Dirac-linearized-Codazzi problem on the link sphere;*
(ii) *closed local branch observables are invariant under changes of zero-mode basis;*
(iii) *the projected low source is non-degenerate in the sense of Definition 2;*
(iv) *the physical non-singlet support is selected by the gap-current Fredholm energy (121).*

The condition is imposed on the punctured normal neighbourhood of Γ and adds no open finite index to the local tensor $Y_{\mu\nu}$. A transverse-frame resolved defect supplies the required integral link period by Proposition 4.

Definition 7 (Primitive visible Fredholm ARC defect)**.** *A regular low-trace Fredholm ARC defect is called primitive visible if it is a non-degenerate second-jet low-source Fredholm defect in the sense of Definition 3, if its degree-one and degree-two non-scalar low responses are primitive two-channel in the sense of Definition 4, and if its support is Hessian-ground-state in the sense of Definition 5.*

Proposition 11 (Fredholm defect reduction)**.** *For a closed ARC branch carrying a regular low-trace Fredholm ARC defect, the no-open-index condition, the projective link sphere, the spin^c zero-mode realization, and the non-degenerate low-source condition hold. If the defect is primitive visible, then the second-jet low-source reduction gives the two principal non-scalar low types, and the gap-current Fredholm energy selects the corresponding minimal active supports.*

Proof. The basis-invariance part of Definition 6 gives the no-open-index condition by Lemma 1. The regular integral spin^c link problem is the normal link problem above; hence the link sphere is the branch projective line and the positive zero-mode sectors are the spaces in (110). For transverse-frame resolved defects, the integral period is supplied by Proposition 4. The non-degenerate low source is the open low-source condition of Definition 2; its non-emptiness follows from Proposition 6 and Corollary 1. For a primitive visible defect, Proposition 7 gives the degree-one and degree-two principal non-scalar low source. Proposition 8 gives the corresponding invariant principal channels, and Proposition 10 gives the active ground-state response. □

Corollary 2 (No lower two-block low-response sector)**.** *A two-block zero-mode sector realizes both the degree-two Codazzi-gap response and the degree-one phase-current response only when $q_C \geq 3$ and $q_W \geq 2$.*

Proof. By Lemma 2, a degree-one response is first visible on E_2, while a degree-two response is first visible on E_3. Thus the dipole channel requires $q_W \geq 2$, and the quadrupole channel requires $q_C \geq 3$. □

Lemma 4 (Exterior parity of a two-block link module)**.** *Let $\mathcal{V} = E_{q_C} \oplus E_{q_W}$ and put $D = q_C + q_W$. After the total top form is fixed, exterior duality gives*

$$(\Lambda^p \mathcal{V})^* \simeq \Lambda^{D-p} \mathcal{V}. \tag{124}$$

Consequently, fixed exterior parity is self-dual when D is even, while even and odd exterior parity are exchanged by duality when D is odd.

Proof. The standard duality is $(\Lambda^p \mathcal{V})^* \simeq \Lambda^{D-p} \mathcal{V} \otimes (\Lambda^D \mathcal{V})^*$. The unimodular top form trivializes the last factor. If D is even, p and $D - p$ have the same parity. If D is odd, they have opposite parity. □

Theorem 3 (Minimal chiral ARC link module)**.** *Let $C \simeq E_{q_C}$ and $W \simeq E_{q_W}$ be the two non-abelian link-index blocks associated with a closed ARC branch mode. Assume that closed local observables are singlet-valued, that the projected low Codazzi-gap source is non-degenerate in the sense of Definition 2, and that its non-scalar*

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part is primitive two-channel in the sense of Definition 4. If the physical non-singlet boundary support minimizes the gap-current Fredholm energy (121), then

$$q_C = 3, \qquad q_W = 2. \tag{125}$$

Equivalently, the minimal finite link-index module is

$$C_\Gamma = \ker D_3^+, \qquad W_\Gamma = \ker D_2^+. \tag{126}$$

On the color block,

$$T_3(\mathcal{H}_1) \oplus T_3(\mathcal{H}_2) = \mathrm{End}_0(E_3). \tag{127}$$

The degree-two color response and the off-diagonal degree-one weak response are expressed by

$$F_C^{(\ell=2)} \neq 0, \qquad \mathcal{L}\langle F_C^{(\ell=1)}, F_C^{(\ell=2)} \rangle = \mathfrak{su}(3), \tag{128}$$

and

$$F_W^{\mathrm{offdiag}} \neq 0, \qquad \mathcal{L}\langle F_W \rangle = \mathfrak{su}(2). \tag{129}$$

Proof. Proposition 6 gives the low trace input. Proposition 7 gives the principal non-scalar types V_1 and V_2, and Proposition 8 separates them into the phase-current dipole channel and the Codazzi-gap quadrupole channel.

By Lemma 2, a degree-one response is visible on E_q exactly when $q \geq 2$, and a degree-two response is visible exactly when $q \geq 3$. Hence the weak channel requires $q_W \geq 2$, and the color channel requires $q_C \geq 3$.

For fixed support, Proposition 10 sets every unforced visible component to zero in the ground state. The finite support comparison is made through the normalized irreducible $SU(2)$ identification recorded in Appendix D; hence passive enlargement of the support does not change the forced source norm. If $q_W > 2$, the additional weak visible summands are unforced and have positive support cost because $\lambda_W > 0$ in (120). Hence the weak minimizer has $q_W = 2$. If $q_C > 3$, the additional color visible summands are unforced and have positive support cost because $\lambda_C > 0$. Hence the color minimizer has $q_C = 3$. This gives (125) and (126).

For $q = 3$, (111) gives $V_1 \oplus V_2 = \mathrm{End}_0(E_3)$, and Lemma 2 identifies these summands with the Toeplitz images of $\mathcal{H}_1$ and $\mathcal{H}_2$. This gives (127). The non-generating dipole-quadrupole pairs are cut out by the corresponding commutator minors, so the generating condition is open on the non-degenerate locus.

Since $q_C + q_W = 5$, the fixed exterior parity is chiral by Lemma 4. The stabilizer, exterior algebra, and one-Higgs channels are applied to the reconstructed object $C \oplus W$ below. Numerical Yukawa matrices and global existence for arbitrary Alena Tensor data remain outside this structural statement. □

The notation C and W is fixed by these low-link roles: C denotes the first Codazzi-gap quadrupole-generating block and W denotes the first phase-current dipole-only block.

4. Finite Zero-Mode Basis Group and Exterior Standard Model Package

The minimal ARC link module of Theorem 3 fixes the finite complex space $C_\Gamma \oplus W_\Gamma$. Here C_Γ is the minimal Codazzi-gap quadrupole-generating block and W_Γ is the minimal phase-current dipole block. The present section records the compact zero-mode basis group, the corresponding exterior-algebra generation, and the one-Higgs Clifford-odd channels. No family multiplicity or numerical Yukawa data are introduced at this stage.

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4.1. Unimodular Zero-Mode Basis Group and Exterior Generation

Proposition 12 (Zero-mode gauge reconstruction)**.** *Let E_q be a positive link zero-mode space as in* (110)*, equipped with its Hermitian zero-mode inner product. If closed local branch observables are invariant under changes of orthonormal zero-mode basis, then the compact internal basis group of this sector is $U(E_q)$. For two link blocks C and W, preservation of the total top form reduces $U(C) \times U(W)$ to $S(U(C) \times U(W))$.*

Proof. By Lemma 1, an open finite index cannot occur in a closed local branch tensor. A unitary change of orthonormal basis in E_q changes the representative of the finite link coordinate but not the closed local branch observable. The compact group preserving the Hermitian zero-mode structure is therefore $U(E_q)$. For $C \oplus W$, fixing the total volume form imposes the unimodular determinant condition and gives $S(U(C) \times U(W))$. □

Theorem 4 (Unimodular group of the minimal link module)**.** *Let $C \oplus W$ be the minimal ARC link module of Theorem 3. Assume that closed local branch observables are invariant under changes of orthonormal zero-mode basis, and that the projected low responses in* (128) *and* (129) *generate $\mathrm{End}_0(C)$ and $\mathrm{End}_0(W)$. Then the full connected compact internal basis group preserving the Hermitian zero-mode structures is $U(C) \times U(W)$. After the total top form on $C \oplus W$ is fixed, this group is reduced to $S(U(C) \times U(W))$. Since $\dim_{\mathbb{C}} C = 3$ and $\dim_{\mathbb{C}} W = 2$, this is $S(U(3) \times U(2))$, with semisimple part $SU(3) \times SU(2)$. If the closed color observable is singlet-valued, then the effective closed color holonomy is projective, namely $PSU(C) = SU(C)/\mathbb{Z}_3$.*

Proof. Theorem 3 gives $C \simeq E_3$ and $W \simeq E_2$. On E_2, the dipole summand is $\mathrm{End}_0(W)$. On E_3, Theorem 3 gives $\mathrm{End}_0(C)$ for the non-degenerate Codazzi-gap color projection. Thus the visible traceless infinitesimal channels contain $\mathfrak{su}(C) \oplus \mathfrak{su}(W)$.

By Proposition 12, invariance of closed local observables under changes of orthonormal zero-mode basis gives the full compact basis group $U(C) \times U(W)$. Its traceless part is the semisimple algebra detected by the gap-current low responses. Fixing the total top form imposes the unimodular determinant condition and gives $S(U(C) \times U(W))$. With $\dim_{\mathbb{C}} C = 3$ and $\dim_{\mathbb{C}} W = 2$, this is $S(U(3) \times U(2))$. The center of $SU(C)$ acts trivially on singlet-valued closed color observables, hence the closed color holonomy is naturally the adjoint form $PSU(C)$. □

Theorem 3 is the finite-sector reconstruction step, and Theorem 4 fixes the corresponding compact basis group. The weak block is selected by the phase-current dipole channel, while the color block is selected by the Codazzi-gap quadrupole channel and contains the full low-mode color algebra. After reconstruction from the ARC link, this module plays the finite internal role familiar from almost-commutative geometry. In the noncommutative-geometric Standard Model the finite internal algebra is an input [24]; the spectral action supplies the dynamical principle [25]. Modern spectral-geometric refinements provide a useful comparison class [57].

Write $C = C_\Gamma$ and $W = W_\Gamma$. The reconstructed complex multiplicity space is

$$\mathcal{V} = C \oplus W, \qquad \dim_{\mathbb{C}} C = 3, \qquad \dim_{\mathbb{C}} W = 2. \tag{130}$$

By Theorem 4, the top-form preserving zero-mode basis group is

$$S(U(3) \times U(2)) \simeq \frac{SU(3)_c \times SU(2)_L \times U(1)_Y}{\mathbb{Z}_6}. \tag{131}$$

The quotient is the usual global Standard Model form detected by line operators [58]. If y_C and y_W are the abelian weights, top-form neutrality gives $3y_C + 2y_W = 0$. With minimal weak normalization $y_W = 1$,

$$C = (3, 1, -2/3), \qquad W = (1, 2, +1). \tag{132}$$

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This is the convention used below; the transverse connection (106) is then the unbroken electromagnetic connection after vacuum selection [28].

The matter representation is the standard $3 + 2$ exterior-algebra package applied to the reconstructed minimal chiral object $\mathcal{V}$ with assignments (132). With the unimodular top form fixed, the even exterior algebra gives the left-handed one-generation module familiar from the $SU(5)/\mathrm{Spin}(10)$ organization [21]:

$$\Lambda^{\mathrm{even}}\mathcal{V} = \nu_L^c \oplus u_L^c \oplus Q_L \oplus e_L^c \oplus d_L^c \oplus L_L. \tag{133}$$

Here ν_L^c denotes the conjugate of the right-handed neutrino. Related Clifford-ideal constructions obtain the same particle-content package from complex Clifford algebras [22] and [23]. The ARC input is the prior reconstruction of $\mathcal{V} = C \oplus W$ from the link; the exterior-algebra package is then applied with fixed exterior parity. This use of exterior parity is compatible with differential-form descriptions of fermions [27].

Proposition 13 (Exterior one-generation module)**.** *For the minimal ARC link module $\mathcal{V} = C \oplus W$ with assignments (132), the even exterior algebra is the left-handed one-generation module recorded in (133).*

Proof. The identity follows by decomposing $\Lambda^{\mathrm{even}}\mathcal{V}$ according to the $C \oplus W$ splitting and using the unimodular top form. The components and hypercharges are those induced by (132). □

4.2. Clifford-Odd Higgs Channels and Anomaly Identities

The weak factor $W = (1, 2, +1)$ has the quantum numbers of the minimal electroweak order parameter. For $\Phi \in W$, exterior multiplication and contraction define the Clifford-odd operator

$$c(\Phi + \Phi^\dagger) = \varepsilon(\Phi) + \iota(\Phi^\dagger). \tag{134}$$

This operator changes exterior parity. Its one-Higgs singlet bilinear channels, after charge conjugation and diagonal locking, are

$$Q_L \leftrightarrow u_L^c, \qquad Q_L \leftrightarrow d_L^c, \tag{135}$$
$$L_L \leftrightarrow \nu_L^c, \qquad L_L \leftrightarrow e_L^c. \tag{136}$$

No other bilinear channel with a single Φ or $\Phi^\dagger$ is invariant under the stabilizer (131). Thus the algebra fixes the representation channels of the Yukawa maps. The numerical entries are spectral data of the Alena-Rainich-Codazzi branch problem through the spin-vorticity gap and the vacuum response.

Proposition 14 (One-Higgs invariant channels)**.** *For the finite module (133) and the weak factor $\Phi \in W$, the one-Higgs Clifford-odd map (134) gives precisely the quark and lepton channels displayed in (135) and (136).*

Proof. Exterior multiplication by Φ and contraction by $\Phi^\dagger$ change the W-degree by one and preserve the color contraction required by the singlet bilinear. With the hypercharge convention (132), the invariant pairings are exactly those in (135) and (136). The remaining one-Higgs pairings have nonzero total hypercharge, an unmatched weak index, or an open color index. □

The anomaly cancellation is used as a consistency check on the minimal sector; the selection of (125) comes from the gap-current link argument. The hypercharges in the sums below are induced

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from (132) on the exterior module (133). With $A(3) = 1$, $A(\bar{3}) = -1$, and $T(3) = T(\bar{3}) = T(2) = 1/2$, the independent local sums are

$$\mathcal{A}_{SU(3)^3} = 2A(3) + A(\bar{3}) + A(\bar{3}) = 0, \tag{137}$$

$$\mathcal{A}_{SU(3)^2U(1)} = 2T(3)\frac{1}{3} + T(\bar{3})\left(-\frac{4}{3}\right) + T(\bar{3})\frac{2}{3} = 0, \tag{138}$$

$$\mathcal{A}_{SU(2)^2U(1)} = 3T(2)\frac{1}{3} + T(2)(-1) = 0, \tag{139}$$

$$\mathcal{A}_{U(1)^3} = 6\left(\frac{1}{3}\right)^3 + 3\left(-\frac{4}{3}\right)^3 + 2^3 + 3\left(\frac{2}{3}\right)^3 + 2(-1)^3 = 0, \tag{140}$$

$$\mathcal{A}_{\mathrm{grav}^2U(1)} = 6\frac{1}{3} + 3\left(-\frac{4}{3}\right) + 2 + 3\frac{2}{3} + 2(-1) = 0. \tag{141}$$

The right-handed neutrino is hypercharge neutral in this convention. Thus the local representation module has been fixed from the closed branch and its link-index sector before any family hypothesis or numerical Yukawa data are introduced. Family multiplicity, mixing, and fermion mass hierarchy require the diagonally locked fermionic branch operator and the global projective color sectors described below.

5. Gap-Locked Fermions and Projective Families

The minimal link module of Theorem 3 fixes the finite representation space. The fermionic sector is obtained by combining this space with the branch spinor bundle and by minimizing the Codazzi-gap locking term compatible with the branch Dirac term, the weak Clifford-odd map, and charge conjugation. The resulting locked operator has a standard low-energy Dirac interface when its physical branch-helicity cluster is isolated. The family refinement is global: the projective color lift gives three central sectors, and central-degree boundary traces generate the finite family algebra.

5.1. Gap-Selected Diagonal Locking and Dirac Interface

The exterior-algebra construction fixes the local Standard Model representation content through (133). The spinorial branch frame fixes the null pair $U \pm N$ through the branch bilinears used below. These two pieces are combined into the associated bundle

$$E_F = S_{\mathrm{branch}} \otimes \Lambda^\bullet \mathcal{V}, \qquad \mathcal{V} = C \oplus W. \tag{142}$$

Let N_C and N_W be the exterior number operators on $\Lambda^\bullet C$ and $\Lambda^\bullet W$, and put $N = N_C + N_W$. The primitive finite parities are $\Gamma_C = (-1)^{N_C}$ and $\Gamma_W = (-1)^{N_W}$, so that $\Gamma_\mathcal{V} = (-1)^N = \Gamma_C\Gamma_W$. Let $\gamma^5_{\mathrm{branch}}$ be the chirality operator defined by the branch tetrad.

For a product-type finite sign T induced by the primitive blocks C and W, put $\Gamma_T = \gamma^5_{\mathrm{branch}} \otimes T$. The local gap-scaled locking energy is

$$\mathcal{E}_{\mathrm{lock}}(T) = m_\Gamma\left(\|\{\Gamma_T, D_E\}\|^2_{\mathrm{pr}} + \left\|\left\{\Gamma_T, c(\Phi + \Phi^\dagger)\right\}\right\|^2_{\mathrm{pr}}\right), \qquad m_\Gamma = m_0|\Delta_C|_\Gamma, \quad m_0 > 0. \tag{143}$$

Here $\|\cdot\|_{\mathrm{pr}}$ denotes the principal link-symbol norm on the frozen low sector. The scale is the same Codazzi gap which appears in (120).

Proposition 15 (Diagonal locking)**.** *Let a grading involution on* (142) *be local, product-type in the branch spinor factor, and induced multiplicatively by signs on the primitive link blocks C and W. If it makes the branch Dirac term odd, makes the weak Clifford-odd map* (134) *odd, and its locked sector is preserved by charge conjugation, then it is, up to the overall sign,*

$$\Gamma_{\mathrm{lock}} = \gamma^5_{\mathrm{branch}} \otimes \Gamma_\mathcal{V}. \tag{144}$$

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Moreover, (144) *is the zero locus of* (143) *among the primitive product-type signs.*

Proof. The branch Dirac term is odd only with the branch chirality factor. Thus the involution has the form $\pm\gamma^5_{\text{branch}} \otimes T$. The weak Clifford-odd map changes total exterior degree by one, hence it anticommutes with $\Gamma_{\mathcal{V}}$ and with no independent primitive block parity compatible with both weak components. Charge conjugation uses the exterior duality of Lemma 4, which fixes $T = \Gamma_{\mathcal{V}}$ up to the common sign. Therefore the principal anticommutators in (143) vanish precisely for (144). □

Thus fixed exterior parity is the finite part of the diagonal grading fixed by Proposition 15 once the weak Clifford-odd interface is imposed. The compatibility with charge conjugation used above is the same exterior-duality mechanism as in Lemma 4. This role of the locking condition is analogous in aim to spectral-geometric Standard Model models in which the fermion-doubling problem is removed by the structure of the finite geometry [59].

The diagonal locking condition is global provided the branch complement carries a spin^c structure and the finite bundle $\mathcal{V} = C \oplus W$ is defined with the unimodular top form. The branch spinor bundle supplies γ^5_{branch}, while exterior degree supplies $\Gamma_{\mathcal{V}} = (-1)^N$ on $\Lambda^\bullet\mathcal{V}$. Their product in (144) is therefore a globally defined involution on (142), and the projectors onto the locked sectors are global. This is the standard spin^c bundle construction [60] and [54].

The locking projector is denoted by Π_{lock} and the mirror projector by $\Pi_{\text{mir}} = 1 - \Pi_{\text{lock}}$. The gap-selected branch-helicity condition is

$$\Pi_{\text{mir}} K_{\text{lock}} \Pi_{\text{mir}} \geq m_\Gamma \Pi_{\text{mir}}, \qquad K_{\text{lock}} = K_F + m_\Gamma \Pi_{\text{mir}}. \tag{145}$$

Equivalently, the mirror complement satisfies the chiral Fredholm gap condition

$$\sigma(\Pi_{\text{mir}} K_{\text{lock}} \Pi_{\text{mir}}) \subset [m_\Gamma, \infty). \tag{146}$$

The comparison class is the localization of chiral zero modes by a defect of a Dirac operator, as in the Jackiw-Rebbi mechanism [61] and in domain-wall fermions [62]. The gap term is imposed on the ARC branch operator (149).

The projective color lift introduced below commutes with the diagonal grading. A central element of $SU(C)$ acts by a scalar on each exterior degree, hence it commutes with $(-1)^N$, and it acts trivially on the branch spinor factor. Thus the central sectors commute with the locking involution. If the branch fermion operator in (148) is odd with respect to this grading, it maps the locked $-$ sector to the locked $+$ sector, so the positive operator (149) is globally defined on the corresponding locked bundle.

The branch Dirac operator on (142) is written as

$$D_E = \gamma^\mu \nabla^E_\mu, \qquad \nabla^E_\mu = \nabla^S_\mu \otimes 1 + 1 \otimes \nabla^{\Lambda\mathcal{V}}_\mu. \tag{147}$$

By Proposition 15, D_E is odd with respect to (144). The odd operator used in the locked sector is

$$Q_F = P_+ \Big(D_E + c(\Phi + \Phi^\dagger) + Q_{\text{hol}} + Q_C \Big) P_-, \tag{148}$$

where $P_\pm$ are the projectors onto the $\pm$ locked sectors, Q_{hol} is the holonomy response, and Q_C is the middle spinorial projection of the linearized Codazzi response. The corresponding positive operator is

$$K_F = Q_F^\dagger Q_F. \tag{149}$$

The gap term in (145) is a boundary-domain term; it does not change the local exterior representation content fixed in Section 4.

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The branch-helicity splitting is obtained from the two principal spinors of the optical branch. If δ_C denotes the spinorial form of the linearized Codazzi response, its extreme and middle projections are

$$\delta_C^{\text{ext}} = P_{\text{ext}} \delta_C P_{\text{ext}}, \tag{150}$$
$$\delta_C^{\text{mid}} = P_{\text{mid}} \delta_C P_{\text{mid}}. \tag{151}$$

Equations (63) and (64) give the branch-geometric origin of (150). The extreme components are the shear-type tensorial leakage of the spinorial Codazzi projection, while the middle components are the two branch-helicity components of a spin-1/2 mode. The corresponding middle Codazzi contribution in (148) is denoted

$$Q_C = P_{\text{mid}}\, \delta_C\, P_{\text{mid}}. \tag{152}$$

The curvature reading of a local mass parameter is seen already in the test branch

$$ds_k^2 = dt^2 - dx^2 - e^{2\chi(t,x)}(dy^2 + dz^2). \tag{153}$$

For $\chi = \chi_0 + \varepsilon f(t,x)$ one obtains

$$R^{(1)} = -4\varepsilon(f_{,tt} - f_{,xx}). \tag{154}$$

If $f_{,tt} - f_{,xx} + m_{\text{eff}}^2 f = 0$, then

$$m_{\text{eff}}^2 = \frac{R^{(1)}}{4\varepsilon f}, \tag{155}$$

where $f \neq 0$. Thus the effective mass parameter is read from the scalar-curvature response of the anisotropic branch. Related Gabor-regularized metric models show explicitly how smoothing metric data can induce effective curvature and stress-energy [63].

Writing $\Omega = (\omega_+, \omega_-)$ in the two middle lines, the local fermion operator has the Dirac form

$$\mathcal{Q}_F = \begin{pmatrix} 0 & Q_- \\ Q_+ & 0 \end{pmatrix}, \qquad \mathcal{Q}_F \Omega = m\Omega. \tag{156}$$

Equivalently,

$$Q_- Q_+ \omega_+ = m^2 \omega_+, \qquad Q_+ Q_- \omega_- = m^2 \omega_-. \tag{157}$$

The mass is the cost of coupling the two branch-helicity components through the odd operator (148). The exterior algebra fixes the allowed representation channel, as in (135) and (136). After Q_{hol} and Q_C have been fixed, the eigenvalues of (149) are branch-response data.

The preceding construction gives a finite locked spinor-link sector before any second-quantized field theory is introduced. This sector has a standard low-energy interface with fermionic QFT when the operator (149) has an isolated low cluster. The Hessian of (29) is kept distinct from (149). The ARC action supplies the closed branch response and the Codazzi channel, while the Dirac-type operator acts on the spinorial and link-zero-mode bundle (142).

Let $\mathcal{M}_\Gamma$ denote a local parameter space of regular deformations of the branch worldline and of its link data for which the positive link spaces have constant dimension. For $q = 2, 3$, the spaces $E_q = \ker D_q^+$ form finite-rank Hermitian bundles over $\mathcal{M}_\Gamma$. If P_q is the corresponding spectral projection, the projected connection is

$$\nabla^{B,q} = P_q\, \mathrm{d}\, P_q. \tag{158}$$

In a local orthonormal zero-mode basis $e_a^{(q)}$, this reads

$$(A_\mu^{B,q})_{ab} = \left\langle e_a^{(q)}, \partial_\mu e_b^{(q)} \right\rangle_{\text{link}}. \tag{159}$$

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For the minimal module this gives the projected C and W connections. This is the Berry-Wilczek-Zee reading of the internal connection on a moving degenerate zero-mode bundle [64] and [65]. The finite labels are the zero-mode coordinates of the moving link. The ordinary gauge-covariant derivative is the local expression of the projected connection on those coordinates, and the Standard Model representation content remains the one reconstructed in (133).

Let P_{low} be the spectral projection of K_F to the isolated physical branch-helicity cluster. The projected first-order operator on this finite-rank coefficient bundle is denoted

$$D_{\text{eff}} = P_{\text{low}} Q_F P_{\text{low}}. \tag{160}$$

The corresponding mass operator is

$$M_{\text{eff}} = P_{\text{low}} K_F^{1/2} P_{\text{low}}. \tag{161}$$

A low branch excitation is written as a spinor-valued coefficient field on the locked finite module, and a change of basis in the zero-mode bundle is read as a gauge transformation. The corresponding effective action has the standard first-order form

$$S_{\text{low}} = \int \bar{\psi}_{\text{low}} \left(i\gamma^\mu D_\mu^{\text{eff}} - M_{\text{eff}} \right) \psi_{\text{low}} \, d\text{vol}_k + S_{\text{higher}}, \tag{162}$$

where D_μ^{eff} contains the branch spin connection and the projected link connection. The term S_{higher} denotes operators suppressed by the gap and by the scales at which the frozen branch and link data are resolved. Quantization of (162) is the standard CAR quantization of the low coefficient field. The reduction separates the structurally fixed data from the branch spectral data: the finite module, locking, and Dirac principal symbol are fixed structurally, while Q_{hol}, Q_C, and the eigenvalues of (149) remain spectral.

Proposition 16 (Locked gap stability)**.** *Let K_F have an isolated low cluster below Δ on the locked bundle and let the mirror complement satisfy (146). If a self-adjoint perturbation R satisfies $\|R\| < \Delta/2$ and $\|R\| < m_\Gamma/2$ on the corresponding spectral subspaces, then the locked low cluster and the mirror gap persist.*

Proof. The statement follows from the usual spectral stability estimate for bounded self-adjoint perturbations of an isolated spectral set, in the sense of Kato [66]. The Hausdorff displacement of the spectrum is bounded by $\|R\|$. Hence a gap larger than $2\|R\|$ cannot close. □

5.2. Projective Color Lift and Finite Family Algebra

After the locking and branch-helicity projection, the family problem is a global spectral problem for (149). A family is a global low mode satisfying the exterior representation condition (133), the locking condition (145), and the branch-helicity condition used in (151). The family number is therefore the dimension of the corresponding low-mode space, or of its protected kernel when an index theorem applies. The APS framework gives the corresponding boundary comparison class for manifolds with boundary or cylindrical ends [67]; the index-theoretic background is the standard one [68].

A minimal global source of family multiplicity is provided by the color lift of a closed elementary branch mode. The closed color holonomy is projective because local closed color observables are singlet-valued. Thus the closed color holonomy lies in $PSU(C)$. Principal bundles over S^2 are classified by the fundamental group of the structure group [69]; for $PSU(3)$ this gives three lift classes. The corresponding central class is denoted

$$\mathfrak{c}_\Gamma \in H^2(S^2_\Gamma, \mathbb{Z}_3) \simeq \mathbb{Z}_3. \tag{163}$$

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Characteristic and secondary classes provide the corresponding global comparison level [70]. The same global form of the gauge group is detected by line operators [58]. Let $\omega_3 = e^{2\pi i/3}$ and let Z and S be the clock and shift matrices on the central-sector space $\mathbb{C}^3$:

$$Ze_a = \omega_3^a e_a, \qquad Se_a = e_{a+1}, \qquad a \in \mathbb{Z}_3. \tag{164}$$

The gap-scaled central response is represented by the finite clock-shift energy

$$\mathcal{E}_{\rm cen} = \nu_\Gamma \sum_{a\in\mathbb{Z}_3} \left|\psi_{a+1} - e^{i\theta_\Gamma}\psi_a\right|^2, \qquad \nu_\Gamma \geq 0. \tag{165}$$

The natural scale is $\nu_\Gamma \sim |\Delta_C|_\Gamma e^{-S_\Gamma}$ when the adjacent central transitions are induced by a finite boundary tunnelling action S_Γ. The case $\nu_\Gamma = 0$ corresponds to a central response blocked by an additional symmetry.

The self-adjoint central-degree family response has the form

$$Y_F = a_0\,1 + a_1 Z + a_1^* Z^\dagger + b_1 S + b_1^* S^\dagger + b_2 ZS + b_2^* S^\dagger Z^\dagger + b_3 Z^\dagger S + b_3^* S^\dagger Z. \tag{166}$$

The adjacent response is called non-degenerate if at least one shift coefficient and at least one clock component are nonzero. The sector with all shift coefficients equal to zero is called central-transition blocked.

Lemma 5 (Generic central-degree response)**.** *Let $\mathcal{B}_{\rm cen}$ be a finite-dimensional space of admissible central-degree boundary traces for the locked low cluster. Assume that the projected trace maps from $\mathcal{B}_{\rm cen}$ to the clock component and to the adjacent-shift component in (166) are linear and not identically zero. Then the non-degenerate central-degree response is open and dense in $\mathcal{B}_{\rm cen}$.*

Proof. The vanishing of the clock component is the kernel of a nonzero linear map, hence a proper closed subspace of $\mathcal{B}_{\rm cen}$. The vanishing of all adjacent-shift components is the kernel of the nonzero adjacent-shift projection, hence also a proper closed subspace. The complement of their union is open and dense. On this complement at least one clock component and at least one adjacent-shift component are nonzero. □

Proposition 17 (Central lift family algebra)**.** *The projective color lift supplies three central sectors. If the central-degree boundary response is non-degenerate, then the finite algebra generated by the central phase and the adjacent shift is $M_3(\mathbb{C})$.*

Proof. The central lift gives the three-dimensional sector space with basis indexed by $\mathbb{Z}_3$. The matrices Z and S in (164) obey $ZS = \omega_3 SZ$. The finite Heisenberg representation generated by Z and S is irreducible on $\mathbb{C}^3$. By Burnside's theorem, the complex algebra generated by this irreducible matrix representation is the full algebra $M_3(\mathbb{C})$. The non-degenerate boundary response activates the phase and adjacent-shift generators in this central-sector algebra. By Lemma 5, this is the generic unblocked central-degree response. □

Let the sector-independent part of the closed fermion operator be $K_0 = 1_{\mathbb{C}^3} \otimes K_*$ and assume

$$\sigma(K_*) \subset \{0\} \cup [\Delta, \infty), \qquad \dim\ker K_* = r, \qquad \Delta > 0. \tag{167}$$

Proposition 18 (Generic locked source simplicity)**.** *Assume that the locked source sector has scalar commutant, so that no residual symmetry protects a multiplicity in* ker K_*. *Then the case $r = 1$ is open and dense among admissible self-adjoint locked source perturbations which preserve the gap in (167).*

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Proof. For a finite-dimensional isolated spectral cluster, multiplicity of the lowest eigenvalue is equivalent to the vanishing of the corresponding spectral discriminant. This is a closed algebraic condition in a local self-adjoint perturbation space. If the commutant is scalar, the degeneracy is not symmetry-protected, so the discriminant is not identically zero. Hence its complement is open and dense. The gap is preserved under sufficiently small bounded perturbations by Proposition 16. □

The unperturbed lowest cluster is $\mathbb{C}^3 \otimes \ker K_*$. A finite family response generated by the boundary algebra is an operator $Y_F = Y_F^\dagger \in M_3(\mathbb{C})$. If $\sigma(Y_F) \subset [0, \Delta)$ and P_* is the projection onto $\ker K_*$, then

$$K_{F,\mathrm{low}} = 1_{\mathbb{C}^3} \otimes K_* + Y_F \otimes P_*. \tag{168}$$

It has exactly $3r$ eigenvalues below Δ, counted with multiplicity:

$$\operatorname{rank} P_{[0,\Delta)}(K_{F,\mathrm{low}}) = 3r. \tag{169}$$

Bounded corrections which do not close the gap preserve this rank by Proposition 16. In the generic simple-source case of Proposition 18,

$$N_{\mathrm{fam}} = 3. \tag{170}$$

The topology fixes the three central sectors. The gap-selected central response supplies the finite family algebra, while the rank statement follows from (169). The entries and eigenvalues of (166) remain spectral data of the full Alena-Rainich-Codazzi branch operator.

The construction belongs to the class of geometric matter models in which particle labels are recovered from geometry and topology rather than introduced as local tensor indices [26].

6. Physical Admissibility and Structural Reconstruction

The preceding sections give the local ARC closure, the translational-current phase, the integral projective link, the gap-current Fredholm support problem, the finite zero-mode module, the locked fermionic interface, and the projective family sector. These pieces are now recorded as one admissible branch sector and then used in the structural reconstruction theorem. The admissibility condition is a compact record of the branch, link, Fredholm, and locking reductions. The structural theorem fixes representation and sector data; numerical spectra remain branch-spectral data.

6.1. Physical ARC Admissibility and Reduction

The branch sector used in the theorem is obtained from the stationary finite-action sector of (29) after the Codazzi-density closure, the phase-current reduction, the single anisotropy reading, the regular integral link resolution, the gap-current support minimization, and the low-cluster isolation have been imposed. The entries below are grouped by the reduction stage at which the corresponding data are fixed.

Definition 8 (Physical ARC admissibility)**.** *A non-null Alena-Rainich branch on $X = M \setminus \Gamma$ is called physically ARC-admissible if its stationary finite-action ARC configuration of* (29) *lies in the following reduction sector:*

(i) ARC closure. *The Hilbert stress response in* (41) *is represented by the branch geometry: the residual scalar ϕ in* (21) *is the scalar Codazzi multiplier, the trace-adjusted tensor satisfies* (51)*, and the matter part satisfies* (34)*;*

(ii) phase-current reduction. *The translational coefficient belongs to the regular phase-current sector of* (22)*, the conservation law* (24) *holds, and the stiffness sector satisfies the frozen-link stability condition* (28)*;*

(iii) single anisotropy reading. *The stationary anisotropy has one branch value: the gauge reading* (16) *and the normalized-vorticity reading* (39) *agree, with nonzero Codazzi gap* (70) *on the link scale;*

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(iv) integral link resolution. *Each non-removable compact-leaf obstruction is resolved by a regular low-trace Fredholm ARC defect in the sense of Definition 6, with integral link period supplied in the transverse-frame resolved sector by Proposition 4;*

(v) low-source Fredholm selection. *The projected low source is non-degenerate second-jet low-source and primitive two-channel in the sense of Definition 4. The physical non-singlet support is the Hessian-ground-state of the gap-current Fredholm energy in the sense of Definition 5;*

(vi) locked low cluster. *The physical fermion space is the lowest isolated branch-helicity cluster after the gap-selected locking term (143); the mirror complement satisfies (146), and the corresponding family gap does not close under admissible branch perturbations.*

Definition 8 records the admissible branch sector used below. Conditions (i)-(iii) are the local closure, current, and anisotropy reductions. Condition (iv) is the regular integral link resolution of the compact-leaf obstruction. Condition (v) fixes the branch origin and low harmonic type of the source; its carrying finite support is selected by (121). Condition (vi) is the spectral isolation condition needed for the fermionic interface. Before the minimization in condition (v), the data contain no choice of $E_3 \oplus E_2$, no finite Standard Model basis group, and no exterior multiplet module. These are obtained from the reductions cited in Proposition 19 and Theorem 5.

Proposition 19 (Admissibility as branch, link, and Fredholm reduction)**.** *A physically ARC-admissible branch carries the reduction outputs needed in Theorem 5: Codazzi closure, translational-current reduction, single anisotropy reading, integral projective link, minimal gap-current link support, and locked low-cluster isolation. The minimal link module and the diagonal locking are consequences of the gap-current Fredholm and locking reductions.*

Proof. Condition (i) identifies the residual scalar in (21) with the scalar Codazzi multiplier, gives (51), and imposes (34). Proposition 1 gives the corresponding density criterion, while Proposition 3 gives the residual divergence closure. Condition (ii) gives the conserved current (24), the Codazzi-current reduction (46), and the transport compatibility (47).

Condition (iii) gives the gauge-matter-vorticity compatibility (40) and the gap scale used in (120). Condition (iv) gives the regular integral link sector. Proposition 11 supplies the no-open-index condition, the projective link, the spin^c zero modes, and the non-degenerate low Codazzi trace.

Condition (v) gives the primitive gap-current low source. Proposition 7 gives the degree-one and degree-two principal non-scalar low types, Proposition 8 separates their invariant channels, and Proposition 10 gives the ground-state response. Theorem 3 then gives the minimal link support.

Condition (vi) gives the isolated low cluster after the gap-locking term. Proposition 15 identifies the diagonal locking operator, and Proposition 16 gives persistence of the locked low cluster under sufficiently small admissible perturbations. □

Proposition 20 (Rank of the generic admissible family cluster)**.** *Assume that a physically ARC-admissible branch has non-degenerate central-degree boundary response in the sense of Proposition 17, and that the locked local source sector has scalar commutant. Then the generic simple-source case has three lowest family modes.*

Proof. Proposition 17 gives the three central sectors and the finite family algebra. Proposition 18 gives the generic rank-one locked source sector. The rank statement is then (169) with $r = 1$, giving (170). □

6.2. Structural ARC Reconstruction Theorem

The following theorem records the structural content of the physically admissible sector. It fixes the closed branch data, the link module, the compact basis group, the exterior representation package, the locking, and the finite family algebra under the stated central-degree condition. Yukawa matrices, CKM and PMNS data, confinement scales, threshold corrections, and full spectra belong to the global spectral problem for the closed branch operator and its boundary data.

Theorem 5 (Structural ARC reconstruction theorem)**.** *Let $Y_{\mu\nu} = -Y_{\mu\nu}$ be a non-null Alena-Rainich branch tensor satisfying the algebraic type (10). Suppose that the ARC action and the reductions summarized in Definition 8 place the branch in the physically ARC-admissible sector. Then the closed branch determines the following structure:*

(i) *the weighted branch stress tensor is divergence closed as in (55), and the principal null directions of the non-degenerate Codazzi branch are geodesic and shear-free as in (63) and (64);*

(ii) *the translational-current coefficient is fixed by the Codazzi-current reduction (46), with transport compatibility (47);*

(iii) *each transverse-frame resolved compact-leaf defect carries an integral spin^c projective link sphere $S^2_\Gamma \simeq \mathbb{CP}^1$;*

(iv) *the positive link zero-mode spaces are the spaces (110), and the principal low source has degree-one and degree-two non-scalar channels;*

(v) *the gap-current Fredholm minimizer is the minimal link module $C_\Gamma \oplus W_\Gamma = E_3 \oplus E_2$;*

(vi) *the top-form preserving zero-mode basis group is $S(U(3) \times U(2))$, with the global form (131);*

(vii) *the even exterior algebra of the reconstructed link module is the one-generation module (133);*

(viii) *the one-Higgs Clifford-odd map has the invariant channels (135) and (136);*

(ix) *the diagonal chirality-exterior parity locking is given by (144), and the isolated locked low cluster has the effective Dirac interface (162);*

(x) *the projective color lift gives three central sectors, and a non-degenerate central-degree boundary response generates the finite family algebra $M_3(\mathbb{C})$.*

Proof. Condition (i) of Definition 8 gives the Codazzi-density closure. The divergence identity follows from Proposition 3, and the optical statement follows from Theorem 1. Condition (ii) gives the translational-current reduction through Proposition 2.

Conditions (iii)-(iv) give the regular integral link sector. Proposition 4 gives the integral period, and Proposition 5 gives the projective link sphere. The zero-mode realization is (110).

Condition (v) gives the non-degenerate primitive low source. The low trace and the second-jet principal channels follow from Propositions 6, 7, and 8. The active ground-state response follows from Proposition 10. The minimal support follows from Theorem 3.

The compact basis group follows from Theorem 4. The exterior module and the one-Higgs channels are those of Section 4.1 and Section 4.2. The diagonal locking follows from Proposition 15; the mirror gap and stability are supplied by (146) and Proposition 16. The low-energy Dirac interface is the reduction in Section 5.1. The projective color lift and the family algebra are supplied by Proposition 17. This proves the stated reconstruction. □

Theorem 5 is a structural statement about the physically ARC-admissible branch sector. The finite module is obtained at the link-support stage, before the exterior-algebra package is applied. The compact basis group is the top-form preserving zero-mode basis group of that reconstructed module. The numerical quantities in the fermionic sector are spectral data of (149): Yukawa matrices, fermion masses, CKM and PMNS matrices, threshold corrections, neutral-sector eigenvalues, confinement energies, and full spectra depend on the global branch solution and on its boundary data.

7. Conclusions and Discussion

Theorem 5 is a structural reconstruction statement inside the physically ARC-admissible branch sector. Starting from a non-null Alena-Rainich branch tensor, it gives a Codazzi-closed anisotropic branch geometry, an integral projective link, the finite zero-mode module $E_3 \oplus E_2$, the compact basis group $S(U(3) \times U(2))$, the exterior one-generation module, the gap-selected diagonal locking, and the

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finite family algebra under a non-degenerate central-degree response. The result does not determine Yukawa matrices, CKM or PMNS data, confinement scales, threshold corrections, neutral-sector eigenvalues, or full spectra. These quantities remain spectral data of the global branch operator and of its boundary conditions.

The construction has three separate layers. The first is the local ARC branch closure: the Alena Tensor stress response is represented by a non-null Rainich branch whose residual scalar acts as a Codazzi multiplier. This gives the divergence closure and the optical branch structure used by the link construction. The second layer is the finite link mechanism: compact-leaf obstructions are resolved by integral spin^c link data, and the gap-current Fredholm energy selects the minimal two-channel support. The third layer is the fermionic and family sector: the reconstructed module is passed to the exterior algebra, the gap-selected locking isolates the physical branch-helicity cluster, and the projective color lift supplies the three central sectors.

The finite group and the one-generation package are therefore reached through the branch link. The module $\mathcal{V} = C \oplus W$ is reconstructed before the exterior-algebra representation theory is applied. This is the sense in which the construction differs from finite-algebra inputs in almost-commutative geometry and from direct algebraic multiplet postulates. The branch does not introduce open finite indices in local closed tensors; finite non-singlet data enter as link, boundary, or sector data.

7.1. Relation to Other Reconstruction Programmes

The comparison with other Standard Model reconstruction programmes is useful, but only at the structural level. Grand-unified notation organizes the same exterior-algebra package efficiently [21]; in conventional unified models the additional vector multiplets and proton-decay channels are part of the enlarged gauge sector [71,72]. In the ARC branch sector, the $3 + 2$ module is obtained from link zero modes and no local X, Y gauge boson is introduced by the minimal reconstruction.

In almost-commutative geometry, the finite algebra and the finite Dirac operator are part of the spectral-geometric input [57]. Lorentzian and no-doubling refinements provide the relevant comparison class [59,73]. In the ARC construction, the finite module is reached from the projective link, while the diagonal locking is selected by a gap term on the branch fermion operator.

Twistor-unification programmes use projective twistor geometry as a primary arena for spacetime and internal data [13]. The corresponding recovery of internal symmetries and fermion content provides a useful comparison point [53]. The present construction uses only the local twistor link of an ARC defect. A global twistor-space reformulation would be an additional structure.

Clifford and division-algebraic models recover multiplets from ideals, ladder operators, or related algebraic modules [22,74]. The one-generation pattern obtained in this way is algebraically efficient, and family structure can then be treated by further algebraic mechanisms [23]. In the ARC branch, the same exterior package is applied after the geometric link module has been reconstructed, and the family sector is tied to the projective color lift.

Supersymmetric extensions enlarge the low-energy field content by supermultiplets and soft-breaking data [75]. Kaluza-Klein and warped extra-dimensional models represent gauge, hierarchy, or force data through additional geometric directions or higher-dimensional bundle data [76,77]. Composite-Higgs and strong-sector models generate the Higgs, and sometimes fermionic partners, from a new strong sector [78]. These possibilities would require branch-sector data beyond Theorem 5. The minimal ARC branch remains four-dimensional, and its finite data come from compact-leaf link geometry.

Dynamical-principal-bundle formulations give another comparison class for gauge and gravity variables [79]. Recent internal-symmetry reconstructions in Kaluza-Klein and related settings give further comparison points [80,81]. The global Standard Model form used here is the usual line-operator-compatible one [58]; the difference is that the finite module is read from link zero modes rather than inserted as a local gauge multiplet.

7.2. Phenomenological Handles

The theorem fixes structural data. Quantitative tests require the relevant spectral operators and matching prescriptions.

First, scalar mass is read through (155); corrections to ordinary massive propagation would have to arise through branch-anisotropy effects or through the spectral terms in (149). Precision tests of gravitational propagation and modified dispersion provide the corresponding comparison class [82].

Second, the neutral chiral sector is the natural place for a seesaw-like effective description. The branch scale extracted in Appendix F gives a branch-level comparison scale. Current global oscillation data provide the numerical target for neutral-sector spectral matching [83]. DUNE and Hyper-Kamiokande give the long-baseline comparison programmes [84] and [85].

Third, the projective color lift fixes the three central sectors, while the family matrices are spectral data of (166). Flavor mixing is therefore a relative alignment problem inside the finite family algebra. Precision flavour fits give the corresponding quark-sector comparison data [86].

Fourth, compact-leaf defects suggest a separate topological-defect problem. Gravitational-wave searches for string-like defects provide the observational comparison scale [87], with the broader framework reviewed in [88]. In the ARC setting such signals would have to be tied to compact-leaf obstructions and transverse-frame resolved link data.

Finally, the reduced one-loop bosonic normalization in Appendix F should be read as a branch-level normalization check after the stated reduced assumptions. The proof of Theorem 5 is independent of this normalization check. The check uses the same finite module and the projective-color quotient, while full threshold matching and higher-loop running remain outside the structural theorem.

7.3. Scope and Open Problems

The branch tensor is restricted to the non-null Rainich stratum, the Alena identification (14), and the physically ARC-admissible sector of Definition 8. The result is a branch-sector theorem rather than a universal rewriting of gauge theory in metric variables. The local warped representative of Theorem 2 shows non-emptiness of the closed sector, while global existence, patching beyond the warped class, and prescribed compact-leaf boundary data remain separate problems.

The open mathematical problems are specific. They include classification of solutions of (40), (51), (65), and (66) beyond the warped class; stability of the Rainich stratum under admissible perturbations; analysis of the trace map (118) on punctured branch neighbourhoods; derivation of the Fredholm support tension from the normal boundary-domain or determinant problem; global obstructions to the locking condition (145); and construction of a quantum measure on the reduced admissible branch space. The discrete projective-color class (163) is topological, so local fluctuations may change spectra inside a fixed sector but cannot remove the three-sector decomposition without changing the topology of the branch complement. These global questions are naturally compared with index and boundary methods [68] and [67], and with characteristic-class techniques [69].

The relation with the Alena Tensor setting is structural. The present article uses the residual density, translational-current coefficient, and vortex terms from the Alena Tensor construction [6,7], together with the Higgs-like branch-potential comparison developed in [8]. In the ARC sector, these ingredients are constrained by the Codazzi closure, the compact-leaf link problem, and the gap-selected fermionic branch operator. The finite module, locking, and family algebra are branch-sector outputs; the numerical spectra are left to the full Alena-Rainich-Codazzi branch dynamics.

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A. Rainich-Codazzi Algebra and Translational-Current Identities

This appendix records the local algebra used in Section 2. The convention (14) is kept throughout, and all branch-covariant derivatives use the Levi-Civita connection of $k_{\mu\nu}$.

The non-null Rainich stratum used in (10) may be written as $Y = \rho J$, where J is η-self-adjoint, $J^2 = 1$, and $\operatorname{tr} J = 0$. If $h = \delta Y$ is a tangent vector, write $h = aJ + \rho K$. Differentiating $J^2 = 1$ gives $JK + KJ = 0$, while the trace conditions give $\operatorname{tr} K = \operatorname{tr}(JK) = 0$. Conversely, if these conditions hold, the curve (13) stays in the same non-null Rainich stratum to first order. This gives (11) and (12). The statement is the tangent-space form of the algebraic Rainich condition; generic symmetric stress perturbations need not satisfy it [31].

The branch tensor components in (8) follow by direct substitution of (2) into (7). The same calculation gives (9), and the mixed tensor has two equal positive and two equal negative eigenvalues. Hence (10) is the non-null Rainich identity for this branch. The self-dual and two-form descriptions of this tensor class are the standard ones used in chiral gauge-gravity variables [32]; the corresponding conformal reconstruction viewpoint is that of Urbantke [33].

The two traces of the pair (η, k) must be kept distinct. The trace used in (3) is $\eta^{\mu\nu}k_{\mu\nu}$. The trace used in $\mathcal{A} = A_{\alpha\beta}k^{\alpha\beta}$ is $k^{\mu\nu}\eta_{\mu\nu}$. In the tetrad of (1)-(2), the inverse branch metric rescales the transverse plane by $e^{-2\chi}$, and therefore the second trace is the quantity $\bar{k}$ appearing in (49).

It remains to verify the trace adjustment used in (50). Put

$$A_{\mu\nu} = \phi p_\Lambda \big(c\, k_{\mu\nu} - \eta_{\mu\nu}\big). \tag{171}$$

Then

$$\mathcal{A} = \phi p_\Lambda \big(4c - \bar{k}\big). \tag{172}$$

Thus

$$A_{\mu\nu} - \mathcal{A}k_{\mu\nu} = \phi p_\Lambda \big((\bar{k} - 3c)k_{\mu\nu} - \eta_{\mu\nu}\big). \tag{173}$$

Comparison with (7) gives $c = (\bar{k} - 4/k)/3$, which is precisely the coefficient in (49). This proves (50).

The divergence identity follows from the Codazzi equation by contraction. Since $\nabla^{(k)}k = 0$, the k-trace of the right-hand side of (51) is the derivative of $\mathcal{A}$. This gives (53) and hence (55). Conversely, (56) shows that residual divergence closure is the vanishing of the contracted Codazzi defect. The full Codazzi equation gives the integrable branch sector used by the optical decomposition.

The eigenvalues in (57) are obtained from (49) in the k-orthonormal frame used before Theorem 1. The Rainich plane gives the double eigenvalue M, and the transverse plane gives the double eigenvalue P. Their difference is the Codazzi gap (70). The non-degenerate link sector is the sector in which this gap is nonzero at the link scale.

The translational-current coefficient enters only through the residual scalar (21) and through the conserved current (22). Put locally $B_\zeta = \mu_\zeta R_\omega$. Then

$$\delta\phi = -2\zeta\,\delta\zeta - \delta B_\zeta. \tag{174}$$

The scalar Codazzi multiplier condition gives $\mathrm{d}\log|\phi| = \theta_B$ on the non-degenerate set. Hence

$$\mathrm{d}(\zeta^2) - \zeta^2\theta_B = -\mathrm{d}B_\zeta - (1 - B_\zeta)\theta_B. \tag{175}$$

If $\theta_B = \mathrm{d}f_B$, integration gives exactly the local form (46). Thus ζ^2 is fixed by the Codazzi multiplier up to the constant multiplier of ϕ.

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The current equation (24) is equivalently

$$U(\zeta^2) + \zeta^2\Big(\nabla^{(k)}_\mu U^\mu + U(\log p_\Lambda)\Big) = 0. \tag{176}$$

Substitution of (46) gives the transport compatibility (47). In this form, the translational density, the stiffness-weighted rotational part, and the Codazzi multiplier are not independent branch data.

The amplitude equation (26) is the local frozen-density Euler equation of the stiffness sector. In the link-frozen reduction, the normal derivative terms are lower order in the finite Toeplitz problem and the algebraic condition is (27). The positive local stiffness condition (28) gives the sign used in the boundary Hessian reduction.

For the link-sector response, the linearized Codazzi equation is used on a punctured neighbourhood of the branch worldline. If $a_{\mu\nu} = \delta A_{\mu\nu}$, its principal part is

$$C_{\rm lin}(a)_{\alpha\mu\nu} = \nabla_\alpha a_{\mu\nu} - \nabla_\mu a_{\alpha\nu}. \tag{177}$$

In normal coordinates at the link, tensors of Hessian type $a_{\mu\nu} = \nabla_\mu\nabla_\nu F$ satisfy (177) up to curvature terms of lower order. On a small link sphere $r = R$, with $F = f_{\ell m}(r)Y_{\ell m}$, the spherical trace has the principal form

$$\operatorname{tr}_{S^2_R} \nabla^2 F = \Big(-\frac{\ell(\ell+1)}{R^2} f_{\ell m}(R) + \frac{2}{R} f'_{\ell m}(R)\Big) Y_{\ell m}. \tag{178}$$

For generic radial data the coefficient in (178) is nonzero. Hence the principal linearized Codazzi equation does not remove the degree-two link response needed in (128), nor the degree-one response needed in (129).

The second-jet low-source condition of Definition 3 is the finite principal version of the same calculation. If $n^i = x^i/r$ is the unit normal to the link sphere, the principal source has the form

$$J^{\rm pr}_\Gamma(n) = J_0 + J_i n^i + J^{\rm tf}_{ij} n^i n^j. \tag{179}$$

The three terms in (179) are the V_0, V_1, and V_2 spherical types. The degree-one source is read through the phase-current trace in (115). The degree-two source is read through the traceless normal Hessian of the Codazzi gap in (117). The annulus calculation in Appendix C records the corresponding nonzero low traces.

For $q = 3$, projection of the degree-two trace to $\ker D_3^+$ gives the quadrupole part of $\operatorname{End}_0(C_\Gamma)$,

$$P_3 M_{Y_{2m}} P_3 \in \operatorname{End}_0(C_\Gamma)_{\ell=2}. \tag{180}$$

For $q = 2$, projection of the degree-one trace gives the dipole part of $\operatorname{End}_0(W_\Gamma)$,

$$P_2 M_{Y_{1m}} P_2 \in \operatorname{End}_0(W_\Gamma)_{\ell=1}. \tag{181}$$

These are the finite projections used in (128) and (129). Higher harmonic types are absent from the principal second-jet source. Passive higher supports are removed by the gap-current Fredholm minimization in (121), as checked in Appendix D.

B. Warped Curvature, Codazzi Gap, and Compact-Leaf Obstruction

This appendix records the warped-product identities from Section 2.2 which are used later. The notation is that of (74)-(79). The curvature convention is the one used in [42] and [43].

For a metric $g = h - a^2\gamma$, with a depending only on the E-variables, the mixed Ricci block vanishes. Applied to (79), this gives (88). The remaining curvature identities are the standard two-dimensional

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warped-product formulas. In the flat-reference branch, (89) eliminates the h-derivatives of the reference warping factor b, while (80) converts the result to the variable u. The scalar curvature of k is

$$R(k) = \frac{K_\gamma}{L^2} H(u), \tag{182}$$

with

$$H(u) = \frac{8(u+5)\left(2u^5 - 4u^4 - 5u^3 + 16u^2 + 13u - 18\right)}{(3-u)^2(4-u-u^2)^3}. \tag{183}$$

The E-block reduces to

$$\mathrm{Ric}^{(k)}_{AB} = \frac{2(5-7u)}{(3-u)(1-u)(1+u)(4-u-u^2)} \nabla_A u \nabla_B u. \tag{184}$$

Consequently, the traceless Rainich-plane part of the Einstein equation with the constant term (87) is

$$\left(G^{(k)}_{AB} + \Lambda h_{AB}\right)^{\mathrm{tf}} = \frac{2}{3-u} (\nabla_A \nabla_B u)^{\mathrm{tf}}. \tag{185}$$

The trace equation fixes the scalar matter trace or the rotational-gradient trace, while (185) fixes the corresponding Rainich-plane channel.

The Codazzi gap in the warped class is controlled by (83). Hence it is nonzero on $0 < u < 1$ whenever $F_0 \neq 0$. The branch radius and the gap satisfy (84), so the support-tension scale in (120) is already present in the closed warped branch. The Fredholm-domain barrier therefore uses the same two-eigenvalue splitting that appears in the Codazzi optical decomposition.

For the hyperbolic representative of Theorem 2, one takes (97) and (98). Since $K_{\gamma_H} = -1$, (182) gives $R(k_H) = -L^{-2} H(u(\tau))$. The flatness of (99) follows directly from (89), with $b = \tau$, $|d\tau|^2_h = 1$, and $\nabla^2_h \tau = 0$. The identities (101) then give (95) and (96) after substituting (102) and (103).

On a compact transverse leaf Σ, the left-hand side of (95) has zero integral. For $D_o \neq 0$ and $0 < u < 1$, this gives the obstruction (104). The distributional form (105) is the corresponding neutrality condition. When the defect is resolved by patching the oriented transverse frame, Proposition 4 supplies the integral spin^c link used in Section 3.

The same compact-leaf obstruction is the boundary-source mechanism for the finite link problem. The resolved link inherits the projective spinor sphere from the optical branch and the support barrier from the nonzero Codazzi gap. The scale relation (84) gives the warped-class consistency check used in the gap-current support comparison.

C. Low Codazzi Traces and Gap-Current Source Projections

This appendix gives the annulus check used in Proposition 6. The purpose is to record non-emptiness of the projected low Codazzi traces, their stability under small collar perturbations, and their assignment to the phase-current and Codazzi-gap source maps in Section 3.2.

Let A_{R_0,R_1} be a normal annulus around the branch worldline, with normal radius r and link spheres S^2_r. In frozen normal coordinates the principal linearized Codazzi operator is the operator (177). The boundary trace is the tangential trace on S^2_R, followed by the finite spherical projection

$$\mathcal{T}^{\le 2}_\Gamma(a) = \Pi_{\le 2} \mathcal{T}_\Gamma(a) \in V_0 \oplus V_1 \oplus V_2. \tag{186}$$

Here $\Pi_{\le 2}$ denotes projection to the spherical degrees $\ell = 0, 1, 2$. The trace is finite-dimensional after this projection.

The model solutions are generated by Hessian-type tensors $a_{\mu\nu} = \nabla_\mu \nabla_\nu F$. In the flat frozen collar these tensors solve the principal Codazzi equation. For $F = f_{\ell m}(r) Y_{\ell m}$, the spherical trace is given by (178). Hence the projected trace is nonzero whenever the coefficient in (178) is nonzero. For $\ell = 0$ this is obtained, for example, by choosing $f'_{00}(R) \neq 0$. For $\ell = 1$ and $\ell = 2$ it is enough to choose $f_{\ell m}(R) \neq 0$

and avoid the single linear relation between $f_{\ell m}(R)$ and $f'_{\ell m}(R)$. Therefore the flat annulus trace map has nonzero image in V_0, V_1, and V_2.

Equivalently, the finite projected trace map

$$T_0^{\leq 2} : \mathcal{B}_0 \longrightarrow V_0 \oplus V_1 \oplus V_2 \tag{187}$$

has nonzero component in each summand. Here $\mathcal{B}_0$ is any finite-dimensional space of frozen annulus data containing the three Hessian test families above. After restricting to a three-dimensional subspace which maps isomorphically to its image, $T_0^{\leq 2}$ may be treated as an invertible finite matrix onto this image.

Let the normal collar geometry be perturbed smoothly by a parameter ε. The projected trace map has the finite-dimensional form

$$T_\varepsilon^{\leq 2} = T_0^{\leq 2} + E_\varepsilon, \qquad \left\| (T_0^{\leq 2})^{-1} E_\varepsilon \right\| = O(\varepsilon). \tag{188}$$

For small ε, the Neumann series applies, and the projected low trace remains nonzero on the corresponding perturbed data. This is the finite-dimensional form of the stability of Calderon and trace data for elliptic boundary problems [89]; the functional-calculus background is standard [90].

The same statement may be expressed without choosing the finite test subspace. Let $\mathcal{B}$ be the Banach space of admissible linearized boundary data used in Corollary 1, and let

$$L_j : \mathcal{B} \longrightarrow V_j, \qquad j = 0, 1, 2, \tag{189}$$

be the projected trace maps. The annulus tests show that no L_j is identically zero. Hence the nonzero-trace locus in Corollary 1 is open and dense.

The second-jet low-source condition is the finite principal version of the annulus model. The normal two-jet of the trace-adjusted branch tensor restricts to the three spherical types displayed in (179). The central term gives the scalar trace, the linear term gives the degree-one response, and the traceless quadratic term gives the degree-two response. The non-degenerate condition in Definition 3 requires the two non-scalar projected traces to be nonzero.

The source projections in (115)-(117) are obtained after applying the frozen boundary-trace map. At the principal level, this gives the finite maps

$$\mathcal{S}_W : \Omega^1(S^2_\Gamma) \longrightarrow V_1 \cap \mathrm{End}_0(W_\Gamma), \tag{190}$$
$$\mathcal{S}_C : \Gamma(N^*_\Gamma \oplus S^2_0 N^*_\Gamma) \longrightarrow (V_1 \oplus V_2) \cap \mathrm{End}_0(C_\Gamma). \tag{191}$$

Here $S^2_0 N^*_\Gamma$ denotes trace-free normal Hessian data. The first map reads the phase-current one-form and supplies the weak dipole source. The second map reads the first and second normal data of $\log|\Delta_C|$ and supplies the color low source. The nonzero image statements above imply that these assignments are non-empty whenever the projected source components in Definition 2 are nonzero.

The linearized anisotropy relation used in Section 3.2 is obtained by varying the first integrals (65) and (66). Let $\chi = \chi_0 + \varepsilon f$, $\zeta = \zeta_0 + \varepsilon z$, and $\mu_\zeta R_\omega = B_0 + \varepsilon b$. With $T_0 = \tanh \chi_0$ and $\phi_0 = 1 - \zeta_0^2 - B_0$, the difference of the two linearized first integrals gives (119). In the real non-degenerate branch, the scalar multiplier in that identity is nonzero. Thus the anisotropy variation is not removed by the linearized Codazzi first-integral difference. Its axisymmetric low components are the Legendre components of the same spherical modes used in the annulus trace map.

The projected low trace used in (118) is therefore non-empty in the flat annulus model and persists under small collar perturbations. This supplies the branch-side low traces used by the phase-current and Codazzi-gap source assignments. The removal of unforced active amplitudes and passive higher supports is the finite variational statement of (121), checked in Appendix D.

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D. Gap-Current Fredholm Minimization and Support Tension

This appendix records the finite-dimensional support comparison used in Theorem 3. The source data are the phase-current dipole source and the Codazzi-gap color source of Section 3.2. The two channels are compared separately: the phase-current source is carried by the weak dipole support, while the Codazzi-gap source is carried by the color quadrupole-capable support.

For every ℓ and $q \geq \ell + 1$, the source component of type V_ℓ is compared through the canonical unitary identification of irreducible $SU(2)$ modules. With this normalization, the source norm does not depend on passive enlargement of the support. On V_ℓ, the fuzzy adjoint Laplacian acts by the scalar $\ell(\ell+1)$, as in (113).

For fixed supports P_W and P_C, the Euler equations are (122) and (123). Thus every forced component is divided by the corresponding positive Casimir eigenvalue and every unforced visible component vanishes. After this elimination of the active response, enlarging a support beyond the smallest block carrying the source changes only the positive support-tension part of (121).

The non-scalar phase-current component relevant to the weak channel is the forced degree-one off-diagonal component, so Lemma 2 gives the minimal weak support E_2. The Codazzi-gap color component has a forced degree-two component, with the degree-one companion allowed in the same color channel, so the minimal color support is E_3. Since $\lambda_W, \lambda_C > 0$ in (120), larger passive supports do not minimize the reduced energy. Hence the minimizing two-channel support is $P_{E_3} \oplus P_{E_2}$, equivalently $C_\Gamma \oplus W_\Gamma = E_3 \oplus E_2$.

The separation of P_W and P_C is part of the finite variational problem. A single unrestricted support could carry both a dipole and a quadrupole on E_3, while the ARC link problem assigns the phase-current dipole and the Codazzi-gap quadrupole to different boundary-source channels. The direct sum is therefore the minimal two-channel support. The numerical values of λ_W and λ_C belong to the full boundary spectral problem.

E. Fermionic Locking, Berry Connection, and Family Algebra Checks

This appendix records the spinorial and finite-family checks used in Section 5. The conventions are those of the branch tetrad, the reconstructed finite module $\mathcal{V} = C \oplus W$, and the gap-selected locking operator (144).

Let (o_A, ι_A) be a spin frame adapted to the principal branch null pair, with o_A representing the ℓ-line and ι_A representing the r-line. Let $L = \mathrm{span}(o_A)$ be the corresponding line. A symmetric two-spinor $\Xi_{AB} \in \mathrm{Sym}^2 S_L$ decomposes as

$$\Xi_{AB} = \Xi_0\, o_A o_B + 2\Xi_1\, o_{(A}\iota_{B)} + \Xi_2\, \iota_A \iota_B. \tag{192}$$

After one primed spinor direction is fixed by the branch optical structure, the covariant derivative of $C\Xi_{AB}$ decomposes into extreme and middle line weights. The two extreme weights are the branch-relative spin-3/2 components, while the middle weights form the branch-relative spinor channel. This is the spin-frame form of the standard two-spinor decomposition [11].

The projectors P_{ext} and P_{mid} used in (150) and (151) are the projections onto the extreme and middle line weights of this decomposition. In the optical ARC branch, the shear-free conclusion (64) removes the tensorial leakage into the extreme projection. The retained middle projection is the branch-helicity channel used in (152). This is the local spinorial reading of the Codazzi optical splitting. Related twistor-line decompositions use the same projective-spinor bookkeeping [12].

The grading in Proposition 15 is checked on the primitive finite generators. Exterior multiplication by $\Phi \in W$ and contraction by $\Phi^\dagger$ satisfy

$$\Gamma_W\, c(\Phi + \Phi^\dagger) = -c(\Phi + \Phi^\dagger)\, \Gamma_W, \qquad \Gamma_C\, c(\Phi + \Phi^\dagger) = c(\Phi + \Phi^\dagger)\, \Gamma_C. \tag{193}$$

Thus the weak Clifford-odd map fixes the W-parity factor and leaves the C-parity factor before charge conjugation. Finite charge conjugation sends p_C to $3 - p_C$ and p_W to $2 - p_W$, hence

$$\mathcal{C}_F \Gamma_C \mathcal{C}_F^{-1} = -\Gamma_C, \qquad \mathcal{C}_F \Gamma_W \mathcal{C}_F^{-1} = \Gamma_W. \tag{194}$$

Branch charge conjugation changes the sign of γ^5_{branch}. The locked sector is preserved for the product grading in (144).

The gap-locking energy (143) is evaluated on primitive product-type signs. The branch Dirac term forces the branch chirality factor. The weak Clifford-odd term then forces the exterior total parity, and the charge-conjugation check above fixes the remaining primitive sign. Therefore the zero locus of (143) is the diagonal product in (144), up to the overall sign. The mirror complement is separated by the positive gap term in (145).

The low-cluster stability used in Proposition 16 can be written with the Riesz projection

$$P_{\text{low}} = \frac{1}{2\pi i} \int_{\mathcal{C}} (z - K_F)^{-1}\, dz, \tag{195}$$

where $\mathcal{C}$ surrounds the isolated low cluster and lies in the resolvent set of K_F. If a self-adjoint perturbation has norm smaller than half the distance from $\mathcal{C}$ to the rest of the spectrum, the contour remains in the perturbed resolvent set and (195) has the same rank. This is the perturbation-theoretic input used in (160) and (162) [66]. For unbounded Dirac-type realizations, the same argument is applied to the closed quadratic forms associated with (149).

The Berry connection (158) is basis covariant. If $e_a^{(q)}$ is changed by a unitary matrix U_q, then the local connection form transforms as

$$A^{B,q} \longmapsto U_q^{-1} A^{B,q} U_q + U_q^{-1} \mathrm{d} U_q. \tag{196}$$

Thus a change of zero-mode basis is read as a gauge transformation of the finite coefficient field. This is the finite-rank version of the projected connection used in (162).

The family algebra in Proposition 17 is generated by the clock-shift pair. With Z and S as in (164), the matrix units are obtained from

$$E_{rs} = \frac{1}{3} \sum_{a=0}^{2} \omega_3^{-ar} Z^a S^{s-r}, \qquad r, s \in \mathbb{Z}_3. \tag{197}$$

Hence the nine operators $Z^a S^b$ span $M_3(\mathbb{C})$. This is the finite Heisenberg representation of the central lift sectors.

The clock-shift energy (165) is diagonalized by the discrete Fourier modes $\psi_a = 3^{-1/2} e^{2\pi i k a/3}$, $k \in \mathbb{Z}_3$. The corresponding eigenvalues are

$$E_k(\theta_\Gamma) = 4\nu_\Gamma \sin^2\!\left(\frac{2\pi k/3 - \theta_\Gamma}{2}\right), \qquad k \in \mathbb{Z}_3. \tag{198}$$

Thus ν_Γ is the adjacent central-response scale. If $\nu_\Gamma = 0$, the central sectors are not mixed by this response. If $\nu_\Gamma > 0$, the finite central-sector operator is nontrivial; together with the central phase it gives the algebra in Proposition 17.

The generic simplicity statement in Proposition 18 is finite-dimensional after the low cluster has been isolated. If H is the compression of the locked source operator to the cluster, then a repeated lowest eigenvalue is detected by the vanishing of the corresponding spectral discriminant. In a local affine space of self-adjoint perturbations,

$$\Delta_{\text{spec}}(H) = \prod_{i<j} (\lambda_i - \lambda_j)^2. \tag{199}$$

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When the commutant is scalar, the discriminant is not identically zero. Thus the complement of its zero set is open and dense. The gap condition in (167) then identifies this finite-dimensional genericity statement with the branch low-cluster statement.

The rank count in (169) is stable under bounded perturbations which preserve the gap. The unperturbed low space is $\mathbb{C}^3 \otimes \ker K_*$, and the family response acts only on the first factor inside that cluster. If $r = \dim \ker K_*$, the low spectral projection has rank $3r$. Proposition 16 preserves this rank under the admissible perturbations used in Definition 8. In the scalar-commutant generic case, Proposition 18 gives $r = 1$, and the rank is three.

The same finite algebra controls the relative flavor alignments. For two channels R and R', the corresponding reduced family-response operators are diagonalized by unitary matrices U_R and $U_{R'}$ on the three-sector space. The relative matrix $U_R^\dagger U_{R'}$ is the mixing matrix of the two channels. Its entries are therefore spectral data of the projected branch operator, while Theorem 5 fixes only the finite family algebra and the rank mechanism.

F. Branch-Level Bosonic Normalization Check

This appendix records a reduced one-loop branch-level normalization check for the bosonic sector. The multiplicity split, the unimodular stabilizer, and the hypercharge assignments are those of (130), (131), and (132). The check is separate from the proof of Theorem 5 and from a threshold calculation for the full Standard Model.

The weak factor W is color neutral and has weak type $(2, +1)$. The color-preserving branch order parameter is therefore taken to be $\Phi \in W$. After a $U(W)$ frame choice, the vacuum representative is

$$\Phi_0 = \frac{v}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{200}$$

The primitive gauge norm is the branch trace on the one-generation module $\Lambda^{\mathrm{even}}(C \oplus W)$. For the normalized abelian generator $T_1 = \sqrt{3/5}\, Y/2$, the three quadratic indices are equal,

$$I_C = I_W = I_1 = 2. \tag{201}$$

Thus the primitive branch norm contains a single gauge coupling,

$$g_C = g_W = g_1 = g_B, \qquad g_1 = \sqrt{\frac{5}{3}}\, g_Y, \tag{202}$$

where g_Y denotes the coupling to $Y/2$. The equality in (202) fixes the relative normalization of the three resolved gauge directions. The absolute normalization used in this appendix is fixed by the unresolved weak projector.

Assumption 1 (Minimal branch-cell normalization)**.** *Let $r_B = e^{\rho_B}$ be the determinant scale of the unresolved weak cell. Since $\Phi \in W \simeq \mathbb{C}^2$, the cell has $N_\Phi = 2\dim_{\mathbb{C}} W = 4$ real components. The finite determinant normalization is taken in the minimal entropy form*

$$\Gamma_{\mathrm{cell}}(r_B) = r_B \log \frac{r_B}{N_\Phi}. \tag{203}$$

The stationary condition gives

$$\frac{d\Gamma_{\mathrm{cell}}}{dr_B} = \log \frac{r_B}{N_\Phi} + 1 = 0, \qquad r_B = \frac{N_\Phi}{e} = \frac{4}{e}, \qquad \rho_B = \log \frac{4}{e}. \tag{204}$$

The primitive transport norm is fixed at the same stationary cell. The unresolved cell carries the determinant scale r_B and its inverse scale r_B^{-1}, because the branch normalization is unimodular. Hence the local generator norm is

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taken to be even in ρ_B. With the weak-generator normalization $T_i = \sigma_i/2$, the unresolved weak-generator norm is $1/2$. The minimal analytic even choice, with no higher powers of ρ_B, is obtained by multiplying this norm by $(r_B + r_B^{-1})/2$. Using (204), this gives

$$g_B = \frac{1}{2}\cosh\rho_B = \frac{1}{2}\cosh\left(\log\frac{4}{\mathrm{e}}\right) = \frac{1}{\mathrm{e}} + \frac{\mathrm{e}}{16}. \tag{205}$$

Thus g_B serves as the matching value of the primitive branch norm at E_B in the branch-level check. This absolute normalization is not used in the structural reconstruction theorem. The scale E_B is defined as the scale at which this primitive norm is read before the lower-energy branch response is resolved. Below this scale the effective inverse couplings are the corresponding branch stiffnesses. In the one-loop spectral approximation,

$$\frac{1}{g_a^2(\mu)} = \frac{1}{g_B^2} + \frac{b_a}{8\pi^2}\log\frac{E_B}{\mu}. \tag{206}$$

The coefficients used in (206) are the reduced one-loop coefficients of the locked ARC branch complex. The propagating gauge directions give the usual adjoint terms. The locked matter sector is the exterior-algebra sector of (132), repeated over the three central projective-color sectors. The order parameter $\Phi \in W$ gives one complex weak doublet. For the weak and abelian branch directions,

$$\begin{aligned} b_W &= -\frac{22}{3} + \frac{2}{3}(6) + \frac{1}{3}\left(\frac{1}{2}\right) = -\frac{19}{6}, \\ b_1 &= \frac{2}{3}(6) + \frac{1}{3}\left(\frac{3}{10}\right) = \frac{41}{10}. \end{aligned} \tag{207}$$

Here $T(2) = 1/2$, and the abelian coefficient is written in the normalization of (202).

The color coefficient receives one additional reduction term. Closed local color observables are singlet-valued, while open color data occur as link, boundary, or sector data. The projective color lift therefore has a closure quotient not present in the weak and abelian factors.

Assumption 2 (Minimal projective-color closure quotient)**.** *The minimal color closure is imposed on the projective lift of a color-singlet branch mode. Its linearized closure operator has one fundamental Dirac-type determinant on $C \oplus C^*$. In the reduced branch measure the non-closed color lifts are divided out. Hence this closure determinant enters as a quotient determinant.*

Proposition 21 (Reduced color coefficient)**.** *Under Assumption 2, the reduced one-loop color coefficient in (206) is*

$$b_C = -11 + \frac{2}{3}(6) - \frac{2}{3} = -\frac{23}{3}. \tag{208}$$

Proof. The propagating color branch gives the adjoint term $-\frac{11}{3}C_2(SU(C)) = -11$. The locked matter sector is repeated over the three central projective-color sectors. In each sector the color part of (132) contributes two fundamental Dirac types, equivalently $\sum T_C = 2$. Hence the three sectors give $\frac{2}{3}(6)$. The closure operator in Assumption 2 has the local coefficient of one fundamental Dirac determinant. In the convention of (206) this coefficient would be $+\frac{4}{3}T(F) = +\frac{2}{3}$ for a propagating determinant, with $T(F) = 1/2$. Since the non-closed lifts are divided out, the quotient determinant contributes with the opposite sign. This gives (208). □

The last term in (208) is the determinant-measure input of the projective-color reduction. A full determinant-line calculation would have to keep the link boundary conditions, the projective stabilizers, and the zero-mode factors.

At fixed E_B, the resolved weak and abelian couplings at $\mu = v$ are obtained from (206) and (205):

$$g_W^2(v) = \left[\frac{1}{g_B^2} + \frac{b_W}{8\pi^2}\log\frac{E_B}{v}\right]^{-1}, \qquad g_1^2(v) = \left[\frac{1}{g_B^2} + \frac{b_1}{8\pi^2}\log\frac{E_B}{v}\right]^{-1}, \qquad g_Y^2(v) = \frac{3}{5}g_1^2(v). \tag{209}$$

Let $T_i = \sigma_i/2$ denote the generators of $\mathfrak{su}(W)$ on W. Since the lower component of (200) has T_3-weight $-1/2$ and $Y/2$-weight $+1/2$, the generator

$$Q = T_3 + \frac{Y}{2} \tag{210}$$

annihilates the vacuum:

$$Q\Phi_0 = 0. \tag{211}$$

The stabilizer of the weak branch vacuum is therefore $U(1)_Q$. The branch transport defect of Φ is

$$D_\mu\Phi = \left(\partial_\mu - ig_W W_\mu^i T_i - ig_Y B_\mu \frac{Y}{2}\right)\Phi. \tag{212}$$

The quadratic transport cost is $\|D_\mu\Phi\|^2$. At (200),

$$\|D_\mu\Phi_0\|^2 = \frac{v^2}{8}\left[g_W^2\left((W_\mu^1)^2 + (W_\mu^2)^2\right) + (g_W W_\mu^3 - g_Y B_\mu)^2\right]. \tag{213}$$

With

$$W_\mu^\pm = \frac{1}{\sqrt{2}}\left(W_\mu^1 \mp iW_\mu^2\right), \tag{214}$$

the charged branch vectors have

$$m_W^2 = \frac{g_W^2 v^2}{4}. \tag{215}$$

In the neutral basis (W_μ^3, B_μ), (213) gives

$$M_0^2 = \frac{v^2}{4}\begin{pmatrix} g_W^2 & -g_W g_Y \\ -g_W g_Y & g_Y^2 \end{pmatrix}. \tag{216}$$

Consequently,

$$m_\gamma^2 = 0, \tag{217}$$

$$m_Z^2 = \frac{v^2}{4}(g_W^2 + g_Y^2). \tag{218}$$

The corresponding neutral branch fields are

$$A_\mu = \sin\theta_B\, W_\mu^3 + \cos\theta_B\, B_\mu, \tag{219}$$

$$Z_\mu = \cos\theta_B\, W_\mu^3 - \sin\theta_B\, B_\mu, \tag{220}$$

where

$$\tan\theta_B = \frac{g_Y}{g_W}. \tag{221}$$

Equations (215), (218), and (221) give

$$m_W^2 = m_Z^2\cos^2\theta_B. \tag{222}$$

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The complex branch W has four real components. Three angular components lie in the broken transport directions and are absorbed by the massive branch vectors. The remaining component is radial. In radial gauge,

$$\Phi(x) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v + h(x) \end{pmatrix}. \tag{223}$$

Substitution into (212) gives

$$\|D_\mu \Phi\|^2 = \frac{1}{2}(\partial_\mu h)^2 + m_W^2\left(1+\frac{h}{v}\right)^2 W_\mu^+ W^{-\mu} + \frac{1}{2}m_Z^2\left(1+\frac{h}{v}\right)^2 Z_\mu Z^\mu. \tag{224}$$

The scalar branch cost is written in terms of the weak projector $P_\Phi = \Phi\Phi^\dagger$. Since $\mathrm{Tr}_W\, P_\Phi = \Phi^\dagger\Phi$ and $\mathrm{Tr}_W(P_\Phi^2) = (\Phi^\dagger\Phi)^2$, the local potential is

$$V_B(\Phi) = \kappa_B(\Phi^\dagger\Phi)^2 - j_B\Phi^\dagger\Phi + V_0. \tag{225}$$

The stationary condition at (200) gives

$$j_B = \kappa_B v^2. \tag{226}$$

Hence

$$V_B(\Phi) = \kappa_B\left(\Phi^\dagger\Phi - \frac{v^2}{2}\right)^2 + \text{const}. \tag{227}$$

Using (223), this becomes

$$V_B(h) = \kappa_B\left(vh + \frac{h^2}{2}\right)^2. \tag{228}$$

The radial mass is therefore

$$m_h^2 = 2\kappa_B v^2 = 2j_B. \tag{229}$$

The leading projector stiffness is fixed by the same branch norm which gives (202). The unimodular compensation between the weak and color blocks gives

$$\kappa_B^{(0)} = \left(\frac{\dim_{\mathbb{C}} W}{\dim_{\mathbb{C}} C}\right)^2 g_B^2 = \frac{4}{9}g_B^2. \tag{230}$$

The radial Hessian also contains the finite determinant contribution of the angular weak-branch modes. Put $X = \Phi^\dagger\Phi$. After the radial gauge choice in (223), the four real components of Φ split into one radial component and three angular components. The angular components are precisely the three broken weak directions. For one such direction, the local fluctuation operator has the form $\mathcal{O}_a(X) = -\partial^2 + g_B^2 X$, up to derivative terms in X. In the four-dimensional local determinant normalization used in (206), the logarithmic part of one real determinant $\frac{1}{2}\,\mathrm{Tr}\log\mathcal{O}_a$ contributes $\frac{1}{2}g_B^4/(16\pi^2)$ to the coefficient of X^2. Summing over the three angular directions gives

$$\Delta\kappa_B = 3\cdot\frac{1}{2}\frac{g_B^4}{16\pi^2} = \frac{3}{2}\frac{g_B^4}{16\pi^2}. \tag{231}$$

Thus

$$\kappa_B = \frac{4}{9}g_B^2 + \frac{3}{2}\frac{g_B^4}{16\pi^2}. \tag{232}$$

Consequently,

$$m_h^2 = 2v^2\left(\frac{4}{9}g_B^2 + \frac{3}{2}\frac{g_B^4}{16\pi^2}\right). \tag{233}$$

For the numerical evaluation one may take $\mu = v$, with

$$v = (\sqrt{2}G_F)^{-1/2}. \tag{234}$$

The low-energy couplings are then obtained from (209). The vector masses are outputs of (215) and (218). The branch scale E_B is fixed here by the color normalization, rather than by the electroweak vector masses. With $g_C^2(m_Z) = 4\pi\alpha_s^{\rm in}(m_Z)$, the color running equation gives

$$\log\frac{E_B}{m_Z} = \frac{8\pi^2}{b_C}\left[\frac{1}{4\pi\alpha_s^{\rm in}(m_Z)} - \frac{1}{g_B^2}\right], \qquad E_B = m_Z \exp\left\{\frac{8\pi^2}{b_C}\left[\frac{1}{4\pi\alpha_s^{\rm in}(m_Z)} - \frac{1}{g_B^2}\right]\right\}. \tag{235}$$

Here m_Z denotes the reference scale at which the color coupling is quoted. Thus the reduced color sector fixes the common branch scale used in the one-loop electroweak check.

Using $G_F = 1.1663785 \times 10^{-5}\,\text{GeV}^{-2}$ and $v = 246.21967\,\text{GeV}$ [91], together with the color input

$$\alpha_s^{\rm in}(m_Z) = 0.11848, \tag{236}$$

one obtains from (235)

$$E_B = 2.64 \times 10^{14}\,\text{GeV}. \tag{237}$$

The value lies in the intermediate, seesaw-compatible range; no neutral-sector spectrum is extracted from it here. With this scale, the resolved low-energy couplings are

$$\log\frac{E_B}{v} = 27.7007, \quad g_B = 0.537772, \quad g_W = 0.652765, \quad g_Y = 0.350061, \quad g_1 = 0.451927. \tag{238}$$

Equations (215), (218), and (233) then give

$$m_W = 80.3618\,\text{GeV}, \qquad m_Z = 91.1881\,\text{GeV}, \qquad m_h = 125.223\,\text{GeV}. \tag{239}$$

Relative to the central experimental values quoted in [91], these correspond approximately to

$$\Delta m_W = -0.56\sigma, \qquad \Delta m_Z = +0.07\sigma, \qquad \Delta m_h = +0.21\sigma. \tag{240}$$

The calculation uses G_F, the primitive normalization (205), the reduced color coefficient of Proposition 21, and the color input (236). The masses in (239) are therefore the electroweak branch-level values obtained after the common scale has been fixed by the reduced color normalization. Threshold matching and higher-loop running are left outside this branch-level check. The check is sensitive to the absolute branch normalization: at (205) and (208), one has $\partial \log E_B/\partial g_B = -132.44$. Thus a one percent shift of g_B changes E_B by a factor of order two. If the projective-color quotient in Assumption 2 is omitted, the corresponding one-loop scale is shifted to $4.06 \times 10^{15}\,\text{GeV}$ for the same color input. If the closure determinant is counted with the ordinary propagating sign, the scale is shifted to $1.11 \times 10^{17}\,\text{GeV}$. The scale E_B fixes the branch normalization used here and leaves the neutral-sector seesaw calculation to the full spectral problem. The weak order parameter $\Phi \in W$ determines the electroweak branch sector: the unbroken generator (210) gives the massless field (219), the broken transport directions give (215) and (218), and the radial projector cost gives (233).

References

1. Hélein, F. Dynamical mechanisms for Kaluza–Klein theories. *Letters in Mathematical Physics* **2022**, *112*, 95, [arXiv:math.DG/2201.01981]. https://doi.org/10.1007/s11005-022-01591-6.
2. Cariglia, M.; Alves, F.K. The Eisenhart lift: a didactical introduction of modern geometrical concepts from Hamiltonian dynamics. *European Journal of Physics* **2015**, *36*, 025018, [arXiv:math-ph/1503.07802]. https://doi.org/10.1088/0143-0807/36/2/025018.
3. Silva, J.E.G. A field theory in Randers–Finsler spacetime. *EPL* **2021**, *133*, 21002, [arXiv:hep-th/2009.03806]. https://doi.org/10.1209/0295-5075/133/21002.
4. Ogonowski, P.; Skindzier, P. Alena Tensor in unification applications. *Physica Scripta* **2024**, *100*, 015018. https://doi.org/10.1088/1402-4896/ad98ca.

5. Ogonowski, P. The halo effect and quantum vortices. Not so dark with Alena Tensor. *Physica Scripta* **2026**, *101*, 155002. https://doi.org/10.1088/1402-4896/ae59ca.
6. Ogonowski, P. Developed method: interactions and their quantum picture. *Frontiers in Physics* **2023**, *11*, 1264925. https://doi.org/10.3389/fphy.2023.1264925.
7. Ogonowski, P. Proposed method of combining continuum mechanics with Einstein Field Equations. *International Journal of Modern Physics D* **2023**, *32*, 2350010. https://doi.org/10.1142/S0218271823500104.
8. Ogonowski, P. Gravitational waves and Higgs-like potential from Alena Tensor. *Physica Scripta* **2025**, *100*, 105026. https://doi.org/10.1088/1402-4896/ae12e2.
9. Rainich, G.Y. Electrodynamics in the General Relativity Theory. *Transactions of the American Mathematical Society* **1925**, *27*, 106–136. https://doi.org/10.1090/S0002-9947-1925-1501302-6.
10. Ferrando, J.J.; Sáez, J.A. Rainich Theory for Type D Aligned Einstein–Maxwell Solutions. *General Relativity and Gravitation* **2007**, *39*, 2039–2051. https://doi.org/10.1007/s10714-007-0500-9.
11. Penrose, R.; Rindler, W. *Spinors and Space-Time. Volume 1: Two-Spinor Calculus and Relativistic Fields*; Cambridge University Press: Cambridge, 1984. https://doi.org/10.1017/CBO9780511564048.
12. Ward, R.S.; Wells, Raymond O., J. *Twistor Geometry and Field Theory*; Cambridge University Press: Cambridge, 1990. https://doi.org/10.1017/CBO9780511524493.
13. Woit, P. Euclidean Twistor Unification, 2021, [arXiv:hep-th/2104.05099]. https://doi.org/10.48550/arXiv.2104.05099.
14. Wu, T.T.; Yang, C.N. Dirac Monopole Without Strings: Monopole Harmonics. *Nuclear Physics B* **1976**, *107*, 365–380. https://doi.org/10.1016/0550-3213(76)90143-7.
15. Jante, R.; Schroers, B.J. Dirac Operators on the Taub–NUT Space, Monopoles and SU(2) Representations. *Journal of High Energy Physics* **2014**, *2014*, 114. https://doi.org/10.1007/JHEP01(2014)114.
16. Bott, R. Homogeneous Vector Bundles. *Annals of Mathematics* **1957**, *66*, 203–248. https://doi.org/10.2307/1969996.
17. Fulton, W.; Harris, J. *Representation Theory: A First Course*; Vol. 129, *Graduate Texts in Mathematics*, Springer: New York, NY, 1991. https://doi.org/10.1007/978-1-4612-0979-9.
18. Bordemann, M.; Meinrenken, E.; Schlichenmaier, M. Toeplitz quantization of Kähler manifolds and gl(N), $N \to \infty$ limits. *Communications in Mathematical Physics* **1994**, *165*, 281–296. https://doi.org/10.1007/BF02099772.
19. Alexanian, G.; Pinzul, A.; Stern, A. Generalized coherent state approach to star products and applications to the fuzzy sphere. *Nuclear Physics B* **2001**, *600*, 531–547. https://doi.org/10.1016/S0550-3213(00)00743-4.
20. Sämann, C.; Szabo, R.J. Branes, Quantization and Fuzzy Spheres. *Proceedings of Science* **2010**, *CNCFG2010*, 005, [arXiv:hep-th/1101.5987]. https://doi.org/10.22323/1.127.0005.
21. Baez, J.C.; Huerta, J. The Algebra of Grand Unified Theories. *Bulletin of the American Mathematical Society* **2010**, *47*, 483–552. https://doi.org/10.1090/S0273-0979-10-01294-2.
22. Stoica, O.C. Leptons, Quarks, and Gauge from the Complex Clifford Algebra $\mathbb{C}\ell_6$. *Advances in Applied Clifford Algebras* **2018**, *28*, 52. https://doi.org/10.1007/s00006-018-0869-4.
23. Gresnigt, N.G. The Standard Model Particle Content with Complete Gauge Symmetries from the Minimal Ideals of Two Clifford Algebras. *The European Physical Journal C* **2020**, *80*, 583. https://doi.org/10.1140/epjc/s10052-020-8141-1.
24. Connes, A.; Lott, J. Particle Models and Noncommutative Geometry. *Nuclear Physics B - Proceedings Supplements* **1991**, *18*, 29–47. https://doi.org/10.1016/0920-5632(91)90120-4.
25. Chamseddine, A.H.; Connes, A. The Spectral Action Principle. *Communications in Mathematical Physics* **1997**, *186*, 731–750, [arXiv:hep-th/hep-th/9606001]. https://doi.org/10.1007/s002200050126.
26. Atiyah, M.F.; Manton, N.S.; Schroers, B.J. Geometric Models of Matter. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences* **2012**, *468*, 1252–1279, [arXiv:hep-th/1108.5151]. https://doi.org/10.1098/rspa.2011.0616.
27. Krasnov, K. Fermions, Differential Forms and Doubled Geometry. *Nuclear Physics B* **2018**, *936*, 36–75, [arXiv:hep-th/1803.06160]. https://doi.org/10.1016/j.nuclphysb.2018.09.006.
28. Tong, D. Gauge Theory, 2018. Lecture notes, University of Cambridge.
29. Jost, J.; Keßler, E.; Wu, R.; Zhu, M. Geometric analysis of the Yang–Mills–Higgs–Dirac model. *Journal of Geometry and Physics* **2022**, *182*, 104669, [arXiv:math-ph/1908.00430]. https://doi.org/10.1016/j.geomphys.2022.104669.
30. Carroll, S.M. *Spacetime and Geometry: An Introduction to General Relativity*, 2 ed.; Cambridge University Press, 2019. https://doi.org/10.1017/9781108770385.

31. Martín-Moruno, P.; Visser, M. Generalized Rainich Conditions, Generalized Stress-Energy Conditions, and the Hawking–Ellis Classification. *Classical and Quantum Gravity* **2017**, *34*, 225014. https://doi.org/10.1088/1361-6382/aa9039.
32. Plebański, J.F. On the separation of Einsteinian substructures. *Journal of Mathematical Physics* **1977**, *18*, 2511–2520. https://doi.org/10.1063/1.523215.
33. Urbantke, H. On integrability properties of SU(2) Yang–Mills fields. I. Infinitesimal part. *Journal of Mathematical Physics* **1984**, *25*, 2321–2324. https://doi.org/10.1063/1.526394.
34. Varshalovich, D.A.; Moskalev, A.N.; Khersonskii, V.K. *Quantum Theory of Angular Momentum*; World Scientific: Singapore, 1988; p. 528. https://doi.org/10.1142/0270.
35. Thorne, K.S. Multipole Expansions of Gravitational Radiation. *Reviews of Modern Physics* **1980**, *52*, 299–339. https://doi.org/10.1103/RevModPhys.52.299.
36. Derdziński, A.; Shen, C.L. Codazzi Tensor Fields, Curvature and Pontryagin Forms. *Proceedings of the London Mathematical Society* **1983**, *47*, 15–26. https://doi.org/10.1112/plms/s3-47.1.15.
37. Mantica, C.A.; Molinari, L.G. Codazzi Tensors and Their Space-Times, and Cotton Gravity. *General Relativity and Gravitation* **2023**, *55*, 62. https://doi.org/10.1007/s10714-023-03106-7.
38. Coll, B.; Ferrando, J.J.; Sáez, J.A. On the Geometry of Killing and Conformal Tensors. *Journal of Mathematical Physics* **2006**, *47*, 062503. https://doi.org/10.1063/1.2207717.
39. Ferrando, J.J.; Sáez, J.A. On the Space-Times Admitting Two Shear-Free Geodesic Null Congruences. *General Relativity and Gravitation* **2007**, *39*, 343–359. https://doi.org/10.1007/s10714-006-0388-9.
40. Penrose, R.; Rindler, W. *Spinors and Space-Time. Volume 2: Spinor and Twistor Methods in Space-Time Geometry*; Cambridge University Press: Cambridge, 1986. https://doi.org/10.1017/CBO9780511524486.
41. Mason, L.J.; Woodhouse, N.M.J. *Integrability, Self-Duality, and Twistor Theory*; Oxford University Press: Oxford, 1996. https://doi.org/10.1093/oso/9780198534983.001.0001.
42. Besse, A.L. *Einstein Manifolds*; Classics in Mathematics, Springer-Verlag: Berlin Heidelberg, 2007. https://doi.org/10.1007/978-3-540-74311-8.
43. Kobayashi, S.; Nomizu, K. *Foundations of Differential Geometry. Volume II*; Interscience Publishers: New York, 1969.
44. Uhlenbeck, K.K. Removable Singularities in Yang–Mills Fields. *Communications in Mathematical Physics* **1982**, *83*, 11–29. https://doi.org/10.1007/BF01947068.
45. Gilbarg, D.; Trudinger, N.S. *Elliptic Partial Differential Equations of Second Order*, 2 ed.; Classics in Mathematics, Springer-Verlag: Berlin, 2001. Reprint of the 1998 edition, https://doi.org/10.1007/978-3-642-61798-0.
46. Atiyah, M.F.; Hitchin, N.J.; Singer, I.M. Self-duality in four-dimensional Riemannian geometry. *Proceedings of the Royal Society of London. Series A* **1978**, *362*, 425–461. https://doi.org/10.1098/rspa.1978.0143.
47. Bilge, A.H.; Özdemir, F. The geometry of self-dual two-forms. *Journal of Mathematical Physics* **1997**, *38*, 4804–4814. https://doi.org/10.1063/1.532125.
48. Krasnov, K. Pure connection action principle for general relativity. *Physical Review Letters* **2011**, *106*, 251103. https://doi.org/10.1103/PhysRevLett.106.251103.
49. Krasnov, K. *Formulations of General Relativity: Gravity, Spinors and Differential Forms*; Cambridge University Press: Cambridge, 2020. https://doi.org/10.1017/9781108772723.
50. Krasnov, K. A Gauge-Theoretic Approach to Gravity. *Proc. R. Soc. A* **2012**, *468*, 2129–2173. https://doi.org/10.1098/rspa.2011.0638.
51. Krasnov, K.; Skvortsov, E. Flat self-dual gravity. *Journal of High Energy Physics* **2021**, *2021*, 082. https://doi.org/10.1007/JHEP08(2021)082.
52. Doplicher, S.; Roberts, J.E. Why there is a field algebra with a compact gauge group describing the superselection structure in particle physics. *Communications in Mathematical Physics* **1990**, *131*, 51–107. https://doi.org/10.1007/BF02097680.
53. Woit, P. Spacetime is Right-handed, 2023, [arXiv:hep-th/2311.00608]. https://doi.org/10.48550/arXiv.2311.00608.
54. Lawson, H.B.; Michelsohn, M.L. *Spin Geometry*; Vol. 38, *Princeton Mathematical Series*, Princeton University Press: Princeton, NJ, 1989. https://doi.org/10.1515/9781400883912.
55. Seeley, R.T. Conic Degeneration of the Dirac Operator. *Colloquium Mathematicum* **1990**, *60/61*, 649–658. https://doi.org/10.4064/cm-60-61-2-649-658.
56. Aharonov, Y.; Casher, A. Ground State of a Spin-$\frac{1}{2}$ Charged Particle in a Two-Dimensional Magnetic Field. *Physical Review A* **1979**, *19*, 2461–2462. https://doi.org/10.1103/PhysRevA.19.2461.

57. Chamseddine, A.H.; Connes, A.; van Suijlekom, W.D. Noncommutativity and Physics: A Non-Technical Review. *European Physical Journal Special Topics* **2023**, *232*, 3581–3588. https://doi.org/10.1140/epjs/s11734-023-00842-4.
58. Tong, D. Line operators in the Standard Model. *Journal of High Energy Physics* **2017**, *2017*, 104. https://doi.org/10.1007/JHEP07(2017)104.
59. Bochniak, A.; Sitarz, A. A Spectral Geometry for the Standard Model without the Fermion Doubling. *Physical Review D* **2020**, *101*, 075038. https://doi.org/10.1103/PhysRevD.101.075038.
60. Atiyah, M.F.; Bott, R.; Shapiro, A. Clifford Modules. *Topology* **1964**, *3*, 3–38. https://doi.org/10.1016/0040-9383(64)90003-5.
61. Jackiw, R.; Rebbi, C. Solitons with Fermion Number 1/2. *Phys. Rev. D* **1976**, *13*, 3398–3409. https://doi.org/10.1103/PhysRevD.13.3398.
62. Kaplan, D.B. A Method for Simulating Chiral Fermions on the Lattice. *Physics Letters B* **1992**, *288*, 342–347, [arXiv:hep-lat/hep-lat/9206013]. https://doi.org/10.1016/0370-2693(92)91112-M.
63. Czuchry, E.; Gazeau, J.P. Finite-resolution measurement induces topological curvature defects in spacetime, 2026, [arXiv:gr-qc/2601.13491].
64. Berry, M.V. Quantal Phase Factors Accompanying Adiabatic Changes. *Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences* **1984**, *392*, 45–57. https://doi.org/10.1098/rspa.1984.0023.
65. Wilczek, F.; Zee, A. Appearance of Gauge Structure in Simple Dynamical Systems. *Physical Review Letters* **1984**, *52*, 2111–2114. https://doi.org/10.1103/PhysRevLett.52.2111.
66. Kato, T. *Perturbation Theory for Linear Operators*; Classics in Mathematics, Springer-Verlag: Berlin Heidelberg, 1995. https://doi.org/10.1007/978-3-642-66282-9.
67. Atiyah, M.F.; Patodi, V.K.; Singer, I.M. Spectral Asymmetry and Riemannian Geometry. I. *Mathematical Proceedings of the Cambridge Philosophical Society* **1975**, *77*, 43–69. https://doi.org/10.1017/S0305004100049410.
68. Atiyah, M.F.; Singer, I.M. The Index of Elliptic Operators: III. *Annals of Mathematics* **1968**, *87*, 546–604. https://doi.org/10.2307/1970717.
69. Milnor, J.W.; Stasheff, J.D. *Characteristic Classes*; Princeton University Press: Princeton, NJ, 1974. https://doi.org/10.1515/9781400881826.
70. Chern, S.S.; Simons, J. Characteristic Forms and Geometric Invariants. *Annals of Mathematics* **1974**, *99*, 48–69. https://doi.org/10.2307/1971013.
71. Senjanović, G. See-saw and Grand Unification. In Proceedings of the SEESAW 25: Proceedings of the International Conference on the Seesaw Mechanism. World Scientific, 2005, pp. 45–64. https://doi.org/10.1142/9789812702210_0004.
72. Senjanović, G. Proton Decay and Grand Unification. In Proceedings of the SUSY09: The 17th International Conference on Supersymmetry and the Unification of Fundamental Interactions. AIP, 2010, Vol. 1200, *AIP Conference Proceedings*, pp. 131–141. https://doi.org/10.1063/1.3327552.
73. Besnard, F.; Brouder, C. Noncommutative geometry, the Lorentzian Standard Model and its $B-L$ extension. *Phys. Rev. D* **2021**, *103*, 035003. https://doi.org/10.1103/PhysRevD.103.035003.
74. Furey, C. $SU(3)_C \times SU(2)_L \times U(1)_Y (\times U(1)_X)$ as a symmetry of division algebraic ladder operators. *The European Physical Journal C* **2018**, *78*, 375. https://doi.org/10.1140/epjc/s10052-018-5844-7.
75. Martin, S.P. A Supersymmetry Primer. *Advanced Series on Directions in High Energy Physics* **1998**, *18*, 1–98. https://doi.org/10.1142/9789812839657_0001.
76. Overduin, J.M.; Wesson, P.S. Kaluza–Klein Gravity Revisited. *International Journal of Modern Physics D* **2013**, *22*, 1330014. https://doi.org/10.1142/S0218271813300144.
77. Randall, L.; Sundrum, R. A Large Mass Hierarchy from a Small Extra Dimension. *Physical Review Letters* **1999**, *83*, 3370–3373. https://doi.org/10.1103/PhysRevLett.83.3370.
78. Panico, G.; Wulzer, A. *The Composite Nambu-Goldstone Higgs*; Vol. 913, *Lecture Notes in Physics*, Springer: Cham, 2016. https://doi.org/10.1007/978-3-319-22617-0.
79. Hélein, F. Gauge and Gravity theories on a dynamical principal bundle. *Annales de la Faculté des Sciences de Toulouse. Mathématiques* **2025**, *34*, 743–849, [arXiv:math.DG/2310.14615]. https://doi.org/10.5802/afst.1823.
80. Baptista, J. Internal symmetries in Kaluza–Klein models. *Journal of High Energy Physics* **2024**, *2024*, 178, [arXiv:hep-th/2306.01049]. https://doi.org/10.1007/JHEP05(2024)178.
81. Gomes, H. Gauge theory is about the geometry of internal spaces, 2024, [arXiv:hep-th/2404.10461]. arXiv preprint, https://doi.org/10.48550/arXiv.2404.10461.
82. Abbott, R.; et al. Tests of general relativity with GWTC-3. *Phys. Rev. D* **2025**, *112*, 084080. https://doi.org/10.1103/PhysRevD.112.084080.

 doi:10.20944/preprints202604.0927.v4

83. Esteban, I.; Gonzalez-Garcia, M.C.; Maltoni, M.; Martinez-Soler, I.; Pinheiro, J.P.; Schwetz, T. NuFit-6.0: updated global analysis of three-flavor neutrino oscillations. *JHEP* **2024**, *2024*, 216. https://doi.org/10.1007/JHEP12(2024)216.
84. Abi, B.; et al. Deep Underground Neutrino Experiment (DUNE), Far Detector Technical Design Report, Volume I: Introduction to DUNE. *JINST* **2020**, *15*, T08008. https://doi.org/10.1088/1748-0221/15/08/T08008.
85. Abe, K.; et al. Hyper-Kamiokande Design Report, 2018. https://doi.org/10.48550/arXiv.1805.04163.
86. Bona, M.; Ciuchini, M.; Derkach, D.; Di Palma, R.; Ferrari, F.; Lubicz, V.; Martinelli, G.; Pierini, M.; Silvestrini, L.; Simula, S.; et al. Overview and theoretical prospects for CKM matrix and CP violation from the UTfit Collaboration. *PoS* **2024**, *WIFAI2023*, 007. https://doi.org/10.22323/1.457.0007.
87. Abbott, R.; et al. Constraints on Cosmic Strings Using Data from the Third Advanced LIGO-Virgo Observing Run. *Phys. Rev. Lett.* **2021**, *126*, 241102. https://doi.org/10.1103/PhysRevLett.126.241102.
88. Sousa, L. Cosmic strings and gravitational waves. *Gen. Relativ. Gravit.* **2024**, *56*, 105. https://doi.org/10.1007/s10714-024-03293-x.
89. Seeley, R.T. Singular Integrals and Boundary Value Problems. *American Journal of Mathematics* **1966**, *88*, 781–809. https://doi.org/10.2307/2373078.
90. Grubb, G. *Functional Calculus of Pseudodifferential Boundary Problems*, 2 ed.; Vol. 65, *Progress in Mathematics*, Birkhäuser Boston: Boston, MA, 1996. https://doi.org/10.1007/978-1-4612-0769-6.
91. Navas, S.; et al. Review of Particle Physics. *Physical Review D* **2024**, *110*, 030001. 2025 update, https://doi.org/10.1103/PhysRevD.110.030001.