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Posted Date: 26 March 2026

doi: 10.20944/preprints202603.2117.v1

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Optimal Control of Steady-State Heat Transfer in Two Solids with Continuous Flow and Thermal Jump at the Interface

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Abstract

This work presents a distributed optimal control problem for steady-state heat conduction in a system made up of two solids in thermal contact, with heat flux continuity and a temperature jump at the interface. The control affects the system's energy source. The existence and uniqueness of optimal control are established, and the corresponding optimality conditions are derived. Additionally, it is shown that optimal control can be considered a fixed point of a well-defined operator. Moreover, an iterative algorithm is introduced to approximate the solution to the optimal control problem, which converges regardless of the initial data. Finally, an explicit solution related to one-dimensional case in Cartesian coordinates is given.

Keywords: elliptic variational inequalities; fixed-point; distributed optimal control; adjoint state; optimality condition; convergence algorithm

MSC: 35J20; 35J57; 35J87; 49J20; 49J40; 49K20

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1. Introduction

In this paper, we examine a distributed optimal control problem related to steady-state heat conduction in a body S, consisting of two solids S_1 and S_2 with thermal contact defined by the continuity of heat fluxes and a temperature jump at the interface between the two solids.

Following [1], we introduce the following domain in $\mathbb{R}^n$:

$$\begin{cases} \Omega = \Omega_1 \cup \Omega_2, \quad \mathring{\Omega}_1 \cap \mathring{\Omega}_2 = \emptyset, \quad \overline{\Omega}_1 \cap \overline{\Omega}_2 = \Gamma_0, \\ \partial\Omega = \Gamma_1 \cup \Gamma_2, \quad \partial\Omega_1 = \Gamma_1 \cup \Gamma_0, \quad \partial\Omega_2 = \Gamma_2 \cup \Gamma_0, \end{cases} \tag{1}$$

where Ω, Ω_i, $\partial\Omega$, and $\partial\Omega_i$ denote the domains, the boundary of the body S, and the solid S_i, respectively. In particular, Γ_0 denotes the interface shared by both solids (see Figure 1).

The phenomenon of steady heat transfer between solids S_1 and S_2 is described by the following equations and conditions:

$$-\kappa_1 \Delta u_1 = g_1 \text{ in } \Omega_1, \tag{2}$$

$$-\kappa_2 \Delta u_2 = g_2 \text{ in } \Omega_2. \tag{3}$$

$$u_1 = 0 \text{ on } \Gamma_1, \tag{4}$$

$$u_2 = 0 \text{ on } \Gamma_2. \tag{5}$$

$$R\,(u_2 - u_1) = \kappa_1 \frac{\partial u_1}{\partial n} \text{ on } \Gamma_0, \tag{6}$$

 doi:10.20944/preprints202603.2117.v1

$$\kappa_1 \frac{\partial u_1}{\partial n} = \kappa_2 \frac{\partial u_2}{\partial n} \quad \text{on } \Gamma_0, \tag{7}$$

where u_i denote the temperature in Ω_i, g_i are source terms and $\kappa_i \in \mathbb{R}^+$ represents the thermal conductivity for the solid S_i $(i = 1, 2)$. We note that, condition (4) and condition (5) indicate that the temperature is zero at the boundary of the entire body. Condition (6) describes the temperature jump at the contact interface, which is proportional to the fluxes, and $R > 0$ corresponds to the resistance at the interface Γ_0. Condition (7) represents the equality of heat fluxes across the same interface. Here, n denotes the unit normal vector to Ω_1 oriented toward Ω_2.

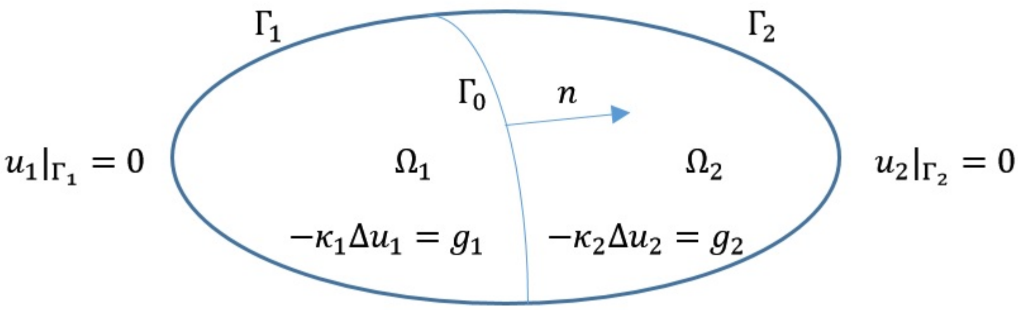

Figure 1. General scheme of the contact problem.

Steady-state heat conduction and mass transfer problems in solids in contact have been extensively studied in recent years [2–6] and this is due to their direct applications in various fields of science, engineering, and industry. For example, studies cover topics such as brain tumor growth [7], microelectronic analysis [8], thermal conduction in composite materials [9], drug release in stents [10], skin permeability studies [11], moisture analysis in composite tissues [12], contamination determination in porous media [13,14], greenhouse gas emission analysis [15], lithium-ion cell analysis [16], innovations in wool cleaning techniques [17], and heat conduction through skin analysis [18], among others.

A variety of mathematical methods have been used to study steady-state heat conduction and mass transfer problems in solids in contact [19–26]. A review of heat and mass transfer in multilayer materials and the mathematical techniques used can be found in [9,23,27]. Recently, in [1], the variational method has been used to obtain existence results of solutions and the continuous dependence of the data for the problem (2)-(7).

We examine the following functional spaces, along with bilinear and linear forms:

$$Q = L^2(\Gamma_0), \quad H_i = L^2(\Omega_i), \quad W_i = H^1(\Omega_i), \tag{8}$$

$$V_i = \{v \in W_i / v\,|_{\Gamma_i} = 0\}, \quad i = 1, 2, \tag{9}$$

$$a_i : W_i \times W_i \longrightarrow \mathbb{R}, \quad \text{such that}$$

$$a_i(u_i, v_i) = \kappa_i \int_{\Omega_i} \nabla u_i \cdot \nabla v_i \, dx, \quad \forall\, u_i, v_i \in W_i, \tag{10}$$

$$\begin{cases} L_{1_{g_1}}(v) = \displaystyle\int_{\Omega_1} g_1 v \, dx - \int_{\Gamma_0} f\, v \, dS, & \forall\, v \in W_1, \\ L_{2_{g_2}}(v) = \displaystyle\int_{\Omega_2} g_2 v \, dx + \int_{\Gamma_0} f\, v \, dS, & \forall\, v \in W_2, \end{cases} \tag{11}$$

where f is defined by [1]:

$$f : \Gamma_0 \longrightarrow \mathbb{R}, \; f = -\kappa_1 \frac{\partial u_1}{\partial n} = -\kappa_2 \frac{\partial u_2}{\partial n} \quad \text{on } \Gamma_0, \tag{12}$$

with u_i in Ω_i, $(i = 1, 2)$ such that u_1 satisfies conditions (2), (4), (12) while u_2 satisfies conditions (3), (5), (12), respectively, for given f on Γ_0.

The elliptic variational equality for the problem (2), (4), and (12) is given by

$$u_1 \in V_1 : a_1(u_1, v) = L_{1_{g_1}}(v), \quad \forall v \in V_1 \tag{13}$$

and we denote by $u_{1_{g_1}}$ the unique solution of the elliptic variational equality (13) for each $g_1 \in H_1$, [28,29].

The elliptic variational equality for the problem (3), (5), and (12) is given by

$$u_2 \in V_2 : a_2(u_2, v) = L_{2_{g_2}}(v), \quad \forall v \in V_2 \tag{14}$$

and we denote by $u_{2_{g_2}}$ the unique solution of the elliptic variational equality (14) for each $g_2 \in H_2$, [28,29].

By classical results [28,30,31], it is known that

$$\|v_i\|_Q \leq \left\|\gamma_{0\Omega_i}\right\| \|v_i\|_{W_i}, \quad \forall v_i \in V_i,\ i = 1,2, \tag{15}$$

where $\gamma_{0\Omega_i}$ is the trace operator of Ω_i on γ_0, and there exists $\lambda_i > 0\,(i = 1,2)$ such that

$$\int_{\Omega_i} \nabla v \cdot \nabla v \, dx \geq \lambda_i \|v\|^2_{W_i} \quad \forall v \in V_i, \tag{16}$$

and then

$$a_i(v, v) \geq \kappa_i \lambda_i \|v\|^2_{W_i}, \quad \forall v \in V_i. \tag{17}$$

Theorem 1. *The expression*

$$R < R_e = \frac{1}{\left(\dfrac{\left\|\gamma_{0\Omega_1}\right\|^2}{\kappa_1 \lambda_1} + \dfrac{\left\|\gamma_{0\Omega_2}\right\|^2}{\kappa_2 \lambda_2}\right)}, \tag{18}$$

provides a sufficient condition for the conduction problem (2)-(7) *to admit a unique solution.*

Proof. See [1]. □

Now, let $g = (g_1, g_2) \in H = H_1 \times H_2$ be the control variable for the functional $J : H \longrightarrow \mathbb{R}_0^+$ given by:

$$\begin{aligned} J(g) &= J(g_1, g_2) \\ &= \frac{1}{2}||u_{1_{g_1}} - z_{d_1}||^2_{H_1} + \frac{1}{2}||u_{2_{g_2}} - z_{d_2}||^2_{H_2} + \frac{M_1}{2}||g_1||^2_{H_1} + \frac{M_2}{2}||g_2||^2_{H_2} \end{aligned} \tag{19}$$

where $u_{1_{g_1}}$ and $u_{2_{g_2}}$ are the unique solutions of the problems (13) and (14), respectively, $z_{d_i} \in H_i$ for $i = 1,2$ is given, and M_i for $i = 1,2$ is a positive constant.

Then, we can formulate the following optimal control problem [32,33]

$$\text{Find} \quad \overline{g} \in H \quad \text{such that} \quad J(\overline{g}) = \min_{g \in H} J(g). \tag{20}$$

We consider the scalar product on H as follows,

$$\begin{aligned} (g, h)_H = ((g_1, g_2), (h_1, h_2))_H &:= (g_1, h_1)_{H_1} + (g_2, h_2)_{H_2}, \\ \forall g = (g_1, g_2),\ h = (h_1, h_2) &\in H \end{aligned} \tag{21}$$

 doi:10.20944/preprints202603.2117.v1

and the corresponding norm on H as follows,

$$||g||_H = ||(g_1, g_2)||_H := \sqrt{||g_1||^2_{H_1} + ||g_2||^2_{H_2}}, \qquad \forall g = (g_1, g_2) \in H \tag{22}$$

which is equivalent to the norm,

$$||g||_1 = ||(g_1, g_2)||_1 := ||g_1||_{H_1} + ||g_2||_{H_2} \tag{23}$$

since we have

$$||g||_1 \leq \sqrt{2}||g||_H \leq \sqrt{2}||g||_1. \tag{24}$$

Similarly, we define the scalar product and norm in the space W:

$$\begin{aligned}(u, v)_W = ((u_1, u_2), (v_1, v_2))_W &:= (u_1, v_1)_{W_1} + (u_2, v_2)_{W_2}, \\ \forall u = (u_1, u_2),\ v = (v_1, v_2) \in W\end{aligned} \tag{25}$$

and the corresponding norm on W,

$$||v||_W := \sqrt{||v_1||^2_{W_1} + ||v_2||^2_{W_2}}, \qquad \forall v = (v_1, v_2) \in W \tag{26}$$

which is equivalent to the norm,

$$||v||_{1,W} = ||(v_1, v_2)||_{1,W} := ||v_1||_{W_1} + ||v_2||_{W_2} \tag{27}$$

taking into account

$$||v||_{1,W} \leq \sqrt{2}||v||_W \leq \sqrt{2}||v||_{1,W}. \tag{28}$$

Optimal control problems related to steady-state heat conduction have been extensively studied. In [34], distributed optimal control problems on the internal energy are formulated and existence, uniqueness and asymptotic behavior of the optimal solutions are established. In [35,36], boundary and simultaneous distributed-boundary optimal control problems on the heat flux, and on the boundary and source term, are studied and similar results are proved.

The aim in this paper is to solve this optimal control problem, and it is three-fold. The first one is to obtain the existence and the uniqueness of the solution to optimal control problem (20). Our second aim is to provide the optimality conditions and the adjoint system. Finally, our third aim is to obtain an iterative algorithm for finding the optimal control which is independent of the choice of the initial data.

The paper is organized as follows. In Section 2, we establish the existence and uniqueness of the solution for the optimal control problem (20). In Section 3, we define the adjoint state and derive the optimality condition. In Section 4, we characterize the unique optimal control as a fixed point of an operator. In Section 5, we present an iterative algorithm to obtain the optimal solution to the problem (20). In Section 6, we provide an explicit solution for the one-dimensional case in Cartesian coordinates. Finally, in Section 7, the conclusions of this paper are given.

2. Existence and Uniqueness of the Optimal Control Problem

In this section, to study the optimal control problem (20) we first need to prove the following result.

Lemma 1. *If $g_1 = 0$ in Ω_1 and $g_2 = 0$ in Ω_2 then we have $u_{1_0} = 0$ in Ω_1 and $u_{2_0} = 0$ in Ω_2.*

Proof. If $g_1 = 0$, and we take $v = u_{1_0} \in V_1$ in the equality (13) we have

$$\kappa_1 \int_{\Omega_1} |\nabla u_{1_0}|^2\, dx = -\int_{\Gamma_0} f\, u_{1_0}\, dS \tag{29}$$

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Similarly, if $g_2 = 0$, and we take $v = u_{2_0} \in V_2$ in the equality (14) we have

$$\kappa_2 \int_{\Omega_2} |\nabla u_{2_0}|^2 \, dx = \int_{\Gamma_0} f \, u_{2_0} \, dS. \tag{30}$$

Since a_i for $i = 1, 2$ is a coercive form, we deduce that

$$\begin{aligned} 0 &\leq \kappa_1 \lambda_1 ||u_{1_0}||^2_{V_1} + \kappa_2 \lambda_2 ||u_{2_0}||^2_{V_2} \leq a_1(u_{1_0}, u_{1_0}) + a_2(u_{2_0}, u_{2_0}) \\ &= \int_{\Gamma_0} f \, (u_{2_0} - u_{1_0}) \, dS = \int_{\Gamma_0} \frac{-1}{R} f^2 \, dS \leq 0 \end{aligned} \tag{31}$$

therefore, the thesis hold. □

Following [32,34,36], we define the control to state operator
$C : H = H_1 \times H_2 \longrightarrow V = V_1 \times V_2$, such that

$$C(g) = C(g_1, g_2) = u_g, \tag{32}$$

considering $u_g = (u_{1_{g_1}}, u_{2_{g_2}})_H \in V_1 \times V_2 = V$ and the applications
$\Pi : H \times H \longrightarrow \mathbb{R}$ and $L : H = H_1 \times H_2 \longrightarrow \mathbb{R}$, defined by

$$\begin{aligned} \Pi(g, h) &= \Pi((g_1, g_2), (h_1, h_2)) \\ &= (C(g), C(h))_H + M_1(g_1, h_1)_{H_1} + M_2(g_2, h_2)_{H_2} \\ &= (u_{1_{g_1}}, u_{1_{h_1}})_{H_1} + M_1(g_1, h_1)_{H_1} + (u_{2_{g_2}}, u_{2_{h_2}})_{H_2} + M_2(g_2, h_2)_{H_2} \end{aligned} \tag{33}$$

and

$$\begin{aligned} L(g) &= L(g_1, g_2) = (C(g), z_d)_H = (u_g, z_d)_H \\ &= \Big((u_{1_{g_1}}, u_{2_{g_2}}), (z_{d_1}, z_{d_2}) \Big)_H = (u_{1_{g_1}}, z_{d_1})_{H_1} + (u_{2_{g_2}}, z_{d_2})_{H_2}. \end{aligned} \tag{34}$$

where we have defined $z_d = (z_{d_1}, z_{d_2}) \ \in \ H = H_1 \times H_2$.

Lemma 2. *We have that*

i) *C is a linear and continuous application. Moreover, we have*

$$||u_g - u_h||_W \leq \frac{1}{b} ||g - h||_H, \quad \forall g, h \in H; \ b = \min\{\kappa_1 \lambda_1, \kappa_2 \lambda_2\}. \tag{35}$$

ii) *Π is a bilinear, continuous, symmetric and coercive form on H, that is,*

$$\Pi(g, g) \geq m||g||^2_H, \quad \forall g \in H; \qquad m = \min\{M_1, M_2\}. \tag{36}$$

iii) *L is a linear and continuous form on H.*

iv) *J can be written as*

$$J(g) = \frac{1}{2} \Pi(g, g) - L(g) + \frac{1}{2} ||z_d||^2_H, \quad \forall g \in H.$$

Proof.

i) C is a linear application since, if we consider $g = (g_1, g_2) \in H$, $h = (h_1, h_2) \in H$ and $\beta \in \mathbb{R}$, we have

$$\begin{aligned} u_{1_{g_1 + \beta h_1}} &= u_{1_{g_1}} + \beta u_{1_{h_1}} \text{ in } \Omega_1, \ \forall \beta \in \mathbb{R} \\ u_{2_{g_2 + \beta h_2}} &= u_{2_{g_2}} + \beta u_{2_{h_2}} \text{ in } \Omega_2, \ \forall \beta \in \mathbb{R} \end{aligned}$$

by using the uniqueness of the solution of the elliptic variational equalities (13) and (14) respectively and then by using the definition of b, (17) and (24), we have

$$\begin{aligned} b||u_g||_W^2 &= b(||u_{1_{g_1}}||_{W_1}^2 + ||u_{2_{g_2}}||_{W_2}^2) \leq k_1\lambda_1||u_{1_{g_1}}||_{W_1}^2 + k_2\lambda_2||u_{2_{g_2}}||_{W_2}^2 \\ &\leq a_1(u_{1_{g_1}}, u_{1_{g_1}}) + a_2(u_{2_{g_2}}, u_{2_{g_2}}) = (g_1, u_{1_{g_1}})_{H_1} + (g_2, u_{2_{g_2}})_{H_2} - \frac{1}{R}\int_{\Gamma_0} f^2 dS \\ &\leq ||g_1||_{H_1}||u_{1_{g_1}}||_{W_1} + ||g_2||_{H_2}||u_{2_{g_2}}||_{W_2} \leq \sqrt{2}||g||_H||u_g||_W. \end{aligned}$$

i.e,

$$||u_g||_W \leq \frac{\sqrt{2}}{b}||g||_H \tag{37}$$

and the continuity of C holds.

Moreover, if we take $v = u_{1_{g_1}} - u_{1_{h_1}}$ in (13) for $u_{1_{h_1}}$, and $v = u_{1_{h_1}} - u_{1_{g_1}}$ in (13) for $u_{1_{g_1}}$ and subtracting, we obtain

$$a_1(u_{1_{g_1}} - u_{1_{h_1}}, u_{1_{g_1}} - u_{1_{h_1}}) = \int_{\Omega_1} (g_1 - h_1)(u_{1_{g_1}} - u_{1_{h_1}})\, dx.$$

Then, by the coerciveness of a_1, we have

$$\kappa_1\lambda_1||u_{1_{g_1}} - u_{1_{h_1}}||_{W_1}^2 \leq ||g_1 - h_1||_{H_1}||u_{1_{g_1}} - u_{1_{h_1}}||_{W_1}$$

and therefore, we conclude that

$$||u_{1_{g_1}} - u_{1_{h_1}}||_{W_1} \leq \frac{1}{\kappa_1\lambda_1}||g_1 - h_1||_{H_1}. \tag{38}$$

Now, if we take $v = u_{2_{g_2}} - u_{2_{h_2}}$ in (14) for $u_{2_{h_2}}$, and $v = u_{2_{h_2}} - u_{2_{g_2}}$ in (14) for $u_{2_{g_2}}$ and subtracting, we obtain

$$a_2(u_{2_{g_2}} - u_{2_{h_2}}, u_{2_{g_2}} - u_{2_{h_2}}) = \int_{\Omega_2} (g_2 - h_2)(u_{2_{g_2}} - u_{2_{h_2}})\, dx.$$

By the coerciveness of a_2, we deduce that

$$\kappa_2\lambda_2||u_{2_{g_2}} - u_{2_{h_2}}||_{W_2}^2 \leq ||g_2 - h_2||_{H_2}||u_{2_{g_2}} - u_{2_{h_2}}||_{W_2}$$

and therefore

$$||u_{2_{g_2}} - u_{2_{h_2}}||_{W_2} \leq \frac{1}{\kappa_2\lambda_2}||g_2 - h_2||_{H_2}. \tag{39}$$

Then, we have

$$\begin{aligned} ||u_g - u_h||_W^2 &= ||(u_{1_{g_1}}, u_{2_{g_2}}) - (u_{1_{h_1}}, u_{2_{h_2}})||_W^2 = ||(u_{1_{g_1}} - u_{1_{h_1}}, u_{2_{g_2}} - u_{2_{h_2}})||_W^2 \\ &= ||u_{1_{g_1}} - u_{1_{h_1}}||_{W_1}^2 + ||u_{2_{g_2}} - u_{2_{h_2}}||_{W_2}^2 \\ &\leq \frac{1}{(\kappa_1\lambda_1)^2}||g_1 - h_1||_{H_1}^2 + \frac{1}{(\kappa_2\lambda_2)^2}||g_2 - h_2||_{H_2}^2 \\ &\leq \frac{1}{b^2}(||g_1 - h_1||_{H_1}^2 + ||g_2 - h_2||_{H_2}^2) \leq \frac{1}{b^2}||(g_1 - h_1, g_2 - h_2)||_H^2 \\ &= \frac{1}{b^2}||(g_1, g_2) - (h_1, h_2)||_H^2 = \frac{1}{b^2}||g - h||_H^2. \end{aligned}$$

i.e. (35) holds.

ii) From the linearity of C and of the scalar product in H, the application Π is a bilinear form and also Π is symmetric. Moreover, Π is a continuous application since

$$\begin{aligned}|\Pi(g,h)| &= |(C(g),C(h))_H + M_1(g_1,h_1)_{H_1} + M_2(g_2,h_2)_{H_2}| \\ &\leq \left(\frac{2}{b^2} + 2M\right)||g||_H||h||_H,\end{aligned}$$

where

$$M = \max\{M_1, M_2\}. \tag{40}$$

Finally, Π is a coercive form. In fact, taking into account that m is defined by (36), then we have

$$\begin{aligned}\Pi(g,g) &= (C(g),C(g))_H + M_1||g_1||^2_{H_1} + M_2||g_2||^2_{H_2} \\ &\geq ||u_g||^2_H + m(||g_1||^2_{H_1} + ||g_2||^2_{H_2}) \geq m||g||^2_H,\end{aligned}$$

iii) From the linearity of C and of the scalar product in H, we obtain that L is a linear and continuous form, by considering

$$|L(g)| \leq \frac{\sqrt{2}N}{b}||g||_H, \; N = \max\{||z_{d_1}||_{H_1}, ||z_{d_2}||_{H_2}\}. \tag{41}$$

iv) This results follows from the following computation:

$$\begin{aligned}J(g) &= \frac{1}{2}(u_{1_{g_1}} - z_{d_1}, u_{1_{g_1}} - z_{d_1})_{H_1} + \frac{1}{2}(u_{2_{g_2}} - z_{d_2}, u_{2_{g_2}} - z_{d_2})_{H_2} \\ &\quad + \frac{M_1}{2}(g_1,g_1)_{H_1} + \frac{M_2}{2}(g_2,g_2)_{H_2} \\ &= \frac{1}{2}(u_{1_{g_1}}, u_{1_{g_1}})_{H_1} - (u_{1_{g_1}}, z_{d_1})_{H_1} + \frac{1}{2}(z_{d_1}, z_{d_1})_{H_1} \\ &\quad + \frac{1}{2}(u_{2_{g_2}}, u_{2_{g_2}})_{H_2} - (u_{2_{g_2}}, z_{d_2})_{H_2} + \frac{1}{2}(z_{d_2}, z_{d_2})_{H_2} \\ &\quad + \frac{M_1}{2}(g_1,g_1)_{H_1} + \frac{M_2}{2}(g_2,g_2)_{H_2} \\ &= \frac{1}{2}\Pi(g,g) - L(g) + \frac{1}{2}||z_d||^2_H.\end{aligned}$$

□

Theorem 2. *There exists a unique optimal control $\overline{g} = (\overline{g_1}, \overline{g_2}) \in H$ to the problem (20).*

Proof. This follows from Lemma 2 and [32]. □

3. The Adjoint State

Following [32,34,36], we define the adjoint states $p_{1_{g_1}}$ and $p_{2_{g_2}}$, for each $g \in H$, as the unique solution of the following mixed elliptic problems

$$-\triangle p_{1_{g_1}} = u_{1_{g_1}} - z_{d_1} \text{ in } \Omega_1, \qquad p_{1_{g_1}}|_{\Gamma_1} = 0, \qquad p_{1_{g_1}}|_{\Gamma_0} = \frac{\partial p_{1_{g_1}}}{\partial n}|_{\Gamma_0} = 0 \tag{42}$$

$$-\triangle p_{2_{g_2}} = u_{2_{g_2}} - z_{d_2} \text{ in } \Omega_2, \qquad p_{2_{g_2}}|_{\Gamma_2} = 0, \qquad p_{2_{g_2}}|_{\Gamma_0} = \frac{\partial p_{2_{g_2}}}{\partial n}|_{\Gamma_0} = 0 \tag{43}$$

whose variational formulations are given by

$$p_{1_{g_1}} \in V_1 : \; a_1(p_{1_{g_1}}, v) = (u_{1_{g_1}} - z_{d_1}, v)_{H_1}, \quad \forall v \in V_1. \tag{44}$$

and

$$p_{2_{g_2}} \in V_2 : \; a_2(p_{2_{g_2}}, v) = (u_{2_{g_2}} - z_{d_2}, v)_{H_2}, \quad \forall v \in V_2. \tag{45}$$

We denote by $p_{1_{g_1}} \in V_1$ and $p_{2_{g_2}} \in V_2$ the unique solutions [28,29] of the elliptic variational inequalities (44) and (45) respectively.

Now, we prove the following properties.

Lemma 3.

i) *The functional J is Gateaux differentiable and J' is given by*

$$\begin{aligned}(J'(g), h) &= \Pi(g, h) - L(h)\\ &= (u_{1_{g_1}} - z_{d_1}, u_{1_{h_1}})_{H_1} + (u_{2_{g_2}} - z_{d_2}, u_{2_{h_2}})_{H_2}\\ &\quad + M_1(g_1, h_1)_{H_1} + M_2(g_2, h_2)_{H_2}, \quad \forall h, g \in H.\end{aligned} \tag{46}$$

ii) *The adjoint states satisfies the following equalities*

$$(p_{1_{g_1}}, h_1)_{H_1} = (u_{1_{g_1}} - z_{d_1}, u_{1_{g_1}})_{H_1} = a_1(p_{1_{g_1}}, u_{1_{g_1}}) \tag{47}$$

and

$$(p_{2_{g_2}}, h_2)_{H_2} = (u_{2_{g_2}} - z_{d_2}, u_{2_{g_2}})_{H_2} = a_2(p_{2_{g_2}}, u_{2_{g_2}}). \tag{48}$$

iii) *The Gateaux derivate of J can be write as*

$$\langle J'(g), h\rangle = \Big((p_{1_{g_1}} + M_1 g_1, p_{2_{g_2}} + M_2 g_2), (h_1, h_2)\Big)_H \quad \forall h \in H. \tag{49}$$

iv) *The optimality condition of the problem (20) is given by*

$$\textit{Find} \quad \overline{g} = (\overline{g_1}, \overline{g_2}) \in H \quad \textit{such that} \qquad J'(\overline{g}) = 0, \tag{50}$$

that is

$$p_{1_{\overline{g_1}}} + M_1\overline{g_1} = 0 \;\; in \;\; \Omega_1 \; and \; p_{2_{\overline{g_2}}} + M_2\overline{g_2} = 0 \;\; in \;\; \Omega_2. \tag{51}$$

Proof. i) Let $g, h \in H$ and $t > 0$, we have

$$\begin{aligned}
&J(g + t(h-g)) - J(g))\\
&= \frac{1}{2}||u_{1_{g_1+t(h_1-g_1)}} - z_{d_1}||^2_{H_1} + \frac{1}{2}||u_{2_{g_2+t(h_2-g_2)}} - z_{d_2}||^2_{H_2}\\
&\quad + \frac{M_1}{2}||g_1 + t(h_1 - g_1)||^2_{H_1} + \frac{M_2}{2}||g_2 + t(h_2 - g_2)||^2_{H_2}\\
&\quad - \frac{1}{2}||u_{1_{g_1}} - z_{d_1}||^2_{H_1} + \frac{1}{2}||u_{2_{g_2}} - z_{d_2}||^2_{H_2} - \frac{M_1}{2}||g_1||^2_{H_1} - \frac{M_2}{2}||g_2||^2_{H_2}\\
&= t\Big[(u_{1_{g_1}} - z_{d_1}, u_{1_{h_1}} - u_{1_{g_1}})_{H_1} + (u_{2_{g_2}} - z_{d_2}, u_{2_{h_2}} - u_{2_{g_2}})_{H_2}\\
&\quad + M_1(g_1, h_1 - g_1)_{H_1} + M_2(g_2, h_2 - g_2)_{H_2}\Big]\\
&\quad + t^2\Big[\frac{1}{2}(u_{1_{h_1}} - u_{1_{g_1}}, u_{1_{h_1}} - u_{1_{g_1}})_{H_1} + \frac{1}{2}(u_{2_{h_2}} - u_{2_{g_2}}, u_{2_{h_2}} - u_{2_{g_2}})_{H_2}\\
&\quad + \frac{M_1}{2}(h_1 - g_1, h_1 - g_1)_{H_1} + \frac{M_2}{2}(h_2 - g_2, h_2 - g_2)_{H_2}\Big]
\end{aligned}$$

Next, dividing by t and making $t \to 0^+$, (46) holds.

ii) On the one hand, we have

$$a_1(p_{1_{g_1}}, u_{1_{h_1}}) = -\kappa_1 \int_{\Omega_1} p_{1_{g_1}} \Delta u_{1_{h_1}}\, dx = \int_{\Omega_1} p_{1_{g_1}} h_1 dx = (p_{1_{g_1}}, h_1)_{H_1}$$

and by using (44), we deduce (47). On the other hand, we obtain

$$a_2(p_{2_{g_2}}, u_{2_{h_2}}) = -\kappa_2 \int_{\Omega_2} p_{2_{g_2}} \Delta u_{2_{h_2}}\, dx = \int_{\Omega_2} p_{2_{g_2}} h_2 dx = (p_{2_{g_2}}, h_2)_{H_2}$$

and by using (45), we deduce (48).

iii) For $g = (g_1, g_2) \in H$, and for all $h = (h_1, h_2) \in H$, we obtain

$$\begin{aligned}\langle J^{'}(g), h\rangle &= (u_{1_{g_1}} - z_{d_1}, u_{1_{h_1}})_{H_1} + (u_{2_{g_2}} - z_{d_2}, u_{2_{h_2}})_{H_2} \\ &\quad + M_1(g_1, h_1)_{H_1} + M_2(g_2, h_2)_{H_2} \\ &= (p_{1_{g_1}}, h_1)_{H_1} + (p_{2_{g_2}}, h_2)_{H_2} + M_1(g_1, h_1)_{H_1} + M_2(g_2, h_2)_{H_1} \\ &= (p_{1_{g_1}} + M_1 g_1, h_1)_{H_1} + (p_{2_{g_2}} + M_2 g_2, h_2)_{H_2} \\ &= \Big((p_{1_{g_1}} + M_1 g_1, p_{2_{g_2}} + M_2 g_2), (h_1, h_2)\Big)_H\end{aligned}$$

that is (49) holds.

iv) From (iii), we obtain (51).

□

4. Point Fixed and Optimal Control

Now, we define the operator $W : H = H_1 \times H_2 \longrightarrow V = V_1 \times V_2$ as

$$W(g) = W(g_1, g_2) = \left(\frac{-1}{M_1} p_{1_{g_1}}, \frac{-1}{M_2} p_{2_{g_2}}\right), \quad \forall g \in H \tag{52}$$

and we prove the following property.

Lemma 4. *W is a Lipschitz operator, that is, there exists $D > 0$ such that*

$$||W(g) - W(h)||_W \le D||g - h||_H, \quad \forall g, h \in H. \tag{53}$$

Proof. If we consider $v = p_{1_{g_1}} - p_{1_{h_1}} \in V_1$ in (44) for solution $p_{1_{h_1}}$, and $v = p_{1_{h_1}} - p_{1_{g_1}} \in V_1$ in (44) for solution $p_{1_{g_1}}$, and adding then, we have

$$\begin{aligned}a_1(p_{1_{h_1}} - p_{1_{g_1}}, p_{1_{h_1}} - p_{1_{g_1}}) &= (u_{1_{h_1}} - z_{d_1} - u_{1_{g_1}} + z_{d_1}, p_{1_{h_1}} - p_{1_{g_1}})_{H_1} \\ &= (u_{1_{h_1}} - u_{1_{g_1}}, p_{1_{h_1}} - p_{1_{g_1}})_{H_1}.\end{aligned}$$

Next, we obtain

$$\begin{aligned}\kappa_1\lambda_1||p_{1_{h_1}} - p_{1_{g_1}}||^2_{W_1} &\le a_1(p_{1_{h_1}} - p_{1_{g_1}}, p_{1_{h_1}} - p_{1_{g_1}}) = (u_{1_{h_1}} - u_{1_{g_1}}, p_{1_{h_1}} - p_{1_{g_1}})_{H_1} \\ &\le ||u_{1_{h_1}} - u_{1_{g_1}}||_{H_1}||p_{1_{h_1}} - p_{1_{g_1}}||_{H_1} \le ||u_{1_{h_1}} - u_{1_{g_1}}||_{H_1}||p_{1_{h_1}} - p_{1_{g_1}}||_{W_1}\end{aligned}$$

then

$$||p_{1_{h_1}} - p_{1_{g_1}}||_{W_1} \le \frac{1}{\kappa_1\lambda_1}||u_{1_{h_1}} - u_{1_{g_1}}||_{H_1}.$$

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Now, if we consider $v = p2_{g_2} - p2_{h_2} \in V_2$ in (45) for solution $p2_{h_2}$, and $v = p2_{h_2} - p2_{g_2} \in V_2$ for solution $p2_{g_2}$ in (45), and adding, we have

$$\begin{aligned} a_2(p2_{h_2} - p2_{g_2}, p2_{h_2} - p2_{g_2}) &= (u2_{h_2} - z_{d_2} - u2_{g_2} + z_{d_2}, p2_{h_2} - p2_{g_2})_{H_2} \\ &= (u2_{h_2} - u2_{g_2}, p2_{h_2} - p2_{g_2})_{H_2}. \end{aligned}$$

Next, we deduce

$$\begin{aligned} \kappa_2\lambda_2 ||p2_{h_2} - p2_{g_2}||^2_{W_2} &\leq a_2(p2_{h_2} - p2_{g_2}, p2_{h_2} - p2_{g_2}) = (u2_{h_2} - u2_{g_2}, p2_{h_2} - p2_{g_2})_{H_2} \\ &\leq ||u2_{h_2} - u2_{g_2}||_{H_2} ||p2_{h_2} - p2_{g_2}||_{H_2} \leq ||u2_{h_2} - u2_{g_2}||_{H_2} ||p2_{h_2} - p2_{g_2}||_{W_2} \end{aligned}$$

then

$$||p2_{h_2} - p2_{g_2}||_{W_2} \leq \frac{1}{\kappa_2\lambda_2} ||u2_{h_2} - u2_{g_2}||_{H_2}.$$

Therefore, we obtain

$$\begin{aligned} ||W(g) - W(h)||_W &= ||W(g_1, g_2) - W(h_1, h_2)||_W \\ &= \left(\frac{1}{M_1}(p1_{g_1} - p1_{h_1}), \frac{1}{M_2}(p2_{g_2} - p2_{h_2}) \right)_W \\ &\leq \frac{1}{M_1} ||(p1_{g_1} - p1_{h_1})||_{W_1} + \frac{1}{M_2} ||(p2_{g_2} - p2_{h_2})||_{W_2} \\ &\leq \frac{1}{M_1\kappa_1\lambda_1} ||u1_{h_1} - u1_{g_1}||_{H_1} + \frac{1}{M_2\kappa_2\lambda_2} ||u2_{h_2} - u2_{g_2}||_{H_2} \\ &\leq B\Big(||u1_{h_1} - u1_{g_1}||_{H_1} + ||u2_{h_2} - u2_{g_2}||_{H_2} \Big) = B||u_h - u_g||_1 \\ &= \sqrt{2}B||u_g - u_h||_W \leq \frac{\sqrt{2}B}{b} ||g - h||_H, \end{aligned}$$

where

$$B = \max\{\frac{1}{M_1\kappa_1\lambda_1}, \frac{1}{M_2\kappa_2\lambda_2}\} > 0 \tag{54}$$

and b is given in (35). Therefore, (53) holds by taking $D = \frac{\sqrt{2}B}{b}$. □

Remark 1. *If*

$$D = \sqrt{2} \max\left\{ \frac{1}{M_1\kappa_1\lambda_1}, \frac{1}{M_2\kappa_2\lambda_2} \right\} \frac{1}{\min(\kappa_1\lambda_1, \kappa_2\lambda_2)} < 1,$$

then the optimal control $\overline{g} = (\overline{g_1}, \overline{g_2}) \in H$ *can be considered as the fixed point of the operator* W.

Lemma 5. *For all* $g = (g_1, g_2) \in H$ *and* $h = (h_1, h_2) \in H$, *we have*

i) *The application* $g \in H \to p_g \in V$ *is strictly monotone. Moreover,*

$$(p_g - p_h, g - h)_H = ||u_g - u_h||^2_H \geq 0.$$

ii) J *is a strictly convex application, that is, there exits* $m > 0$, *defined in* (36), *such that*

$$\begin{aligned} &(1-t)J(g) + tJ(h) - J((1-t)g + th) \\ &\geq \frac{1}{2}t(1-t)m||g - h||^2_H. \end{aligned} \tag{55}$$

iii) *The application* J' *is Lipschitz and strictly monotone, that is, there exits* $E > 0$ *such that*

$$||J'(g) - J'(h)||_H \leq E||g - h||_H$$

doi:10.20944/preprints202603.2117.v1

and

$$(J'(g) - J'(h), g - h)_H \geq m||g - h||_H^2,$$

where

$$E = \max\{E_1, E_2\},\ E_1 = \frac{1}{k_1\lambda_1} + M_1,\ E_2 = \frac{1}{k_2\lambda_2} + M_2. \tag{56}$$

Proof. i) On the one hand

$$\begin{aligned}
&(p_{1_{g_1}} - p_{1_{h_1}}, g_1 - h_1)_{H_1} = (p_{1_{g_1}}, g_1)_{H_1} - (p_{1_{g_1}}, h_1)_{H_1} - (p_{1_{h_1}}, g_1) + (p_{1_{h_1}}, h_1)_{H_1} \\
&= (u_{1_{g_1}} - z_{d_1} - u_{1_{h_1}} + z_{d_1}, u_{1_{g_1}})_{H_1} + (u_{1_{h_1}} - z_{d_1} - u_{1_{g_1}} + z_{d_1}, u_{1_{h_1}})_{H_1} \\
&= (u_{1_{g_1}} - u_{1_{h_1}}, u_{1_{g_1}} - u_{1_{h_1}})_{H_1} = ||u_{1_{g_1}} - u_{1_{h_1}}||_{H_1}^2.
\end{aligned}$$

On the other hand

$$\begin{aligned}
&(p_{2_{g_2}} - p_{2_{h_2}}, g_2 - h_2)_{H_2} = (p_{2_{g_2}}, g_2) - (p_{2_{g_2}}, h_2)_{H_2} - (p_{2_{h_2}}, g_2)_{H_2} + (p_{2_{h_2}}, h_2)_{H_2} \\
&= (u_{2_{g_2}} - z_{d_2} - u_{2_{h_2}} + z_{d_2}, u_{2_{g_2}})_{H_2} + (u_{2_{h_2}} - z_{d_2} - u_{2_{g_2}} + z_{d_2}, u_{2_{h_2}})_{H_2} \\
&= (u_{2_{g_2}} - u_{2_{h_2}}, u_{2_{g_2}} - u_{2_{h_2}})_{H_2} = ||u_{2_{g_2}} - u_{2_{h_2}}||_{H_2}^2.
\end{aligned}$$

Then, we obtain

$$\begin{aligned}
(p_g - p_h, g - h)_H &= (p_{1_{g_1}} - p_{1_{h_1}}, g_1 - h_1)_{H_1} + (p_{2_{g_2}} - p_{2_{h_2}}, g_2 - h_2)_{H_2} \\
&= ||u_{1_{g_1}} - u_{1_{h_1}}||_{H_1}^2 + ||u_{2_{g_2}} - u_{2_{h_2}}||_{H_2}^2 = ||u_g - u_h||_H^2
\end{aligned}$$

and the stricty monotony holds.

ii) We have for all $g, h \in H$ and $t \in [0,1]$:

$$\begin{aligned}
&(1-t)J(g) + tJ(h) - J((1-t)g + th) \\
&= (1-t)\left[\frac{1}{2}\|u_{1_{g_1}} - z_{d_1}\|_{H_1}^2 + \frac{1}{2}\|u_{2_{g_2}} - z_{d_2}\|_{H_2}^2 + \frac{M_1}{2}\|g_1\|_{H_1}^2 + \frac{M_2}{2}\|g_2\|_{H_2}^2\right] \\
&\quad + t\left[\frac{1}{2}\|u_{1_{h_1}} - z_{d_1}\|_{H_1}^2 + \frac{1}{2}\|u_{2_{h_2}} - z_{d_2}\|_{H_2}^2 + \frac{M_1}{2}\|h_1\|_{H_1}^2 + \frac{M_2}{2}\|h_2\|_{H_2}^2\right] \\
&\quad - \left[\frac{1}{2}\|u_{1_{(1-t)g_1+th_1}} - z_{d_1}\|_{H_1}^2 + \frac{1}{2}\|u_{2_{(1-t)g_2+th_2}} - z_{d_2}\|_{H_2}^2\right. \\
&\quad \left. + \frac{M_1}{2}\|(1-t)g_1 + th_1\|_{H_1}^2 + \frac{M_2}{2}\|(1-t)g_2 + th_2\|_{H_2}^2\right] \\
&= \frac{1}{2}t(1-t)||u_{1_{g_1}} - u_{1_{h_1}}||_{H_1}^2 + \frac{1}{2}t(1-t)||u_{2_{g_2}} - u_{2_{h_2}}||_{H_2}^2 \\
&\quad + \frac{1}{2}M_1 t(1-t)||g_1 - h_1||_{H_1}^2 + \frac{1}{2}M_2 t(1-t)||g_2 - h_2||_{H_2}^2 \\
&\geq \frac{1}{2}t(1-t)\left(||u_{1_{g_1}} - u_{1_{h_1}}||_{H_1}^2 + ||u_{2_{g_2}} - u_{2_{h_2}}||_{H_2}^2\right) \\
&\quad + \frac{1}{2}t(1-t)m\left(||g_1 - h_1||_{H_1}^2 + ||g_2 - h_2||_{H_2}^2\right) \\
&\geq \frac{1}{2}t(1-t)m\left(||g_1 - h_1||_{H_1}^2 + ||g_2 - h_2||_{H_2}^2\right).
\end{aligned}$$

and (55) holds.

iii) J' is Lipschitz, since for all $g, h \in H$, we have:

$$\begin{aligned}
&||(p_{1_{g_1}} + M_1 g_1) - (p_{1_{h_1}} + M_1 h_1)||_{H_1} \leq ||p_{1_{g_1}} - p_{1_{h_1}}||_{H_1} + M_1||g_1 - h_1||_{H_1} \\
&\leq \frac{1}{k_1\lambda_1}||g_1 - h_1||_{H_1} + M_1||g_1 - h_1||_{H_1} = E_1||g_1 - h_1||_{H_1},
\end{aligned}$$

 doi:10.20944/preprints202603.2117.v1

and

$$\begin{aligned}&||(p_{2_{g_2}}+M_2g_2)-(p_{2_{h_2}}+M_2h_2)||_{H_2}\leq ||p_{2_{g_2}}-p_{2_{h_2}}||_{H_2}+M_2||g_2-h_2||_{H_2}\\&\leq \frac{1}{k_2\lambda_2}||g_2-h_2||_{H_2}+M_2||g_2-h_2||_{H_2}=E_2||g_2-h_2||_{H_2}.\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}||J'(g)-J'(h)||_H&\leq E_1||g_1-h_1||_{H_1}+E_2||g_2-h_2||_{H_2}\\&\leq E\big(||g_1-h_1||_{H_1}+||g_2-h_2||_{H_2}\big)=E||g-h||_H\end{aligned}$$

where E is given in (56).
J' is a strictly monotone application. In fact, for all $g,h\in H$, we have

$$\begin{aligned}&((p_{1_{g_1}}+M_1g_1)-(p_{1_{h_1}}+M_1h_1),g_1-h_1)_{H_1}\\&=(p_{1_{g_1}}-p_{1_{h_1}},g_1-h_1)_{H_1}+M_1(g_1-h_1,g_1-h_1)_{H_1}\\&=||u_{1_{g_1}}-u_{1_{h_1}}||^2_{H_1}+M_1||g_1-h_1||^2_{H_1}\geq M_1||g_1-h_1||^2_{H_1}\end{aligned}$$

and

$$\begin{aligned}&((p_{2_{g_2}}+M_2g_2)-(p_{2_{h_2}}+M_2h_2),g_2-h_2)_{H_2}\\&=(p_{2_{g_2}}-p_{2_{h_2}},g_2-h_2)_{H_2}+M_2(g_2-h_2,g_2-h_2)_{H_2}\\&=||u_{2_{g_2}}-u_{2_{h_2}}||^2_{H_2}+M_2||g_2-h_2||^2_{H_2}\geq M_2||g_2-h_2||^2_{H_2}.\end{aligned}$$

Therefore

$$\begin{aligned}&(J'(g)-J'(h),g-h)_H\\&=((p_{1_{g_1}}+M_1g_1)-(p_{1_{h_1}}+M_1h_1),g_1-h_1)_{H_1}\\&\quad+((p_{2_{g_2}}+M_2g_2)-(p_{2_{h_2}}+M_2h_2),g_2-h_2)_{H_2}\\&\geq M_1||g_1-h_1||^2_{H_1}+M_2||g_2-h_2||^2_{H_2}\geq m\Big(||g_1-h_1||^2_{H_1}+||g_2-h_2||^2_{H_2}\Big)\\&\geq m||g-h||^2_H\end{aligned}$$

where m is given in (36).
□

5. Convergent Algorithm

Now, similar to [34], we introduce an iterative algorithm to find the optimal solution $\overline{g}=(\overline{g_1},\overline{g_2})$ for problem (20). For each $\rho>0$, we define the following sequence $g^n=(g_1^n,g_2^n)$, with

$$g^0=(g_1^0,g_2^0)\quad\text{given arbitrary}$$

and for all $n\geq 0$, we define

$$g^{n+1}=(g_1^{n+1},g_2^{n+1})=\Big((1-\rho M_1)g_1^n-\rho p_{g_1^n},(1-\rho M_2)g_2^n-\rho p_{g_2^n}\Big)$$

which will converge to $\overline{g}=(\overline{g_1},\overline{g_2})$ for a suitable ρ.

Lemma 6. *If ρ satisfies the inequality $0<\rho<\frac{m}{E^2}$, then the previous algorithm converges strongly in H to the optimal control $\overline{g}$ for the problem (20) regardless of the initial data g^0.*

Proof. The operator $S : H \to H$ defined by

$$S(g) = S(g_1, g_2) = \big((1-\rho M_1)g_1 - \rho p_{g_1}, (1-\rho M_2)g_2 - \rho p_{g_2}\big)$$

is a Lipschitz operator, that is

$$||S(g) - S(h)||_H \le \sqrt{\gamma(\rho)}||g-h||_H$$

where $\gamma(\rho)$ is given by

$$\gamma(\rho) = 1 - \rho m + E^2\rho^2.$$

In fact

$$\begin{aligned}
&||S(g) - S(h)||_H^2 = ||S(g_1, g_2) - S(h_1, h_2)||_H^2 \\
&= ||\big((g_1 - h_1) - \rho[M_1(g_1 - h_1) + (p_{g_1} - p_{h_1})], (g_2 - h_2) - \rho[M_2(g_2 - h_2) + (p_{g_2} - p_{h_2})]\big)||_H^2 \\
&= ||(g_1 - h_1, g_2 - h_2)||_H^2 \\
&\quad - 2\rho\big((g_1 - h_1, g_2 - h_2), (M_1(g_1 - h_1) + (p_{g_1} - p_{h_1}), M_2(g_2 - h_2) + (p_{g_2} - p_{h_2}))\big)_H \\
&\quad + \rho^2||(M_1(g_1 - h_1) + (p_{g_1} - p_{h_1}), M_2(g_2 - h_2) + (p_{g_2} - p_{h_2}))||_H^2 \\
&= ||(g_1 - h_1, g_2 - h_2)||_H^2 - 2\rho\big((g_1 - h_1, g_2 - h_2), J'(g_1, g_2) - J'(h_1, h_2)\big) \\
&\quad + \rho^2||J'(g_1, g_2) - J'(h_1, h_2)||_H^2 \\
&= ||g-h||_H^2 - 2\rho(g-h, J'(g) - J'(h)) + \rho^2||J'(g) - J'(h)||_H^2 \\
&\le ||g-h||_H^2 - \rho m||g-h||_H^2 + \rho^2E^2||g-h||_H^2 = \gamma(\rho)||g-h||_H^2.
\end{aligned}$$

Therefore, S is a contraction operator if only if, $0 \le \gamma(\rho) < 1$, that is $0 < \rho < \frac{m}{E^2}$. Moreover, the optimal parameter $\overline{\rho}$ and the optimal Lipschitz constant $\overline{\gamma} = \gamma(\overline{\rho})$ are given by

$$\overline{\rho} = \frac{m}{2E^2} \qquad \text{and} \qquad \overline{\gamma} = 1 - \frac{m^2}{4E^2}.$$

□

6. Study of Particular Case

We now present a specific case with explicit solution related to one-dimensional case in Cartesian coordinates.

We consider the following domains and boundaries:

$$\Omega_1 = (-l_1, 0),\ \Omega_2 = (0, l_2),\ \Gamma_0 = \{0\},\ \Gamma_1 = \{-l_1\}\ and\ \Gamma_2 = \{l_2\}. \tag{57}$$

The steady-state heat transfer problem, which includes continuity of heat flow and a thermal jump at the interface is modeled by:

$$\begin{aligned}
&-k_1u_1''(x) = g_1, \quad -l_1 < x < 0 \\
&-k_2u_2''(x) = g_2, \quad 0 < x < l_2 \\
&u_1(-l_1) = \theta_1,\ u_2(l_2) = \theta_2 \\
&k_1u_1'(0) = k_2u_2'(0) \\
&R(u_2(0) - u_1(0)) = k_1u_1'(0)
\end{aligned} \tag{58}$$

The explicit solution to problem (58) is given by [1]:

$$\begin{cases} u_1(x) = \theta_1 + \frac{g_1}{2k_1}(l_1^2 - x^2) + \frac{R}{k_1}\frac{\left((\theta_2-\theta_1)+\left(\frac{g_2 l_2^2}{2k_2}-\frac{g_1 l_1^2}{2k_1}\right)\right)}{\mu}(x+l_1) \;\; in \; -l_1 < x < 0 \\ u_2(x) = \theta_2 + \frac{g_2}{2k_2}(l_2^2 - x^2) - \frac{R}{k_2}\frac{\left((\theta_2-\theta_1)+\left(\frac{g_2 l_2^2}{2k_2}-\frac{g_1 l_1^2}{2k_1}\right)\right)}{\mu}(l_2-x) \;\; in \; 0 < x < l_2 \end{cases} \tag{59}$$

where the parameter μ is defined by:

$$\mu = \mu(R) = 1 + R\left(\frac{l_1}{k_1} + \frac{l_2}{k_2}\right). \tag{60}$$

The function J is given by:

$$J(g_1, g_2) = Ag_1^2 + 2Bg_1g_2 + Cg_2^2 + Dg_1 + Eg_2 + F$$

where the coefficients are:

$$A = \frac{1}{2}M_1 l_1 + \frac{1}{24}\frac{R^2 l_1^4 l_2^3}{k_1^2 k_2^2 \mu^2} + \frac{7}{24}\frac{R^2 l_1^7}{k_1^4 \mu^2} + \frac{11}{48}\frac{R l_1^6}{k_1^3 \mu} + \frac{1}{15}\frac{l_1^5}{k_1^2}$$

$$B = \frac{-1}{24}\frac{R l_1^2 l_2^2}{k_1 k_2 \mu}\left[\frac{R l_2^3}{k_2^2 \mu} + 7\frac{R l_1^3}{k_1^2 \mu} - \frac{5}{4}\frac{l_2^2}{k_2} + \frac{11}{4}\frac{l_1^2}{k_1}\right]$$

$$C = \frac{1}{2}M_2 l_2 + \frac{1}{24}\frac{R^2 l_2^7}{k_2^4 \mu^2} + \frac{7}{24}\frac{R^2 l_1^3 l_2^4}{k_1^2 k_2^2 \mu^2} - \frac{5}{48}\frac{R l_2^6}{k_2^3 \mu} + \frac{1}{15}\frac{l_2^5}{k_2^2}$$

$$\begin{aligned} D = \frac{-1}{12}\frac{R l_1}{k_1 \mu}\Bigg[& 2\frac{R l_1}{\mu}(\theta_2 - \theta_1)\left(\frac{l_2^3}{k_2^2} + 7\frac{l_1^3}{k_1^2}\right) - 3\frac{l_1 l_2^2}{k_2}(\theta_2 - z_2) \\ & - 4\frac{l_1^2 \mu}{R}(\theta_1 - z_1) + \frac{l_1^3}{2k_1}(11\theta_2 - 29\theta_1) + 9\frac{l_1^3 z_1}{k_1}\Bigg] \end{aligned}$$

$$\begin{aligned} E = \frac{1}{6}\frac{R l_2}{k_2 \mu}\Bigg[& \frac{R l_2}{\mu}(\theta_2 - \theta_1)\left(\frac{l_2^3}{k_2^2} + 7\frac{l_1^3}{k_1^2}\right) - \frac{1}{4}\frac{l_2^3}{k_2}(11\theta_2 - 5\theta_1) \\ & - \frac{9}{2}\frac{l_1^2 l_2}{k_1}(\theta_1 - z_1) + 2\frac{l_2^2 \mu}{R}(\theta_2 - z_2) + \frac{3}{2}\frac{l_2^3 z_2}{k_2}\Bigg] \end{aligned}$$

$$\begin{aligned} F = & \frac{1}{6}\frac{R^2}{\mu^2}(\theta_1^2 + \theta_2^2)\left(\frac{l_2^3}{k_2^2} + 7\frac{l_1^3}{k_1^2}\right) - \frac{1}{3}\frac{R^2 \theta_1 \theta_2}{k_1^2 k_2^2 \mu^2}(l_2^3 k_1^2 + 7 l_1^3 k_2^2) \\ & + \frac{1}{2}\frac{R}{\mu}(\theta_2 - \theta_1)\left(\frac{l_2^2}{k_2}(z_2 - \theta_2) + 3\frac{l_1^2}{k_1}(z_1 - \theta_1)\right) \\ & - (l_1\theta_1 z_1 + l_2\theta_2 z_2) + \frac{1}{2}(l_1 z_1^2 + l_2 z_2^2) + \frac{1}{2}(l_1\theta_1^2 + l_2\theta_2^2) \end{aligned}$$

Taking into account that $A > 0$ and $AC - B^2 > 0$, we obtain that minimum of J is given by

$$\overline{g_1} = \frac{1}{2}\frac{-CD + BE}{AC - B^2} \quad \text{and} \quad \overline{g_2} = \frac{1}{2}\frac{-AE + BD}{AC - B^2}.$$

7. Conclusions

This paper has presented an extensive study of the distributed optimal control problem related to steady-state heat conduction in a system made up of two solids in thermal contact. The system is

 doi:10.20944/preprints202603.2117.v1

characterized by the continuity of heat fluxes and a temperature jump at the interface between the solids. We have established the existence and uniqueness of the solution, provided the first order optimality condition, and formulated an iterative algorithm to find the optimal control. Finally, we obtain an explicit analytical solution for the one-dimensional case in Cartesian coordinates.

Author Contributions: Conceptualization, Tarzia D.A.; methodology, Gariboldi C.M., Rossani , M., and Tarzia D.A.; original draft preparation, Rossani M.; review and editing, Gariboldi C.M., Rossani , M., and Tarzia D.A.

Funding: This research was funded by Projects $006-25CI2002$ from Universidad Austral, Rosario, Argentina and PPI $18C/614-1$ from SeCyT, Universidad Nacional de Río Cuarto, Río Cuarto, Argentina.

Conflicts of Interest: The authors declare no conflicts of interest. The funders had no role in the design of the study, the collection, analysis, or interpretation of data, the writing of the manuscript, or the decision to publish the results.

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