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Posted Date: 1 August 2025

doi: 10.20944/preprints202508.0048.v1

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The Studies of (Dual) Fusion Frames on Hilbert Space and Generalization of the Index Set

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Abstract

We introduce orthogonal projection P between Hilbert space H and $\theta(H)$ (the range of the frame transform θ of traditional tight frame) firstly, and study the relationship between θ and P, then we explore the fusion frame and extend the index set to infinite set through an example. Secondly, we study the dual fusion frames starting with an example which illustrate how the traditional dual frames recover the original signal in the case of data loss. Finally, We obtain some important conclusions mainly including the necessary and sufficient condition, the stability of dual fusion frames and the relationship between canonical dual and alternate dual fusion frames especially the relationship between their respective frame operators.

Keywords: (Dual) fusion frame; index set; positive term series; frame operator

MSC: 46H25; 42C15

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1. Introduction

Frame theory plays an important role in signal processing, image processing, data compression, sampling theory, and other fields. In 1952, Duffin and Schaeffer [4] introduced the concept of frames in Hilbert space when they study non-harmonic Fourier series, However, this idea did not attract much attention at that time. Until 1986, Daubechies, Grossman and Meyer [5] made groundbreaking research in this area, after which frame theory began to be extensively studied.

Casazza and Kutyniok [6] introduced fusion frames and studied their properties, perturbations and approximation of inverse operators of fusion frames. In recent years, the theory of fusion frames has also developed rapidly [7–10]. Fusion frames are generalized form of traditional frames and they are used to process block data or multi-channel signals by combining orthogonal projections of subspaces with weights.

Nowadays, many scholars have conducted research on dual frames, for example, J. Lopez, Han Deguang and Sun Wenchang, et al. [11–19] studied the problems of selection of the optimal dual frame from different aspects and their studies have enabled frame theory to achieve breakthrough progress in practical applications.

The main contents are as follows. In section 2.1, we will study the discovery of fusion frames on Hilbert Spaces and generalization of index set by given an example. In section 2.2, the properties of dual fusion frames which different from traditional dual frames will be given.

First, we present the preliminary definitions and theorems used in this paper and throughout this paper, H is a Hilbert spaces, we write $W=\{(W_i,w_i)\}_{i\in I}$ as $W=\{(W_i,w_iP_{W_i})\}_{i\in I}$ and write $\|f\|^2$ as $\langle f,f\rangle$. I_H denotes the identity operator on H.

Definition 1 (see [1]) A sequence $\{f_i, j\in J\}$ of Hilbert space H is called a frame if there exist constants $a,b>0$ such that for all $f\in H$,

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$$a\|f\|^2 \le \sum_{j\in J}\left\|\langle f, f_j\rangle\right\|^2 \le b\|f\|^2$$

The numbers a, b are called the lower and the upper frame bounds respectively. the frame is called a tight frame if $a=b$ and a normalized tight frame if $a=b=1$.

Definition 2 (see [1]) Let $\{f_i, i\in I\}$ be a frame for H, and let $\{e_i, i\in I\}$ be its orthonormal basis, the frame transform θ is defined by

$$\theta : H \to H \text{ with } \theta(f)=\sum_{i\in I}\langle f, f_i\rangle e_i \text{ for any } f\in H.$$

Then θ is adjointable, and

$$\theta^* : H \to H \text{ with } \theta^*(f)=\sum_{i\in I}\langle f, e_i\rangle f_i, \text{ and } \theta^*(e_i)=f_i.$$

Moreover, we have

$$S(f)=\theta^*\theta(f)=\theta^*(\sum_{i\in I}\langle f, f_i\rangle e_i)=\sum_{i\in I}\langle f, f_i\rangle \theta^*(e_i)=\sum_{i\in I}\langle f, f_i\rangle f_i.$$

A direct calculation now yields

$$\langle S(f), f\rangle=\left\langle \sum_{i\in I}\langle f, f_i\rangle f_i, f\right\rangle=\sum_{i\in I}\langle f, f_i\rangle\langle f_i, f\rangle.$$

If $\{f_i, i\in I\}$ is a frame for H, then $S=\theta^*\theta$ is a positive, self-adjoint and invertible operator on H, called the frame operator.

It follows from Definition 2 that $\{f_i, i\in I\}$ is $a-$tight frame if and only if $S=\theta^*\theta=aI_H$ and a normalized tight frame if and only if $S=\theta^*\theta=I_H$.

Since $S(f)=\sum_{i\in I}\langle f, f_i\rangle f_i$, using $S^{-1}(f)$ to replace f, we have

$$f=SS^{-1}(f)=\sum_{i\in I}\langle S^{-1}(f), f_i\rangle f_i=\sum_{i\in I}\langle f, S^{-1}(f_i)\rangle f,$$

and

$$f=S^{-1}S(f)=S^{-1}(\sum_{i\in I}\langle f, f_i\rangle f_i)=\sum_{i\in I}\langle f, f_i\rangle S^{-1}(f).$$

Then,

$$f=\sum_{i\in I}\langle x, x_i\rangle S^{-1}(x_i)=\sum_{i\in I}\langle x, S^{-1}(x_i)\rangle x_i.$$

We have the following definition of the dual frame.

Definition 3 (see [1]) Let $\{f_i, i\in I\}$ be a frame for H with the frame operator S .Then $\{S^{-1}(f_i), i\in I\}$ is called the canonical dual frame of $\{f_i, i\in I\}$. If a frame $\{g_i, i\in I\}$ for H satisfies that

$$\sum_{i\in I}\langle f, f_i\rangle g_i=\sum_{i\in I}\langle f, g_i\rangle f_i, \text{ for any } f\in H,$$

then $\{g_i, i \in I\}$ is called an alternate dual frame of $\{f_i, i \in I\}$.

Definition 4 (see [2]) Let I be a countable index set, let $\{W_i\}_{i\in I}$ be a family of closed subspaces of H, and let $\{w_i\}_{i\in I}$ be a family of weights, i.e., $w_i > 0$ for all $i \in I$. Then $W = \{(W_i, w_i P_{W_i})\}_{i\in I}$ is a fusion frame if there exist constants $0 < A < B < +\infty$ such that

$$A\langle f, f\rangle \le \sum_{i\in I}\langle w_i P_{W_i}(f), w_i P_{W_i}(f)\rangle \le B\langle f, f\rangle \quad \text{for all } f \in H.$$

where $P_{W_i}: H \to W_i$ is the orthogonal projection. We call A and B are the fusion frame bound.

The family $W = \{(W_i, w_i P_{W_i})\}_{i\in I}$ is called $A-$ tight fusion frame if $A = B$, a parseval fusion frame provided that $A = B = 1$, and a orthonormal fusion frame if $H = \oplus_{i\in I} W_i$. If only the right-hand inequality holds, we call it a Bessel fusion sequence with Bessel frame bound B. Family $W = \{(W_i, 1P_{W_i})\}_{i\in I}$ is called $1-$ consistent Parseval fusion frame if

$$\sum_{i\in I}\langle P_{W_i}(f), P_{W_i}(f)\rangle = \langle f, f\rangle \quad \text{for all } f \in H,$$

or $w-$ consistent Parseval fusion frame if

$$\sum_{i\in I}\langle w P_{W_i}(f), w\, P_{W_i}(f)\rangle = \langle f, f\rangle \quad \text{for all } f \in H.$$

Theorem 1 (see [3]) *Let H be a Hilbert space, and let $\{P_i\}_{i\in I}$ be orthogonal projections, if $P_i(f) = f_i$ for any $f \in H$, $W_i = \overline{span}\{f_i\}_{i\in I}$. Then $\{f_i, i \in I\}$ is a frame for H if and only if $W = \{(W_i, w_i)\}_{i\in I}$ is fusion frame for H with the frame bounds unchanged and the weight set being $w_i = \|f_i\| > 0$.*

Theorem 2 (see [3]) *Let $\{W_i\}_{i\in I}$ be a sequence of closed subspaces of H, $\{w_i > 0\}_{i\in I}$ be the weight set, and let $\{\varphi_{ij}, j \in J\}$ be frames for W_i. Then $W = \{(W_i, w_i)\}_{i\in I}$ is a fusion frame for H if and only if $\{w_i\varphi_{ij}, i \in I, j \in J\}$ is a frame for H.*

2. Main Results

2.1. Discovery of Fusion Frames on Hilbert Spaces and Generalization of the Index Set

In this section, starting with traditional tight frame. We introduce orthogonal projection between the Hilbert space and the range of the frame transform of the traditional tight frame, and study the relationship between the frame transform and the orthogonal projection. On this basis, we explore fusion frame, and further visualize traditional frames, fusion frames, and complementable closed subspaces. Secondly, we obtain example by finding convergent positive term series and combining them with the orthonormal basis of Hilbert space, where the square root of the general term of the positive term series is taken as the weight set of the fusion frame. More importantly, through this example, the index set is generalized to an infinite set.

Theorem 3 *Let $\{f_i, i \in I\}$ be $a-$ tight frame for H with the frame transform θ, and let $P: H \to \theta(H)$ be an orthogonal projection. Then $P(e_i) = \frac{1}{a}\theta(f_i)$ and $\theta\theta^* = aP$. where $\{e_i, i \in I\}$ is the orthonormal basis for H.*

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Proof Firstly, since $\{f_i, i\in I\}$ is $a-$ tight frame for H, $\theta^*\theta = aI_H$. And since P is an orthogonal projection from H onto $\theta(H)$, $P = I_H$ on $\theta(H)$, i.e., $P\theta(H) = \theta(H)$.

Then, for any $f \in H$,

$$\begin{aligned}\langle \theta(f), P(e_i)\rangle &= \langle P\theta(f), e_i\rangle = \langle \theta(f), e_i\rangle = \langle f, \theta^*(e_i)\rangle \\ &= \langle f, f_i\rangle = \frac{1}{a}\langle \theta(f), \theta(f_i)\rangle = \left\langle \theta(f), \frac{1}{a}\theta(f_i)\right\rangle,\end{aligned}$$

therefore, $P(e_i) = \frac{1}{a}\theta(f_i)$ or $\theta(f_i) = aP(e_i)$.

Secondly, since $f = \sum_{i\in I}\langle f, e_i\rangle e_i$, we have

$$\begin{aligned}\theta\theta^*(f) &= \theta\theta^*(\sum_{i\in I}\langle f, e_i\rangle e_i) = \theta(\sum_{i\in I}\langle f, e_i\rangle \theta^*(e_i)) = \theta(\sum_{i\in I}\langle f, e_i\rangle f_i) \\ &= \sum_{i\in I}\langle f, e_i\rangle \theta(f_i) = \sum_{i\in I}\langle f, e_i\rangle aP(e_i) = aP(\sum_{i\in I}\langle f, e_i\rangle e_i) = aP(f).\end{aligned}$$

By the arbitrariness of f, it follows that $\theta\theta^* = aP.$

In addition, from $P = \frac{1}{a}\theta\theta^*$ and $\theta^*\theta = aI$, it follows that

$$P^2 = (\frac{1}{a}\theta\theta^*)(\frac{1}{a}\theta\theta^*) = \frac{1}{a^2}\theta\theta^*\theta\theta^* = \frac{1}{a^2}\theta aI\theta^* = \frac{1}{a}\theta^*\theta = p.$$

Its self-adjointness is obvious. Thus, $P^2 = P = P^*$, which fully verifies that P acts as an orthogonal projection in the relevant context.

In particular, when $\{f_i, i\in I\}$ is a normalized tight frame, i.e., $a = 1$, it is clear that $P(e_i) = \theta(f_i)$ and $\theta\theta^* = P$.

Let $\{f_{ij}, i\in I, j\in J\}$ be $a_i -$ tight frame sequence for H with the frame transforms θ_i (where i is the number of frames and j is the dimension of the frames), and let $P_i : H \to \theta_i(H)$ be the orthogonal projections. Then $\theta_i\theta_i^* = a_iP_i$, and

$$\sum_{i\in I}(\sqrt{a_i})^2 P_i = \sum_{i\in I}a_iP_i = \sum_{i\in I}\theta_i\theta_i^*.$$

Therefore, $\{(\theta_i(H), \sqrt{a_i}P_{\theta_i(H)})\}_{i\in I}$ is a fusion frame if and only if there exist $0 < A < B < +\infty$ such that

$$AI_H \le \sum_{i\in I}(\sqrt{a_i})^2 P_i = \sum_{i\in I}\theta_i\theta_i^* \le BI_H.$$

Obviously, since $\sum_{i\in I}\theta_i\theta_i^* \le \sum_{i\in I}\left\|\theta_i^*\right\|^2 I_H$, when I is a finite set, $\{(\theta_i(H), \sqrt{a_i}P_{\theta_i(H)})\}_{i\in I}$ must be a Bessel fusion sequence.

□

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Our questions are: Do such traditional frames and fusion frames exist? and can I here be extended to an infinite set, i.e., $i \in N^+$? Our answers are affirmative, and we will first illustrate this with example below.

Example 1 Let $f_{ij} = \sqrt{a_i} e_j$ with $a_i > 0$ and $\sum_{i=1}^{+\infty} a_i = s$, and let $\{e_j, j \in J\}$ is the standard orthonormal basis for H. Then $\{f_{ij}, i \in N^+, j \in J\}$ are $a_i -$ tight frames for H, and $\{(\theta_i(H), \sqrt{a_i} P_{\theta_i(H)})\}_{i \in N^+}$ is a $s -$ tight fusion frame for H.

Proof for any $f \in H$, we have

$$\sum_{j \in J} \langle f, f_{ij} \rangle \langle f_{ij}, f \rangle = \sum_{j \in J} \langle f, \sqrt{a_i} e_j \rangle \langle \sqrt{a_i} e_j, f \rangle = a_i \sum_{j \in J} \langle f, e_j \rangle \langle e_j, f \rangle = a_i \langle f, f \rangle,$$

Therefore, $\{f_{ij}, i \in N^+, j \in J\}$ are $a_i -$ tight frames for H.

Since $\theta_i(f) = \sum_{j \in J} \langle f, \sqrt{a_i} e_j \rangle e_j = \sqrt{a_i} \sum_{j \in J} \langle f, e_j \rangle e_j = \sqrt{a_i}(f)$, i.e., $\theta_i = \sqrt{a_i} I_H$, $\theta_i^* = (\sqrt{a_i} I_H)^* = \sqrt{a_i} I_H$, then $\theta \theta_i^* = (\sqrt{a_i} I_H)(\sqrt{a_i} I_H) = a_i I_H = a_i P_i$, which means $P_i = I_H$ satisfying $P^2 = P = P^*$ and

$$\sum_{i=1}^{+\infty} (\sqrt{a_i})^2 P_i = \sum_{i=1}^{+\infty} a_i P_i = \sum_{i=1}^{+\infty} \theta_i \theta_i^* = \sum_{i=1}^{+\infty} a_i I_H = s I_H$$

Therefore, $\{(\theta_i(H), \sqrt{a_i} P_{\theta_i(H)})\}_{i \in N^+}$ is a $s -$ tight fusion frame.

Furthermore, we obtain that $\{(\theta_i(H), \sqrt{a_i} P_{\theta_i(H)})\}_{i \in N^+}$ is a $s -$ tight fusion frame if and only if the infinite series $\sum_{i=1}^{+\infty} \theta_i \theta_i^*$ composed of the operators converges to $s I_H$.

However, $\{(\theta_i(H), \sqrt{a_i} P_{\theta_i(H)})\}_{i \in N^+}$ cannot be $1-$ uniform fusion frame or $w-$ uniform fusion frame, because the constant series $\sum_{i=1}^{+\infty} w (w > 0)$ diverges forever unless I is a finite set. We present the following theorem.

Theorem 4 *Let* H *be a Hilbert space.Then there exist a sequence of* $a_i -$ *tight frames* $\{f_{ij}, i \in N^+, j \in J\}$ *for* H *such that* $\{(\theta_i(H), \sqrt{a_i} P_{\theta_i(H)})\}_{i \in N^+}$ *is a tight fusion frame for* H, *where* θ_i *are the frame transforms of* $\{f_{ij}, i \in I, j \in J\}$, $P_i : H \to \theta_i(H)$ *are the orthogonal projections.*

Remark 1

(1) Example 1 make use of the convergence of positive term series. For instance, if we take $a_i = \frac{1}{2^i}$, then $\sum_{i=1}^{+\infty} a_i = 1$, and $\{(\theta_i(H), \sqrt{a_i} P_{\theta_i(H)})\}_{i \in N^+}$ is a Parseval fusion frame, and in this case, $\sum_{i=1}^{+\infty} \theta_i \theta_i^* = I_H$. In fact, for the positive term series here, it is sufficient that it converges, and there is no need to find its sum. Moreover, many series can only be judged

for their convergence, without being able to calculate their specific sums. Even in such cases, $\{(\theta_i(H), \sqrt{a_i} P_{\theta_i(H)})\}_{i\in N^+}$ is still a tight fusion frame.

(2) The conclusion only holds for traditional tight frames, because the relationship between the frame transform and orthogonal projection exists only for tight frames.

(3) Example 1 and Theorem 4 closely connect traditional frames with fusion frames, and also materializes the closed subspaces W_i of H by taking $W_i = P_{\theta_i(H)}$. It is even more valuable that the index set I is generalized to infinite sets.

Next, focusing on Theorem 1, we will provide an example regarding the relationship between traditional frame and fusion frame on Hilbert space.

Example 2 Let $H = \mathbb{R}^3$, $P_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $P_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $P_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Then for any $f = (x, y, z)^T \in \mathbb{R}^3$, by $f_i = P_i(f), i = 1,2,3$, we have $f_1 = (x,0,0)^T$, $f_2 = (0, y, 0)^T$, $f_3 = (0,0,z)^T$, $W_i = \overline{span}\{f_i\}_{i=1}^3$ and $w_1^2 = \|f_1\|^2 = x^2$, $w_2^2 = \|f_2\|^2 = y^2$, $w_3^2 = \|f_3\|^2 = z^2$.

Therefore,

$$\sum_{i=1}^{3} |\langle f, f_i \rangle|^2 = x^4 + y^4 + z^4 = \|f_1\|^2 x^2 + \|f_2\|^2 y^2 + \|f_3\|^2 z^2$$
$$= w_1^2 \langle P_1(f), P_1(f) \rangle + w_2^2 \langle P_2(f), P_2(f) \rangle + w_3^2 \langle P_3(f), P_3(f) \rangle$$
$$= \sum_{i=1}^{3} w_i^2 \langle P_i(f), P_i(f) \rangle,$$

and

$$\min_{1\le i\le 3}\{\|f_i\|^2\}\|f\|^2 = \min_{1\le i\le 3}\{\|f_i\|^2\}(x^2+y^2+z^2) \le \|f_1\|^2 x^2 + \|f_2\|^2 y^2 + \|f_3\|^2 z^2$$
$$\le \max_{1\le i\le 3}\{\|f_i\|^2\}(x^2+y^2+z^2) = \max_{1\le i\le 3}\{\|f_i\|^2\}\|f\|^2,$$

That is to say,

$$\min_{1\le i\le 3}\{\|f_i\|^2\}\|f\|^2 \le \sum_{i=1}^{3} |\langle f, f_i \rangle|^2 = \sum_{i=1}^{3} \|f_i\|^2 \langle P_i(f), P_i(f) \rangle \le \max_{1\le i\le 3}\{\|f_i\|^2\}\|f\|^2.$$

Thus, $\{f_i\}_{i=1}^3$ is a frame for H if and only if $W = \{(W_i, w_i P_i)\}_{i\in I}$ is a fusion frame for H, and (fusion) frame bounds are the same which is $\min_{1\le i\le 3}\{\|f_i\|^2\}$ and $\max_{1\le i\le 3}\{\|f_i\|^2\}$ respectively.

In addition, since W_1, W_2 and W_3 are the x-axis, y-axis and z-axis respectively, they are pairwise orthogonal and span $\mathbb{R}^3$. Then we have $W_1 \perp W_2 \perp W_3$ and $W_1 + W_2 + W_3 = \mathbb{R}^3$, i.e., $W_1 \oplus W_2 \oplus W_3 = \mathbb{R}^3$. This demonstrates the orthogonal complementability of $\mathbb{R}^3$ and reveals a construction method for orthogonal complement subspaces of real spaces.

For Theorem 2, we have more detailed proof process. Due to space constraints, it will not be elaborated on here.

2.2. Research on Dual Fusion Frames on Hilbert Space

In this section, we start with an example to explore how traditional dual frames recover original signals in cases of data loss, demonstrating the importance and research significance of dual frames.

 doi:10.20944/preprints202508.0048.v1

Then we find that alternate dual fusion frames are not mutually alternate dual fusion frames which different from traditional frames, so we study the necessary and sufficient conditions of mutually alternate dual fusion frames firstly, then investigate the stability of alternate dual fusion frames and finally examine the relationship between canonical dual fusion frames and alternate dual fusion frames especially the relationship between their respective frame operators.

Example 3 Let $x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $x_2 = \begin{bmatrix} -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{5}}{2} \end{bmatrix}$, $x_3 = \begin{bmatrix} -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{5}}{2} \end{bmatrix}$.

Then

$$\theta_X = \begin{bmatrix} 1 & 0 \\ -\frac{\sqrt{3}}{2} & \frac{\sqrt{5}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{\sqrt{5}}{2} \end{bmatrix}$$

$$S = \theta_X^* \theta_X = \theta_X^T \theta_X = \begin{bmatrix} 1 & -\frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{5}}{2} & -\frac{\sqrt{5}}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{\sqrt{3}}{2} & \frac{\sqrt{5}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{\sqrt{5}}{2} \end{bmatrix} = \begin{bmatrix} \frac{5}{2} & 0 \\ 0 & \frac{5}{2} \end{bmatrix} = \frac{5}{2} I$$

so $X = \{x_1, x_2, x_3\}$ is a tight frame for $\mathbb{R}^2$ with frame bound $\frac{5}{2}$, where θ_X is the frame transform of X and S is its frame operator.

Let the original signal vector be $x = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Then the frame coefficients are $c_1 = \langle x, x_1 \rangle = x^T x_1 = 1$, $c_2 = \langle x, x_2 \rangle = -\frac{\sqrt{3}+\sqrt{5}}{2}$, $c_2 = \langle x, x_3 \rangle = \frac{\sqrt{5}-\sqrt{3}}{2}$. Suppose that during transmission, the original signal loses c_3, how can we recover the original signal?

We try to recover it using the alternate dual frame $Y = \{y_1, y_2, y_3\}$ of X. Since c_3 is lost, let us set $y_3 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $y_1 = \begin{bmatrix} a \\ b \end{bmatrix}$, $y_2 = \begin{bmatrix} c \\ d \end{bmatrix}$. Then $\theta_Y = \begin{bmatrix} a & b \\ c & d \\ 0 & 0 \end{bmatrix}$. Since Y is the alternate dual frame of X, they are mutually alternate dual frames and $\theta_X^* \theta_Y = \theta_X^T \theta_Y = I$ or $\theta_Y^* \theta_X = \theta_Y^T \theta_X = I$. Then we have

$$\theta_X^T \theta_Y = \begin{bmatrix} 1 & -\frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{5}}{2} & -\frac{\sqrt{5}}{2} \end{bmatrix} \begin{bmatrix} a & b \\ c & d \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

doi:10.20944/preprints202508.0048.v1

It is solved that $a=1$, $b=\frac{\sqrt{3}}{\sqrt{5}}$, $c=0$, $d=\frac{2}{\sqrt{5}}$. So $y_1=\begin{bmatrix}1\\ \frac{\sqrt{3}}{\sqrt{5}}\end{bmatrix}$, $y_2=\begin{bmatrix}0\\ \frac{2}{\sqrt{5}}\end{bmatrix}$, $y_3=\begin{bmatrix}0\\0\end{bmatrix}$, and

$$c_1y_1+c_2y_2=1\begin{bmatrix}1\\ \frac{\sqrt{3}}{\sqrt{5}}\end{bmatrix}+(-\frac{\sqrt{3}+\sqrt{5}}{2})\begin{bmatrix}0\\ \frac{2}{\sqrt{5}}\end{bmatrix}=\begin{bmatrix}1\\-1\end{bmatrix}=x.$$

i.e., the alternate dual frame Y of X can recover the original signal.

The canonical dual frame $Z=\{z_1,z_2,z_3\}$ of X can also recover the original signal. Since $S_X=\frac{5}{2}I$, $S_X^{-1}=\frac{2}{5}I$. Then we have $z_1=S^{-1}(x_1)=\frac{2}{5}I\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}\frac{2}{5}\\0\end{bmatrix}$,

$$z_2=S^{-1}(x_2)=\frac{2}{5}I\begin{bmatrix}-\frac{\sqrt{3}}{2}\\ \frac{\sqrt{5}}{2}\end{bmatrix}=\begin{bmatrix}-\frac{\sqrt{3}}{5}\\ \frac{\sqrt{5}}{5}\end{bmatrix},\quad z_3=S^{-1}(x_3)=\frac{2}{5}I\begin{bmatrix}-\frac{\sqrt{3}}{2}\\ -\frac{\sqrt{5}}{2}\end{bmatrix}=\begin{bmatrix}-\frac{\sqrt{3}}{5}\\ -\frac{\sqrt{5}}{5}\end{bmatrix},$$

Taking

$$c_1'=\langle x,z_1\rangle=x^Tz_1=\frac{2}{5},\quad c_2'=\langle x,z_2\rangle=x^Tz_2=-\frac{\sqrt{3}+\sqrt{5}}{5},\quad c_3'=\langle x,z_3\rangle=x^Tz_2=\frac{\sqrt{5}-\sqrt{3}}{5}.$$

Then

$$c_1'x_1+c_2'x_2+c_3'x_3=\frac{2}{5}\begin{bmatrix}1\\0\end{bmatrix}-\frac{\sqrt{3}+\sqrt{5}}{5}\begin{bmatrix}-\frac{\sqrt{3}}{2}\\ \frac{\sqrt{5}}{2}\end{bmatrix}+\frac{\sqrt{5}-\sqrt{3}}{5}\begin{bmatrix}-\frac{\sqrt{3}}{2}\\ -\frac{\sqrt{5}}{2}\end{bmatrix}=\begin{bmatrix}1\\-1\end{bmatrix}=x.$$

This means that the canonical dual frame Z of X can also recover the original signal.

Meanwhile, let $y_i=z_i+u_i\ (i=1,2,3)$, then $u_1=\begin{bmatrix}\frac{3}{5}\\ \frac{\sqrt{3}}{\sqrt{5}}\end{bmatrix}$, $u_2=\begin{bmatrix}\frac{\sqrt{3}}{5}\\ \frac{1}{\sqrt{5}}\end{bmatrix}$, $u_3=\begin{bmatrix}\frac{\sqrt{3}}{5}\\ \frac{1}{\sqrt{5}}\end{bmatrix}$, and

$$c_1u_1+c_2u_2+c_3u_3=1\begin{bmatrix}\frac{3}{5}\\ \frac{\sqrt{3}}{\sqrt{5}}\end{bmatrix}+(-\frac{\sqrt{3}+\sqrt{5}}{2})\begin{bmatrix}\frac{\sqrt{3}}{5}\\ \frac{1}{\sqrt{5}}\end{bmatrix}+\frac{\sqrt{5}-\sqrt{3}}{2}\begin{bmatrix}\frac{\sqrt{3}}{5}\\ \frac{1}{\sqrt{5}}\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}.$$

We obtain $\sum_{i=1}^{3}\langle x,x_i\rangle u_i=0$, i.e., $\theta_X^*\theta_U=0$, and of course $\theta_U^*\theta_X=0$ .Where $U=\{u_1,u_2,u_3\}$. It follows that the frames X and U are disjoint, meaning that any alternate dual frame of a frame can be expressed as the sum of its canonical dual frame and a frame disjoint from itself, and the canonical dual frame is the "minimal" dual frame.

Next, we study dual fusion frames on Hilbert spaces.

Definition 5 (see [2]) Let $W=\{(W_i,w_iP_{W_i})\}_{i\in I}$ be a fusion frame for H, the analysis operator is defined by

$$T:H\to(\sum_{i\in I}\oplus W_i)_{l^2} \text{ with } T(f)=\{w_iP_{W_i}(f)\}_{i\in I} \text{ for any } f\in H,$$

where $(\sum_{i\in I}\oplus W_i)_{l^2}=\{\{f_i\}_{i\in I}\mid f_i\in W_i \; and \; \{\|f_i\|\}_{i\in I}\in l^2(I)\}$. It can easily be shown that the synthesis operator T^*, which is defined to be the adjoint operator, is given by

$$T^*:(\sum_{i\in I}\oplus W_i)_{l^2}\to H \text{ with } T^*(f_i)=\sum_{i\in I}w_iP_{W_i}(f_i) \text{ for any } f_i\in(\sum_{i\in I}\oplus W_i)_{l^2}.$$

The fusion frame operator S for $W=\{(W_i,w_iP_{W_i})\}_{i\in I}$ is defined by

$$S:H\to H \text{ with } S(f)=T^*T(f)=\sum_{i\in I}w_i^2P_{W_i}(f).$$

If $W=\{(W_i,w_iP_{W_i})\}_{i\in I}$ be a fusion frame for H with fusion frame bound A and B, then the associated fusion frame operator S is self-adjoint, positive and invertible operator on H, and $AI_H\le S=\sum_{i\in I}w_i^2P_{W_i}\le BI_H$

It follows from Definition 5 that $W=\{(W_i,w_iP_{W_i})\}_{i\in I}$ is a $B-$tight fusion frame if and only if $S=BI_H$, and is a Parseval fusion frame if and only if $S=I_H$. Obviously, if $W=\{(W_i,w_iP_{W_i})\}_{i\in I}$ is a fusion frame for H with fusion frame operator S, then $\{(W_i,w_iP_{W_i}S_1^{-\frac{1}{2}})\}_{i\in I}$ is a Parseval fusion frame for H.

In fact, $\sum_{i\in I}w_i^2(P_{W_i}S^{-\frac{1}{2}})^*(P_{W_i}S^{-\frac{1}{2}})=S^{-\frac{1}{2}}(\sum_{i\in I}w_i^2P_{W_i})S^{-\frac{1}{2}}=S^{-\frac{1}{2}}SS^{-\frac{1}{2}}=I_H$.

Theorem 5 (see [4]) *Let T be a bounded linear operator on H, and let V be a closed subspace of H. Then $P_VT^*=P_VT^*P_{\overline{TV}}$. Moreover if T is a unitary operator, Then $P_{\overline{TV}}T=TP_V$, where P is the orthogonal projection from H to V.*

It follows from theorem5 that $(P_VT^*)^*=(P_VT^*P_{\overline{TV}})^*$, i.e., $TP_V=P_{\overline{TV}}TP_V$. So if $W=\{(W_i,w_iP_{W_i})\}_{i\in I}$ is the fusion frame with fusion frame operator S, then $S^{-1}P_{W_i}=P_{S^{-1}(W_i)}S^{-1}P_{W_i}$. Therefore, for any $f\in H$, $S(f)=\sum_{i\in I}w_i^2P_{W_i}(f)$, and

$$f=S^{-1}S(f)=\sum_{i\in I}w_i^2S^{-1}P_{W_i}(f)=\sum_{i\in I}w_i^2P_{S^{-1}(W_i)}S^{-1}P_{W_i}(f).$$

We call $\mathrm{S}^{-1}(W)=\{(\mathrm{S}^{-1}(W_i),w_iP_{S^{-1}(W_i)})\}_{i\in I}$ the canonical dual fusion frame of $W=\{(W_i,w_iP_{W_i})\}_{i\in I}$.

Corollary 1 *Let $W=\{(W_i,w_iP_{W_i})\}_{i\in I}$ is the fusion frame for H with fusion frame operator S, and let $\mathrm{S}^{-1}(W)=\{(\mathrm{S}^{-1}(W_i),w_iP_{S^{-1}(W_i)})\}_{i\in I}$ be its the canonical dual fusion frame.Then $S_{\mathrm{S}^{-1}(W)}=S^{-1}$.*

Proof for any $f\in H$,

 doi:10.20944/preprints202508.0048.v1

$$S_{S^{-1}(W)}(f)=\sum_{i\in I}P_{S^{-1}(W_i)}(w_iP_{W_i}S^{-1})^*(w_iP_{W_i}S^{-1})P_{S^{-1}(W_i)}(f)$$
$$=\sum_{i\in I}P_{S^{-1}(W_i)}S^{-1}w_iP_{W_i}w_iP_{W_i}S^{-1}P_{S^{-1}(W_i)}(f)=\sum_{i\in I}w_i^2(P_{W_i}S^{-1}P_{S^{-1}(W_i)})^*(P_{W_i}S^{-1}P_{S^{-1}(W_i)})(f)$$
$$=\sum_{i\in I}w_i^2(P_{W_i}S^{-1})^*(P_{W_i}S^{-1})=\sum_{i\in I}w_i^2S^{-1}P_{W_i}P_{W_i}S^{-1}(f)$$
$$S^{-1}(\sum_{i\in I}w_i^2P_{W_i})S^{-1}(f)=S^{-1}SS^{-1}(f)=S^{-1}(f).$$

Therefore, the frame operator of the canonical dual fusion frame is the inverse of its own frame operator. It is obvious that if $W=\{(W_i,w_iP_{W_i})\}_{i\in I}$ is a parseval fusion frame, i.e., $S=I_H$, then its canonical dual fusion frame is itself.

□

Definition 6 Let $W=\{(W_i,w_iP_{W_i})\}_{i\in I}$ and $V=\{(V_i,v_iQ_{V_i})\}_{i\in I}$ be fusion frames for H. If for any $f\in H$, $f=\sum_{i\in I}w_iv_iQ_{V_i}S^{-1}P_{W_i}(f)$, then $V=\{(V_i,v_iQ_{V_i})\}_{i\in I}$ is called an alternate dual fusion frame of $W=\{(W_i,w_iP_{W_i})\}_{i\in I}$, where S is the fusion frame operator of $W=\{(W_i,w_iP_{W_i})\}_{i\in I}$.

Remark 2.

(1) The canonical dual fusion frame of $W=\{(W_i,w_iP_{W_i})\}_{i\in I}$ must be its alternate dual fusion frame. In particular, when $S^{-1}(W_i)=V_i$, and $w_i=v_i$, these two dual fusion frames are the same fusion frame.

(2) Traditional alternate dual frames are mutually alternate dual frames, while alternate dual fusion frames are not mutually alternate dual fusion frames.

In fact, If $V=\{(V_i,v_iQ_{V_i})\}_{i\in I}$ is the alternate dual fusion frame of $W=\{(W_i,w_iP_{W_i})\}_{i\in I}$, then $f=\sum_{i\in I}w_iv_iQ_{V_i}S_1^{-1}P_{W_i}(f)$ for any $f\in H$, where S_1 is the fusion frame operator of $W=\{(W_i,w_iP_{W_i})\}_{i\in I}$. If $W=\{(W_i,w_iP_{W_i})\}_{i\in I}$ is an alternate dual fusion frame of $V=\{(V_i,v_iQ_{V_i})\}_{i\in I}$, then $f=\sum_{i\in I}w_iv_iP_{W_i}S_2^{-1}Q_{V_i}(f)$ for any $f\in H$, where S_2 is the fusion frame operator of $V=\{(V_i,v_iQ_{V_i})\}_{i\in I}$.

In particular, if $S_1=S_2=S$, then the alternate dual fusion frames are mutually alternate dual fusion frames.

In fact, if $V=\{(V_i,v_iQ_{V_i})\}_{i\in I}$ is an alternate dual fusion frame of $W=\{(W_i,w_iP_{W_i})\}_{i\in I}$, then for any $f\in H$, $f=\sum_{i\in I}w_iv_iQ_{V_i}S^{-1}P_{W_i}(f)$, thus

$$\langle f,f\rangle=\left\langle\sum_{i\in I}w_iv_iQ_{V_i}S^{-1}P_{W_i}(f),f\right\rangle=\sum_{i\in I}w_iv_i\left\langle Q_{V_i}S^{-1}P_{W_i}(f),f\right\rangle$$
$$=\sum_{i\in I}w_iv_i\left\langle f,P_{W_i}S^{-1}Q_{V_i}(f)\right\rangle=\left\langle f,\sum_{i\in I}w_iv_iP_{W_i}S^{-1}Q_{V_i}(f)\right\rangle.$$

We have $f=\sum_{i\in I} w_i v_i P_{W_i} S^{-1} Q_{V_i}(f)$, which means $W=\{(W_i, w_i P_{W_i})\}_{i\in I}$ is also an alternate dual fusion frame of $V=\{(V_i, v_i Q_{V_i})\}_{i\in I}$.

The following are the necessary and sufficient conditions of mutually alternate dual fusion frames and their stability.

Theorem 6 *Let* $W=\{(W_i, w_i P_{W_i})\}_{i\in I}$ *and* $V=\{(V_i, v_i Q_{V_i})\}_{i\in I}$ *be fusion frames for* H, *let* T_1 *and* T_2 *be their analysis operators respectively, let* T_1^* *and* T_2^* *be their synthesis operators respectively, and let* S_1 *and* S_1 *be their frame operators respectively. Then*

1. $W=\{(W_i, w_i P_{W_i})\}_{i\in I}$ *and* $V=\{(V_i, v_i Q_{V_i})\}_{i\in I}$ *are mutually alternate dual fusion frames if and only if* $T_2^* S_1^{-1} T_1 = T_1^* S_2^{-1} T_2 = I_H$.
2. *If* $S_1=S_2=S$ *and* $V=\{(V_i, v_i Q_{V_i})\}_{i\in I}$ *is alternate dual fusion frames of* $W=\{(W_i, w_i P_{W_i})\}_{i\in I}$, *then* $W+V=\{(W_i+V_i, w_i S^{-\frac{1}{2}} P_{W_i} + v_i S^{-\frac{1}{2}} Q_{V_i})\}_{i\in I}$ *is also a fusion frame for* H.

Proof (1) Since for any $f\in H$,

$$T_2^* S_1^{-1} T_1(f) = T_2^* S_1^{-1}\{(w_i P_{W_i}(f)\}_{i\in I} = \sum_{i\in I} v_i Q_{V_i} S_1^{-1} w_i P_{W_i}(f) = \sum_{i\in I} w_i v_i Q_{V_i} S_1^{-1} P_{W_i}(f);$$

$$T_1^* S_2^{-1} T_2(f) = T_1^* S_2^{-1}\{(v_i Q_{V_i}(f)\}_{i\in I} = \sum_{i\in I} w_i P_{W_i} S_2^{-1} v_i Q_{V_i}(f) = \sum_{i\in I} w_i v_i P_{W_i} S_2^{-1} Q_{V_i}(f).$$

Thus, if $V=\{(V_i, v_i Q_{V_i})\}_{i\in I}$ is an alternate dual fusion frame of $W=\{(W_i, w_i P_{W_i})\}_{i\in I}$, then $f=\sum_{i\in I} w_i v_i Q_{V_i} S_1^{-1} P_{W_i}(f)$, i.e., $T_2^* S_1^{-1} T_1(f)=f$; if $W=\{(W_i, w_i P_{W_i})\}_{i\in I}$ is an alternate dual fusion frame of $V=\{(V_i, v_i Q_{V_i})\}_{i\in I}$, then $f=\sum_{i\in I} w_i v_i P_{W_i} S_2^{-1} Q_{V_i}(f)$, i.e., $T_1^* S_2^{-1} T_2(f)=f$.

From above discussions, we obtain that $W=\{(W_i, w_i P_{W_i})\}_{i\in I}$ and $V=\{(V_i, v_i Q_{V_i})\}_{i\in I}$ are mutually alternate dual fusion frames if and only if $T_2^* S_1^{-1} T_1(f) = T_1^* S_2^{-1} T_2(f) = f$, by the arbitrariness of f, we have

$$T_2^* S_1^{-1} T_1 = T_1^* S_2^{-1} T_2 = I_H.$$

(2) Since $S_1=S_2=S$, $W=\{(W_i, w_i P_{W_i})\}_{i\in I}$ and $V=\{(V_i, v_i Q_{V_i})\}_{i\in I}$ are mutually alternate dual fusion frames. Combining with (1), for any $f\in H$, we have

$$T_2^* S^{-1} T_1(f) = \sum_{i\in I} w_i v_i Q_{V_i} S^{-1} P_{W_i}(f) = f,$$

$$T_1^* S^{-1} T_2(f) = \sum_{i\in I} w_i v_i P_{W_i} S^{-1} Q_{V_i}(f) = f.$$

and

$$T_1^* S^{-1} T_1(f) = \sum_{i\in I} w_i^2 P_{W_i} S^{-1} P_{W_i}(f),\quad T_2^* S^{-1} T_2(f) = \sum_{i\in I} v_i^2 Q_{V_i} S^{-1} Q_{V_i}(f).$$

Therefore,

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$$\sum_{i\in I}\left\langle (w_iS^{-\frac{1}{2}}P_{W_i}+v_iS^{-\frac{1}{2}}Q_{V_i})(f),(w_iS^{-\frac{1}{2}}P_{W_i}+v_iS^{-\frac{1}{2}}Q_{V_i})(f)\right\rangle$$
$$=\sum_{i\in I}\left\langle w_iS^{-\frac{1}{2}}P_{W_i}(f),w_iS^{-\frac{1}{2}}P_{W_i}(f)\right\rangle+\sum_{i\in I}\left\langle w_iS^{-\frac{1}{2}}P_{W_i}(f),v_iS^{-\frac{1}{2}}Q_{V_i}(f)\right\rangle$$
$$+\sum_{i\in I}\left\langle v_iS^{-\frac{1}{2}}Q_{V_i}(f),w_iS^{-\frac{1}{2}}P_{W_i}(f)\right\rangle+\sum_{i\in I}\left\langle v_iS^{-\frac{1}{2}}Q_{V_i}(f),v_iS^{-\frac{1}{2}}Q_{V_i}(f)\right\rangle$$
$$=\left\langle\sum_{i\in I}w_i^2P_{W_i}S^{-1}P_{W_i}(f),f\right\rangle+\left\langle\sum_{i\in I}w_iv_iQ_{V_i}S^{-1}P_{W_i}(f),f\right\rangle$$
$$+\left\langle\sum_{i\in I}w_iv_iP_{W_i}S^{-1}Q_{V_i}(f),f\right\rangle+\left\langle\sum_{i\in I}v_i^2Q_{V_i}S^{-1}Q_{V_i}(f),f\right\rangle$$
$$=\left\langle T_1^*S^{-1}T_1(f),f\right\rangle+\left\langle T_2^*S^{-1}T_1(f),f\right\rangle+\left\langle T_1^*S^{-1}T_2(f),f\right\rangle+\left\langle T_2^*S^{-1}T_2(f),f\right\rangle$$
$$=\left\langle T_1^*S^{-1}T_1(f),f\right\rangle+\left\langle f,f\right\rangle+\left\langle f,f\right\rangle+\left\langle T_2^*S^{-1}T_2(f),f\right\rangle$$
$$=\left\langle (T_1^*S^{-1}T_1+T_2^*S^{-1}T_2+2I_H)(f),f\right\rangle,$$

that is to say,

$$\sum_{i\in I}(w_iS^{-\frac{1}{2}}P_{W_i}+v_iS^{-\frac{1}{2}}Q_{V_i})^*(w_iS^{-\frac{1}{2}}P_{W_i}+v_iS^{-\frac{1}{2}}Q_{V_i})=T_1^*S^{-1}T_1+T_2^*S^{-1}T_2+2I_H.s$$

Moreover, since $W=\{(W_i,w_iP_i)\}_{i\in I}$ and $V=\{(V_i,v_iQ_i)\}_{i\in I}$ are fusion frames for H, there exist constants $0<A<B<+\infty$ such that $AI_H\le S_1=S_2=S\le BI_H$ .Therefore, $\frac{1}{B}I_H\le S^{-1}\le\frac{1}{A}I_H$, and

$$\frac{A}{B}I_H\le\frac{1}{B}S=\frac{1}{B}T_1^*T_1\le T_1^*S^{-1}T_1\le\frac{1}{A}T_1^*T_1=\frac{1}{A}S\le\frac{B}{A}I_H.$$

Similarly, since $\frac{A}{B}I_H\le T_2^*S^{-1}T_2\le\frac{B}{A}I_H$, we have

$$2(\frac{A}{B}+1)I_H\le T_1^*S^{-1}T_1+T_2^*S^{-1}T_2+2I_H\le 2(\frac{B}{A}+1)I_H,$$

so $W+V=\{(W_i+V_i,w_iS^{-\frac{1}{2}}P_{W_i}+v_iS^{-\frac{1}{2}}Q_{V_i})\}_{i\in I}$ is also a fusion frame with frame operator $S_{W+V}=T_1^*S^{-1}T_1+T_2^*S^{-1}T_2+2I_H$ and $2(\frac{A}{B}+1)I_H\le S_{W+V}\le 2(\frac{B}{A}+1)I_H$.

□

Theorem 7 *Let* $W=\{(W_i,w_iP_{W_i})\}_{i\in I}$ *be a fusion frame for* H *with fusion frame operator* S_1, *let* $S_1^{-1}(W)=\{(S_1^{-1}(W_i),w_iP_{S_1^{-1}(W_i)})\}_{i\in I}$ *and* $V=\{(V_i,v_iQ_{V_i})\}_{i\in I}$ *be its canonical dual fusion frame and alternate dual fusion frame respectively. Then* $S_1^{-1}\le\left\|S_1^{-1}\right\|^2S_2$, *where* S_2 *is the fusion frame operator of* $V=\{(V_i,v_iQ_{V_i})\}_{i\in I}$.

Proof Since for any $f\in H$,

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$$\sum_{n=1}^{+\infty} w_i^2 P_{S_1^{-1}(W_i)} S_1^{-1} P_{W_i}(f) = f = \sum_{i\in I} w_i v_i Q_{V_i} S_1^{-1} P_{W_i}(f),$$

and

$$\sum_{n=1}^{+\infty} w_i^2 P_{S_1^{-1}(W_i)} S_1^{-1} P_{W_i}(f) = \sum_{n=1}^{+\infty} w_i^2 S_1^{-1} P_{W_i}(f),$$

we have

$$\sum_{n=1}^{+\infty} w_i^2 S_1^{-1} P_{W_i}(f) = f = \sum_{i\in I} w_i v_i Q_{V_i} S_1^{-1} P_{W_i}(f).$$

Replacing f with $S_1^{-1}(f)$,

$$\sum_{n=1}^{+\infty} w_i^2 S_1^{-1} P_{W_i}(S_1^{-1}(f)) = S_1^{-1}(f) = \sum_{i\in I} w_i v_i Q_{V_i} S_1^{-1} P_{W_i}(S_1^{-1}(f)).$$

Therefore, $\left\langle f, S_1^{-1}(f)\right\rangle = \left\langle f, \sum_{i\in I} w_i v_i Q_{V_i} S_1^{-1} P_{W_i}(S_1^{-1}(f))\right\rangle = \left\langle f, \sum_{n=1}^{+\infty} w_i^2 S_1^{-1} P_{W_i}(S_1^{-1}(f))\right\rangle$, then $\sum_{i\in I}\left\langle v_i S_1^{-1} Q_{V_i}(f), w_i P_{W_i} S_1^{-1}(f)\right\rangle = \sum_{i\in I}\left\langle w_i P_{W_i} S_1^{-1}(f), w_i P_{W_i} S_1^{-1}(f)\right\rangle$, i.e., $\sum_{i\in I}\left\langle (v_i S_1^{-1} Q_{V_i}(f) - w_i P_{W_i} S_1^{-1})(f), w_i P_{W_i} S_1^{-1}(f)\right\rangle = 0.$

Moreover, since

$$\begin{aligned}&\sum_{i\in I}\left\langle w_i P_{W_i} S_1^{-1}(f), (v_i S_1^{-1} Q_{V_i}(f) - w_i P_{W_i} S_1^{-1})(f)\right\rangle\\ &= \overline{\sum_{i\in I}\left\langle (v_i S_1^{-1} Q_{V_i}(f) - w_i P_{W_i} S_1^{-1})(f), w_i P_{W_i} S_1^{-1}(f)\right\rangle} = \overline{0} = 0,\end{aligned}$$

and

$$\begin{aligned}&\sum_{i\in I}\left\langle (v_i S_1^{-1} Q_{V_i}(f) - w_i P_{W_i} S_1^{-1})(f), (v_i S_1^{-1} Q_{V_i}(f) - w_i P_{W_i} S_1^{-1})(f)\right\rangle\\ &= \sum_{i\in I}\left\| (v_i S_1^{-1} Q_{V_i}(f) - w_i P_{W_i} S_1^{-1})(f)\right\|^2 \geq 0,\end{aligned}$$

we obtain

$$\begin{aligned}&\sum_{i\in I}\left\langle v_i S_1^{-1} Q_{V_i}(f), v_i S_1^{-1} Q_{V_i}(f)\right\rangle\\ &= \sum_{i\in I}\left\langle (v_i S_1^{-1} Q_{V_i}(f) - w_i P_{W_i} S_1^{-1})(f) + w_i P_{W_i} S_1^{-1}(f), (v_i S_1^{-1} Q_{V_i}(f) - w_i P_{W_i} S_1^{-1})(f) + w_i P_{W_i} S_1^{-1}(f)\right\rangle\\ &= \sum_{i\in I}\left\langle (v_i S_1^{-1} Q_{V_i}(f) - w_i P_{W_i} S_1^{-1})(f), (v_i S_1^{-1} Q_{V_i}(f) - w_i P_{W_i} S_1^{-1})(f)\right\rangle\\ &+ \sum_{i\in I}\left\langle ((v_i S_1^{-1} Q_{V_i}(f) - w_i P_{W_i} S_1^{-1})(f), w_i P_{W_i} S_1^{-1}(f)\right\rangle\\ &+ \sum_{i\in I}\left\langle w_i P_{W_i} S_1^{-1}(f), (v_i S_1^{-1} Q_{V_i}(f) - w_i P_{W_i} S_1^{-1})(f)\right\rangle + \sum_{i\in I}\left\langle w_i P_{W_i} S_1^{-1}(f), w_i P_{W_i} S_1^{-1}(f)\right\rangle\\ &= \sum_{i\in I}\left\langle (v_i S_1^{-1} Q_{V_i}(f) - w_i P_{W_i} S_1^{-1})(f), (v_i S_1^{-1} Q_{V_i}(f) - w_i P_{W_i} S_1^{-1})(f)\right\rangle + \sum_{i\in I}\left\langle w_i P_{W_i} S_1^{-1}(f), w_i P_{W_i} S_1^{-1}(f)\right\rangle\\ &\geq \sum_{i\in I}\left\langle w_i P_{W_i} S_1^{-1}(f), w_i P_{W_i} S_1^{-1}(f)\right\rangle.\end{aligned}$$

Thus,

$$\sum_{i\in I}\left\langle w_iP_{W_i}S_1^{-1}(f), w_iP_{W_i}S_1^{-1}(f)\right\rangle \le \sum_{i\in I}\left\langle v_iS_1^{-1}Q_{V_i}(f), v_iS_1^{-1}Q_{V_i}(f)\right\rangle$$

$$\le \left\|S_1^{-1}\right\|^2 \left(\sum_{i\in I}\left\langle v_iQ_{V_i}(f), v_iQ_{V_i}(f)\right\rangle\right) = \left\|S_1^{-1}\right\|^2 \left\langle \sum_{i\in I} v_i^2 Q_{V_i}(f), f\right\rangle.$$

Furthermore, we have $\sum_{i\in I}(w_iP_{W_i}S_1^{-1})^*(w_iP_{W_i}S_1^{-1}) \le \left\|S_1^{-1}\right\|^2 \sum_{i\in I} v_i^2 Q_{V_i} = \left\|S_1^{-1}\right\|^2 S_2$, and $\sum_{i\in I}(w_iP_{W_i}S_1^{-1})^*(w_iP_{W_i}S_1^{-1}) = S_1^{-1}(\sum_{i\in I} w_i^2 P_{W_i})S_1^{-1} = S_1^{-1}S_1S_1^{-1} = S_1^{-1}$, So $S_1^{-1} \le \left\|S_1^{-1}\right\|^2 S_2$.

Since S_1^{-1} is the frame operator of the canonical dual fusion frame of $W = \{(W_i, w_iP_{W_i})\}_{i\in I}$, Theorem 7 reveals the relationship between the canonical dual fusion frame and the alternate dual fusion frame, especially the relationship between their respective fusion frame operators.

3. Conclusions

In this paper, we discovery the implicit fusion frames and obtain an example by combining convergent positive series with the standard orthonormal basis Firstly, then we extend the index set of fusion frames to infinite set through this example. Secondly, we give an example to explore how traditional dual frames recover original signals in cases of data loss, and study the relationship between the frame operator of the canonical dual fusion frame of a fusion frame and its own frame operator, the necessary and sufficient conditions of mutually alternate dual fusion frames, the stability of alternate dual fusion frames. Finally, the relationship between canonical dual and alternate dual fusion frames is studied.

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