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Article

Spectral Synthesis on Direct Products

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Abstract: In a former paper we introduced the concept of localization of ideals in the Fourier algebra of a locally compact Abelian group. It turns out that localizability of a closed ideal in the Fourier algebra is equivalent to the synthesizability of the annihilator of that closed ideal which corresponds to this ideal in the measure algebra. This equivalence provides an effective tool to prove synthesizability of varieties on locally compact Abelian groups. In another paper we used this method to show that when investigating synthesizability of a variety, roughly speaking, compact elements of the group can be neglected. Here we show, using localization, that the extension of a synthesizable locally compact Abelian group G to the direct product $G \times \mathbb{Z}$, where $\mathbb{Z}$ is the integers, is synthesizable as well. These results are used to provide a complete characterization of synthesizable locally compact Abelian groups in [9].

Keywords: spectral analysis; spectral synthesis; locally compact Abelian groups

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1. Introduction

Let G be a locally compact Abelian group. Spectral synthesis deals with uniformly closed translation invariant linear spaces of continuous complex valued functions on G. Such a space is called a *variety*. We say that a variety is *synthesizable*, if its finite dimensional subvarieties span a dense subspace in the variety. If every subvariety of a variety is synthesizable, then we say that *spectral synthesis* holds for the variety. If every variety on a topological Abelian group is synthesizable, then we say that *spectral synthesis holds* on the group. On commutative topological groups finite dimensional varieties of continuous functions are completely characterized: they consist of exponential polynomials. *Exponential polynomials* on a topological Abelian group are defined as the elements of the complex algebra of continuous complex valued functions generated by all continuous homomorphisms into the multiplicative group of nonzero complex numbers (*exponentials*), and all continuous homomorphisms into the additive group of all complex numbers (*additive functions*). An *exponential monomial* is a function of the form

$$x \mapsto P\big(a_1(x), a_2(x), \ldots, a_n(x)\big)m(x),$$

where P is a complex polynomial in n variables, the a_i's are additive functions, and m is an exponential. Every exponential polynomial is a linear combination of exponential monomials. For more about spectral analysis and synthesis on groups see [5,6].

In [4], the authors characterized those discrete Abelian groups having spectral synthesis: spectral synthesis holds on the discrete Abelian group if and only if it has finite torsion-free rank. In particular, from this result it follows, that if spectral synthesis holds on G and H, then it holds on $G \times H$. Unfortunately, such a result does not hold in the non-discrete case. Namely, by the fundamental result of L. Schwartz [1], spectral synthesis holds on $\mathbb{R}$, but D. I. Gurevich showed in [3] that spectral synthesis fails to hold on $\mathbb{R} \times \mathbb{R}$. In this paper we enlighten this mysterious situation by proving that if spectral synthesis holds on the locally compact Abelian group G, then it holds on $G \times \mathbb{Z}$ as well, however, by [3], it does not hold with $\mathbb{R}$ instead of $\mathbb{Z}$. Using this result we characterize those locally compact Abelian groups having spectral synthesis. The idea is that starting with G, which has spectral synthesis, we can extend it by either $\mathbb{Z}$, or, due to our result in [7], by a compact Abelian group so that the resulting group has spectral synthesis as well. Starting with $\mathbb{R}$, and applying this process, by the

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basic structure theory of locally compact Abelian groups we can reach any locally compact Abelian group with spectral synthesis, in finitely many steps. Our main tool is the concept of localizability of ideals in the Fourier algebra (see [8]).

2. Localization

In our former paper [8] we introduced the concept of localization of ideals in the Fourier algebra of a locally compact Abelian group. We recall this concept here.

Let G be a locally compact Abelian group and let $\mathcal{A}(G)$ denote its Fourier algebra, that is, the algebra of all Fourier transforms of compactly supported complex Borel measures on G. This algebra is topologically isomorphic to the measure algebra $\mathcal{M}_c(G)$. For the sake of simplicity, if the annihilator $\operatorname{Ann} I$ of the ideal I in $\mathcal{M}_c(G)$ is synthesizable, then we say that the corresponding ideal $\widehat{I}$ in $\mathcal{A}(G)$ is synthesizable. For each derivation D on $\mathcal{A}(G)$, we introduce (see [8]) the set $\widehat{I}_{D,m}$ as the set of all functions $\widehat{\mu}$ in $\mathcal{A}(G)$ for which

$$D\widehat{\mu}(m) = \int \Delta_{x,y_1,y_2,\dots,y_k} * f_{D,m}(0)\check{m}(x)\,d\mu(x) = \widehat{\mu}(m) = 0$$

holds for each $k = 1, 2, \dots$ and $y_1, y_2, \dots, y_k$ in G. Then $\widehat{I}_{D,m}$ is a closed ideal in $\mathcal{A}(G)$. For a family $\mathcal{D}$ of derivations we write

$$\widehat{I}_{\mathcal{D},m} = \bigcap_{D\in\mathcal{D}} \widehat{I}_{D,m}.$$

Clearly, $\widehat{I}_{\mathcal{D},m}$ is a closed ideal as well. In other words, $\widehat{I}_{\mathcal{D},m}$ is the ideal of those functions in $\mathcal{A}(D)$ which are annihilated at m by all derivations in the family of derivations $\mathcal{D}$.

The dual concept is the following: given an ideal $\widehat{I}$ in $\mathcal{A}(G)$ and an exponential m, the set of all derivations on $\mathcal{A}(G)$ which annihilate $\widehat{I}$ at m is denoted by $\mathcal{D}_{\widehat{I},m}$. The subset of $\mathcal{D}_{\widehat{I},m}$ consisting of all polynomial derivations is denoted by $\mathcal{P}_{\widehat{I},m}$. We have the basic inclusion

$$\widehat{I} \subseteq \bigcap_m \widehat{I}_{\mathcal{D}_{\widehat{I},m},m} \subseteq \bigcap_m \widehat{I}_{\mathcal{P}_{\widehat{I},m},m}. \tag{1}$$

We note that if m is not a root of the ideal $\widehat{I}$, then $\mathcal{D}_{\widehat{I},m} = \mathcal{P}_{\widehat{I},m} = \{0\}$, consequently $\widehat{I}_{\mathcal{D}_{\widehat{I},m},m} = \widehat{I}_{\mathcal{P}_{\widehat{I},m},m} = \mathcal{A}(G)$, hence those terms have no affect on the intersection.

The ideal $\widehat{I}$ is called *localizable*, if we have equalities in (1). The main result in [8] is that $\widehat{I}$ is synthesizable if and only if it is localizable. We shall use this result in the subsequent paragraphs.

3. The Fourier Algebra of $G \times \mathbb{Z}$

It is known that every exponential $e : \mathbb{Z} \to \mathbb{C}$ has the form

$$e(k) = \lambda^k$$

for k in $\mathbb{Z}$,where λ is a nonzero complex number, which is uniquely determined by e. For this exponential we use the notation e_λ. It follows that for every commutative topological group G, the exponentials on $G \times \mathbb{Z}$ have the form $m \otimes e_\lambda : (g,k) \mapsto m(g)e_\lambda(k)$, where m is an exponential on G, and λ is a nonzero complex number. Hence the Fourier transforms in $\mathcal{A}(G \times \mathbb{Z})$ can be thought as two variable functions defined on the pairs (m, λ), where m is an exponential on G, and λ is a nonzero complex number.

Let G be a locally compact Abelian group. For each measure μ in $\mathcal{M}_c(G \times \mathbb{Z})$ and for every k in $\mathbb{Z}$ we let:

$$S_k(\mu) = \{g : g \in G \text{ and } (g,k) \in \operatorname{supp}\mu\}.$$

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This is the k-projection of the support of μ onto G. As μ is compactly supported, there are only finitely many k's in $\mathbb{Z}$ such that $S_k(\mu)$ is nonempty. We have

$$\operatorname{supp}\mu = \bigcup_{k\in\mathbb{Z}} (S_k(\mu) \times \{k\}),$$

and

$$S_k(\mu) \times \{k\} = (G \times \{k\}) \cap \operatorname{supp}\mu.$$

It follows that the sets $S_k(\mu) \times \{k\}$ are pairwise disjoint compact sets in $G \times \mathbb{Z}$, and they are nonempty for finitely many k's only. The restriction of μ to $S_k(\mu) \times \{k\}$ is denoted by μ_k. Then, by definition

$$\langle \mu_k, f \rangle = \int f \cdot \chi_k \, d\mu$$

for each f in $\mathcal{C}(G \times \mathbb{Z})$, where χ_k denotes the characteristic function of the set $S_k(\mu) \times \{k\}$. In other words,

$$\int f \, d\mu_k = \int f(g,k) \, d\mu(g,l)$$

holds for each k in $\mathbb{Z}$ and for every f in $\mathcal{C}(G \times \mathbb{Z})$. Clearly, $\mu = \sum_{k\in\mathbb{Z}} \mu_k$, and this sum is finite.

Lemma 1. *Let μ be in $\mathcal{M}_c(G \times \mathbb{Z})$. Then, for each k in $\mathbb{Z}$, we have*

$$\mu_k = \mu_0 * \delta_{(0,k)}.$$

Here $\delta_{(0,k)}$ denotes the Dirac measure at the point $(0,k)$ in $G \times \mathbb{Z}$.

Proof. For each f in $\mathcal{C}(G \times \mathbb{Z})$, we have:

$$\langle \mu_0 * \delta_{(0,k)}, f \rangle = \int\int f(g+h, l+n) \, d\mu_0(g,l) \, d\delta_{(0,k)}(h,n) =$$

$$\int f(g, l+k) \, d\mu_0(g,l) = \int f(g,k) \, d\mu(g,l) = \langle \mu_k, f \rangle.$$

□

Given a measure μ in $\mathcal{M}_c(G \times \mathbb{Z})$ we define μ_G in $\mathcal{M}_c(G)$ by

$$\langle \mu_G, \varphi \rangle = \int \varphi(g) \, d\mu(g,l)$$

whenever φ is in $\mathcal{C}(G)$. Clearly, every φ in $\mathcal{C}(G)$ can be considered as a function in $\mathcal{C}(G \times \mathbb{Z})$, hence this definition makes sense, further we have

$$\langle \mu_G, \varphi \rangle = \int \varphi(g) \, d\mu_0(g,l).$$

Lemma 2. *If I is a closed ideal in $\mathcal{M}_c(G \times \mathbb{Z})$, then the set I_G of all measures μ_G with μ in I, is a closed ideal in $\mathcal{M}_c(G)$.*

Proof. Clearly $\mu_G + \nu_G = (\mu + \nu)_G$ and $\lambda \cdot \mu_G = (\lambda \cdot \mu)_G$. Let μ_G be in I and ξ in $\mathcal{M}_c(G)$. Then we have for each φ in $\mathcal{C}(G)$:

$$\langle \xi * \mu_G, \varphi \rangle = \int\int \varphi(g+h) \, d\xi(g) \, d\mu_G(h) = \int\int \varphi(g+h) \, d\xi(g) \, d\mu(h,l).$$

On the other hand, we extend ξ from $\mathcal{M}_c(G)$ to $\mathcal{M}_c(G \times \mathbb{Z})$ by the definition

$$\langle \tilde{\xi}, f \rangle = \int f(g,0) \, d\xi(g)$$

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whenever f is in $\mathcal{C}(G \times \mathbb{Z})$. Then

$$\langle \tilde{\xi}_G, \varphi \rangle = \int \varphi(g)\, d\tilde{\xi}_0(g, l) = \int \varphi(g)\, d\xi(g) = \langle \xi, \varphi \rangle,$$

that is $\tilde{\xi}_G = \xi$. Finally, a simple calculation shows that

$$\langle \xi * \mu_G, \varphi \rangle = \langle (\tilde{\xi} * \mu)_G, \varphi \rangle,$$

hence $\xi * \mu_G = (\tilde{\xi} * \mu)_G$ is in I_G, as $\tilde{\xi} * \mu$ is in I.

Now we show that the ideal I_G is closed. Assume that (μ_α) is a generalized sequence in I such that the generalized sequence $(\mu_{\alpha,G})$ converges to ξ in $\mathcal{M}_c(G)$. This means that

$$\lim_\alpha \int \varphi(g)\, d\mu_{\alpha,G}(g) = \int \varphi(g)\, d\xi(g)$$

holds for each φ in $\mathcal{C}(G)$. In particular, for each exponential m on G we have

$$\lim_\alpha \int \check{m}(g)\, d\mu_{\alpha,0}(g, l) = \lim_\alpha \int \check{m}(g)\, d\mu_{\alpha,G}(g) = \int \check{m}(g)\, d\xi(g) = \int \check{m}(g)\, d\tilde{\xi}_0(g, l).$$

In other words, $\lim_\alpha \widehat{\mu}_{\alpha,0} = \widehat{\tilde{\xi}}_0$ holds, which implies $\lim_\alpha \mu_{\alpha,0} = \tilde{\xi}_0$, consequently

$$\tilde{\xi}_k = \tilde{\xi}_0 * \delta_{(0,k)} = \lim_\alpha \mu_{\alpha,0} * \delta_{(0,k)} = \lim_\alpha \mu_{\alpha,k},$$

hence we infer

$$\tilde{\xi} = \sum_k \tilde{\xi}_k = \sum_k \lim_\alpha \mu_{\alpha,k} = \lim_\alpha \sum_k \mu_{\alpha,k} = \lim_\alpha \mu_\alpha,$$

as each sum is finite. Since I is closed, hence $\tilde{\xi}$ is in I, which proves that $\tilde{\xi} = \tilde{\xi}_G$ is in I_G, that is, I_G is closed. □

Now we can derive the following theorem.

Theorem 1. *Let G be a locally compact Abelian group. If spectral synthesis holds on G then it holds on $G \times \mathbb{Z}$.*

Proof. If spectral synthesis holds on G, then, by the results in [8], every closed ideal in the Fourier algebra of G is localizable, and we need to show the same for all closed ideals of the Fourier algebra of $G \times \mathbb{Z}$.

We consider the closed ideal $\widehat{I}$ in the Fourier algebra $\mathcal{A}(G \times \mathbb{Z})$, and we assume that $\widehat{I}$ is non-localizable, that is, there is a measure ν in $\mathcal{M}_c(G \times \mathbb{Z})$ such that $\widehat{\nu}$ is annihilated by $\mathcal{P}_{\widehat{I},m,\lambda}$ for each m and λ, but $\widehat{\nu}$ is not in $\widehat{I}$. We show that $\widehat{\nu}_G$ is in $\widehat{I}_G$; then it will follow that $\widehat{\nu}$ is in $\widehat{I}$, a contradiction.

Suppose that a polynomial derivation d annihilates $\widehat{I}_G$ at m. Then we have

$$d\widehat{\mu}_G(m) = \int p_{d,m}(g)\check{m}(g)\, d\mu_G(g) = \int p_{d,m}(g)\check{m}(g)\, d\mu(g, l) = 0$$

for each $\widehat{\mu}$ in $\widehat{I}_G$ and for every exponential m on G, where $p_{d,m} : G \to \mathbb{C}$ is the generating polynomial of d at m. Then we define the polynomial derivation D on the Fourier algebra $\mathcal{A}(G \times \mathbb{Z})$ by

$$D\widehat{\mu}(m, \lambda) = \int p_{d,m}(g)\check{m}(g)\lambda^{-l}\, d\mu(g, l).$$

If $\widehat{\mu}$ is in $\widehat{I}$, then we have

$$D\widehat{\mu}_k(m, \lambda) = \int p_{d,m}(g)\check{m}(g)\lambda^{-l}\, d\mu_k(g, l) = \lambda^{-k} \cdot \int p_{d,m}(g)\check{m}(g)\, d\mu(g, l) = 0$$

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for each k in $\mathbb{Z}$. As $\widehat{\mu} = \sum_{k\in\mathbb{Z}} \widehat{\mu}_k$, it follows that $D\widehat{\mu}(m,\lambda) = 0$ for each $\widehat{\mu}$ in $\widehat{I}$. In other words, D is in $\mathcal{P}_{\widehat{I},m,\lambda}$ for each exponential m and nonzero complex number λ. In particular, $\widehat{\nu}$ is annihilated by D:

$$D\widehat{\nu}(m,\lambda) = \int p_{d,m}\check{m}(g)\lambda^{-l}\,d\nu(g,l) = 0.$$

It follows

$$d\widehat{\nu}_G(m) = D\widehat{\nu}_0(m,\lambda) = \int p_{d,m}(g)\check{m}(g)\,d\nu(g,l) = 0.$$

As d is an arbitrary polynomial derivation which annihilates $\widehat{I}_G$ at m, we have that $\widehat{\nu}_G$ is annihilated by $\mathcal{P}_{\widehat{I}_G,m}$ for each m. As spectral synthesis holds on G, the ideal $\widehat{I}_G$ is localizable, consequently $\widehat{\nu}_G$ is in $\widehat{I}_G$, which implies that $\widehat{\nu}$ is in $\widehat{I}$, and our theorem is proved. □

Using this theorem and the structure theory, in [9] we proved the following results, which completely describe those locally compact Abelian groups on which spectral synthesis holds.

Corollary 1. *Let G be a compactly generated locally compact Abelian group. Then spectral synthesis holds on G if and only if G is topologically isomorphic to $\mathbb{R}^a \times \mathbb{Z}^b \times F$, where $a \leq 1$ and b are nonnegative integers, and F is an arbitrary compact Abelian group.*

Corollary 2. *Let G be a locally compact Abelian group. Let B denote the closed subgroup of all compact elements in G. Then spectral synthesis holds on G if and only if G/B is topologically isomorphic to $\mathbb{R}^n \times F$, where $n \leq 1$ is a nonnegative integer, and F is a discrete torsion free Abelian group of finite rank.*

4. Statements and Declarations

Data sharing not applicable to this article as no datasets were generated or analysed during the current study. There are no financial or non-financial interests that are directly or indirectly related to the work submitted for publication.

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