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Computing Nash equilibria for multiplayer symmetric games based on tensor form

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Abstract: In an m-person symmetric game, all players are identical and indistinguishable. In this paper, we find that the payoff tensor of the player k in an m-person symmetric game is k-mode symmetric, and the payoff tensors of two different individuals are the transpose of each other. Furthermore, we reformulate the m-person symmetric game as a tensor complementary problem and demonstrate that locating a symmetric Nash equilibrium is equivalent to finding a solution to the resulting tensor complementary problem. Finally, we use the hyperplane projection algorithm to solve the resulting tensor complementary problem, and we present some numerical results to find the symmetric Nash equilibrium.

Keywords: k-mode symmetric tensor; tensor complementary problem; m-person game; symmetric Nash equilibrium

1. Introduction

A game is symmetric if all players have the same strategy set and the payoff of a given strategy is determined merely by the strategy itself and has nothing to do with who plays it. Symmetric games have been widely studied since the dawn of game theory, and together with zero-sum games, they form one of the most classical subclasses of games. For example, Nash [1,2] proposed a very important concept of equilibrium, called Nash equilibrium, which is a strategy profile in which each player's strategy is an optimal response to the strategies of the other players. This concept has a nice property that there is at least one Nash equilibrium for every finite game, and every finite symmetric game leads to a symmetric Nash equilibrium (an equilibrium in which all players use the same strategy) [1–4]. Some researchers were devoted to symmetric games; for additional statements, one can refer to [5–15].

The symmetric game, in particular, is important in the study of economic and biological models [16–19]. As we know, many interactions have been modeled in terms of one-shot and some of them have been reported as symmetric two-player cooperative dilemmas, including the Prisoner's Dilemma [20,21], Hawk Dove Game [21], Snowdrift Game [22], and Stag-Hunt Game [23]. If each player has n actions to choose in a 2-player symmetric game, the game can be represented by two n-by-n matrices [24,25]. It is known that the 2-player symmetric game can be formulated as a linear complementary problem. The Lemke-Howson algorithm [24] is a well known method which was designed to solve the above mentioned linear complementary problem.

However, in many real-life situations, decisions are made by groups of people that include more than two individuals [19]. This type of collective action problem is better to be described as an m-person symmetric game [26–28]. We prefer to find symmetric equilibria for an m-person symmetric game because asymmetric behavior appears relatively unintuitive [6] and difficult to explain in a one-shot interaction [29]. Nash proved that there is a symmetric equilibrium in every m-person symmetric game [2], but the Nash

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equilibrium equations for m-person games are non-linear which are difficult to be solved analytically in general.

To better address the m-person game, Huang and Qi [30] extended the classic form of a game to a tensor form by using a tensor to represent the utility function of each player. They reformulated the m-person game as a tensor complementary problem, and showed that finding a Nash equilibrium of the m-person is equivalent to finding a solution to the resulting tensor complementary problem. Abdou et al. [31] presented an efficient method to solve the pure Nash equilibrium by using tensor operations. However, they did not take the Nash equilibrium of a multiplayer symmetric game into consideration.

In this paper, we consider the tensor-based m-person symmetric games. We demonstrate that the payoff tensors of the k-th player in an m-person symmetric game are k-mode symmetric and that the payoff tensors between different players are the transpose of each other. We also reformulate the m-person symmetric games as a tensor complementary problem. It is worth noting that the order of tensors in the tensor complementary problem is less than that resulting in [30]. In addition, we show that finding a symmetric Nash equilibrium of the m-person symmetric game is equivalent to finding a solution to the resulting tensor complementary problem. Finally, we apply a hyperplane projection algorithm to solve the resulting tensor complementary problem and provide some numerical results for solving the symmetric Nash equilibrium.

The rest of this paper is organized as follows: In Section 2, we present some definitions and notations that will be frequently used in the following parts. In Section 3, we provide a brief description of the m-person symmetric game. Some experimental results that can be used to explain our proposed theory's reliability are shown in Secciton 4. Finally, we provide a brief summary in Section 5.

2. Preliminaries

In this section, we introduce some definitions and notations, which will be used in the sequel. Throughout this paper, we use small letters (e.g.,a), small bold letters (e.g.,$\mathbf{a}$), capital letters (e.g.,A), and calligraphic letters (e.g.,$\mathcal{A}$) to denote scalars, vectors, matrices, and tensors, respectively. For a positive integer n, let $[n] = \{1, 2, \cdots, n\}$.

A real m-th order $n_1 \times n_2 \times \cdots \times n_m$-dimensional tensor is a multidimensional array, and its elementwise form can be denoted as

$$\mathcal{A} = (a_{i_1 i_2 \cdots i_m}), \quad i_j \in [n_j], \quad j = 1, 2, \cdots, m.$$

When $m = 2$, $\mathcal{A}$ is an n_1-by-n_2 matrix. If $n_1 = n_2 = \cdots = n_m = n$, $\mathcal{A}$ is called a real m-th order n-dimensional tensor. We denote the set of all real m-th order $n_1 \times n_2 \times \cdots \times n_m$-dimensional tensors by $\mathbb{R}^{n_1 \times n_2 \times \cdots \times n_m}$ and denote the set of all real m-th order n-dimensional tensors by $\mathbb{R}^{[m,n]}$. When $m = 1$, $\mathbb{R}^{[1,n]}$ is simplified as $\mathbb{R}^n$, which is the set of all n-dimensional real vectors. We denote the set of all nonnegative n-dimensional real vectors by $\mathbb{R}^n_+$. Let $\mathbf{1}_n$, $\mathbf{e}_i$, and $\mathbf{0}$ denote the n-dimensional vector of all ones, the i-th column of the n-dimensional identity matrix, and the zero vector, respectively. The order $\mathbf{x} \geqslant \mathbf{0}$ means that each component of $\mathbf{x}$ is nonnegative.

The definition of the k-mode product of a tensor with a vector is recalled as follows.

Definition 1. [32] *The k-mode (vector) product of a tensor $\mathcal{A} = (a_{i_1 \cdots i_k \cdots i_m}) \in \mathbb{R}^{n_1 \times \cdots \times n_k \times \cdots \times n_m}$ multiplied by a vector $\mathbf{v} = (v_i) \in \mathbb{R}^{n_k}$ is denoted by $\mathcal{A} \bar{\times}_k \mathbf{v}$, which leads to a real $(m-1)$-th order $n_1 \times \cdots \times n_{k-1} \times n_{k+1} \times \cdots \times n_m$-dimensional tensor with*

$$(\mathcal{A} \bar{\times}_k \mathbf{v})_{i_1 \cdots i_{k-1} i_{k+1} \cdots i_m} = \sum_{i_k=1}^{n_k} a_{i_1 \cdots i_k \cdots i_m} v_{i_k},$$

for any $i_j \in [n_j]$ with $j \in [m] \setminus \{k\}$.

Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{R}^{n_1 \times n_2 \times \cdots \times n_m}$ and $\mathbf{u}^{(k)} = (u^{(k)}_{i_j}) \in \mathbb{R}^{n_k}$ with $k \in [m]$. According to Definition 1, it follows that

$$\mathcal{A} \bar{\times}_m \mathbf{u}^{(m)} \bar{\times}_{m-1} \cdots \bar{\times}_2 \mathbf{u}^{(2)} \bar{\times}_1 \mathbf{u}^{(1)} = \sum_{i_1=1}^{n_1} \sum_{i_2=1}^{n_2} \cdots \sum_{i_m=1}^{n_m} a_{i_1 i_2 \cdots i_m} u^{(m)}_{i_m} \cdots u^{(2)}_{i_2} u^{(1)}_{i_1},$$

where $\mathcal{A} \bar{\times}_m \mathbf{u}^{(m)} \bar{\times}_{m-1} \cdots \bar{\times}_2 \mathbf{u}^{(2)}$ is an n_1-dimensional vector with entries

$$(\mathcal{A} \bar{\times}_m \mathbf{u}^{(m)} \bar{\times}_{m-1} \cdots \bar{\times}_2 \mathbf{u}^{(2)})_i = \sum_{i_2=1}^{n_2} \cdots \sum_{i_m=1}^{n_m} a_{i i_2 \cdots i_m} u^{(m)}_{i_m} \cdots u^{(2)}_{i_2}.$$

For simplicity of notion, we use $\mathcal{A}\mathbf{u}^{(m)} \cdots \mathbf{u}^{(2)}\mathbf{u}^{(1)}$ to denote the scale

$$\mathcal{A} \bar{\times}_m \mathbf{u}^{(m)} \bar{\times}_{m-1} \cdots \bar{\times}_2 \mathbf{u}^{(2)} \bar{\times}_1 \mathbf{u}^{(1)},$$

and use $\mathcal{A}\mathbf{u}^{(m)} \cdots \mathbf{u}^{(2)}$ to denote the real n_1-dimensional vector

$$\mathcal{A} \bar{\times}_m \mathbf{u}^{(m)} \bar{\times}_{m-1} \cdots \bar{\times}_2 \mathbf{u}^{(2)}.$$

Particularly, when $n_1 = n_2 = \cdots = n_m = n$, and $\mathbf{u}, \mathbf{u}^* \in \mathbb{R}^n$, we take the symbols $\mathcal{A}\mathbf{u}^m$, $\mathcal{A}\mathbf{u}^{*m-1}\mathbf{u}$, and $\mathcal{A}\mathbf{u}^{*m-k}\mathbf{u}\mathbf{u}^{*k-1}$ to denote the scales

$$\mathcal{A} \bar{\times}_m \mathbf{u} \bar{\times}_{m-1} \cdots \bar{\times}_2 \mathbf{u} \bar{\times}_1 \mathbf{u},$$

$$\mathcal{A} \bar{\times}_m \mathbf{u}^* \bar{\times}_{m-1} \cdots \bar{\times}_2 \mathbf{u}^* \bar{\times}_1 \mathbf{u},$$

and

$$\mathcal{A} \bar{\times}_m \mathbf{u}^* \bar{\times}_{m-1} \cdots \bar{\times}_{k+1} \mathbf{u}^* \bar{\times}_k \mathbf{u} \bar{\times}_{k-1} \mathbf{u}^* \cdots \bar{\times}_2 \mathbf{u}^* \bar{\times}_1 \mathbf{u}^*,$$

respectively. The symbol $\mathcal{A}\mathbf{u}^{m-1}$ is short for the n-dimensional vector $\mathcal{A} \bar{\times}_m \mathbf{u} \bar{\times}_{m-1} \cdots \bar{\times}_2 \mathbf{u}$.

The definitions of symmetric and semi-symmetric tensors are given as follows.

Definition 2. [33,34] *A tensor* $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{R}^{[m,n]}$ *is called symmetric, if*

$$a_{i_1 i_2 \cdots i_m} = a_{i_{\sigma(1)} i_{\sigma(2)} \cdots i_{\sigma(m)}}, \quad \forall \sigma \in \Pi_m,$$

where Π_m *is the set of permutations of length* m.

Definition 3. [32] *A tensor* $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{R}^{[m,n]}$ *is called semi-symmetric, if for any* $i_1 \in [n]$,

$$a_{i_1 i_2 \cdots i_m} = a_{i_1 i_{\sigma(2)} \cdots i_{\sigma(m)}}, \quad \forall \sigma \in \Pi_{m-1}.$$

We provide a more general definition about k-mode symmetric of a tensor.

Definition 4. *A tensor* $\mathcal{A} \in \mathbb{R}^{[m,n]}$ *is called k-mode symmetric, if for any* $i_k \in [n]$,

$$a_{i_1 \cdots i_{k-1} i_k i_{k+1} \cdots i_m} = a_{i_{\sigma(1)} \cdots i_{\sigma(k-1)} i_k i_{\sigma(k+1)} \cdots i_{\sigma(m)}}, \quad \forall \sigma \in \Pi_{m-1}.$$

Remark 1. *When* $k = 1$, $\mathcal{A}$ *is* 1*-mode symmetric if and only if* $\mathcal{A}$ *is semi-symmetric. An m-th order n-dimensional tensor* $\mathcal{A}$ *has* n^m *independent entries. If* $\mathcal{A}$ *is k-mode symmetric, then* $\mathcal{A}$ *has* $n\binom{m+n-2}{m-1}$ *independent entries. If* $\mathcal{A}$ *is symmetric, then* $\mathcal{A}$ *has only* $\binom{m+n-1}{m}$ *independent entries.*

By Definitions 1, 2, and 4, we have the following lemma.

Lemma 1. *Let* $\mathcal{A} \in \mathbb{R}^{[m,n]}$. *Then* $\mathcal{A}$ *is k-mode symmetric if and only if* $\mathcal{A} \bar{\times}_k \mathbf{e}_i \in \mathbb{R}^{[m-1,n]}$ *is a symmetric tensor for all* $i \in [n]$.

Proof. From Definition 1, we get

$$(\mathcal{A}\bar{\times}_k\mathbf{e}_i)_{j_1\cdots j_{k-1}j_{k+1}\cdots j_m} = \sum_{j_k=1}^{n} a_{j_1\cdots j_{k-1}j_kj_{k+1}\cdots j_m}(\mathbf{e}_i)_{j_k} = a_{j_1\cdots j_{k-1}ij_{k+1}\cdots j_m},$$

which implies that for all $i \in [n]$,

$$a_{j_1\cdots j_{k-1}ij_{k+1}\cdots j_m} = a_{j_{\sigma(1)}\cdots j_{\sigma(k-1)}ij_{\sigma(k+1)}\cdots j_{\sigma(m)}}, \quad \forall\sigma \in \Pi_{m-1},$$

if and only if

$$(\mathcal{A}\bar{\times}_k\mathbf{e}_i)_{j_1\cdots j_{k-1}j_{k+1}\cdots j_m} = (\mathcal{A}\bar{\times}_k\mathbf{e}_i)_{j_{\sigma(1)}\cdots j_{\sigma(k-1)}j_{\sigma(k+1)}\cdots j_{\sigma(m)}}, \quad \forall\sigma \in \Pi_{m-1}.$$

According to Definitions 2 and 4, it comes with that $\mathcal{A}$ is k-mode symmetric if and only if $\mathcal{A}\bar{\times}_k\mathbf{e}_i \in \mathbb{R}^{[m-1,n]}$ is a symmetric tensor for all $i \in [n]$. □

In [35], the definition of transpose of a tensor was introduced as follows:

Definition 5. [35] *Let $\mathcal{A} = (a_{i_1i_2\cdots i_m}) \in \mathbb{R}^{[m,n]}$ and $\sigma \in \Pi_m$. The σ-transpose of $\mathcal{A}$ is an m-th order n-dimensional tensor, denoted by $\mathcal{A}^{<\sigma>} = (b_{i_1i_2\cdots i_m}) \in \mathbb{R}^{[m,n]}$, with entries*

$$b_{i_1i_2\cdots i_m} = a_{i_{\sigma(1)}i_{\sigma(2)}\cdots i_{\sigma(m)}}.$$

Remark 2. *When $m = 2$, $\mathcal{A}$ is an n-by-n matrix. Let $\sigma \in \Pi_2$ with $\sigma(1) = 2$ and $\sigma(2) = 1$, then $\mathcal{A}^{<\sigma>}$ reduces to its matrix transpose.*

We recall the psdudomonotone mapping and projection operator, which will be used in Section 4.

Definition 6. *A mapping $F : \mathbb{R}^n_+ \to \mathbb{R}^n$ is said to be pseudomonotone on $\mathbb{R}^n_+$ if for every x and y in $\mathbb{R}^n_+$, and it follows that if $(x-y)^\top F(y) \geqslant 0$ is true then $(x-y)^\top F(x) \geqslant 0$ is true.*

Definition 7. *Let $\mathbb{X}$ be a nonempty convex subset of $\mathbb{R}^n$. $P_{\mathbb{X}}$ denotes the projection operator that maps from $\mathbb{R}^{(n)}$ to $\mathbb{X}$ and is defined as*

$$P_{\mathbb{X}}(\mathbf{y}) = \arg\min\{\parallel \mathbf{x}-\mathbf{y} \parallel : \mathbf{x} \in \mathbb{X}\}, \quad \mathbf{y} \in \mathbb{R}^n,$$

where $\|\cdot\|$ is l_2-norm in $\mathbb{R}^n$.

Remark 3. *When $\mathbb{X} = \mathbb{R}^n_+$, $P_{\mathbb{R}^n_+}(\mathbf{y})$ is simplified as $[\mathbf{y}]_+$. The projection operator $[\mathbf{y}]_+$ can be computed as follows*

$$[\mathbf{y}]_+ = \max\{\mathbf{0}, \mathbf{y}\},$$

where the max *operator denotes the componentwise maximum of two vectors.*

3. Description of the m-person symmetric game

In this section, we first provide the definition of a tensor form of an m-person game, adapted from [30] and [31].

Definition 8. [30,31] *An m-person game in a tensor form is a tuple*

$$G = ([m]; \{[n_k]\}_{k=1}^m; \{\mathcal{A}^{(k)}\}_{k=1}^m),$$

where $[m]$ is the set of players, $[n_k]$ is the pure strategy set of player k, $\mathcal{A}^{(k)} = (a^{(k)}_{i_1i_2\cdots i_m}) \in \mathbb{R}^{n_1\times n_2\times\cdots\times n_m}$ is the payoff tensor of player k, that is, for any $i_j \in [n_j]$ with any $j \in [m]$, if the player 1 plays his i_1-th pure strategy, the player 2 plays his i_2-th pure strategy, $\cdots$, and the player

m plays his i_m-th pure strategy, then the payoffs of player 1, player 2, $\cdots$, and player m are $a^{(1)}_{i_1 i_2 \cdots i_m}$, $a^{(2)}_{i_1 i_2 \cdots i_m}$, $\cdots$, and $a^{(m)}_{i_1 i_2 \cdots i_m}$, respectively.

Given an m-person game $G = ([m]; \{[n_k]\}_{k=1}^m; \{\mathcal{A}^{(k)}\}_{k=1}^m)$. A mixed strategy of player k is the probability distribution on pure strategy set $[n_k]$. Let

$$\Omega_k = \{\mathbf{u} \in \mathbb{R}^{n_k} : \mathbf{u} \geqslant \mathbf{0} \quad \text{and} \quad \mathbf{1}_{n_k}^\top \mathbf{u} = 1\}$$

denote the set of the all probability distributions on $[n_k]$. For $\mathbf{u}^{(k)} = (u^{(k)}_{i_j}) \in \Omega_k$, the probability assigned to i_j-th pure strategy of player k is $u^{(k)}_{i_j}$.

Using

$$\Omega := \times_{k \in [m]} \Omega_k = \{(\mathbf{u}^{(1)}, \cdots, \mathbf{u}^{(m)}) : \mathbf{u}^{(k)} \in \Omega_k, k = 1, \cdots, m\}$$

denote the set of strategy profiles. We say that $(\mathbf{u}^{(1)}, \cdots, \mathbf{u}^{(m)}) \in \Omega$ is a mixed strategy combination if $\mathbf{u}^{(k)} \in \Omega_k$ is a mixed strategy of player k for any $k \in [m]$. If a mixed strategy combination $(\mathbf{u}^{(1)}, \cdots, \mathbf{u}^{(m)}) \in \Omega$ is played, then the expected payoff of player k is

$$\mathcal{A}^{(k)} \mathbf{u}^{(m)} \cdots \mathbf{u}^{(2)} \mathbf{u}^{(1)} = \sum_{i_1=1}^{n_1} \cdots \sum_{i_m=1}^{n_m} a^{(k)}_{i_1 i_2 \cdots i_m} u^{(m)}_{i_m} \cdots u^{(2)}_{i_2} u^{(1)}_{i_1}.$$

Definition 9. [2,30] *A mixed strategy combination $(\mathbf{u}^{(1^*)}, \cdots, \mathbf{u}^{(m^*)}) \in \Omega$ is a Nash equilibrium of the m-person game, if for each strategy combination $(\mathbf{u}^{(1)}, \cdots, \mathbf{u}^{(m)}) \in \Omega$ and $k \in [m]$, it holds that*

$$\mathcal{A}^{(k)} \mathbf{u}^{(m^*)} \cdots \mathbf{u}^{(1^*)} \geqslant \mathcal{A}^{(k)} \mathbf{u}^{(m^*)} \cdots \mathbf{u}^{(k+1^*)} \mathbf{u}^{(k)} \mathbf{u}^{(k-1^*)} \cdots \mathbf{u}^{(1^*)}.$$

What we consider here is the m-person symmetric games. In such games, all players play the same role in the game and moreover the payoff of any player depends only upon its strategy and the combination of the strategies of its opponents.

Definition 10. *An m-person game $G = ([m]; \{[n_k]\}_{k=1}^m; \{\mathcal{A}^{(k)}\}_{k=1}^m)$ is symmetric if the players have identical pure strategy set, that is $n_1 = n_2 = \cdots = n_m = n$, and every payoff tensor $\mathcal{A}^{(k)} \in \mathbb{R}^{[m,n]}$ satisfies*

$$a^{(k)}_{i_1 i_2 \cdots i_m} = a^{(\sigma^{-1}(k))}_{i_{\sigma(1)} i_{\sigma(2)} \cdots i_{\sigma(m)}}, \quad \forall \sigma \in \Pi_m,$$

where σ^{-1} is the inverse of the permutation σ.

It is well know that the payoff matrices of the 2-person symmetric game are transpose of each other [20]. Furthermore, we come to a similar conclusion for the m-person symmetric game.

Theorem 1. *Suppose that $G = ([m]; \{[n_k]\}_{k=1}^m; \{\mathcal{A}^{(k)}\}_{k=1}^m)$ be an m-person game. Then G is symmetric if and only if*

(i) $\mathcal{A}^{(k)}$ is k-mode symmetric for all $k \in [m]$;

(ii) if $k \neq j$, then $(\mathcal{A}^{(k)})^{<\sigma>} = \mathcal{A}^{(j)}$, for all $\sigma \in \Pi_m$ satisfies $\sigma(k) = j$.

Proof. Firstly, we show the necessity. By Definition 10, it comes with $n_1 = n_2 = \cdots = n_m = n$. Combing it with Definition 1, we have

$$\begin{aligned}(\mathcal{A}^{(k)} \bar{\times}_k \mathbf{e}_i)_{j_1\cdots j_{k-1}j_{k+1}\cdots j_m} &= \sum_{j_k=1}^{n} a^{(k)}_{j_1\cdots j_{k-1}j_kj_{k+1}\cdots j_m} (\mathbf{e}_i)_{j_k} \\ &= a^{(k)}_{j_1\cdots j_{k-1}ij_{k+1}\cdots j_m}. \end{aligned} \tag{1}$$

Let $p \in \Pi_{m-1}$ be a permutation on set $[m] \setminus \{k\}$ and $\pi \in \Pi_m$ satisfies

$$\pi(i) = \begin{cases} p(i), & \text{if} \quad i \in [m] \setminus \{k\}, \\ k, & \text{if} \quad i = k. \end{cases}$$

By (1) and Definition 10, we have

$$\begin{aligned}(\mathcal{A}^{(k)} \bar{\times}_k \mathbf{e}_i)_{j_{p(1)}\cdots j_{p(k-1)}j_{p(k+1)}\cdots j_{p(m)}} &= a^{(k)}_{j_{p(1)}\cdots j_{p(k-1)}ij_{p(k+1)}\cdots j_{p(m)}} \\ &= a^{(\pi^{-1}(k))}_{j_{\pi(1)}\cdots j_{\pi(k-1)}ij_{\pi(k+1)}\cdots j_{\pi(m)}} \\ &= a^{(k)}_{j_1\cdots j_{k-1}ij_{k+1}\cdots j_m} \\ &= (\mathcal{A}^{(k)} \bar{\times}_k \mathbf{e}_i)_{j_1\cdots j_{k-1}j_{k+1}\cdots j_m}.\end{aligned}$$

Therefore, $\mathcal{A}^{(k)} \bar{\times}_k \mathbf{e}_i$ is a symmetric tensor for all $i \in [n]$. By Lemma 1, $\mathcal{A}^{(k)}$ is k-mode symmetric. Hence, the statement (i) holds.

Since $\sigma \in \Pi_m$ satisfies $\sigma(k) = j$. From Definition 5, we have

$$\begin{aligned}((\mathcal{A}^{(k)})^{<\sigma>})_{l_1\cdots l_j\cdots l_k\cdots l_m} &= a^{(k)}_{l_{\sigma(1)}\cdots l_{\sigma(j)}\cdots l_{\sigma(k)}\cdots l_{\sigma(m)}} \\ &= a^{(\sigma^{-1}(j))}_{l_{\sigma(1)}\cdots l_{\sigma(j)}\cdots l_{\sigma(k)}\cdots l_{\sigma(m)}} \\ &= a^{(j)}_{l_1\cdots l_j\cdots l_k\cdots l_m} \\ &= (\mathcal{A}^{(j)})_{l_1\cdots l_j\cdots l_k\cdots l_m},\end{aligned}$$

which implies $(\mathcal{A}^{(k)})^{<\sigma>} = \mathcal{A}^{(j)}$. Hence, the statement (ii) holds.

Now, we show the sufficiency. It is comes to $n_1 = n_2 = \cdots = n_m$ for $\mathcal{A}^{(k)}$ is k-mode symmetric. For any $\sigma \in \Pi_m$ and fixed index j, we have $\sigma(j) = j$ or $\sigma(k) = j (k \neq j)$. To prove our statement we have to talk about it in the following two cases.

Case 1: $\sigma(j) = j$. Let $p \in \Pi_{m-1}$ be a permutation on set $[m] \setminus \{j\}$ that satisfies $p(i) = \sigma(i)$. Since $\mathcal{A}^{(k)}$ is k-mode symmetric, we have

$$\begin{aligned} a^{(j)}_{i_1\cdots i_{j-1}i_ji_{j+1}\cdots i_m} &= a^{(j)}_{i_{p(1)}\cdots i_{p(j-1)}i_ji_{p(j+1)}\cdots i_{p(m)}} \\ &= a^{(j)}_{i_{\sigma(1)}\cdots i_{\sigma(j-1)}i_ji_{\sigma(j+1)}\cdots i_{\sigma(m)}} \\ &= a^{(\sigma^{-1}(j))}_{i_{\sigma(1)}\cdots i_{\sigma(j-1)}i_{\sigma(j)}i_{\sigma(j+1)}\cdots i_{\sigma(m)}}.\end{aligned}$$

Case 2: $\sigma(k) = j (k \neq j)$. According to $(\mathcal{A}^{(k)})^{<\sigma>} = \mathcal{A}^{(j)}$, we obtain

$$\begin{aligned} a^{(j)}_{i_1\cdots i_{j-1}i_ji_{j+1}\cdots i_m} &= a^{(k)}_{i_{\sigma(1)}\cdots i_{\sigma(j-1)}i_{\sigma(j)}i_{\sigma(j+1)}\cdots i_{\sigma(m)}} \\ &= a^{(\sigma^{-1}(j))}_{i_{\sigma(1)}\cdots i_{\sigma(j-1)}i_{\sigma(j)}i_{\sigma(j+1)}\cdots i_{\sigma(m)}}.\end{aligned}$$

By Definition 10, we have $G = ([m]; \{[n_k]\}_{k=1}^m; \{\mathcal{A}^{(k)}\}_{k=1}^m)$ is an m-person symmetric game. $\square$

Remark 4. *(I) Consider the m-player Volunteer's Dilemma [36,37], m individuals observe an approaching predator and need to decide whether or not to sound the alarm, independently and without coordination. If an individual alarms, the predator's ambush may be ruined. Each individual has two choices: volunteer (alarm) or ignore (no alarm). An alarm call would benefit everyone because it would deter the predator from attacking again; each player, however, prefers that the other players report the presence of the predator because giving the alarm has a cost $c > 0$. If one sets the alarm, he has a payoff of $b - c$, while the others who do not set the alarm have a payoff of b. If nobody raises the alarm, the predator attacks, inflicting damage $a > c$ and a payoff of $b - a$ on everyone. Herein, we use the digits "1" and "2" to denote the alarm and non-alarm strategies, respectively. Therefore, the tensor form of the m-player Volunteer's Dilemma can be represented as $G = ([m], \{[2]\}_{k=1}^m, \{\mathcal{A}^{(k)}\}_{k=1}^m)$, where*

$$a^{(k)}_{i_1 \cdots i_{k-1} 1 i_{k+1} \cdots i_m} = b - c$$

and

$$a^{(k)}_{i_1 \cdots i_{k-1} 2 i_{k+1} \cdots i_m} = \begin{cases} b - a, & \text{if } i_1 = \cdots = i_{k-1} = i_{k+1} = \cdots = i_m = 2, \\ b, & \text{otherwise.} \end{cases}$$

By calculation, we get that the statements (i) and (ii) of Theorem 1 are held. Therefore, the m-player Volunteer's Dilemma is symmetric.

(II) From Theorem 1, we have the first player's payoff tensor $\mathcal{A}^{(1)}$ is 1-mode symmetric. The payoffs in m-person symmetric games are uniquely determined by the first player's payoff tensor $\mathcal{A}^{(1)}$. Therefore, in the rest of this paper, we denote an m-person symmetric game by the tuple $G = ([m]; [n]; \mathcal{A}^{(1)})$, and let

$$\Omega_1 = \cdots = \Omega_m = \{\mathbf{u} \in \mathbb{R}^n : \mathbf{u} \geqslant \mathbf{0} \quad \text{and} \quad \mathbf{1}_n^\top \mathbf{u} = 1\}.$$

Definition 11. [2] *A Nash equilibrium $(\mathbf{u}^{(1^*)}, \mathbf{u}^{(2^*)}, \cdots, \mathbf{u}^{(m^*)}) \in \Omega$ is symmetric if all players take the same strategy, that is, $\mathbf{u}^{(1^*)} = \mathbf{u}^{(2^*)} = \cdots = \mathbf{u}^{(m^*)}$.*

Nash proved the existence of equilibrium for m-person games and symmetric equilibrium for symmetric m-person games in 1951 [2], and the statement is summarized in the following Lemma.

Lemma 2. [2] *Every m-person symmetric game has a symmetric Nash equilibrium.*

Now, we are going to propose two equivalent conditions to ensure that a mixed strategy combination is a symmetric Nash equilibrium, which are shown in Theorems 2 and 3.

Lemma 3. *Suppose that $G = ([m]; [n]; \mathcal{A}^{(1)})$ be an m-person symmetric game, then $(\mathbf{x}^*, \mathbf{x}^*, \cdots, \mathbf{x}^*) \in \Omega$ is a symmetric Nash equilibrium if and only if $\mathbf{x}^*$ is an optimal solution to the following optimization problem,*

$$\begin{aligned} &\max \mathcal{A}^{(1)} \mathbf{x}^{*m-1} \mathbf{x} \\ &s.t. \, \mathbf{x} \in \{\mathbf{x} = (x_i) \in \mathbb{R}^n : \mathbf{x} \geqslant \mathbf{0} \quad \text{and} \quad \mathbf{1}_n^\top \mathbf{x} = 1\}. \end{aligned} \tag{2}$$

Proof. By Definition 9, $(\mathbf{x}^*, \mathbf{x}^*, \cdots, \mathbf{x}^*) \in \Omega$ is a symmetric Nash equilibrium if and only if for any strategy combination $(\mathbf{u}^{(1)}, \cdots, \mathbf{u}^{(n)}) \in \Omega$ and any $k \in [m]$, namely,

$$\mathcal{A}^{(k)} \mathbf{x}^{*m} \geqslant \mathcal{A}^{(k)} \mathbf{x}^{*m-k} \mathbf{u}^{(k)} \mathbf{x}^{*k-1}. \tag{3}$$

Let $\sigma \in \Pi_m$ with $\sigma(k) = 1, \sigma(1) = k$, and $\sigma(i) = i$ for all $i \in [m] \setminus \{1, k\}$. By Definition 1, we have

$$\mathcal{A}^{(k)}\mathbf{x}^{*m} = (\mathcal{A}^{(k)})^{<\sigma>}\mathbf{x}^{*m}$$

and

$$\mathcal{A}^{(k)}\mathbf{x}^{*m-k}\mathbf{u}^{(k)}\mathbf{x}^{*k-1} = (\mathcal{A}^{(k)})^{<\sigma>}\mathbf{x}^{*m-1}\mathbf{u}^{(k)}.$$

Together with Theorem 1, we can get $(\mathcal{A}^{(k)})^{<\sigma>} = \mathcal{A}^{(1)}$. Therefore, the inequality (3) is equivalent to

$$\mathcal{A}^{(1)}\mathbf{x}^{*m} \geqslant \mathcal{A}^{(1)}\mathbf{x}^{*m-1}\mathbf{u}^{(k)}.$$

The proof is completed. □

Let

$$\mathcal{A} = \eta\mathcal{E} - \mathcal{A}^{(1)}, \tag{4}$$

where $\eta = \max\limits_{i_1, i_2, \cdots, i_m \in [n]} a^{(1)}_{i_1 i_2 \cdots i_m} + 1$ and $\mathcal{E} \in \mathbb{R}^{[m,n]}$ is an m-th order n-dimensional tensor whose all entries are ones. According to $\mathcal{A}^{(1)}$ is 1-mode symmetric we can see that $\mathcal{A}$ is also a 1-mode symmetric tensor with all entries are positive.

Theorem 2. *Suppose that $G = ([m]; [n]; \mathcal{A}^{(1)})$ be an m-person symmetric game, and $\mathcal{A}$ is defined as (4), then $(\mathbf{x}^*, \mathbf{x}^*, \cdots, \mathbf{x}^*) \in \Omega$ is a symmetric Nash equilibrium if and only if $\mathbf{x}^*$ is an optimal solution to the following optimization problem:*

$$\begin{aligned} &\min \mathcal{A}\mathbf{x}^{*m-1}\mathbf{x}. \\ &s.t.\, \mathbf{x} \in \{\mathbf{x} = (x_i) \in \mathbb{R}^n : \mathbf{x} \geqslant \mathbf{0} \quad \text{and} \quad \mathbf{1}_n^\top \mathbf{x} = 1\} \end{aligned} \tag{5}$$

Proof. For any mix strategy $(\mathbf{x}^*, \cdots, \mathbf{x}^*, \mathbf{x}) \in \Omega$, we have

$$\begin{aligned} \mathcal{A}\mathbf{x}^{*m-1}\mathbf{x} &= \sum_{i_1,\cdots,i_{m-1},i_m=1}^{n} a_{i_1\cdots i_{m-1}i_m}\mathbf{x}^*_{i_1}\cdots\mathbf{x}^*_{i_{m-1}}\mathbf{x}_{i_m} \\ &= \sum_{i_1,\cdots,i_{m-1},i_m=1}^{n} (\eta - a^{(1)}_{i_1\cdots i_{m-1}i_m})\mathbf{x}^*_{i_1}\cdots\mathbf{x}^*_{i_{m-1}}\mathbf{x}_{i_m} \\ &= \eta - \sum_{i_1,\cdots,i_{m-1},i_m=1}^{n} a^{(1)}_{i_1\cdots i_{m-1}i_m}\mathbf{x}^*_{i_1}\cdots\mathbf{x}^*_{i_{m-1}}\mathbf{x}_{i_m} \\ &= \eta - \mathcal{A}^{(1)}\mathbf{x}^{*m-1}\mathbf{x}. \end{aligned}$$

Therefore, $\mathbf{x}^*$ is an optimal solution to the optimization problem (5) if and only if $\mathbf{x}^*$ is an optimal solution to the optimization problem (2). □

We consider the following model: Find a mixed strategy combination $(\mathbf{x}^*, \mathbf{x}^*, \cdots, \mathbf{x}^*) \in \Omega$ such that $\mathbf{x}^*$ is an optimal solution to the optimization problem (5). Given a tensor $\mathcal{B} \in \mathbb{R}^{[m,n]}$ and a vector $\mathbf{q} \in \mathbb{R}^n$, the tensor complementary problem, denoted by TCP$(\mathbf{q}, \mathcal{B})$, is to find a vector $\mathbf{z} \in \mathbb{R}^n$ such that

$$\mathbf{z} \geqslant \mathbf{0}, \quad \mathcal{B}\mathbf{z}^{m-1} + \mathbf{q} \geqslant \mathbf{0}, \quad \mathbf{z}^\top(\mathcal{B}\mathbf{z}^{m-1} + \mathbf{q}) = 0. \tag{6}$$

The TCP$(\mathbf{q}, \mathcal{B})$ was introduced by Song and Qi [38] and was further studied by many researchers [39–48]. The tensor complementary problem is a generalization of the linear complementary problem (corresponding to $m = 2$) [49] and also a special instance of a nonlinear complementary problem, as well as a particular case of a variational inequality problem corresponding to the close convex cone $\mathbb{R}^n_+$ [50]. Denote the solution set of the TCP$(\mathbf{q}, \mathcal{B})$ by SOL$(\mathbf{q}, \mathcal{B})$. In this section, we show that the m-person symmetric game can be reformulated as a specific tensor complementary problem.

Let $G = ([m]; [n]; \mathcal{A}^{(1)})$ be an m-person symmetric game and $\mathcal{A}$ defined as (4). We construct a tensor complementary problem as follows,

$$\mathbf{y} \geqslant \mathbf{0}, \quad \mathcal{A}\mathbf{y}^{m-1} - \mathbf{1}_n \geqslant \mathbf{0}, \quad \mathbf{y}^\top(\mathcal{A}\mathbf{y}^{m-1} - \mathbf{1}_n) = 0. \tag{7}$$

There is an explicit corresponding relation between the solutions of a symmetric Nash equilibrium of the m-person symmetric game and a solution to the TCP$(-\mathbf{1}_n, \mathcal{A})$ (7).

Theorem 3. *Suppose that $G = ([m]; [n]; \mathcal{A}^{(1)})$ be an m-person symmetric game and $\mathcal{A}$ defined as (4). If $(\mathbf{x}^*, \mathbf{x}^*, \cdots, \mathbf{x}^*) \in \Omega$ is a symmetric Nash equilibrium of the game $G = ([m]; [n]; \mathcal{A}^{(1)})$, then $\mathbf{y}^* \in \mathrm{SOL}\,(-\mathbf{1}_n, \mathcal{A})$, where*

$$\mathbf{y}^* = \frac{\mathbf{x}^*}{\sqrt[m-1]{\mathcal{A}\mathbf{x}^{*m}}}. \tag{8}$$

Conversely, if $\mathbf{y}^* \in \mathrm{SOL}(-\mathbf{1}_n, \mathcal{A})$, then $\mathbf{y}^* \neq \mathbf{0}$ and $(\mathbf{x}^*, \mathbf{x}^*, \cdots, \mathbf{x}^*) \in \Omega$ with

$$\mathbf{x}^* = \frac{\mathbf{y}^*}{\mathbf{1}_n^\top \mathbf{y}^*} \tag{9}$$

is a symmetric Nash equilibrium of the game $G = ([m]; [n]; \mathcal{A}^{(1)})$.

Proof. Firstly, we show the necessity. Suppose that $(\mathbf{x}^*, \mathbf{x}^*, \cdots, \mathbf{x}^*) \in \Omega$ is a symmetric Nash equilibrium of the m-person symmetric game $G = ([m]; [n]; \mathcal{A}^{(1)})$. By the Karush-Kuhn-Tucker conditions of problem (5), there exist a number $\lambda^* \in \mathbb{R}$ and a nonnegative vector $\mathbf{b}^* \in \mathbb{R}^n$ such that

$$\mathcal{A}\mathbf{x}^{*m-1} - \lambda^*\mathbf{1}_n - \mathbf{b}^* = \mathbf{0} \tag{10}$$

and

$$\mathbf{1}_n^\top \mathbf{x}^* = 1, \quad \mathbf{x}^* \geqslant \mathbf{0}, \quad \mathbf{b}^* \geqslant \mathbf{0}, \quad \mathbf{b}^{*\top}\mathbf{x}^* = 0. \tag{11}$$

By (10), we have

$$\mathcal{A}\mathbf{x}^{*m} - \lambda^*\mathbf{1}_n^\top\mathbf{x}^* - \mathbf{b}^{*\top}\mathbf{x}^* = \mathbf{0},$$

which together with (11) it yields

$$\lambda^* = \mathcal{A}\mathbf{x}^{*m}.$$

Thus,

$$\mathbf{y}^* = \frac{1}{\sqrt[m-1]{\lambda^*}}\mathbf{x}^* \geqslant \mathbf{0}. \tag{12}$$

According to (10), we obtain

$$\mathcal{A}\mathbf{y}^{*m-1} - \mathbf{1}_n = \frac{1}{\lambda^*}(\lambda^*\mathbf{1}_n + \mathbf{b}^*) - \mathbf{1}_n = \frac{1}{\lambda^*}\mathbf{b}^* \geqslant \mathbf{0}. \tag{13}$$

Combing (13) with (11), we get

$$\mathbf{y}^{*\top}(\mathcal{A}\mathbf{y}^{*m-1} - \mathbf{1}_n) = \frac{1}{\sqrt[m-1]{\lambda^*}}\mathbf{x}^{*\top}\mathbf{b}^* = 0. \tag{14}$$

Apply the results of (13) and (14) to (12) we obtain that $\mathbf{y}^*$ defined by (8) is a solution to the TCP$(-\mathbf{1}_n, \mathcal{A})$. That is, $\mathbf{y}^* \in \mathrm{SOL}(-\mathbf{1}_n, \mathcal{A})$.

Now, we show the sufficiency. Suppose that $\mathbf{y}^* \in \mathrm{SOL}(-\mathbf{1}_n, \mathcal{A})$, then

$$\mathbf{y}^* \geqslant \mathbf{0}, \quad \mathcal{A}\mathbf{y}^{*m-1} - \mathbf{1}_n \geqslant \mathbf{0}, \quad \mathbf{y}^{*\top}(\mathcal{A}\mathbf{y}^{*m-1} - \mathbf{1}_n) = 0. \tag{15}$$

By the first and second inequalities of (15), we have $\mathbf{y}^* \neq \mathbf{0}$. Next, we prove that $(\mathbf{x}^*, \mathbf{x}^*, \cdots, \mathbf{x}^*) \in \Omega$ defined by (9) is a symmetric Nash equilibrium of the m-person symmetric game $G = ([m]; [n]; \mathcal{A}^{(1)})$. For this purpose, we need to prove that there exists a number $\lambda^* \in \mathbb{R}$ and a nonnegative vector $\mathbf{b}^* \in \mathbb{R}^n$ such that (10) and (11) hold.

We get the following equation from the iequalities of (15).

$$\mathcal{A}\mathbf{y}^{*m} - \mathbf{1}_n^\top \mathbf{y}^* = 0.$$

Since $\mathbf{y}^* \neq \mathbf{0}$ and $\mathbf{y}^* \geqslant \mathbf{0}$ follow, this implies that $\mathbf{1}_n^\top \mathbf{y}^* > 0$ follows, and then

$$\mathcal{A}\left(\frac{\mathbf{y}^*}{\mathbf{1}_n^\top \mathbf{y}^*}\right)^m - \frac{1}{(\mathbf{1}_n^\top \mathbf{y}^*)^{m-1}} = 0.$$

By (9), the above equality becomes

$$\mathcal{A}\mathbf{x}^{*m} - \frac{1}{(\mathbf{1}_n^\top \mathbf{y}^*)^{m-1}} = 0. \tag{16}$$

From $\mathbf{y}^* \geqslant \mathbf{0}$, $\mathbf{1}_n^\top \mathbf{y}^* > 0$, and the definition of $\mathbf{x}^*$, it follows that $\mathbf{x}^* \geqslant \mathbf{0}$ and $\mathbf{1}_n^\top \mathbf{x}^* = 1$. In addition, by the second inequality of (15), we have

$$\mathcal{A}\mathbf{x}^{*m-1} - \frac{\mathbf{1}_n}{(\mathbf{1}_n^\top \mathbf{y}^*)^{m-1}} \geqslant \mathbf{0},$$

which implies that there exists a nonnegative vector $\mathbf{b}^* \in \mathbb{R}^n$ such that

$$\mathbf{b}^* = \mathcal{A}\mathbf{x}^{*m-1} - \frac{\mathbf{1}_n}{(\mathbf{1}_n^\top \mathbf{y}^*)^{m-1}}.$$

The above result together with (16), yields

$$\begin{aligned}
\mathbf{b}^{*\top}\mathbf{x}^* &= \mathbf{x}^{*\top}\left(\mathcal{A}\mathbf{x}^{*m-1} - \frac{\mathbf{1}_n}{(\mathbf{1}_n^\top \mathbf{y}^*)^{m-1}}\right) \\
&= \mathcal{A}\mathbf{x}^{*m} - \frac{\mathbf{x}^{*\top}\mathbf{1}_n}{(\mathbf{1}_n^\top \mathbf{y}^*)^{m-1}} \\
&= \mathcal{A}\mathbf{x}^{*m} - \frac{1}{(\mathbf{1}_n^\top \mathbf{y}^*)^{m-1}} \\
&= 0.
\end{aligned}$$

So, we obtain that (10) and (11) hold with

$$\lambda^* = \frac{1}{(\mathbf{1}_n^\top \mathbf{y}^*)^{m-1}}.$$

Therefore, $(\mathbf{x}^*, \mathbf{x}^*, \cdots, \mathbf{x}^*) \in \Omega$ defined by (9) is a symmetric Nash equilibrium of the m-person symmetric game $G = ([m]; [n]; \mathcal{A}^{(1)})$. $\square$

4. Algorithm and numerical results

It is well known that the hyperplane projection algorithm [50] is a class of effective methods for solving variational inequalities and complementary problems. In this section, we apply the hyperplane projection algorithm to solve the TCP$(-\mathbf{1}_n, \mathcal{A})$ and provide some preliminary numerical results for solving the m-person symmetric game.

Let $G = ([m]; [n]; \mathcal{A}^{(1)})$ be an m-person symmetric game and $\mathcal{A}$ defined as (4). Denote $F(y) = \mathcal{A}\mathbf{y}^{m-1} - \mathbf{1}_n$, then we can rewrite the TCP$(-\mathbf{1}_n, \mathcal{A})$ (7) as follows,

$$\mathbf{y} \geqslant \mathbf{0}, \quad F(\mathbf{y}) \geqslant \mathbf{0}, \quad \mathbf{y}^\top F(\mathbf{y}) = 0. \tag{17}$$

It is well know that (17) is equivalent to the variational inequality problem: Find a nonnegative vector $\mathbf{y} \in \mathbb{R}^n_+$ such that

$$(\mathbf{x} - \mathbf{y})^\top F(\mathbf{y}) \geqslant 0, \quad \forall \mathbf{x} \in \mathbb{R}^n_+, \tag{18}$$

and it is equivalent to finding a root of the following equations

$$\mathbf{y} = [\mathbf{y} - \tau F(\mathbf{y})]_+, \quad \forall \tau > 0.$$

We use the hyperplane projection algorithm, see Algorithm 1, to solve the problem (17), which can be described geometrically as follows. Let $\tau \in \mathbb{R}$ be a fixed scale, and $\mathbf{y}^{(k)} \in \mathbb{R}^n$ is a given vector. Firstly, we compute the point $[\mathbf{y}^{(k)} - \tau F(\mathbf{y}^{(k)})]_+$. Secondly, based on the Armijo-type search routine, we search the line segment joining the $\mathbf{y}^{(k)}$ and $[\mathbf{y}^{(k)} - \tau F(\mathbf{y}^{(k)})]_+$ and get a point $\mathbf{u}^{(k)}$ such that the hyperplane

$$\mathbb{H}^k := \{\mathbf{y} \in \mathbb{R}^n : F(\mathbf{u}^{(k)})^\top (\mathbf{y} - \mathbf{u}^{(k)}) = 0\}$$

strictly separates $\mathbf{y}^{(k)}$ from SOL$(-\mathbf{1}_n, \mathcal{A})$. We project $\mathbf{y}^{(k)}$ onto $\mathbb{H}^k$ and get a point $\mathbf{w}^{(k)}$, and project the point $\mathbf{w}^{(k)}$ onto $\mathbb{R}^n_+$, then $\mathbf{y}^{(k+1)}$ is obtained. It can be shown that $\mathbf{y}^{(k+1)}$ is closer to SOL$(-\mathbf{1}_n, \mathcal{A})$ than $\mathbf{y}^{(k)}$ [50].

Algorithm 1

Input: The first player payoff tensor $\mathcal{A}^{(1)}$, a starting vector $\mathbf{y}^{(0)} \in \mathbb{R}^n_+$, some scales $\tau > 0, 0 < \sigma < 1$.
Output: The symmetric Nash equilibrium $(\mathbf{x}^*, \mathbf{x}^*, \cdots, \mathbf{x}^*)$.

1: Compute $\eta = \max\limits_{i_1, i_2, \cdots, i_m \in [n]} a^{(1)}_{i_1 i_2 \cdots i_m} + 1$, deonte $\mathcal{A} = \eta \mathcal{E} - \mathcal{A}^{(1)}$. Set $k = 0$.

2: Compute

$$\mathbf{z}^{(k)} = [\mathbf{y}^{(k)} - \tau F(\mathbf{y}^{(k)})]_+$$

and find the smallest nonnegative integer i_k such that with $i = i_k$

$$F(2^{-i}\mathbf{z}^{(k)} + (1 - 2^{-i})\mathbf{y}^{(k)})^\top (\mathbf{y}^{(k)} - \mathbf{z}^{(k)}) \geqslant \frac{\sigma}{\tau} \parallel \mathbf{y}^{(k)} - \mathbf{z}^{(k)} \parallel^2 . \tag{19}$$

3: Set

$$\mathbf{u}^{(k)} = 2^{-i_k}\mathbf{z}^{(k)} + (1 - 2^{-i_k})\mathbf{y}^{(k)}$$

and

$$\mathbf{w}^{(k)} = P_{H^k}(\mathbf{y}^{(k)}) = \mathbf{y}^{(k)} - \frac{F(\mathbf{u}^{(k)})^\top (\mathbf{y}^{(k)} - \mathbf{u}^{(k)})}{\parallel F(\mathbf{u}^{(k)}) \parallel^2} F(\mathbf{u}^{(k)}).$$

4: Set $\mathbf{y}^{(k+1)} = [\mathbf{w}^{(k)}]_+$.

5: If $\mathbf{y}^{(k+1)} = \mathbf{y}^{(k)}$, then $\mathbf{x}^* = \frac{\mathbf{y}^{(k)}}{\mathbf{1}_n^\top \mathbf{y}^{(k)}}$, stop; otherwise replace k by $k + 1$, go to Step 2.

From [50], we obtain the following theorems.

Theorem 4. *Suppose that $\mathbf{y}^{(k)}$ is not a solution to the TCP$(-\mathbf{1}_n, \mathcal{A})$ (17). Then there exists a finite integer $i_k \geqslant 0$ such that (19) holds and*

$$F(\mathbf{u}^{(k)})^\top (\mathbf{y}^{(k)} - \mathbf{u}^{(k)}) > 0. \tag{20}$$

Remark 5. *If $F(\mathbf{y})$ is pseudomonotone on $\mathbb{R}^n_+$, the inequality (20) shows that the hyperplane $\mathbb{H}^k$ strictly separates $\mathbf{y}^{(k)}$ from SOL$(-\mathbf{1}_n, \mathcal{A})$. Based on $\mathbf{u}^{(k)} \in \mathbb{R}^n_+$ and the inequality (18), it follows that for any solution $\mathbf{y}^*$ of the TCP$(-\mathbf{1}_n, \mathcal{A})$, we have*

$$(\mathbf{u}^{(k)} - \mathbf{y}^*)^\top F(\mathbf{y}^*) \geqslant 0. \tag{21}$$

Based on the fact that $F(\mathbf{y})$ is pseudomonotone on $\mathbb{R}^n_+$, then combing (20) and (21) it yields

$$(\mathbf{u}^{(k)} - \mathbf{y}^*)^\top F(\mathbf{u}^{(k)}) \geqslant 0 > (\mathbf{u}^{(k)} - \mathbf{y}^{(k)})^\top F(\mathbf{u}^{(k)}).$$

Theorem 5. *Suppose that $F(\mathbf{y})$ is pseudomonotone on $\mathbb{R}^n_+$, then there exists $\mathbf{y}^* \in SOL(-\mathbf{1}_n, \mathcal{A})$ such that*

$$\lim_{k \to \infty} \mathbf{y}^{(k)} = \mathbf{y}^*.$$

Table 1. The numerical results of the problem in Example 1

m	b	a	c	No.Iter	CPU(s)	Res	SNE($\mathbf{x}^{*\top}$)
3	200	4	1	10	0.0582	4.18e-7	(0.5000,0.5000)
		4	2	41	0.2277	9.99e-7	(0.2929,0.7071)
		16	2	47	0.2644	7.62e-7	(0.6465,0.3235)
		32	1	50	0.3912	7.42e-7	(0.8232,0.1768)
		64	3	21	0.1919	9.82e-7	(0.7835,0.2165)
		192	3	72	0.6176	9.19e-7	(0.8750,0.1250)
6	200	4	1	121	0.8751	9.18e-7	(0.2422,0.7578)
		4	2	101	0.7523	9.17e-7	(0.1295,0.8705)
		16	2	63	0.5111	8.43e-7	(0.3403,0.6597)
		32	1	9	0.1503	3.53e-7	(0.5000,0.5000)
		64	3	11	0.0818	3.54e-7	(0.4578,0.5422)
		192	3	14	0.1152	4.32e-7	(0.5647,0.4353)
9	600	64	0.25	14	0.1891	5.95e-7	(0.5000,0.5000)
		64	0.5	59	0.5382	8.54e-7	(0.4548,0.5452)
		64	1	30	0.2826	5.90e-7	(0.4054,0.5946)
		512	0.25	88	0.7381	8.84e-7	(0.6145,0.3855)
		512	0.5	77	0.7354	6.79e-7	(0.5796,0.4204)
		512	1	54	0.6315	7.15e-7	(0.5415,0.4585)
12	600	64	0.25	206	2.1890	9.43e-7	(0.3960,0.6040)
		64	0.5	143	1.8373	9.61e-7	(0.3567,0.6433)
		64	1	114	1.2770	9.67e-7	(0.3149,0.6851)
		512	0.25	11	0.5486	5.97e-7	(0.5000,0.5000)
		512	0.5	53	0.7763	9.67e-7	(0.4675,0.5325)
		512	1	27	0.5974	8.64e-7	(0.4329,0.5671)
15	2500	2048	0.125	8	3.1455	3.90e-7	(0.5000,0.5000)
		2048	64	63	3.6172	4.56e-7	(0.2193,0.7807)
		2048	128	65	3.8518	9.35e-7	(0.1797,0.8203)
		1024	0.125	177	4.6659	9.93e-7	(0.4747,0.5253)
		1024	64	47	3.3835	8.62e-7	(0.1797,0.8203)
		1024	128	85	4.4102	7.11e-7	(0.1380,0.8620)

In the following, we give some preliminary numerical results of Algorithm 1 for solving the symmetric Nash equilibrium of the m-person symmetric game. Throughout our experiments, the parameters used in Algorithm 1 are chosen as

$$\tau := 0.5, \sigma := 0.01, \mathbf{y}^{(0)} := 0.1 * \mathbf{1}_n.$$

We use $\| \mathbf{y}^{(k+1)} - \mathbf{y}^{(k)} \| \leqslant 10^{-6}$ as the stopping rule. All tests are conducted in MATLAB R2015a with the configuration: Intel(R) Core(TM)i7-7500U CPU 2.70GHz and 8.00G RAM.

Example 1. *According to Remark 4. The tensor form of the m-player Volunteer's Dilemma can be represented as $G = ([m], [2], \mathcal{A}^{(1)})$, where*

$$a^{(1)}_{1i_2\cdots i_m} = b - c, \quad a^{(1)}_{2i_2\cdots i_m} = \begin{cases} b-a, & \text{if} \quad i_2 = i_3 = \cdots = i_m = 2, \\ b, & \text{otherwise.} \end{cases}$$

We use Algorithm 1 to solve the symmetric Nash equilibrium of the m-player Volunteer's Dilemma and the numerical results are reported in Table 1. In this table, the number of iteration steps are denoted by No.Iter, the CPU time in seconds are denoted by CPU(s). As the algorithm stops the value of $\| \mathbf{y}^{(k+1)} - \mathbf{y}^{(k)} \|$ is computed and denoted by Res and the first player's strategy in symmetric Nash equilibrium is denoted by SNE($\mathbf{x}^{*\top}$).

Table 2. The numerical results of the problem in Example 2

m	b	c	No.Iter	CPU(s)	Res	SNE($\mathbf{x}^{*\top}$)
3	8	1	71	0.3320	8.84e-7	(0.7686,0.2314)
	8	2	34	0.1822	8.49e-7	(0.6496,0.3504)
	10	5	26	0.1606	8.69e-7	(0.4418,0.5582)
	10	8	161	0.8231	8.92e-7	(0.1886,0.8114)
	20	1	20	0.1729	9.22e-7	(0.8611,0.1389)
	20	10	27	0.1719	7.46e-7	(0.4417,0.5583)
6	8	1	136	0.8696	8.79e-7	(0.4653,0.5347)
	8	2	134	0.7635	9.08e-7	(0.3595,0.6405)
	10	5	132	0.8305	9.62e-7	(0.2162,0.7838)
	10	8	157	0.9867	8.08e-7	(0.0817,0.9183)
	20	1	92	0.5705	8.62e-7	(0.5711,0.4289)
	20	10	92	0.6909	7.21e-7	(0.2162,0.7838)
9	8	1	95	0.7384	2.12e-7	(0.3291,0.6709)
	8	2	68	0.5151	8.84e-7	(0.2466,0.7534)
	10	5	59	0.4447	3.20e-7	(0.1436,0.8574)
	10	8	84	0.7784	8.17e-7	(0.0520,0.9480)
	20	1	164	1.2335	9.53e-7	(0.4179,0.5821)
	20	10	56	0.5454	4.96e-7	(0.1426,0.8574)
12	8	1	370	3.4376	9.99e-7	(0.2542,0.7458)
	8	2	283	2.6579	9.51e-7	(0.1876,0.8124)
	10	5	191	1.8678	7.55e-7	(0.1065,0.8935)
	10	8	198	1.9516	7.56e-7	(0.0383,0.9617)
	20	1	319	3.0998	9.39e-7	(0.3283,0.6717)
	20	10	161	1.5762	9.82e-7	(0.1066,0.8934)
15	8	1	357	6.1214	9.91e-7	(0.2068,0.7932)
	8	2	249	5.3642	9.38e-7	(0.1512,0.8488)
	10	5	199	4.5910	9.94e-7	(0.0850,0.9150)
	10	8	214	4.7155	8.20e-7	(0.0304,0.9696)
	20	1	338	6.1258	9.69e-7	(0.2699,0.7301)
	20	10	180	4.7217	9.65e-7	(0.0850,0.9150)

Example 2. *In the m-player Snowdrift Game* [51]*, m individuals are driving on a crossroad that is blocked by a snowdrift. Each individual has the option to cooperate by shoveling the snow or not. The snow need to be removed before they continue their journey home. Again, everyone wants to go home, but not everyone is willing to shovel. If the benefit of getting home is defined as b and the cost of shoveling is defined as c. Assuming that the benefit exceeds the cost, i.e., $b > c$. If everyone shovels, then everyone gets $b - \frac{c}{m}$. But if only k individuals shovel, they get $b - \frac{c}{k}$ whereas those who refuse to shovel get home for free and get b. Nobody gets anything if everyone refuses to shovel. Let 1 and 2 denote the shovel snow and do not shovel snow strategies, respectively. It is easy to obtain that the m-player Snowdrift Game is symmetric. Let $\phi(1)$ denote the number of 1 in $i_2 i_3 \cdots i_m$. The tensor form of the m-player Snowdrift Game can be represented as $G = ([m], [2], \mathcal{A}^{(1)})$, where*

$$a^{(1)}_{1i_2\cdots i_m} = b - \frac{c}{1+\phi(1)}, \quad a^{(1)}_{2i_2\cdots i_m} = \begin{cases} 0, & \text{if } \phi(1) = 0, \\ b, & \text{if } \phi(1) \neq 0. \end{cases}$$

We use Algorithm 1 to solve the symmetric Nash equilibrium of m-player Snowdrift Game. The numerical results are reported in Table 2, where No.Iter, CPU(s), Res and SNE($\mathbf{x}^{*\top}$) are the same as those used in Table 1.

Example 3. *Consider the bimatrix symmetric game "Rock-Paper-Scissors" [9,52]. The tensor form of "Rock-Paper-Scissors" can be represented as $G = ([2], [3], A^{(1)})$, where*

$$A^{(1)} = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}.$$

We use Algorithm 1 to solve the symmetric Nash equilibrium of "Rock-Paper-Scissors". A symmetric Nash equilibrium $(\mathbf{x}^*, \mathbf{x}^*)$ with

$$\mathbf{x}^* = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)^\top$$

is obtained with 23 iteration steps in 0.2921 seconds.

5. Conclusions

In this paper, we show that the payoff tensor of the kth player in an m-person symmetric game is k-mode symmetric, and the payoff tensors of two different individuals are the transpose of each other. We also reformulate the m-person symmetric games as tensor complementary problems and demonstrate that finding a symmetric Nash equilibrium of the m-person symmetric game is equivalent to finding a solution to the resulting tensor complementary problem. Finally, we apply the hyperplane projection algorithm to solve the resulting tensor complementary problem and provide some numerical results for solving the symmetric Nash equilibrium.

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