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Exact Projections of the Real Part of $\zeta(s)$ Arithmetic Grids, Identities, Bounds, and a Mean-Square Formula in $\Re(s) > 1$

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Abstract

We present a deterministic framework for decomposing the real part of the Riemann zeta function $\mathrm{Re}\,\zeta(s)$ in the region $\Re(s) > 1$ by means of *arithmetic grids* (structured subsets of $\mathbb{N}$). We deduce exact identities for multiplicative, additive, and double grids, along with elementary bounds on their contributions and a treatment of the primorial mollifier as a finite Dirichlet polynomial. As a *concrete* application, we establish a **mean-square formula** for series of the form $b(n) = \sum_{d|n} a(d)$: for $\sigma > 1$,

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \left| \sum_{n\geq 1} \frac{b(n)}{n^{\sigma+it}} \right|^2 dt = \zeta(2\sigma) \sum_{d,e\geq 1} \frac{a(d)\overline{a(e)}}{\mathrm{lcm}(d,e)^{2\sigma}}.$$

We provide explicit corollaries for the *additive grid* (multiples of a finite set of primes) and for the *prime power grid* with weights. These tools allow for the quantification, isolation, and bounding of layers of $\mathrm{Re}\,\zeta(s)$ without resorting to unproven hypotheses.

Keywords: Riemann zeta function; arithmetic grids; Dirichlet series; prime zeta function; von Mangoldt function; primorial; mollifier; mean-square formula

MSC (2020): 11M06; 11N05; 11M45

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1. Introduction

For $s = \sigma + it$ with $\sigma > 1$, the Dirichlet expansion

$$\zeta(s) = \sum_{n\geq 1} n^{-s} = \sum_{n\geq 1} \frac{1}{n^{\sigma}} e^{-it\log n} \tag{1.1}$$

converges absolutely. Taking the real part,

$$\mathrm{Re}\,\zeta(\sigma + it) = \sum_{n\geq 1} \frac{\cos(t\log n)}{n^{\sigma}}. \tag{1.2}$$

This observation enables an *exact decomposition* of $\mathrm{Re}\,\zeta$ by regrouping terms according to disjoint subsets of $\mathbb{N}$ ("arithmetic grids"). In this note, we formalize this decomposition and provide useful identities and bounds for classical grids (prime powers, multiples of primes) and for their *double grid* (products $p^m n$).

Contributions.

- *Exact projection by grids.* If $\mathbb{N} = \bigsqcup_j R_j$, then $\mathrm{Re}\,\zeta(\sigma + it) = \sum_j \sum_{n\in R_j} n^{-\sigma}\cos(t\log n)$ (Lemma 3.1).
- *Additive layer.* The identity $\sum_{n\geq 1} \omega(n) n^{-s} = \zeta(s)P(s)$ with $P(s) = \sum_{p\in\mathbb{P}} p^{-s}$ (Lemma 3.2) links the grid $\{np\}$ to the count of distinct prime factors of n.

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- *Multiplicative spine.* $\sum_{n\geq 1} \Lambda(n) n^{-s} = -\zeta'(s)/\zeta(s)$ (Lemma 3.3).
- *Grid of prime powers.* $\sum_{n\geq 1} 1_{\mathrm{pp}}(n) n^{-s} = \sum_p p^{-s}/(1-p^{-s})$ (Proposition 3.4).
- *Convolution of grids.* An exact equality for $1_A * 1_B$ and its Dirichlet series (Lemma 3.5).
- *Double grid and primorial mollifier.* Exact factorization of the double grid generator and a finite representation of the primorial mollifier (Proposition 4.1, Theorem 4.2).
- *Elementary bounds.* Universal and specific bounds for multiplicative and additive grids (Lemmas 6.1 to 6.3).
- *Quantitative application: mean-square formula.* A mean-square formula for series with coefficients $b(n) = \sum_{d|n} a(d)$ and corollaries for grids (Theorem 5.1, Corollaries 5.2 and 5.3).

Related Work

The Euler product, the von Mangoldt function, and their links to $-\zeta'/\zeta$ are classical [1–3]. The *prime zeta function* $P(s) = \sum_p p^{-s}$ is standard in the literature (e.g., [4,5]). Here, we emphasize its role as an exact additive layer through $\sum \omega(n) n^{-s}$. The use of finite Dirichlet polynomials for mollifiers (like the primorial) is equally elementary; we systematize their closed form and bounds for $\sigma > 1$. Our additional contribution is to package these layers into an explicit *mean-square formula* that quantifies their contribution in L^2 norms on vertical lines.

2. Notation

- $\mathbb{P}$ denotes the set of prime numbers; $\mathbb{N} = \{1, 2, 3, \ldots\}$.
- $\omega(n)$: number of *distinct prime factors* of n; $\Lambda(n)$: von Mangoldt function.
- $P(s) = \sum_{p\in\mathbb{P}} p^{-s}$; $\zeta(s) = \sum_{n\geq 1} n^{-s}$ for $\Re(s) > 1$.
- 1_S denotes the indicator function of a subset $S \subset \mathbb{N}$.
- For a prime P, $M_P(s) = \prod_{p\leq P}(1-p^{-s})$ (primorial mollifier).

3. Basic Results and Decomposition

Lemma 3.1 (Decomposition by partitions)**.** *Let $\sigma > 1$ and a disjoint partition $\mathbb{N} = \bigsqcup_{j=1}^{k} R_j$. Then*

$$\operatorname{Re} \zeta(\sigma + it) = \sum_{j=1}^{k} \sum_{n\in R_j} \frac{\cos(t \log n)}{n^{\sigma}}. \tag{3.1}$$

Proof. The series (1.1) converges absolutely for $\sigma > 1$, allowing for rearrangement by blocks R_j; taking the real part yields the identity. □

Lemma 3.2 (Additive layer as $\omega(n)$)**.** *For $\Re(s) > 1$,*

$$\sum_{n\geq 1} \frac{\omega(n)}{n^s} = \zeta(s)\, P(s), \qquad P(s) = \sum_{p\in\mathbb{P}} \frac{1}{p^s}. \tag{3.2}$$

Proof. Each integer n has $\omega(n)$ decompositions of the form $n = pm$ with p a prime and $m \in \mathbb{N}$. By direct counting, $\sum_n \omega(n) n^{-s} = \sum_p \sum_m (pm)^{-s} = P(s)\zeta(s)$. □

Lemma 3.3 (Multiplicative spine)**.** *For $\Re(s) > 1$,*

$$\sum_{n\geq 1} \frac{\Lambda(n)}{n^s} = -\frac{\zeta'(s)}{\zeta(s)}. \tag{3.3}$$

Proof. Logarithmic differentiation of the Euler product for $\zeta(s)$. □

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Proposition 3.4 (Grid of prime powers)**.** *Let $1_{\mathrm{pp}}(n) = 1$ if $n = p^m$ for some $p \in \mathbb{P}$ and $m \ge 1$, and 0 otherwise. For $\Re(s) > 1$,*

$$\sum_{n\ge1} \frac{1_{\mathrm{pp}}(n)}{n^s} = \sum_{p\in\mathbb{P}} \frac{p^{-s}}{1-p^{-s}}. \tag{3.4}$$

Proof. Sum of independent geometric series over primes p. □

Lemma 3.5 (Convolution of grids)**.** *Let $A, B \subset \mathbb{N}$ and $c(n) = (1_A * 1_B)(n) = \sum_{ab=n} 1_A(a)1_B(b)$. For $\Re(s) > 1$,*

$$\sum_{n\ge1} \frac{c(n)}{n^s} = \Big(\sum_{a\in A} a^{-s}\Big)\Big(\sum_{b\in B} b^{-s}\Big). \tag{3.5}$$

Proof. This is a standard identity for Dirichlet convolutions. □

4. Double Grid and Primorial Mollifier

We define the *double grid* as

$$\mathcal{D} = \{\, p^m n : \; p \in \mathbb{P},\; m \ge 1,\; n \in \mathbb{N} \,\}. \tag{4.1}$$

Proposition 4.1 (Generator of the double grid)**.** *Let $w_{p,m}$ be a weight independent of n. For $\Re(s) > 1$,*

$$\sum_{p\in\mathbb{P}} \sum_{m\ge1} \sum_{n\ge1} \frac{w_{p,m}}{(p^m n)^s} = \Big(\sum_{p\in\mathbb{P}} \sum_{m\ge1} \frac{w_{p,m}}{p^{ms}}\Big)\zeta(s). \tag{4.2}$$

Proof. Factor out the sum $\sum_{n\ge1} n^{-s} = \zeta(s)$. □

Theorem 4.2 (Primorial as a finite Dirichlet polynomial)**.** *For a prime P and $\Re(s) > 0$,*

$$M_P(s) := \prod_{p\le P}(1-p^{-s}) = \sum_{\substack{n\ge1 \\ \mu^2(n)=1 \\ P^+(n)\le P}} \frac{\mu(n)}{n^s}, \tag{4.3}$$

where μ is the Möbius function and $P^+(n)$ is the largest prime factor of n.

Proof. Expansion of the finite product; only square-free integers n with $P^+(n) \le P$ appear. □

Lemma 4.3 (Elementary bounds for the mollifier)**.** *For $\sigma = \Re(s) > 1$,*

$$\log|M_P(s)| \le \sum_{p\le P} \log(1+p^{-\sigma}) \le \sum_{p\le P} p^{-\sigma}, \qquad \log|M_P(s)| \ge -\sum_{p\le P} \frac{p^{-\sigma}}{1-p^{-\sigma}}. \tag{4.4}$$

Proof. Use $\log|1-z| \le \log(1+|z|)$ and $\log(1-x) \ge -x/(1-x)$ for $x \in (0,1)$. □

5. Application: A Mean-Square Formula

The following identity quantifies (in the L^2 norm on vertical lines) the contribution of layers constructed by *sum over divisors*.

Theorem 5.1 (Mean-square for series with sum over divisors)**.** *Let $\sigma > 1$ and $a : \mathbb{N} \to \mathbb{C}$ with finite support (or absolutely summable with weight $d^{-2\sigma}$). Define $b(n) = \sum_{d|n} a(d)$ and*

$$F(s) = \sum_{n\ge1} \frac{b(n)}{n^s}, \qquad \Re(s) = \sigma.$$

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Then

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\left|F(\sigma+it)\right|^2 dt=\zeta(2\sigma)\sum_{d,e\ge 1}\frac{a(d)\,\overline{a(e)}}{\mathrm{lcm}(d,e)^{2\sigma}}. \tag{5.1}$$

Proof. By absolute convergence, we can interchange the sum and the integral. Expanding the square and using

$$\frac{1}{2T}\int_{-T}^{T}e^{-it(\log m-\log n)}\,dt\to\begin{cases}1, & m=n,\\ 0, & m\ne n,\end{cases}$$

we obtain the orthogonality limit. Now, $b(n)=\sum_{d|n}a(d)$ implies

$$\sum_{n\ge 1}\frac{|b(n)|^2}{n^{2\sigma}}=\sum_{d,e\ge 1}a(d)\overline{a(e)}\sum_{\substack{n\ge 1\\ d|n,\,e|n}}\frac{1}{n^{2\sigma}}=\sum_{d,e\ge 1}a(d)\overline{a(e)}\sum_{k\ge 1}\frac{1}{(\mathrm{lcm}(d,e)\,k)^{2\sigma}},$$

which is equal to $\zeta(2\sigma)\sum_{d,e}a(d)\overline{a(e)}\,\mathrm{lcm}(d,e)^{-2\sigma}$. □

Corollary 5.2 (Additive layer: multiples of a finite set of primes)**.** *Let $S\subset\mathbb{P}$ be a finite set and let $a(d)=1$ if $d\in S$, and 0 otherwise. Then $b(n)=\sum_{p\in S}1_{p|n}$ and, for $\sigma>1$,*

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\left|\sum_{n\ge 1}\frac{b(n)}{n^{\sigma+it}}\right|^2 dt=\zeta(2\sigma)\left(\sum_{p\in S}\frac{1}{p^{2\sigma}}+\sum_{\substack{p,q\in S\\ p\ne q}}\frac{1}{(pq)^{2\sigma}}\right). \tag{5.2}$$

Proof. In (5.1), $\mathrm{lcm}(p,q)=pq$ if $p\ne q$ and $\mathrm{lcm}(p,p)=p$. □

Corollary 5.3 (Prime powers with weights)**.** *Let $a(p^m)=\beta_m\log p$ (and $a(d)=0$ if d is not a prime power). Then $b(n)=\sum_{p^m|n}\beta_m\log p$ and*

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\left|\sum_{n\ge 1}\frac{b(n)}{n^{\sigma+it}}\right|^2 dt=\zeta(2\sigma)\sum_{p,q\in\mathbb{P}}\sum_{m,\ell\ge 1}\frac{\beta_m\beta_\ell\log p\,\log q}{\mathrm{lcm}(p^m,q^\ell)^{2\sigma}}. \tag{5.3}$$

Remark 1. The preceding identities quantify, in the mean-square norm, the contribution of additive and multiplicative grids along the vertical line $\Re(s)=\sigma$. They provide, for instance, a basis for comparing *mollifiers* or filters constructed from such grids under an L^2 criterion.

6. Bounds for Projections of Re ζ

Lemma 6.1 (Universal bound)**.** *Let $R\subset\mathbb{N}$ and $w:\mathbb{N}\to[0,W]$. For $\sigma>1$ and all $t\in\mathbb{R}$,*

$$\left|\sum_{n\in R}\frac{w(n)\cos(t\log n)}{n^\sigma}\right|\le W\sum_{n\in R}n^{-\sigma}. \tag{6.1}$$

Proof. From $|\cos(x)|\le 1$. □

Lemma 6.2 (Multiplicative grids)**.** *If R is multiplicatively closed and generated by $S\subset\mathbb{P}$, then*

$$\sum_{n\in R}n^{-\sigma}=\prod_{p\in S}\frac{1}{1-p^{-\sigma}},\qquad\left|\sum_{n\in R}\frac{w(n)\cos(t\log n)}{n^\sigma}\right|\le W\prod_{p\in S}\frac{1}{1-p^{-\sigma}}. \tag{6.2}$$

Proof. Restrict the Euler product to S and apply Lemma 6.1. □

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Lemma 6.3 (Finite additive layer)**.** *Let $R = \{np : n \geq 1, p \in S\}$ with $S \subset \mathbb{P}$ being a finite set. Then*

$$\sum_{np \in R} (np)^{-\sigma} = \zeta(\sigma) \sum_{p \in S} p^{-\sigma}, \qquad \Big| \sum_{np \in R} \frac{w(np)\cos(t \log(np))}{(np)^{\sigma}} \Big| \leq W \zeta(\sigma) \sum_{p \in S} p^{-\sigma}. \tag{6.3}$$

Proof. Factor out $\sum_n n^{-\sigma}$ and sum over $p \in S$; apply Lemma 6.1. □

7. Final Remarks

The results above provide a *deterministic toolkit* for isolating, quantifying, and bounding contributions from arithmetic grids to Re $\zeta(\sigma + it)$ for $\sigma > 1$. In particular: (i) the additive layer is identified with $\omega(n)$ and the prime zeta function, (ii) the prime power layer is related to the multiplicative spine via Λ, and (iii) the double grid is factored exactly. The mean-square formula Theorem 5.1 and its corollaries Corollaries 5.2 and 5.3 offer an explicit L^2 criterion for evaluating filters and mollifiers based on these grids.

Perspectives.

Extending these identities to $\Re(s) \geq 1$ requires tools from analytic continuation and control of truncation errors; we do not address this here. Another natural direction is to incorporate Dirichlet characters to separate parities and arithmetic progressions, which is straightforward: $\sum_{n \geq 1} \chi(n) n^{-s} = L(s, \chi)$ for $\Re(s) > 1$.

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