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Edgeworth-Cornish-Fisher Expansions for the Mean When Sampling from a Stationary Process

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Abstract: We give the Edgeworth-Cornish-Fisher expansions for the distribution, density and quantiles of the sample mean of a stationary process.

Keywords: Edgeworth-Cornish-Fisher expansions; sample mean; stationary

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1. Introduction and Summary

The behaviour of a standard estimate, as described by its Edgeworth-Cornish-Fisher expansions, is governed by the coefficients obtained by expanding its cumulants. For the simplest case, the mean of independent identically distributed random variables, the cumulant expansion has only one term. In Section 2 we summarise Edgeworth-Cornish-Fisher expansions for a standard estimate.

In Section 3 and Section 4 we apply this to the mean of a sample from a stationary process for univariate and multivariate series. Remarkably, we show that for the sample mean of a stationary process, its cumulant expansion has exactly 2 terms.

Suppose that $\bar{X}$ is the mean of a sample $X_1, \ldots, X_n$ from a stationary process $\{X_i\}$ in R^p. So $\bar{X}$ is an unbiased estimate of $\mu = E\, X_0$. In Section 2 we show that when $p = 1$, for $r > 1$, its rth cumulant has the form

$$\kappa_r(\bar{X}) = a_{nr,r-1}\, n^{1-r} + a_{nrr}\, n^{-r}. \tag{1}$$

where a_{nri} are bounded as n increases, and a_{n21} is bounded away from 0. This makes it a special case of a standard estimate, so that Section 2 applies with $a_{ri} = a_{nri}$ for $i = r-1, r$ and $a_{ri} = 0$ for $i > r$.

If $a_{nri} = a_{ri} + O(e^{-n\lambda_r})$ where $\lambda_{ri} > 0$, then a_{nri} can be replaced by a_{ri}. Here $x_n = O(y_n)$ means that x_n/y_n is bounded.

We also consider the case where observations are not sequential, as for missing data. And we consider unbiased weighted means.

2. Edgeworth-Cornish-Fisher Theory

Here we summarise the expansions of Withers (1984) for the distribution and quantiles of a standard estimate. In Section 3 we shall show that (1) holds, so that the sample mean is a special case of a standard estimate.

Univariate estimates. An estimate $\hat{w}$ of an unknown $w \in R$ is said to be a *standard estimate* with respect to n, if $E\,\hat{w} \to w$ as $n \to \infty$, and for $r \geq 1$, its rth cumulant can be expanded as

$$\kappa_r(\hat{w}) \approx \sum_{i=r-1}^{\infty} a_{ri}\, n^{-i}. \tag{1}$$

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The *cumulant coefficients* a_{ri} may depend on n but are bounded as $n \to \infty$, and a_{21} is bounded away from 0. Here and below $\approx$ indicates an asymptotic expansion that need not converge. That is, (1) holds in the sense that

$$\text{for } I \geq r,\ \kappa_r(\hat{w}) = \sum_{i=r-1}^{I-1} a_{ri}\, n^{-i} + O(n^{-I}).$$

For non-lattice estimates, the distribution and quantiles of

$$Y_n = (n/a_{21})^{1/2}(\hat{w} - w)$$

have asymptotic expansions in powers of $n^{-1/2}$:

$$P_n(x) = P(Y_n \leq x) \approx \Phi(x) - \phi(x) \sum_{r=1}^{\infty} h_r(x)\, n^{-r/2}, \tag{2}$$

$$p_n(x) = dP_n(x)/dx \approx \phi(x)\ [1 + \sum_{r=1}^{\infty} \overline{h}_r(x)\, n^{-r/2}], \tag{3}$$

$$\Phi^{-1}(P_n(x)) \approx x - \sum_{r=1}^{\infty} f_r(x)\, n^{-r/2},\ P_n^{-1}(\Phi(x)) \approx x + \sum_{r=1}^{\infty} g_r(x)\, n^{-r/2}, \tag{4}$$

where Φ and φ are the unit normal distribution and density of $N \sim \mathcal{N}(0,1)$, and $h_r(x), \overline{h}_r(x), f_r(x)$, $g_r(x)$ are polynomials in x and the standardized cumulant coefficients $\{A_{ri}\}$,

$$A_{ri} = a_{ri}/a_{21}^{r/2}. \tag{5}$$

The expansions (2), (4), are given in Withers (1984):

$$\begin{aligned}
h_1(x) &= f_1(x) = g_1(x) = A_{11} + A_{32}H_2/6, \qquad (6)\\
\overline{h}_1(x) &= A_{11}H_1 + A_{32}H_3/6,\\
h_2(x) &= (A_{11}^2 + A_{22})H_1 + (A_{11}A_{32} + A_{43})H_3/6 + A_{32}^2 H_5/72,\\
f_2(x) &= (A_{22}/2 - A_{11}A_{32}/3)H_1 + A_{43}H_3/24 - A_{32}^2(4x^3 - 7x)/36,\\
g_2(x) &= A_{22}H_1/2 + A_{43}H_3/24 - A_{32}^2(4x^3 - 7x)/36,\\
\overline{h}_2(x) &= (A_{11}^2 + A_{22})H_2 + (A_{11}A_{32} + A_{43})H_4/6 + A_{32}^2 H_6/72,
\end{aligned}$$

where H_k is the kth Hermite polynomial,

$$\begin{aligned}
H_k &= H_k(x) = \phi(x)^{-1}(-d/dx)^k \phi(x)\\
&= E(x + iN)^k \text{ for } k \geq 0,\ i = \sqrt{-1}: \qquad (7)\\
H_0 &= 1,\ H_1 = x,\ H_2 = x^2 - 1,\ H_3 = x^3 - 3x,\ H_4 = x^4 - 6x^2 + 3,\\
H_5 &= x^5 - 10x^3 + 15x, H_6 = x^6 - 15x^4 + 45x^2 - 15,\ \cdots
\end{aligned}$$

See Withers (2000) for (7). The log density has a simpler form than the density:

$$\ln\,[p_n(x)/\phi(x)] = \sum_{r=1}^{\infty} b_r(x)\, n^{-r/2},\ b_1(x) = h_1(x),$$

$$b_2(x) = -A_{11}^2/2 + (A_{22} - A_{32}A_{11})H_2/2 - A_{32}^2(3x^4 - 12x^2 + 5)/24 + A_{43}H_4/24,$$

and for $r > 1$, $b_r(x)$ is a polynomial of order only $r + 2$, while $\overline{h}_r(x)$ is of order $3r$. See Withers and Nadarajah (2010a) for $\overline{h}_r(x)$ and $b_r(x)$.

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Notation 1. *The original Edgeworth expansion was for $\hat{w}$ the mean of n independent identically distributed random variables from a distribution with rth cumulant κ_r. So (1) holds with $a_{ri} = \kappa_r I(i = r-1)$, and $A_{ii} \equiv 0$. An explicit formula for its general term was given in Withers and Nadarajah (2009).*

Ordinary Bell polynomials. For a sequence $e = (e_1, e_2, \ldots)$, *the partial ordinary Bell polynomial* $\tilde{B}_{rs} = \tilde{B}_{rs}(e)$, is defined by the identity

$$S^s = \sum_{r=s}^{\infty} z^r \tilde{B}_{rs}(e) \text{ where } S = \sum_{r=1}^{\infty} z^r e_r,\ z \in R. \tag{8}$$

$$\text{So, } \tilde{B}_{r0} = \delta_{r0},\ \tilde{B}_{r1} = e_r,\ \tilde{B}_{rr} = e_1^r,\ \tilde{B}_{21} = 2e_1e_2,$$

where $\delta_{00} = 1,\ \delta_{r0} = 0$ for $r \neq 0$. They are tabled on p309 of Comtet (1974). *The complete ordinary Bell polynomial*, $\tilde{B}_r(e)$ is defined in terms of S by

$$e^S = \sum_{r=0}^{\infty} z^r \tilde{B}_{rs}(e). \text{ So } \tilde{B}_r(e) = \sum_{s=0}^{r} \tilde{B}_{rs}(e)/s!: \tag{9}$$

$$\tilde{B}_0(e) = 1,\ \tilde{B}_1(e) = e_1,\ \tilde{B}_2(e) = e_2 + e_1^2/2,\ \tilde{B}_3(e) = e_3 + e_1e_2 + e_1^3/6. \tag{10}$$

Multivariate estimates. Suppose that $\hat{w}$ is a *standard estimate* of $w \in R^p$ with respect to n. That is, $E\,\hat{w} \to w$ as $n \to \infty$, and for $r \geq 1, 1 \leq i_1, \ldots, i_r \leq p$, the rth order cumulants of $\hat{w}$ can be expanded as

$$\bar{k}^{1-r} = \kappa(\hat{w}^{i_1}, \ldots, \hat{w}^{i_r}) = \sum_{e=r-1}^{\infty} \bar{k}_e^{1-r}\, n^{-e},\ \bar{k}_e^{1-r} = k_e^{i_1-i_r}, \tag{11}$$

where the *cumulant coefficients* $\bar{k}_e^{1-r} = k_e^{j_1-j_r}$ may depend on n but are bounded as $n \to \infty$. So $\bar{k}_0^1 = w^{i_1}$. Set $V = (k_2^{i_1i_2}),\ p \times p$. Y_n converges in law to the multivariate normal $\mathcal{N}_p(0, V)$ with $p \times p$ covariance V and distribution and density $\Phi_V(x)$ and $\phi_V(x)$. So V may depend on n, but we assume that $det(V)$ is bounded away from 0. By Withers and Nadarajah (2010b) or Withers (2024), $Y_n = n^{1/2}(\hat{w} - w)$ has distribution and density

$$Prob.(Y_n \leq x) \approx \sum_{r=0}^{\infty} n^{-r/2} P_r(x),\ p_{Y_n}(x) \approx \sum_{r=0}^{\infty} n^{-r/2} p_r(x),\ x \in R^p, \tag{12}$$

$$\text{where } (b_1)_i = b_1^i,\ P_0(x) = \Phi_V(x),\ p_0(x) = \phi_V(x), \tag{13}$$

$$P_r(x) = \tilde{B}_r(e(-\partial/\partial x))\, \Phi_V(x),\ p_r(x) = \tilde{B}_r(e(-\partial/\partial x))\, \phi_V(x),\ r \geq 1, \tag{14}$$

$$e_j(s) = \sum_{r=1}^{j+2} \bar{b}_{r+j}^{1\ldots r}\, s_{i_1} \ldots s_{i_r}/r!, \tag{15}$$

$$\bar{b}_{r+j}^{1\ldots r} = b_{r+j}^{i_1\ldots i_r},\ b_{2d+1}^{i_1\ldots i_r} = 0,\ b_{2d}^{i_1\ldots i_r} = k_d^{i_1\ldots i_r}:$$

$$e_1 = \bar{k}_1^1 \bar{t}_1 + \bar{k}_2^{1-3} \bar{t}_1 \bar{t}_2 \bar{t}_3/6,\ e_2 = \bar{k}_2^{12} \bar{t}_1 \bar{t}_2/2 + \bar{k}_3^{1-4} \bar{t}_1 \ldots \bar{t}_4/24,$$

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This gives the Edgeworth expansion for the distribution of Y_n to $O(n^{-3/2})$. See Withers (2024) for more terms. (15) uses *the tensor summation convention* of implicitly summing $i_1, \ldots, i_r$ over their range $1, \ldots, p$. For example,

$$\text{for } \partial_i = \partial/\partial x_i \text{ and } \bar\partial_k = \partial_{i_k},$$

$$P_1(x) = e_1(-\partial/\partial x))\, \Phi_V(x) = \sum_{r=1}^{3} \bar b_{r+1}^{1\ldots r}\, (-\bar\partial_1)\ldots(-\bar\partial_r)\, \Phi_V(x)/r!$$

$$= \bar k_1^1\, (-\bar\partial_1)\, \Phi_V(x) + \bar k_2^{1-3}\, (-\bar\partial_1)(-\bar\partial_2)(-\bar\partial_3)\, \Phi_V(x)/6,$$

$$p_1(x) = \bar k_1^1\, (-\bar\partial_1)\, \phi_V(x) + \bar k_2^{1-3}\, (-\bar\partial_1)(-\bar\partial_2)(-\bar\partial_3)\, \phi_V(x)/6.$$

$$(-\bar\partial_1)\ldots(-\bar\partial_k)\, \phi_V(x) = \bar H^{1-k}(x, V)\, \phi_V(x),$$

for $\bar H^{1-k} = \bar H^{1-k}(x, V)$ the multivariate Hermite polynomial. For their dual form see Withers and Nadarajah (2014). By Withers (2020), for $i = \sqrt{-1}$,

$$\bar H^{1-k}(x, V) = E\, \Pi_{j=1}^{k}(\bar y_j + i\bar Y_j) \text{ where } \bar y_j = y_{i_j},\ \bar Y_j = Y_{i_j},\ y = V^{-1}x,$$

$$Y \sim \mathcal{N}_p(0, V^{-1}). \text{ So, } H^1 = y_1,\ \bar H^1 = \bar y_1,\ H^{12} = y_1 y_2 - V^{12},\ \bar H^{12} = \bar y_1 \bar y_2 - \bar V^{12},$$

$$H^{1-3} = y_1 y_2 y_3 - \sum^{3} V^{12} y_3,$$

$$H^{1-4} = y_1 \ldots y_4 - \sum^{6} V^{12} y_3 y_4 + \sum^{3} V^{12} V^{34},$$

$$H^{1-5} = y_1 \ldots y_5 - \sum^{10} V^{12} y_3 \ldots y_5 + \sum^{15} V^{12} V^{34} y_5,$$

$$H^{1-6} = y_1 \ldots y_6 - \sum^{15} V^{12} y_3 \ldots y_6 + \sum^{45} V^{12} V^{34} y_5 y_6 - \sum^{45} V^{12} V^{34} V^{56},$$

where $V^{i_1 i_2}$ is the (i_1, i_2) element of V^{-1}, $\bar V_2^{j_1 j}$ is the (i_{j_1}, i_{j_2}) element of V^{-1}, and for example,

$$\sum^{3} V^{12} y_3 = V^{12} y_3 + V^{13} y_2 + V^{23} y_1.$$

For $r \geq 1$, we can write

$$\tilde{B}_r(e(s)) = \sum_{k=1}^{3r} [\bar{B}_r^{1-k}\ \bar{s}_1 \ldots \bar{s}_k :\ k-r \text{ even}],$$
where $\bar{B}_1^1 = \bar{k}_1^1,\ \bar{B}_1^{1-3} = \bar{k}_2^{1-3}/6,\ \bar{B}_2^{12} = \bar{k}_1^1\bar{k}_1^2 + \bar{k}_2^{12}/2,$
$\bar{B}_2^{1-4} = \bar{k}_3^{1-4}/24 + \bar{k}_1^1\bar{k}_2^{2-4}/6 + \bar{k}_1^4\bar{k}_2^{1-3}/6,\ \bar{B}_2^{1-6} = \bar{k}_2^{1-3}\bar{k}_2^{4-6}/36.$

So, $P_r(x) = \sum_{k=1}^{3r} [\bar{B}_r^{1-k}\ (-\bar{\partial}_1) \ldots (-\bar{\partial}_k)\ \Phi_V(x) :\ k-r \text{ even}],$

$$p_r(x)/\phi_V(x) = \sum_{k=1}^{3r} [\bar{B}_r^{1-k}\bar{H}^{1-k}(x,V) :\ k-r \text{ even}] = \tilde{p}_r(x) \text{ say}.$$

So, $\tilde{p}_1(x) = \bar{k}_1^1\ \bar{H}^1(x,V) + \bar{k}_2^{1-3}\ \bar{H}^{1-3}(x,V)/6,$

$$P_2(x) = \sum_{k=2,4,6} \bar{B}_2^{1-k}\ (-\bar{\partial}_1) \ldots (-\bar{\partial}_k)\ \Phi_V(x),$$

$$\tilde{p}_2(x) = \sum_{k=2,4,6} \bar{B}_2^{1-k}\ \bar{H}^{1-k}(x,V).$$

The log density can be expanded as

$$\ln\,[p_n(x)/\phi_V(x)] \approx \sum_{r=1}^{\infty} n^{-r/2} b_r(x). \tag{16}$$

$$\text{So } p_n(x)/\phi_V(x)] \approx \sum_{r=0}^{\infty} n^{-r/2}\tilde{B}_r(b(x)) \text{ where } b = (b_2, b_2, \ldots). \tag{17}$$

See Withers and Nadarajah (2016). Cornish-Fisher expansions for parametric and nonparametric standard estimates were first given in Withers (1984) and Withers (1983), and extended to functions of them in Withers (1982). In Withers and Nadarajah (2012) we gave Cornish-Fisher expansions for smooth functions of the sample cross-moments of a *linear process*. In Section 3 we show that this extends easily to a stationary process.

3. The Cumulants of the Sample Mean

Consider the general real stationary process $\cdots, X_{-1}, X_0, X_1, \cdots$ with finite mean and cross-cumulants,

$$\mu = E\ X_0,\ k(i_1, \cdots, i_r) = \kappa(X_{i_1}, \cdots, X_{i_r}). \tag{1}$$

Given a sequence of integers $i_1, \cdots, i_r$, set

$$i_0 = \min_{k=1}^{r} i_k,\ I_k = i_k - i_0 \geq 0,\ n(i_1 \ldots i_r) = I_0 = \max_{k=1}^{r} I_k = \max_{k=1}^{r} i_k - i_0. \tag{2}$$

Since $\{X_i\}$ is stationary,

$$k(i_1 \cdots i_r) = k(I_1 \cdots I_r). \tag{3}$$

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These are not changed by permuting subscripts. Also at least one I_k is zero. $E\ \bar{X} = \mu$. For $r \geq 2$, transforming from i_k to $T_k = i_k - i_1$ for $k = 2, \cdots, r$,

$$n^r \kappa_r(\bar{X}) = \sum_{i_1,\cdots,i_r=1}^{n} k(i_1, \cdots, i_r) = \sum_{|T_k|<n,\ k=2,\cdots,r} (n - \delta_r(\mathbf{T}))\ k(0, T_2, \cdots, T_r) \quad (4)$$

$$\text{where } \delta_r(\mathbf{T}) = \max(0, T_2, \cdots, T_r) - \min(0, T_2, \cdots, T_r). \quad (5)$$

For example, $\delta_2(\mathbf{T}) = |T_2|$,

$$\delta_3(\mathbf{T}) = T_3\ I(0 \leq T_2 < T_3) + (T_3 - T_2)\ I(T_2 < 0 < T_3) - T_2\ I(T_2 < T_3 \leq 0).$$

So for $r \geq 2$, $\kappa_r(\bar{X}) = \sum_{i=r-1}^{r} a_{nri}\, n^{-i}$ where

$$a_{nr,r-1} = \sum_{|T_i|<n,\ i=2,\cdots,r} k(0, T_2, \cdots, T_r),$$

$$a_{nrr} = -\sum_{|T_i|<n,\ i=2,\cdots,r} \delta_r(\mathbf{T})\ k(0, T_2, \cdots, T_r). \quad (6)$$

This proves that (1) holds. So the Edgeworth-Cornish-Fisher expansions of Section 2 apply to $(\hat{w}, w) = (\bar{X}, \mu)$ with a_{ri} in (5) replaced by these a_{nri}.

If the cross-cumulants $k(0, T_2, \cdots, T_r)$ decrease exponentially in T_2, as is true for a stationary ARMA process by Withers and Nadarajah (2012), then for $r \geq 2$,

$$a_{nr,r-1} = a_{r,r-1} + O(e^{-n\lambda_r}),\ a_{nrr} = a_{rr} + O(e^{-n\lambda_r}) \text{ where } \lambda_r > 0,$$

$$a_{r,r-1} = \sum_{|T_i|<\infty,\ i=2,\cdots,r} k(0, T_2, \cdots, T_r),$$

$$a_{rr} = -\sum_{|T_i|<\infty,\ i=2,\cdots,r} \delta_r(\mathbf{T})\ k(0, T_2, \cdots, T_r),$$

so that the Edgeworth-Cornish-Fisher expansions of Section 2 apply to $(\hat{w}, w) = (\bar{X}, \mu)$ with a_{ri} in (5) replaced by these a_{ri}.

For convergence in law of $n^{1/2}(\bar{X} - \mu))$ to $\mathcal{N}(0, a_{21})$ with $a_{21} = \sum_{T=-\infty}^{\infty} k(0, T) < \infty$, under mixing conditions on a stationary process, see Sections 18.4, 18.5 of Ibragimov and Linnik (1971). They also show how to express a_{21n} and a_{21} in terms of the spectral distribution and density.

Missing values. Now suppose that we only have observations at times $t_1, \ldots, t_n$. Our estimate of $\mu = E\ X_0$ is then

$$\hat{\mu} = \hat{\mu}(t_1, \ldots, t_n) = n^{-1} \sum_{i=1}^{n} X_{t_i}. \text{ So } E\ \hat{\mu} = \mu, \text{ and for } r \geq 2,\ S_k = t_{i_k} - t_{i_1},$$

$$n^r \kappa_r(\hat{\mu}) = \sum_{i_1,\cdots,i_r=1}^{n} k(t_{i_1}, \cdots, t_{i_r}) = \sum_{i_1,\cdots,i_r=1}^{n} k(0, S_2, \cdots, S_r) = n a_{r,r-1} \text{ say.}$$

So if a_{21} is bounded away from 0 and $a_{r,r-1}$ is bounded in n, we can apply Section 2 with $a_{ri} = 0$ for $i \geq r$.

Weighted means. Let $w_{n1}, \ldots, w_{nn}$ be given real numbers adding to n. For example the standardized form of the Chernoff weight i/n is $w_{ni} = 2i/(n+1)$. See Chernoff and Zacks(1964). An unbiased estimate of μ is *the weighted sample mean*, $\hat{\mu}_w = \sum_{i=1}^{n} w_{ni} X_i$.

$$\text{For } r \geq 2,\ n^r \kappa_r(\hat{\mu}_w) = \sum_{i_1,\cdots,i_r=1}^{n} w_{ni_1} \ldots w_{ni_r} k(t_{i_1}, \cdots, t_{i_r}) = n a_{r,r-1}$$

say. So if a_{21} is bounded away from 0 and $a_{r,r-1}$ is bounded in n, we can apply Section 2 with $a_{ri} = 0$ for $i \geq r$.

4. Multivariate Edgeworth Expansions for $\bar{X}$

Suppose that $\cdots, X_{-1}, X_0, X_1, \cdots$ lie in R^p and are stationary with finite moments. For $j = 1, \cdots, p$, denote the jth component of X_i by X_i^j and the and the cross-cumulants, by

$$\mu = E\ X_0,\ \mu^j = E\ X_0^j, \text{ and for } 1 \leq j_1, \ldots, j_r \leq p,$$
$$k\begin{pmatrix} j_1 \cdots j_r \\ i_1, \cdots, i_r \end{pmatrix} = \kappa(X_{i_1}^{j_1}, \cdots, X_{i_r}^{j_r}). \tag{1}$$

Given a sequence of integers $i_1, \cdots, i_r$, define i_0, I_k as in (2), and again transform from i_k to $T_k = i_k - i_1$ for $k = 2, \cdots, r$. (3) becomes

$$k\begin{pmatrix} j_1 \cdots j_r \\ i_1 \cdots i_r \end{pmatrix} = k\begin{pmatrix} j_1 \cdots j_r \\ I_1 \cdots I_r \end{pmatrix}. \tag{2}$$

In general $k(\substack{j_1 j_2 \\ 0,I}) \neq k(\substack{j_1 j_2 \\ 0,-I})$. By (4),

$$n^r \kappa(\bar{X}^{j_1}, \ldots, \bar{X}^{j_r}) = \sum_{i_1,\cdots,i_r=1}^{n} k\begin{pmatrix} j_1 \cdots j_r \\ i_1, \cdots, i_r \end{pmatrix}$$
$$= \sum_{|T_k|<n,\ k=2,\cdots,r} (n - \delta_r(\mathbf{T}))\ k\begin{pmatrix} j_1 \cdots j_r \\ 0, T_2, \cdots, T_r \end{pmatrix}. \tag{3}$$

So for $r \geq 2$, $\kappa(\bar{X}^{j_1}, \ldots, \bar{X}^{j_r}) = \sum_{e=r-1}^{r} k_{ne}^{j_1 \cdots j_r}\ n^{-e}$ where

$$k_{n,r-1}^{j_1 \cdots j_r} = \sum_{|T_i|<n,\ i=2,\cdots,r} k\begin{pmatrix} j_1 \cdots j_r \\ 0, T_2, \cdots, T_r \end{pmatrix},$$
$$\text{and } k_{nr}^{j_1 \cdots j_r} = -\sum_{|T_i|<n,\ i=2,\cdots,r} \delta_r(\mathbf{T})\ k\begin{pmatrix} j_1 \cdots j_r \\ 0, T_2, \cdots, T_r \end{pmatrix}.$$

This proves that a 2 term version of (11) holds. So the expansions (12)–(14) and (16) hold for the distribution and density of $n^{1/2}(\bar{X} - \mu)$.

If the cross-cumulants $(\substack{k(j_1 \cdots j_r \\ 0, T_2, \cdots, T_r)})$ decrease exponentially in T_2, then for $r \geq 2$,

$$k_{n,r-1}^{j_1 \cdots j_r} = k_{r-1}^{j_1 \cdots j_r} + O(e^{-n\lambda_r}), \text{ and } k_{nr}^{j_1 \cdots j_r} = k_r^{j_1 \cdots j_r} + O(e^{-n\lambda_r}) \text{ where } \lambda_r > 0,$$
$$k_{r-1}^{j_1 \cdots j_r} = \sum_{|T_i|<\infty,\ i=2,\cdots,r} k\begin{pmatrix} j_1 \cdots j_r \\ 0, T_2, \cdots, T_r \end{pmatrix},$$
$$\text{and } k_r^{j_1 \cdots j_r} = -\sum_{|T_i|<\infty,\ i=2,\cdots,r} \delta_r(\mathbf{T})\ k\begin{pmatrix} j_1 \cdots j_r \\ 0, T_2, \cdots, T_r \end{pmatrix}.$$

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