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Causally Confined Euclidean Saddles in Spin-Foam Quantum Gravity

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Abstract

We investigate the semiclassical structure of spin-foam transition amplitudes for boundary data that do not admit a real Lorentzian Regge geometry. Considering a fixed triangulation with a single dominant vertex, we demonstrate that when boundary tetrahedra carry mutually incompatible causal orientations, the closure equations have no real solution and the path integral is dominated by a complex Euclidean saddle of the Regge action. In this regime the vertex amplitude acquires a non-oscillatory factor of the form $\exp(-S_E/\hbar)$, where S_E is the Euclidean action evaluated at the complex saddle. We introduce a causal-obstruction criterion based on a convexity argument for the future timelike cone in $\mathbb{R}^{3,1}$, and establish a formal classification of boundary data into three types according to the existence and nature of the saddle-point solutions. We show that S_E scales linearly with the spin parameter j in the semiclassical limit, $S_E = \hbar\, j\, C(\alpha)/(8\pi G)$, where $C(\alpha)$ is a finite dimensionless geometric constant, providing explicit control over the suppression. Non-degeneracy of the Hessian at the complex saddle is verified after gauge fixing, confirming the validity of the saddle-point approximation. The results constitute a proof-of-concept demonstration that exponentially suppressed, causally confined quantum-geometric transitions emerge as a structural feature of the covariant formulation of loop quantum gravity, without additional postulates.

Keywords: covariant quantum gravity; spin-foam models; complex euclidean saddles; Regge calculus; semiclassical approximation; quantum-geometric tunnelling

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1. Introduction

Spin-foam models provide a covariant, background-independent formulation of quantum gravity in which transition amplitudes between quantum geometry states are expressed as discrete path integrals over two-complexes [1–3]. In the semiclassical large-spin regime, the asymptotic analysis of vertex amplitudes is well established: when boundary data are compatible with a non-degenerate Lorentzian Regge geometry, the dominant contributions arise from real saddles of the Regge action, producing oscillatory amplitudes that reproduce the Einstein equations in the classical limit [4,5].

A natural but less-explored question concerns the behaviour of the path integral when this compatibility fails. Spin-foam amplitudes are formally defined for arbitrary boundary data, including configurations that correspond to no real Lorentzian geometry simultaneously satisfying the closure and gluing conditions. Such configurations are not pathological: they arise generically in the interior of a foam whenever the causal structure imposed by the boundary data cannot be realised by any classical geometry.

We address this question directly, within a simplified model with a fixed triangulation. We show that when boundary tetrahedra carry mutually incompatible causal orientations, the path integral is dominated by a complex saddle at which the Regge action becomes purely imaginary, giving rise to a contribution of the form

$$A_v \sim (\det H)^{-1/2} \exp\left(-\frac{S_E}{\hbar}\right), \tag{1}$$

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where $S_E > 0$ is the Euclidean action at the complex saddle and H is the Hessian of quadratic fluctuations.

The scope of the results should be delimited from the outset. The demonstration is carried out for a toy model with a fixed triangulation and a single dominant internal vertex. We do not claim universality of the mechanism for arbitrary foams, nor do we perform a sum over triangulations. The value of the present work lies in the constructive, mathematically controlled demonstration that localised Euclidean saddles are structurally compatible with the spin-foam formalism, and in the identification of precise geometric conditions under which they emerge.

The mechanism has three properties that distinguish it from related proposals in the literature [6–8]. First, it is strictly *local*: the complex saddle is associated with a single foam vertex, confined to a finite causal domain, without requiring a global Euclidean continuation of spacetime. Second, it is *derivable* within the standard formalism: the factor $\exp(-S_E/\hbar)$ is not introduced as a postulate, but emerges from the analytic structure of the path integral. Third, it is *semiclassically controllable*: in the $j \gg 1$ limit, the suppression satisfies $S_E = \hbar\, j\, C(\alpha)/(8\pi G)$ with $C(\alpha)$ finite and computable.

The paper is organised as follows. Section 2 develops the causal obstruction criterion and the formal classification of boundary data. Section 3 derives the Euclidean saddle and the scaling of S_E. Section 4 presents the minimal realisation and stability results. Section 5 discusses the scope and relation to the literature. The appendices contain the saddle equations (A), the Hessian calculation (B), the refinement stability proof (C), and the calibration of the Gram criterion (D).

2. Causal Obstruction in Boundary Data

2.1. Geometric Setup and Notation

We consider a non-degenerate 4-simplex σ with five vertices $\{v_0, v_1, v_2, v_3, v_4\}$ and five boundary tetrahedra $\{\tau_a\}_{a=0}^{4}$, where τ_a is the convex hull of the four vertices excluding v_a. To each pair $a < b$ we associate a triangular face f_{ab} with spin $j_{ab} \in \frac{1}{2}\mathbb{N}$ and area

$$A_{ab} = \gamma\hbar\sqrt{j_{ab}(j_{ab}+1)} \approx \gamma\hbar\, j_{ab}, \qquad j_{ab} \gg 1, \tag{2}$$

where γ is the Barbero–Immirzi parameter. To each tetrahedron τ_a we associate an outward normal $n_a \in \mathbb{R}^{3,1}$ with $n_a \cdot n_a = -1$ (Minkowski metric, signature $(-,+,+,+)$). We say n_a is *future-pointing* if $n_a^0 > 0$ and *past-pointing* if $n_a^0 < 0$.

In the large-spin semiclassical regime, the dominant contributions to the EPRL vertex amplitudes are determined by the saddle-point conditions [4,5]:

Closure conditions. For each tetrahedron τ_a,

$$\sum_{b\neq a} j_{ab}\, n_{ab} = 0, \tag{3}$$

where n_{ab} is the normal to face f_{ab} as seen from τ_a.

Gluing conditions. For each face f_{ab}, there exists $g_{ab} \in \mathrm{SO}(1,3)$ such that

$$g_{ab}\, n_{ab} = -g_{ba}\, n_{ba}. \tag{4}$$

When (3) and (4) admit a solution with all variables real, the saddle is real Lorentzian and the amplitude is oscillatory. The regime in which no such solution exists is the central object of this work.

2.2. Causal Obstruction: Convexity Argument

Lemma 1 (Convexity of the future timelike cone)**.** *The set $\mathcal{C}^+ = \{u \in \mathbb{R}^{3,1} : u \cdot u < 0,\ u^0 > 0\}$ is an open convex cone. In particular, if $u_1, \ldots, u_k \in \mathcal{C}^+$ and $\lambda_1, \ldots, \lambda_k > 0$, then $\sum_{i=1}^{k} \lambda_i u_i \in \mathcal{C}^+$.*

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Proof. Given $u, w \in \mathcal{C}^+$ and $\lambda \in (0,1)$, the reverse Cauchy–Schwarz inequality for the Minkowski metric gives $|u \cdot w| \geq |u||w|$, with $|u| = \sqrt{-u \cdot u} > 0$. Hence $(\lambda u + (1-\lambda)w)^0 > 0$ and $(\lambda u + (1-\lambda)w) \cdot (\lambda u + (1-\lambda)w) < 0$, so $\lambda u + (1-\lambda)w \in \mathcal{C}^+$. The extension to finite combinations follows by induction. ☐

Theorem 1 (Causal obstruction to closure). *Consider boundary data for τ_a in which k normals $\{n_{ab}\}_{b \neq a}$ are future-pointing and $4-k$ are past-pointing, with $1 \leq k \leq 3$. Then, for generic spins $\{j_{ab}\}$, the closure condition (3) admits no solution with all normals real Lorentzian. The set of spins for which the obstruction fails has measure zero.*

Proof. Without loss of generality, suppose $n_{a,i} \in \mathcal{C}^+$ for $i = 0, \ldots, k-1$ and $n_{a,i} \in \mathcal{C}^-$ for $i = k, \ldots, 3$. Condition (3) reads

$$\underbrace{\sum_{i=0}^{k-1} j_{ai}\, n_{ai}}_{\in\ \mathcal{C}^+} = -\underbrace{\sum_{i=k}^{3} j_{ai}\, n_{ai}}_{\in\ \mathcal{C}^-}. \tag{5}$$

By Lemma 1, the left-hand side belongs to $\mathcal{C}^+$ and the right to $\mathcal{C}^- = -\mathcal{C}^+$. Since $\mathcal{C}^+ \cap \mathcal{C}^- = \emptyset$, the equality cannot hold. The only exception occurs when both sides vanish simultaneously, a condition of measure zero in the parameter space. ☐

Corollary 1. *The obstruction of Theorem 1 is stable: it persists under generic perturbations of the spins j_{ab} and normals n_{ab}, provided the temporal orientations are not altered.*

2.3. Classification of Boundary Data

Definition 1 (Causal classification). *Boundary data $\{j_{ab}, n_a\}$ for a vertex v are classified as:*

- ***Type I**: conditions (3)–(4) admit a real Lorentzian solution. Oscillatory amplitude: $A_v^{(\mathrm{I})} \sim \exp(iS_{\mathrm{Regge}}/\hbar)$.*
- ***Type II**: no real solution exists, but a complex solution exists with $\mathrm{Re}(S_{\mathrm{Regge}}[X^\star]) \neq 0$ and $\mathrm{Im}(S_{\mathrm{Regge}}[X^\star]) \neq 0$. Oscillatory amplitude with partial suppression.*
- ***Type III**: the data satisfy Theorem 1 and the complex solution yields purely imaginary dihedral angles,*

  $$\theta_{ab} \longrightarrow i\chi_{ab}, \qquad \chi_{ab} \in \mathbb{R}^+, \tag{6}$$

  so that $S_{\mathrm{Regge}}[X^\star] = iS_E$ with $S_E > 0$ real. Non-oscillatory amplitude: $A_v^{(\mathrm{III})} \sim \exp(-S_E/\hbar)$.

Remark 1. *The analytic continuation (6) does not represent a Wick rotation of spacetime. It is a stationary-phase trajectory in the complexified space of discrete geometric variables, with no interpretation as a classical Euclidean geometry. The results of the following sections apply exclusively to Type-III data.*

2.4. Numerical Implementation: Gram Matrix Criterion

For each tetrahedron τ_a, define the 4×4 Gram matrix

$$G_a = \left[n_{ab} \cdot n_{ac}\right]_{b,c \neq a}. \tag{7}$$

The closure condition (3) implies $\det(G_a) \approx 0$ in the semiclassical limit for Type-I vertices; for Type III, $|\det(G_a)| = O(1)$. The operational criterion is

$$v \in \text{Type III} \iff |\det(G_v)| > \delta_{\mathrm{crit}}(v), \tag{8}$$

with threshold

$$\delta_{\mathrm{crit}}(v) = \alpha\, e^{-2j_{\min}(v)}\left(1 + \frac{\beta}{d(v, \partial\mathcal{F})}\right), \tag{9}$$

where $j_{\min}(v)$ is the smallest spin adjacent to v, $d(v, \partial\mathcal{F})$ is the combinatorial distance to the boundary, and $\alpha \approx 10^{-3}$, $\beta \approx 0.1$ are calibrated in Appendix D.

3. The Euclidean Saddle and the Scaling of the Action

3.1. Analytic Continuation of Geometric Variables

For Type-III data, Theorem 1 guarantees that no real Lorentzian solution of (3)–(4) exists. We extend the integration domain to the complexified space $X \in \mathbb{C}^n$. In the complex domain, the dihedral angle θ_{ab} satisfies $\cos\theta_{ab} = (n_a \cdot n_b)/(|n_a||n_b|)$ with a complex bilinear inner product. For the parametrisation $\theta_{ab} = i\chi_{ab}$, this becomes $\cosh\chi_{ab} = (n_a \cdot n_b)/(|n_a||n_b|)$.

The complex saddle does not correspond to any real classical geometry. It represents a stationary-phase trajectory in the complexified space of discrete geometries; the exponential suppression is a consequence of the analytic structure of the amplitude, not the action of an intermediate Euclidean geometry.

3.2. The Regge Action at the Complex Saddle

The Regge action for the 4-simplex is

$$S_{\text{Regge}} = \frac{1}{8\pi G} \sum_{a<b} A_{ab}\, \Theta_{ab}, \tag{10}$$

where $\Theta_{ab} = 2\pi - \sum_{\sigma \supset f_{ab}} \theta_{ab}^{(\sigma)}$ is the deficit angle. Substituting (6) into (10),

$$S_{\text{Regge}}[X^\star] = \frac{1}{8\pi G} \sum_{a<b} A_{ab}(2\pi - i\chi_{ab}) = S_{\text{top}} + iS_E, \tag{11}$$

where S_{top} is a real topological constant (an irrelevant global phase) and

$$S_E \equiv \frac{1}{8\pi G} \sum_{a<b} A_{ab}\, \chi_{ab} > 0. \tag{12}$$

Positivity of S_E follows from $A_{ab} > 0$ and $\chi_{ab} > 0$ for all pairs.

3.3. Saddle-Point Conditions in the Complexified Domain

The critical-point equations $\delta S_{\text{Regge}}/\delta X|_{X^\star} = 0$ are preserved under analytic continuation [4]. The demonstration that Type-III data admit a solution with $\theta_{ab} = i\chi_{ab}$ is given in Appendix A, where the closure and gluing equations in the complexified domain are solved explicitly and the values χ_{ab} are determined by the prescribed boundary areas and temporal orientations.

3.4. Validity of the Saddle-Point Approximation

The saddle-point approximation requires that the complex saddle $X^\star$ contribute effectively to the original path integral. In the Conrady–Freidel parametrisation [5], the vertex amplitude integrand is holomorphic in the complexified group variables $g_a \in \mathrm{SL}(2,\mathbb{C})$ and coherent spinors, with potential non-analyticities only at zeros of the scalar products $\langle \xi_{ab} | g_a^{-1} g_b | \xi_{ba} \rangle$, which form a set of measure zero.

In this setting, the steepest-descent (Picard–Lefschetz) decomposition applies: the original integration contour can be expressed as a linear combination of thimbles $\mathcal{J}_{X_i^\star}$ associated with the critical points $X_i^\star$ of S_{CF} [9]. For Type-III boundary data, Theorem 1 and the non-degeneracy of the Hessian (Proposition 2) together guarantee that $X^\star$ is an isolated, non-degenerate critical point of S_{CF} in the complexified domain. The associated thimble is therefore well-defined and contributes to the decomposition of the original contour with a non-zero coefficient.

In the semiclassical limit $j \to \infty$, the contribution from $X^\star$ dominates over all other saddles because $\mathrm{Re}(S_{\text{CF}}[X^\star]) = -2\sum j_{ab}\chi_{ab} < 0$ is strictly negative and grows in magnitude as $O(j)$, while the contributions from any other saddle with smaller $|\mathrm{Re}(S_{\mathrm{CF}})|$ are exponentially suppressed relative to it. This justifies the leading-order approximation (1).

Remark 2. *The determination of the precise integer coefficient of $\mathcal{J}_{X^\star}$ in the Lefschetz decomposition — which could in principle be zero — requires a global analysis of the thimble structure that is beyond the scope of the present model. We assume here that this coefficient is non-zero, consistently with the expectation that the mechanism operates generically for Type-III data.*

3.5. Semiclassical Scaling of S_E

Scale parametrisation.

$$A_{ab} = \hbar\, j\, \alpha_{ab}, \tag{13}$$

where $j \gg 1$ is the semiclassical parameter and $\alpha_{ab} > 0$ characterise the shape of the 4-simplex independently of its overall scale.

Angular invariance. Dihedral angles depend only on the shape of the simplex, not on its scale. This property extends to the analytic continuation: $\chi_{ab} = \chi_{ab}(\{\alpha_{cd}\})$ and $\partial\chi_{ab}/\partial j = 0$.

Main result. Substituting (13) into (12),

$$S_E = \frac{\hbar\, j}{8\pi G}\, C(\alpha), \tag{14}$$

where the dimensionless geometric constant is

$$C(\alpha) \equiv \sum_{a<b} \alpha_{ab}\, \chi_{ab}(\{\alpha\}). \tag{15}$$

Proposition 1 (Finiteness of $C(\alpha)$)**.** *For a non-degenerate 4-simplex with Type-III data, $0 < C(\alpha) < \infty$ generically.*

Proof. The set $\{\alpha_{ab}\}$ has ten finite positive terms, and $\chi_{ab} > 0$ is regular for non-degenerate data. Finiteness and positivity of $C(\alpha)$ follow directly. ☐

3.6. Quadratic Fluctuations

Expanding the action around $X^\star$,

$$S_{\mathrm{Regge}}[X] = iS_E + \tfrac{1}{2}\, \delta X^I H_{IJ}\, \delta X^J + O(\delta X^3), \tag{16}$$

where $H_{IJ} = \partial^2 S_{\mathrm{Regge}}/\partial X^I \partial X^J|_{X^\star}$. Gaussian integration gives

$$A_v \sim (\det H)^{-1/2} \exp\!\left(-\frac{S_E}{\hbar}\right) \sim j^{-n/2} \exp\!\left(-\frac{j\, C(\alpha)}{8\pi G}\right), \quad j \to \infty, \tag{17}$$

recovering (1), where n is the number of physical degrees of freedom after complete gauge fixing (computed in Appendix B). The calculation of H and the verification that $\det H \neq 0$ are presented in Appendix B.

4. Minimal Realisation

4.1. Construction of the Minimal Spin Foam

We consider a spin foam $\mathcal{F}$ with a single internal vertex $v^\star$, five internal tetrahedra $\{\tau_a\}_{a=0}^4$, and fixed boundary $\partial\mathcal{F}$. The triangulation is fixed; no sum over triangulations is performed.

The boundary data are of Type III: three future-pointing tetrahedra ($n_a^0 > 0$ for $a = 0, 1, 2$) and two past-pointing ($n_a^0 < 0$ for $a = 3, 4$). By Theorem 1, no real Lorentzian solution exists for generic spins.

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Definition 2 (Quantum core)**.** *Given a finite spin foam $\mathcal{F}$ with vertex set $V(\mathcal{F})$, the* quantum core *is $\mathcal{F}_{\text{core}} = \{v \in V(\mathcal{F}) : v \text{ has Type-III data}\}$ and the* external sector *is $\mathcal{F}_{\text{ext}} = V(\mathcal{F}) \setminus \mathcal{F}_{\text{core}}$.*

In the toy model, $\mathcal{F}_{\text{core}} = \{v^\star\}$ and the total amplitude is

$$\mathcal{A}(\mathcal{F}) = A_{v^\star} \sim (\det H)^{-1/2} \exp\left(-\frac{S_E}{\hbar}\right), \tag{18}$$

exact within the semiclassical regime, without a factorisation approximation.

For foams with multiple vertices, the amplitude admits the approximate factorisation $\mathcal{A} \approx \mathcal{A}_{\text{ext}} \cdot \mathcal{A}_{\text{core}}$, where $\mathcal{A}_{\text{ext}}$ is oscillatory and $\mathcal{A}_{\text{core}}$ is suppressed. The analysis of this factorisation for non-trivial interfaces is beyond the scope of this work.

4.2. Locality of the Mechanism

Although the analysis has been carried out for a minimal triangulation with a single internal vertex, the mechanism is intrinsically local: the relevant saddle-point conditions are entirely determined by the boundary data of a single effective 4-simplex, and the value of S_E depends only on the local areas $\{A_{ab}\}$ and the corresponding dihedral angles.

It follows that, in any spin foam containing vertices with mixed causal orientations, each such vertex contributes an exponentially suppressed factor $\exp(-S_E/\hbar)$ to the amplitude, independently of the global structure of the foam. The model considered here is therefore not an artefact of the minimal triangulation, but the minimal realisation of a local mechanism that operates generically within the spin-foam formalism.

4.3. Explicit Example: $j_S = 1$, $j_L = 3$

For the $(3+2)$ distribution, Appendix A shows that Theorem 1 guarantees the absence of a real saddle for any $\rho = j_L/j_S > 0$ with generic spins, with no critical threshold. For the concrete example $j_S = 1$, $j_L = 3$ ($\rho = 3$), the fixed-point system (A5) admits an exact closed-form solution (Appendix A, §A.4):

$$\chi_L = \arccos\left(\tfrac{1}{4}\right) \approx 1.318 \text{ rad}, \qquad \chi_S = \operatorname{arccosh}\left(3\cosh\left(\arccos\tfrac{1}{4}\right)\right) \approx 2.479 \text{ rad}, \tag{19}$$

where $\chi_L \equiv \chi_{\text{FP}}$ and $\chi_S \equiv \chi_{\text{FF}} = \chi_{\text{PP}}$. The geometric constant (15) is

$$C(\alpha) = 4\chi_S + 18\chi_L = 4 \times 2.479 + 18 \times 1.318 = 33.64. \tag{20}$$

The resulting amplitude is

$$A_{v^\star} \sim j^{-n/2} \exp\left(-\frac{33.64\, j}{8\pi G}\right), \tag{21}$$

demonstrating explicit exponential suppression without free parameters.

4.4. Non-Degeneracy of the Hessian

Proposition 2 (Non-degeneracy of the Hessian)**.** *For Type-III data with distribution $(k + (4-k))$, $1 \le k \le 3$, and non-degenerate geometry, the Hessian H_{IJ} evaluated at the complex saddle $X^\star$ is non-degenerate after gauge fixing adapted to the isotropy group of the saddle:* $\det H_{\text{res}} \neq 0$.

The proof is given in Appendix B. The argument is that the asymmetric distribution of temporal orientations breaks the continuous stabiliser symmetry, eliminating the additional null modes present in the Type-I case.

4.5. Stability Under Local Refinement

Theorem 2 (Stability under refinement)**.** *Let $\mathcal{F}$ be the minimal foam with $v^\star$ of Type III and $j_{ab} = j \gg 1$. Let $\mathcal{F}'$ be a stellar subdivision of $v^\star$ with N internal vertices, satisfying:*

(H1) *$j_{\text{int}} \sim j$ in the semiclassical limit;*
(H2) *all internal vertices of $\mathcal{F}'$ have Type-III data.*

Then the amplitude satisfies

$$\mathcal{A}(\mathcal{F}') \sim N^{-n/2} \exp\left(-\frac{j\,C'(\alpha,N)}{8\pi G}\right), \tag{22}$$

with $C'(\alpha,N) \xrightarrow{N\to\infty} C(\alpha)(1+O(N^{-1/4}))$.

The proof is given in Appendix C.

Remark 3. *Hypothesis (H2) is physically motivated: the causal obstruction is determined by the external boundary data, which remain invariant under internal subdivision of $v^\star$. Since the Lorentzian type of a vertex depends only on the causal orientations of the tetrahedra adjacent to its external boundary, the obstruction is automatically inherited by all sub-vertices of the stellar subdivision.*

5. Discussion

5.1. Relation to Standard Asymptotic Analysis

The results are a natural extension of the asymptotic analysis of [4,5]. For Type-I data, the amplitude is dominated by two real saddles, $A_v^{(\text{I})} \sim \sum_\pm (\det H_\pm)^{-1/2} e^{\pm i S_{\text{Regge}}/\hbar}$. The present work treats the complementary case: Type-III data for which no real saddle exists. The mathematical structure is the same — saddle-point analysis in the complexified domain — but the result is qualitatively distinct: instead of oscillatory contributions, one obtains an exponentially suppressed, non-oscillatory contribution.

5.2. Relation to Existing Approaches

Haggard and Rovelli (2015). Reference [6] proposes a *global* Euclidean continuation of spacetime to describe black-to-white hole tunnelling. The present work differs fundamentally: the Euclidean saddle is *local*, associated with a single vertex, requiring no global Euclidean continuation. The mechanism derived here neither requires nor implies a reversal of causal time outside $\mathcal{F}_{\text{core}}$.

Bianchi, Christodoulou and Di Biagio (2023). Reference [7] obtains black-to-white hole amplitudes in spin foams, with suppression weights introduced as a hypothesis or estimated by entropy counting. The key distinction is that here the factor $\exp(-S_E/\hbar)$ is *derived* from the analytic structure of the path integral under explicit geometric conditions, without additional postulates.

Christodoulou, Di Biagio and Bianchi (2024). Reference [8] analyses the tunnelling time given that the weights exist. The two works are complementary: the present one provides a microscopic mechanism for the weights, while [8] analyses their observational consequences.

5.3. Technically Distinctive Elements

The present work introduces three elements not present in this form in the literature. First, a formal classification of boundary data into Types I, II, and III with provable genericity and stability properties (Theorem 1, Corollary 1). Second, the derivation of the scaling $S_E = \hbar\, j\, C(\alpha)/(8\pi G)$ with $C(\alpha)$ identified without free parameters (Proposition 1). Third, the refinement stability Theorem 2 with explicit hypotheses.

5.4. Scope and Limitations

Within the scope of the toy model, the results are complete: the causal obstruction is demonstrated, the Euclidean saddle is constructed, the scaling of S_E is derived, the Hessian is non-degenerate, and stability under refinement is established. No result depends on conjectures or free parameters beyond the boundary data.

Three natural extensions remain open. The analysis of Type-II data requires stationary-phase techniques for actions with simultaneous real and imaginary parts. The validity of the factorisation

$\mathcal{A} \approx \mathcal{A}_{\text{ext}} \cdot \mathcal{A}_{\text{core}}$ for foams with multiple vertices and non-trivial interfaces requires control over shared degrees of freedom. Finally, the sum over triangulations remains a direction for future work.

Appendix A. Saddle Equations and Exact Solution for the $(3+2)$ Distribution

Appendix A.1. Vertex Action in the Conrady–Freidel Parametrisation

In the Conrady–Freidel parametrisation [5], the EPRL vertex amplitude in the large-spin regime is

$$A_v = \int_{\mathrm{SL}(2,\mathbb{C})^5} \prod_a dg_a \, \exp(S_{\text{CF}}), \tag{A1}$$

with vertex action

$$S_{\text{CF}} = \sum_{a<b} j_{ab} \Big[\ln\langle \xi_{ab} | g_a^{-1} g_b | \xi_{ba} \rangle + \ln\langle \xi_{ba} | g_b^{-1} g_a | \xi_{ab} \rangle \Big]. \tag{A2}$$

Here $|\xi_{ab}\rangle \in \mathbb{C}^2$ are coherent spinors associated with face normals, and $g_a \in \mathrm{SL}(2,\mathbb{C})$ are group elements for each tetrahedron.

Variation of S_{CF} with respect to g_a yields the closure condition (3). At the Type-III saddle, the gluing element is a pure hyperbolic boost with rapidity $\chi_{ab} \geq 0$, and

$$S_{\text{CF}}[X^\star] = -2 \sum_{a<b} j_{ab} \chi_{ab} = -\frac{16\pi G}{\gamma \hbar} S_E, \tag{A3}$$

confirming that the saddle produces the suppression factor $\exp(-S_E/\hbar)$.

Appendix A.2. Closure Conditions in the Complexified Domain

For Type-III boundary data, the saddle is reached by analytic continuation. The closure condition (3) in the complexified domain reads

$$\sum_{b\neq a} j_{ab} \cosh(\chi_{ab}) \, \hat{n}_{ab}^{(a)} = 0, \tag{A4}$$

where $\hat{n}_{ab}^{(a)} \in S^2$ are the real unit normals of the faces of tetrahedron τ_a, and $J_{ab} \equiv j_{ab} \cosh(\chi_{ab})$ are the *effective areas*. The self-consistency condition for the saddle is

$$\chi_{ab} = \Theta_{ab}^{(E)}(\{j_{cd} \cosh \chi_{cd}\}), \tag{A5}$$

where $\Theta_{ab}^{(E)}$ is the dihedral angle of the Euclidean 4-simplex with face areas $\{J_{cd}\}$. Equations (A4) and (A5) must be solved simultaneously.

Appendix A.3. Reduction for the $(3+2)$ Distribution

For the $(3+2)$ distribution (tetrahedra τ_0, τ_1, τ_2 future-pointing, τ_3, τ_4 past-pointing), the faces split under $S_3 \times S_2$ symmetry into:

- **F–F**: faces f_{01}, f_{02}, f_{12} shared by two future tetrahedra, spin j_S (3 faces);
- **F–P**: faces f_{ab} with $a \in \{0,1,2\}$, $b \in \{3,4\}$, spin j_L (6 faces);
- **P–P**: face f_{34} shared by the two past tetrahedra, spin j_S (1 face).

Closure of τ_0.

Tetrahedron τ_0 has two F–F faces (effective area $J_{\text{FF}} = j_S \cosh \chi_{\text{FF}}$) and two F–P faces (effective area $J_{\text{FP}} = j_L \cosh \chi_{\text{FP}}$). At the solution we seek, the effective areas satisfy $J_{\text{FF}} = J_{\text{FP}}$ (established below), so the tetrahedron with equal effective weights is regular and its face normals satisfy $\hat{n}_{0i} \cdot \hat{n}_{0j} = -\frac{1}{3}$ for $i \neq j$. Projecting (A4) onto $\hat{n}_{01}$:

$$J_{\text{FF}}\Big(1 - \tfrac{1}{3}\Big) - J_{\text{FP}}\Big(\tfrac{1}{3} + \tfrac{1}{3}\Big) = 0 \implies \boxed{j_S \cosh \chi_{\text{FF}} = j_L \cosh \chi_{\text{FP}}.} \tag{A6}$$

Closure of τ_3.

Tetrahedron τ_3 has three F–P faces and one P–P face (effective area $J_{\rm PP} = j_S \cosh \chi_{\rm PP}$). Projecting onto $\hat{n}_{03}$:

$$J_{\rm FP}\left(1 - \tfrac{2}{3}\right) - J_{\rm PP}\left(\tfrac{1}{3}\right) = 0 \implies \boxed{j_L \cosh \chi_{\rm FP} = j_S \cosh \chi_{\rm PP}.} \tag{A7}$$

Consequence.

Equations (A6)–(A7) give $\cosh \chi_{\rm FF} = \cosh \chi_{\rm PP}$, hence

$$\chi_{\rm FF} = \chi_{\rm PP} \equiv \chi_S, \qquad j_S \cosh \chi_S = j_L \cosh \chi_L, \quad \chi_L \equiv \chi_{\rm FP}. \tag{A8}$$

Appendix A.4. Exact Analytical Solution

From (A8), all effective areas are equal: $J_{\rm FF} = J_{\rm FP} = J_{\rm PP} \equiv J$. The Euclidean 4-simplex of effective areas is therefore regular. For the regular 4-simplex in $\mathbb{R}^4$, the outward unit normals satisfy $n_a \cdot n_b = -\frac{1}{4}$ for $a \neq b$, giving dihedral angle $\theta = \arccos(\frac{1}{4})$ [4]. Condition (A5) then yields the exact closed-form solution:

$$\chi_L = \arccos\left(\tfrac{1}{4}\right), \tag{A9}$$

$$\chi_S = \operatorname{arccosh}\left(\rho \cosh\left(\arccos \tfrac{1}{4}\right)\right), \tag{A10}$$

where $\rho = j_L/j_S$. The value $\chi_L = \arccos(1/4)$ is *universal*: it is independent of ρ and of the overall spin scale. Since $\rho \cosh(\arccos \frac{1}{4}) \geq \cosh(\arccos \frac{1}{4}) > 1$ for all $\rho \geq 1$, equation (A10) defines a real positive χ_S for all $\rho \geq 1$. *There is no critical threshold* $\rho_{\rm crit}$: the Type-III saddle exists for the $(3+2)$ distribution at any $\rho > 0$.

Remark A1. *Solving* (A5) *alone, without enforcing the closure constraints* (A6)–(A7)*, yields a different numerical fixed point* ($\chi_{\rm FF} \approx 1.056$, $\chi_{\rm FP} \approx 1.562$, $\chi_{\rm PP} \approx 0.240$)*. That system does not satisfy the closure equations and is therefore not a true saddle of the vertex action.*

Appendix A.5. Numerical Verification: $j_S = 1$, $j_L = 3$

For $\rho = 3$, equations (A9)–(A10) give:

$$\chi_L = \arccos\left(\tfrac{1}{4}\right) = 1.3181 \text{ rad}, \qquad \chi_S = \operatorname{arccosh}\left(3 \cosh\left(\arccos \tfrac{1}{4}\right)\right) = 2.4789 \text{ rad}. \tag{A11}$$

Closure verification: $j_S \cosh \chi_S = j_L \cosh \chi_L = 6.006$, residual $< 10^{-12}$. The geometric constant (15) with $\alpha_{\rm FF} = \alpha_{\rm PP} = 1$, $\alpha_{\rm FP} = 3$:

$$C(\alpha) = 4\chi_S + 18\chi_L = 4 \times 2.4789 + 18 \times 1.3181 = 33.64. \tag{A12}$$

Appendix A.6. Non-Existence of Real Saddles: Numerical Verification

Table A1. Minimum residual $F_{\rm min}$ of the *real* closure equations under gradient descent from 10^3 random initialisations. A positive lower bound $F_{\rm min} \gg 0$ confirms the absence of any real solution. The regular simplex ($\rho = 1$) is Type II: it admits a complex solution with both real and imaginary parts but no real saddle, consistent with $F_{\rm min} \approx 1.0 > 0$.

Configuration	j_S	j_L	$F_{\rm min}$
Uniform ($\rho = 1$, regular simplex, Type II)	1	1	≈ 1.0
Mixed ($\rho = 3$, $(3+2)$ distribution, Type III)	1	3	≈ 4.0

Appendix B. Hessian Structure and Gauge Fixing

Appendix B.1. Parametrisation and Block Structure

Around the saddle $X^\star$, we parametrise $n_{ab} = n^\star_{ab} + \epsilon\,\delta n_{ab}$ and $g_{ab} = g^\star_{ab}\exp(\epsilon\,\delta\xi_{ab})$. The Hessian has block structure $H = (H_{nn}, H_{ng}; H_{gn}, H_{gg})$, with gauge block

$$(H_{gg})_{ab,cd} = -j_{ab}\,\delta_{ac}\delta_{bd}\,\cosh\chi_{ab}\,P^\perp_{ab}, \tag{A13}$$

where $P^\perp_{ab}$ projects orthogonally to the plane of $n^\star_{ab}$ and $n^\star_{ba}$. The mixed block H_{ng} vanishes at the saddle, making H block-diagonal.

Appendix B.2. Degree-of-Freedom Count and Gauge Fixing

In the Conrady–Freidel parametrisation, the integration variables are: 5 group elements $g_a \in \mathrm{SL}(2,\mathbb{C})$ (real dimension 6 each) and, for each of the 10 pairs (a,b), two coherent spinors $|\xi_{ab}\rangle, |\xi_{ba}\rangle \in \mathbb{C}P^1 \cong S^2$ (real dimension 2 each). Total: $5\times 6 + 10\times 2\times 2 = 70$ real variables.

The gauge symmetries before fixing are: the diagonal action $g_a \mapsto hg_a$ of $\mathrm{SL}(2,\mathbb{C})$ ($\dim = 6$); the local $\mathrm{SU}(2)$ action per tetrahedron $|\xi_{ab}\rangle \mapsto R_a|\xi_{ab}\rangle$ ($5\times 3 = 15$); and the individual $\mathrm{U}(1)$ phases of the spinors ($2\times 10 = 20$). Total gauge modes: $6+15+20 = 41$.

Physical degrees of freedom after complete gauge fixing:

$$n = 70 - 41 = 29. \tag{A14}$$

The prefactor therefore scales as $(\det H_{\mathrm{res}})^{-1/2} \sim j^{-29/2}$, subdominant relative to the exponential factor for $j \gg 1$.

Appendix B.3. Non-Degeneracy at the Type-III Saddle

Proposition A1. *For the Type-III saddle with distribution* $(k + (4-k))$, $1 \le k \le 3$, *and non-degenerate geometry,* $\dim(\mathrm{Stab}(X^\star)) = 0$, *hence* $\det H_{\mathrm{res}} \neq 0$.

Proof. In the Type-I case, the stabiliser contains the diagonal $\mathrm{SL}(2,\mathbb{C})$ acting on all tetrahedra ($\dim = 6$). For the Type-III saddle, a transformation $g \in \mathrm{SL}(2,\mathbb{C})$ must simultaneously preserve the future and past normals. For $k \neq 2$ and generic geometry, no $g \neq \mathrm{id}$ has this property: $\mathrm{Stab}(X^\star)$ is trivial. For $k = 2$, there is a discrete $\mathbb{Z}_2$ symmetry but no continuous one: $\dim(\mathrm{Stab}(X^\star)) = 0$ in all cases. □

Appendix C. Proof of the Refinement Stability Theorem

Appendix C.1. Convergence of Saddle Angles

Let $\mathcal{F}'$ be the stellar subdivision with N sub-simplices $\{\sigma_i\}$ satisfying (H1)–(H2). The power $N^{-1/4}$ in the convergence estimate reflects the scaling relation between edge length and volume in 4 dimensions: in a subdivision into N equal parts of volume Vol_4/N, the typical edge length scales as $(\mathrm{Vol}_4/N)^{1/4}$, producing an $O(N^{-1/4})$ correction to the shape coefficients $\alpha^{(i)}_{ab}$ relative to the original values α_{ab}. By regularity of $\chi_{ab}(\{\alpha_{cd}\})$:

$$\chi^{(i)}_{ab} = \chi_{ab} + O(N^{-1/4}). \tag{A15}$$

Appendix C.2. Scaling of the Total Action and Prefactor

The total Euclidean action is $S^{\mathrm{tot}}_E = \frac{1}{8\pi G}\sum_i\sum_{a<b} A^{(i)}_{ab}\chi^{(i)}_{ab}$. Since $\sum_i A^{(i)}_{ab} = A_{ab} + O(N^{-1/4})$ by area additivity and using (A15):

$$S^{\mathrm{tot}}_E = S_E\big(1 + O(N^{-1/4})\big), \tag{A16}$$

establishing the result of Theorem 2 with exponent $1/4$. The prefactor $N^{-n/2}$ arises from the N factors $(\det H^{(i)})^{-1/2} \sim j^{-29/2}$, subdominant relative to the exponential factor. □

Appendix D. Calibration of the Gram Matrix Criterion

Appendix D.1. Asymptotic Behaviour for Type-I Vertices

For Type-I vertices with a real Lorentzian saddle, the linear dependence of the weighted normals implies $\det(G_v) = 0$ exactly at the saddle. In the finite semiclassical regime, the departure from the saddle is exponentially suppressed in j.

Proposition A2 (Asymptotic scaling of the Gram determinant)**.** *Consider a vertex with boundary data in the semiclassical regime $j \gg 1$.*

1. *For Type-I vertices (with a real Lorentzian saddle), the Gram matrix determinant satisfies*

$$|\det(G_v)| = O\Big(e^{-c\, j_{\min}(v)}\Big), \quad c > 0, \tag{A17}$$

reflecting the exponential approach to the exact closure condition.

2. *For Type-III vertices, the determinant is generically of order unity, $|\det(G_v)| = O(1)$.*

Proof. For Type-I vertices, the existence of a real Lorentzian saddle implies exact linear dependence of the weighted normal vectors at the critical point. In the semiclassical regime, the dominant contribution to the path integral comes from a Gaussian neighbourhood of that saddle, with fluctuations suppressed as $\exp(-j\,\delta^2)$. Since the Gram determinant measures the deviation from exact linearity, its magnitude inherits this exponential suppression, giving $|\det(G_v)| \sim e^{-cj}$ for some geometry-dependent $c > 0$.

For Type-III vertices, Theorem 1 guarantees the non-existence of any real configuration satisfying closure. The deviation from linear dependence is therefore finite and cannot be reduced by increasing j, implying $|\det(G_v)| = O(1)$ generically. □

Remark A2. *The above result is not an independent theorem but an asymptotic consequence of the saddle analysis, combined with the geometric interpretation of the Gram determinant as a measure of non-closure. Its validity is supported by the Gaussian structure of the semiclassical fluctuations and by the numerical verification of Appendix A.*

Appendix D.2. Discrimination Window

For $j_{\min} = 1$, the base threshold is $\alpha e^{-2} \approx 1.35 \times 10^{-4}$, well below the typical Type-I value $|\det(G_v)|_{\text{Type I}} \sim e^{-2} \approx 0.135$ and well above numerical noise ($\sim 10^{-15}$). The factor $(1 + \beta/d(v, \partial\mathcal{F}))$ with $\beta \approx 0.1$ raises the threshold near the boundary, where partially fixed data may produce intermediate values of $|\det(G_v)|$.

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