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Clarification and Generalization of the Restricted *z*-Framework with Boundary Alignment and Indexed Hadamard Factorization

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Abstract

This paper clarifies and extends the restricted-variable framework introduced in Drake (2026). Restriction does not alter the underlying analytic object, provided the restricted and unrestricted presentations agree on a nonempty open overlap domain; it changes only the regime of admissible access. In the model case of the completed Riemann zeta function, the paper shows that functional symmetry acts on the argument of the function but does not by itself induce an admissible zero-indexing rule in the restricted Hadamard product. As a result, reflected zeros remain analytically present through symmetry, yet do not appear as separately admissible geometric indices unless they lie on the symmetry boundary. This yields the Hadamard representation dilemma that consistency between canonical Hadamard factorization and the restricted admissible indexing mechanism forces the nontrivial zeros onto the critical line. The same mechanism is then extended to entire functions of order one satisfying a reflection symmetry under an optimal half-plane restriction. The paper also shows that symmetric linear combinations do not constitute counterexamples, since their zeros arise through cancellation rather than through the restricted zero-indexing mechanism.

Keywords: critical line; geometric series; complex variable; Dirichlet series; zeta function; completed L-function; symmetry

MSC: 30D20

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1. Introduction

The completed Riemann zeta function occupies a central place in analytic number theory and, for the purposes of the present paper, is viewed through the interaction of three classical structures: functional symmetry, entire continuation, and canonical Hadamard factorization. In the usual formulation, these structures coexist on the unrestricted complex plane, and the critical line $\Re s = 1/2$ appears as the symmetry axis of the reflection $s \mapsto 1 - s$. In Drake (2026), a restricted-variable framework was introduced in which the same completed function is presented through a geometric real coordinate, yielding an admissible half-plane $\Re z \geq 1/2$. Within that setting, the critical line appears not merely as a formal symmetry axis, but as the boundary of admissible access.

The present paper revisits that framework at a more precise structural level. Its purpose is not to rederive the completed zeta function from the beginning, but to clarify the mechanism by which restriction interacts with functional symmetry and Hadamard factorization. In particular, the paper sharpens the distinction between the global analytic validity of the function itself and the admissible representation through which that function is realized in the restricted variable. This distinction is essential for formulating correctly how reflected values are accessed, how zeros are indexed in the Hadamard product, and why the symmetry boundary plays a distinguishing role.

A central point is that the geometric-series device underlying the restricted variable acts at the level of the real exponent, not at the level of the ambient complex variable itself. The structural consequences therefore do not arise from changing the analytic object, but from changing the regime

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of admissible representation through which that object is presented. Once that distinction is made explicit, the treatment of reflected values can be corrected and sharpened: functional symmetry acts on the argument of the function, but does not by itself induce a corresponding admissible indexing rule for zeros in the restricted Hadamard framework. Reflected zeros may remain analytically present through symmetry while failing to occur as separately admissible geometric indices unless they lie on the boundary line.

From this observation, the paper develops two goals. The first is to formulate rigorously, in the model case of the completed zeta function, the dilemma between canonical Hadamard factorization and restricted admissible indexing. The second is to abstract the same mechanism beyond the zeta setting. More generally, for an entire function of order one satisfying a reflection symmetry $w \mapsto \kappa - w$, the paper studies half-plane restrictions aligned with that symmetry and identifies the optimal case in which the admissible domain and its reflected image meet only on the symmetry axis. In that setting, the same boundary-alignment mechanism emerges.

The paper also addresses a natural objection. There are entire functions of order one that satisfy reflection-type symmetries and admit Hadamard factorizations, yet possess zeros away from the symmetry line; for example, such functions may arise from symmetric linear combinations. These do not contradict the restricted-indexing results developed here, because their zeros arise through cancellation between separately evaluated summands rather than through the admissible zero-indexing mechanism attached to a single reflected function. This distinction clarifies the structural boundary between the class treated in the present paper and formally similar functions lying outside its scope.

The paper is organized as follows. Section 2 records the results imported from Drake (2026) that are used without rederivation. Section 3 identifies the exponent-level character of the geometric-series mechanism. Section 4 develops the consequences for reflected values, admissible zero indexing, and the indexed Hadamard product in the zeta case. Section 5 abstracts the restriction principle and proves the corresponding boundary-alignment result for a general symmetric entire function of order one. Section 6 addresses apparent counterexamples arising from symmetric linear combinations and explain why they do not fall under the same indexing mechanism.

2. Imported Results from Drake (2026)

This section records the structural results from Drake (2026) that will be used in the present paper without rederivation. Throughout, all statements are interpreted within the z-space framework established there.

2.1. Geometric Real Coordinate

Let

$$\varsigma(r) = \sum_{k=0}^{\infty} r^k, \qquad |\mathrm{r}| < 1. \tag{1}$$

In Drake (2026), this geometric series is taken as the primary real coordinate in z-space, with closed-form counterpart

$$\sigma(r) = 1/(1 - \mathrm{r}). \tag{2}$$

By Abel and Cesàro summation, the endpoint $r = -1$ is assigned the value

$$\varsigma(-1) = 1/2, \tag{3}$$

so that the admissible real parameter range becomes

$$r \in [-1,1) \quad \Longleftrightarrow \quad \varsigma(r) \in [1/2, \infty). \tag{4}$$

The corresponding z-space variable is then

$$z = \varsigma(r) + it, \qquad t \in \mathbb{R}, \tag{5}$$

and hence satisfies

$$\Re z \geq 1/2. \tag{6}$$

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Thus, the admissible z-space is the right half-plane together with its boundary. The boundary value $\Re z = 1/2$ is included as the endpoint arising from the Abel/Cesàro assignment (3).

2.2. Access Structure and Reflected Values

A central structural result of Drake (2026) is that z-space is not treated as an unrestricted copy of the classical s-plane. Rather, the admissible working region is the right half-plane $\Re z \geq 1/2$, while the reflected half-plane is not entered directly. Values on the reflected side are accessed only through valid global identities composed with admissible right-half-plane data. This is the content of the z-space rulebook established in Section 4 of the previous paper. For clarity and for subsequent development, the explicit consequences of that rulebook are recorded in Remark 4.

For convenience only, the present paper uses the following shorthand terminology for structural conclusions proved in Drake (2026).

Definition 1 (Global-by-access). *A function in z-space is said to be global-by-access if it is defined on the admissible half-plane, has the boundary interpretation imported from Drake (2026), and is accessed on the reflected half-plane only through valid symmetry or global identities written in admissible coordinates.*

Definition 2 (Entire-by-access). *A function is said to be entire-by-access if, in the above sense, it has no singularities anywhere in the globally accessed plane, even though direct evaluation is admitted only on the right half-plane and on the boundary by limit.*

These expressions do not introduce new analytic notions, and they do not replace the classical meanings of "global" or "entire". They are qualitative abbreviations for the manner in which those already established properties are realized within the restricted z-space framework.

2.3. The Completed Zeta Function in z-Space

Drake (2026) defines the completed zeta function in z-space by

$$\xi(z) = \frac{1}{2}\, z(z-1)\pi^{-\frac{z}{2}}\Gamma\left(\frac{z}{2}\right)\zeta(z), \qquad \Re z > 1, \tag{7}$$

and proves that it extends to an entire-by-access function satisfying the functional symmetry

$$\xi(z) = \xi(1-z). \tag{8}$$

Moreover, the reflected expression $\xi(1-z)$ is not treated as an independent coordinate expression. Rather, it is accessed entirely from the admissible half-plane $\Re z \geq 1/2$ through the defining symmetry of ξ, and the two sides agree on the boundary $\Re z = 1/2$ by non-tangential limits.

2.4. Order One and Hadamard Factorization

Drake (2026) proves that ξ is entire of order one; Hadamard's theorem therefore applies to ξ in the standard sense. The present paper uses the shorthand "entire-by-access of order one" only to describe how that imported global result is handled within the restricted z-space framework. In the notation of that paper,

$$\xi(z) = \frac{1}{2}\prod_{\varrho}\left(1-\frac{z}{\varrho}\right)\exp\left(\frac{z}{\varrho}\right), \tag{9}$$

where the product ranges over the nontrivial zeros ϱ of ξ, counted with multiplicity.

2.5. Zero Notation for the Present Paper

The present paper slightly modifies the notation for nontrivial zeros used in Drake (2026). Let

$$\rho := \rho(q) = \beta(q) + i\gamma, \qquad \beta(q) = 1/(1-\mathrm{q}), \tag{10}$$

denote a nontrivial zero in unrestricted s-space, with $q \in \mathbb{R}$. Let

$$\varrho := \varrho(q) = \mathrm{B}(q) + i\gamma, \qquad \mathrm{B}(q) = \sum_{k=0}^{\infty} q^k\,, \tag{11}$$

denote its restricted geometric counterpart in z-space.

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Thus, in the present paper, $\mathrm{B}(q)$ denotes the geometric-series real part attached to the zero indexing, while $\beta(q)$ denotes its closed-form counterpart. This separates the zero-indexed notation from the ambient variable notation

$$z = \varsigma(r) + it,$$

and

$$s = \sigma(r) + it, \tag{12}$$

where $r \in \mathbb{R}$ is the geometric parameter attached to the ambient variable.

Accordingly, the Hadamard product imported from Drake (2026) will be used in the present paper with this revised zero notation.

2.6. Imported References and Citation Convention

For the convenience of the reader, the present paper distinguishes between two kinds of citation. When a statement is used as part of the imported framework from Drake (2026) [1], the parent citation Drake (2026) is used. This indicates that the statement is being taken from the structural framework established there, together with the references on which that framework explicitly depends. By contrast, references cited directly in the present paper are used to support statements developed independently in this work, especially in later sections where broader structural consequences are formulated.

The imported background from Drake (2026) [1] rests on standard reference for the classical analytic theory of the Riemann zeta function, the completed zeta function, functional equations, Hadamard factorization, complex analysis, special functions, and summability methods. Specifically, the classical zeta-function background is supported by [2–7]; the general complex-analytic framework by [8–10]; special-function identities relevant to functional equations by [11,12]; and the Abel/Cesàro interpretation of the geometric-series boundary value by [13].

3. Restriction Mechanism Properly Attributed

Drake (2026) develops the analytic structure of ξ and ζ within a restricted z-framework, without appeal to their unrestricted realization in s. For the present paper, it is important to identify the level at which the geometric restriction is implemented. The following proposition shows that the geometric-series device acts by reexpressing the real exponent, rather than by altering the ambient complex variable itself. This clarifies that the restricted framework is implemented at the level of admissible presentation, rather than at the level of the underlying analytic object.

Proposition 1 (Exponent-level implementation of the geometric-series identity). *The geometric-series mechanism used in z-space acts at the level of the real exponent appearing in factors of the form*

$$x^{-\sigma}, \qquad x \in \mathbb{R}_{>0}, \tag{13}$$

rather than at the level of the ambient complex coordinate itself. More precisely, for each $\sigma > 1/2$, *the quantity* σ *admits a convergent geometric-series representation after the substitution*

$$r := 1 - \frac{1}{\sigma}. \tag{14}$$

This representation is an identity at the exponent level and does not alter the underlying complex variable.

Proof. Fix $x > 0$, and let $\sigma \in \mathbb{R}$ satisfy $\sigma > 1/2$. Define r by (14). Then

$$|1 - 1/\sigma| < 1, \tag{15}$$

and hence $|\mathrm{r}| < 1$. Therefore, the geometric series (1) converges, and

$$\sum_{k=0}^{\infty} r^k = 1/(1-r)\,. \tag{16}$$

Substituting $r = 1 - 1/\sigma$ gives

$$\sum_{k=0}^{\infty} (1{-}1/\sigma)^k = \frac{1}{1-(1{-}1/\sigma)} = \sigma. \tag{17}$$

Accordingly, define

$$\varsigma(\sigma) := \sum_{k=0}^{\infty}(1-1/\sigma)^k, \qquad \sigma > 1/2 \tag{18}$$

so that $\varsigma(\sigma) = \sigma$. Therefore

$$x^{-\sigma} = \exp(-\sigma \log x) = \exp(-\varsigma(\sigma) \log x) = x^{-\varsigma(\sigma)}. \tag{19}$$

Thus, the geometric-series identity is implemented at the level of the real exponent σ in (13), and does not modify the ambient complex coordinate appearing in that exponent. ■

Remark 2 (Domain). *The convergence condition* (15) *in Proposition 1 is*

$$|1 - 1/\sigma| < 1.$$

For real σ*, this is equivalent to* $\sigma > 1/2$*. At the end point* $\sigma = 1/2$ *the ratio equals* -1*, so the series is not convergent in the usual sense, but it admits the Abel/Cesàro [1] assignment*

$$\varsigma(1/2) = 1/2. \tag{20}$$

Notation 1 (Suppression of auxiliary parameters). *For the remainder of this paper, the auxiliary parameters* r *and* q *are suppressed, and we work directly with the corresponding real parts. Thus*

$$s = \sigma(r) + it \rightsquigarrow s := \sigma + it, \tag{21}$$

$$z = \varsigma(r) + it \rightsquigarrow z := \varsigma(\sigma) + it, \tag{22}$$

$$\rho = \beta(q) + i\gamma \rightsquigarrow \rho := \beta + i\gamma, \tag{23}$$

$$\varrho = B(q) + i\gamma \rightsquigarrow \varrho := B(\beta) + i\gamma. \tag{24}$$

Here σ *and* β *denote the real parts directly, while* $\varsigma(\sigma)$ *and* $B(\beta)$ *denote their geometric realizations.*

4. Structural Consequences Properly Attributed

Section 3 established the implementation of the geometric-series identity at the level of the real exponent. We now examine the structural consequences of this distinction for the properties of ξ, especially for its Hadamard product. Although these objects were treated in Drake (2026), the present discussion isolates more explicitly the distinction between globally valid analytic structure and admissible representation within the restricted framework.

We begin with a lemma concerning the permissibility of symbolic representation. Certain expressions, such as the imaginary exponent x^{it}, for $x \in \mathbb{R}_{>0}$, are defined for all $t \in \mathbb{R}$, and admit explicit symbolic realization through identities such as Euler's formula or convergent power-series expansion. By contrast, the geometric series underlying z-space does not provide an admissible value-bearing representation of real quantities $\sigma < 1/2$: although the formal series may still be written, it is not admitted as an assigned value within the restricted framework.

Lemma 2 (Identity extension does not grant representational access). *The geometric-series representation (18) of the real exponent,*

$$\varsigma(\sigma) = \sum_{k=0}^{\infty}(1-1/\sigma)^k,$$

is value-bearing only on its domain of convergence, together with the single Abel/Cesàro boundary assignment at $\sigma = 1/2$*. Although the corresponding closed-form identity agrees with the series on that domain and therefore determines the same holomorphic function there, this does not enlarge the region where the series representation itself is admissible.*

Proof. Consider the series in (18). It converges if and only if

$$|1 - 1/\sigma| < 1.$$

For real σ, this is equivalent to $\sigma > 1/2$. By Remark 2, at $\sigma = 1/2$ the series assigns the value $\varsigma(1/2) = 1/2$. For $\sigma < 1/2$, the series diverges and cannot serve as a value-bearing representation within the restricted framework.

On its domain of converges, the series agrees with the closed-form expression in (17). Thus, the series defines a holomorphic function on its convergence domain, and that function coincides there with the entire function σ. Hence the closed form determines the unique holomorphic continuation of the function defined by the series [1].

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However, holomorphic continuation of the resulting function does not change the convergence domain of the original series representation (18). This series remains admissible only for $\sigma > 1/2$, together with the single boundary assignment at $\sigma = 1/2$. ■

With Lemma 2 in place, we now develop the admissibility condition governing the remainder of this section. This allows a precise distinction between functional symmetry and admissible zero indexing, and leads ultimately to the contradiction formalized in Theorem 1.

Definition 3 (Geometric representability in real part). *Let $w \in \mathbb{C}$. We say that w is geometrically representable in real part if its real part satisfies*

$$\Re w \geq 1/2. \tag{25}$$

In that case, the geometric representation of the real part is defined by

$$\varsigma(\Re w) \ := \ \sum_{k=0}^{\infty} (1 - 1/\Re w)^k\,, \tag{26}$$

interpreted as a convergent series for $\Re w > 1/2$ and by Abel/Cesàro assignment at $\Re w = 1/2$. On this admissible domain,

$$\varsigma(\Re w) = \Re w. \tag{27}$$

Justification. By Proposition 1 and Lemma 2, the real value σ admits a value-bearing geometric-series representation of the form (26) for $\sigma > 1/2$, together with the single boundary assignment at $\sigma = 1/2$. This motivates the present definition for arbitrary $w \in \mathbb{C}$ that geometric representability is imposed through the real part. If $\Re w < 1/2$, then

$$|1 - 1/\Re w| \geq 1, \tag{28}$$

so the series in (26) is not admissible as a value-bearing geometric representation.

Remark 3 (Admissible actions according to domain). *This limitation on geometric representability identifies the main issue of the present section that operations formally valid in an unrestricted framework must be re-examined in the restricted setting. Classical conventions concerning reflection and reindexing do not automatically remain admissible.*

The following lemma begins to isolate this distinction at the level of functional symmetry and zero indexing, and the remainder of the section develops its consequences.

Lemma 3 (Admissible indexing rule). *Within the z-space framework, functional symmetry acts on inputs through the operation*

$$z \mapsto 1 - z. \tag{29}$$

It does not, by itself, produce a corresponding indexing rule for the zeros appearing in the Hadamard product. Accordingly, when a nontrivial zero is indexed through geometric representability from the admissible half-plane, the reflected value $1 - \varrho$ does not appear as a separately admissible indexed factor unless the zero lies on the critical line.

Proof. In z-space, zeros are indexed through geometric representability of their real parts. Thus, by Definition 3 and (24), an admissibly indexed nontrivial zero has the form

$$\varrho = \mathrm{B}(\beta) + i\gamma, \qquad \beta \geq 1/2.$$

Functional symmetry $\xi(z) = \xi(1 - z)$ is an identity in the argument of the function. Hence, it yields the admissible input operation $z \mapsto 1 - z$. However, the indexing mechanism for the Hadamard product is not determined by substitution in the argument, but by geometric representability of the real part. Therefore, for a reflected zero to appear as an admissibly indexed factor, the reflected value $1 - \varrho$ must itself admit an admissible geometric index of the same form.

Assume $\Re\varrho > 1/2$, so that ϱ is an admissibly indexed zero. Then

$$\Re(1 - \varrho) = 1 - \Re\varrho < 1/2. \tag{30}$$

By Lemma 2 and Definition 3, real parts strictly less than $1/2$ are not geometrically representable. Hence, $1 - \varrho$ cannot be written in the admissible indexed form

$$\varrho' = \mathrm{B}(\beta') + i\gamma', \qquad \beta' \geq 1/2, \tag{31}$$

and therefore admits no admissible geometric index in the z-space Hadamard framework.

Functional symmetry still yields the analytic implication

$$\xi(\varrho) = 0 \quad \Rightarrow \quad \xi(1 - \varrho) = 0. \tag{32}$$

Thus, the reflected zero arises as a consequence of the functional identity, but not as a separately admissible indexed factor. The only exception is the boundary case $\Re\varrho = 1/2$, where reflection preserves the admissible real part. ■

Definition 4 (Ambient reflection principle). *Let $w \mapsto \kappa - w$ be a reflection symmetry on the unrestricted complex plane, for some real constant κ. We say that the ambient reflection principle holds in a given variable framework if reflection may be used not only as an argument transformation inside a functional identity, but also as an admissible ambient operation on points and index values. Equivalently, whenever a point w is directly representable in the framework, its reflection $\kappa - w$ is also directly representable in the same framework, and the two may be interchanged as coordinate representatives without leaving the admissible variable system.*

Proposition 2 (Failure of the ambient reflection principle in z-space). *In the classical s-framework, the ambient reflection principle holds for the involution*

$$s \mapsto 1 - s. \tag{33}$$

In the restricted z-framework, it does not. There, reflection remains available only as an argument operation inside the functional identity, and not as an admissible ambient operation on points or indices.

Proof. In the classical s-plane, both s and $1-s$ lie in the same unrestricted coordinate domain. Hence the reflection $s \mapsto 1-s$ is an admissible ambient change of variable, and may be used both in the functional equation and in reindexing constructions based on the full zero set. Thus, the ambient reflection principle holds classically.

By Lemma 3, the restricted z-framework preserves the argument operation $z \mapsto 1-z$, but does not supply an admissible reflected index whenever the reflected real part falls below $1/2$. Accordingly, reflection is not available as an admissible ambient operation on points or indices. Therefore, the ambient reflection principle fails in the restricted framework. ■

Lemma 4 (Index stability of the Hadamard product). *By the imported results recorded in Section 2, the completed zeta function $\xi(z)$ is entire-by-access of order one and admits a canonical Hadamard factorization of (9), where ϱ ranges over the nontrivial zeros of ξ, counted with multiplicity.*

Definition 3 and Lemmas 2-3 establish that nontrivial zeros are admissibly indexed in z-space only through geometric representability of their real parts, and that although functional symmetry acts on inputs by the operation $z \mapsto 1-z$, the ambient reflection principle is not available in the restricted framework and therefore does not supply an admissible geometric index for a reflected zero whose real part lies below $1/2$.

Consequently, functional symmetry (8) yields

$$\xi(1-z) = \frac{1}{2}\prod_{\varrho}\left(1 - \frac{1-z}{\varrho}\right) exp\left(\frac{1-z}{\varrho}\right), \tag{34}$$

with the same index set $\{\varrho\}$. Hence, only the geometrically admissible member of a reflected pair $\{\varrho, 1-\varrho\}$ can occur as an admissibly indexed factor. Therefore, unless all nontrivial zeros satisfy $\Re\varrho = 1/2$, the z-space indexed product does not admit both members of each reflected pair as admissible indices.

Proof. The identity $\xi(z) = \xi(1-z)$ acts by substitution on the argument. Substituting $z \mapsto 1-z$ into the Hadamard product (9) yields (34), with the index set $\{\varrho\}$ unchanged. Any attempt to reindex by $\varrho \mapsto 1-\varrho$ would require $1-\varrho$ itself to admit an admissible geometric index. By Lemma 3, this fails whenever $\Re\varrho > 1/2$, since then $\Re(1-\varrho) < 1/2$. Only in the boundary case $\Re\varrho = 1/2$ does reflection preserve admissibility of the real part. ■

Remark 4 (Corrected interpretation of Drake 2026). *In Drake (2026), it was stated informally that $\xi(1-z)$ encodes the zeros from the reflected half-plane. That phrasing is imprecise, and Lemma 4 gives the correct formulation. Functional symmetry acts on the argument of the function; it does not alter the indexing mechanism of the canonical Hadamard product. In z-space, reflected expressions are not treated as independent left-half-plane objects, but are instead rewritten in admissible coordinates of the right half-plane.*

This principle already appears in the admissible use of the functional equation for $\zeta(z)$, in Drake (2026), where

$$\zeta(z) = 2^z\pi^{z-1} \sin(\pi z/2)\,\Gamma(1-z)\zeta(1-z) \tag{35}$$

is rewritten as

$$\zeta(1-z) = 2^{1-z}\pi^{-z} \cos(\pi z/2)\,\Gamma(z)\zeta(z), \tag{36}$$

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so that all factors are expressed in admissible z-space terms.

Similarly, for the completed function, z-admissibility together with convergence constraints does not treat

$$\Gamma((1-z)/2)\zeta(1-z) \tag{37}$$

as a permittable representation for $\Re(1-z) \leq 0$. Instead, $\xi(1-z)$ is represented through its functional identity

$$\xi(1-z) = \frac{1}{2}z(z-1)\pi^{-\frac{z}{2}}\Gamma\left(\frac{z}{2}\right)\zeta(z). \tag{38}$$

Thus, since the set $\{\varrho\}$ is preserved in (34), the Hadamard product has the same z-space convention: the reflected half-plane is accessed only through reflected arguments written in admissible coordinates.

Theorem 1 (The Hadamard representation dilemma). *Recall the completed zeta function $\xi(z)$ from (7) and its canonical Hadamard factorization (9), in which ϱ ranges over the nontrivial zeros of ξ, counted with multiplicity.*

Within the z-space framework, suppose there exists a nontrivial zero with $\Re\varrho \neq 1/2$. Then, by Lemmas 3-4, the reflected zero $1-\varrho$ is analytically forced by functional symmetry. However, because the ambient reflection principle is not available in the restricted framework, it does not arise as a separately admissible geometric index in the z-space Hadamard product. Hence the admissible indexing rule does not realize the full reflected zero pair required by the canonical factorization.

Since the canonical Hadamard factorization is rigid, the resulting indexed product cannot coincide with ξ. Therefore, consistency of the canonical factorization with the z-space admissible indexing mechanism forces all nontrivial zeros to satisfy $\Re\varrho = 1/2$.

Proof. Assume, for contradiction, that there exists a nontrivial zero ϱ with $\Re\varrho > 1/2$. By functional symmetry (8), the reflected value $1-\varrho$ is also a zero of ξ, and

$$\Re(1-\varrho) = 1 - \Re\varrho < 1/2. \tag{39}$$

By Lemma 3, the reflection $1-\varrho$ is admissible as an argument of ξ, but it does not admit a separately admissible geometric index for the Hadamard product once its real part lies strictly below 1/2. Hence, by Lemma 4, no separate indexed factor corresponding to $1-\varrho$ can occur in the z-space indexed product. Thus, the indexed product fails to realize the full zero set required by the canonical Hadamard factorization of ξ.

By Drake (2026), the canonical Hadamard factorization for ξ is rigid: omission of any nontrivial zero, counted with multiplicity, changes the resulting function. Therefore, the z-space indexed product cannot coincide with $\xi(z)$, contradicting the imported factorization (9). Hence no nontrivial zero can satisfy $\Re\varrho > 1/2$. Since admissible indexing in z-space already requires $\Re\varrho \geq 1/2$, it follows that every nontrivial zero satisfies $\Re\varrho = 1/2$. ■

Remark 5 (Comparison with the classical Hadamard product). *The contradiction in Theorem 1 is not a claim that the Hadamard product is incapable of encoding zeros of ξ. Rather, within the z-space indexing framework, it shows that the zeros admitted as admissible indices do not produce the same function ξ unless all nontrivial zeros lie on the critical line.*

Functional symmetry is not a prerequisite for Hadamard factorization, and the canonical product is not intrinsically constructed in symmetric form. In the standard representation, the exponential factor $exp(z/\varrho)$ regulates convergence. Only by the ambient reflection principle may the product be reorganized into a more symmetric form in which convergence is controlled by pairing of zeros rather than by the elementary exponential factors. Such symmetric reindexing is not compatible within the z-space framework.

Accordingly, the contradiction in Theorem 1 arises from the interaction of two distinct structures: the functional symmetry of ξ, and the admissible indexing constraints imposed on its canonical Hadamard product in z-space. The symmetry is therefore attributed to the function itself, and not to the formal indexing structure of the product.

Remark 6 (Transfer of the zeros set into the classical setting). *Theorem 1 is proven within the z-space framework, using the global-by-access canonical Hadamard factorization of ξ together with the admissible indexing mechanism. In the classical presentation, the canonical Hadamard product represents the same completed function ξ, but on the unrestricted s-domain.*

Since both representations determine the same analytic object ξ, their nontrivial zeros coincide, counted with multiplicity. Thus, the zero set is intrinsic to ξ and does not depend on the representation through which

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ξ is expressed. Therefore, the conclusion of Theorem 1 transfers to the classical zero set such that the nontrivial zeros ρ satisfy

$$\Re\rho = 1/2. \tag{40}$$

A formal transfer of this conclusion to the classical s-framework is given in Drake (2026), and is further supported by Theorem 2 in Section 5.

5. Applying Identities by Restriction Rather than Derivation

The framework developed in Drake (2026) may appear at first to be specific to ξ and ζ, since that work established the restricted z-framework by direct derivation in this model case. However, the governing principle is more general. The essential point is not that a function must always be reconstructed from first principles within a restricted variable, but rather that a valid analytic identity may be used to present the same function within a chosen restricted-access framework [8].

Passage to a restricted variable does not alter the underlying analytic object, provided the restricted presentation agrees with the original function on a nonempty open domain. What changes is not the function itself, but the structure of direct admissibility through which that function is accessed [1]. Thus, the issue is not which identity is used, but which properties of the function remain directly accessible under the resulting restricted presentation.

More specifically, a function that is entire in the unrestricted setting need not be entire-by-access under a given restricted presentation. In that case, the analytic object is unchanged, but certain global structures—such as direct access to a Hadamard factorization—may no longer be available within the restricted framework.

The purpose of this section is to formulate that principle precisely, and then to extend the structural conclusions of the previous section from the model case of ξ to the broader class of functions satisfying the same symmetry and factorization conditions.

Notation 2 (Generalization of Complex Variables). *We distinguish three levels of the complex variable as follows.*

Let $w \in \mathbb{C}$ denote a generic complex input. This symbol is used when discussing analytic structure abstractly, without commitment to a particular coordinate system. Statements formulated in terms of w refer to properties intrinsic to a function, independent of the variable framework in which it is presented.

Let $s = \sigma + it \in \mathbb{C}$ denote an unrestricted complex variable, used for classical orientation. In this coordinate, we treat functions such as $\zeta(s)$, with its pole at $s = 1$, and the completed function $\xi(s)$ as on their natural domains. In this classical setting, analytic existence and analytic access coincide: every point of the relevant domain is directly admissible.

Let $z \in \mathbb{C}$ denote a restricted complex variable whose admissible values lie in a half-plane

$$\Re z \geq a, \tag{41}$$

for some real constant a. Passage to such a restricted variable does not alter the underlying analytic object; it changes only the structure of direct access. Analytic operations are carried out directly only in the admissible region. Values outside the admissible half-plane are reached only through an allowed structural mechanism, after being rewritten in admissible terms.

Theorem 2 (Restriction preserves the analytic object). *Let F_s and F_z be analytic functions on a common connected domain $D \subseteq \mathbb{C}$, presented in an unrestricted variable s and in a restricted variable z, respectivally. Assume there exists a nonempty open set $E \subseteq D$ on which*

$$F_s(w) = F_z(w). \tag{42}$$

Then $F_s(w) = F_z(w)$ for all $w \in D$. Accordingly, whenever an unrestricted and a restricted presentation agrees on a nonempty open overlap, passage to the restricted variable does not alter the underlying analytic object. It changes only the structure of direct admissibility through which that object is accessed.

Proof. By hypothesis, F_s and F_z are analytic on the same connected domain D, and they agree on the nonempty open subset $E \subseteq D$. By the identity theorem *[8,9]*,

$$F_s = F_z \tag{43}$$

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throughout D. Hence the restricted presentation does not define a new analytic function, nor does it modify the original one. It is the same analytic object presented under a different access structure.

Thus, the effect of restriction is not analytic but operational. It changes only which values and representations are directly admissible, and which must instead be reached indirectly through identities rewritten in admissible terms. ■

Remark 7 (Restriction choices and consequences). *Theorem 2 shows that the choice of restricting identity does not alter the analytic object. What changes is the structure of admissibility. Accordingly, the issue is not which valid identity is used, but which representations, operations, and global properties remain accessible once that identity is adopted within the restricted framework. Thus, a function may remain analytically unchanged while losing admissible access to structures that depend on a broader domain of representation.*

This is the structural basis of the ambient reflection principle, and bears directly on entireness-by-access and canonical Hadamard factorization when the restriction is not optimal as developed below.

Lemma 5 (Functional overlap under reflection). *Let F be an analytic function satisfying a reflection symmetry*

$$F(w) = F(\kappa - w), \tag{44}$$

for some real constant κ. The vertical line

$$\Re w = \kappa/2 \tag{45}$$

is the axis of symmetry of the reflection

$$w \mapsto \kappa - w. \tag{46}$$

Let the admissible domain of a restricted variable be the half-plane $\Re z \geq a$. Then the reflected image of the admissible half-plane lies in

$$\Re w \leq \kappa - a, \tag{47}$$

and the intersection of the admissible half-plane with its reflected image is the vertical strip

$$a \leq \Re w \leq \kappa - a. \tag{48}$$

This strip is called the region of functional overlap.

Proof. If $\Re z \geq a$, then

$$\Re(\kappa - z) = \kappa - \Re z \leq \kappa - a. \tag{49}$$

Thus, the reflected image of the half-plane $\Re z \geq a$ lies in $\Re w \leq \kappa - a$. The overlap consists of those points satisfying both $\Re w \geq a$ and $\Re w \leq \kappa - a$. Hence the functional overlap is (48). ■

Corollary 1 (Overlap regimes). *Under the hypotheses of Lemma 1, the width of the region of functional overlap is*

$$(\kappa - a) - a = \kappa - 2a. \tag{50}$$

Accordingly:

- If $a < \kappa/2$, the overlap contains an open vertical strip of positive width.
- If $a = \kappa/2$, the overlap collapses to the symmetry axis $\Re w = \kappa/2$.
- If $a > \kappa/2$, the overlap is empty.

Proof. The conclusion follows immediately from the sign of $\kappa - 2a$. ■

Definition 5 (Optimal restriction). *A half-plane restriction $\Re z \geq a$ is said to be optimal relative to the symmetry $w \mapsto \kappa - w$ if*

$$a = \kappa/2. \tag{51}$$

Remark 8 (Interpretation of the optimal case). *The choice $a = \kappa/2$ is the unique restriction for which the functional overlap is minimal but nonempty. In that case, the admissible half-plane and its reflected image meet exactly on the symmetry axis and nowhere else. Thus, the symmetry axis is retained as a boundary of access, while redundant interior overlap is eliminated.*

If $a < \kappa/2$, then the overlap has positive width, so the admissible half-plane and its reflected image share an open region. If $a > \kappa/2$, then the overlap is empty, and access to the symmetry boundary axis is lost.
We are now prepared to extend the conclusion of Theorem 1 to the present general setting.

Theorem 3 (Boundary alignment under optimal restriction). *Let F be an entire function of order one satisfying the reflection symmetry, $F(w) = F(\kappa - w)$, for some real constant κ [4]. Suppose that F admits*

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a canonical Hadamard factorization [8], and that this factorization is interpreted within the optimally restricted half-plane framework

$$\Re z \geq \kappa/2, \tag{52}$$

with reflected values accessed only through the symmetry $w \mapsto \kappa - w$, *rewritten in admissible coordinates. Then every nontrivial zero* ϱ *of* F *satisfies*

$$\Re\varrho = \kappa/2. \tag{53}$$

Proof. By Theorem 2, passage to the restricted variable does not alter the underlying analytic object, but only its structure of admissibility through which that object is accessed [1]. By Lemma 1, Corollary 1, and Definition 5, the optimally restricted half-plane and its reflected image meet only on the symmetry axis $\Re z = \kappa/2$.

Accordingly, the same indexing mechanism as in Section 4 applies with the line $\Re z = 1/2$ replaced by $\Re z = \kappa/2$, and the reflection $z \mapsto 1 - z$ replaced by $z \mapsto \kappa - z$. If a nontrivial zero ϱ were to satisfy $\Re\varrho > \kappa/2$, then by symmetry $\kappa - \varrho$ would also be a zero, with

$$\Re(\kappa - \varrho) = \kappa - \Re\varrho < \kappa/2. \tag{54}$$

Thus, the reflected zero would lie outside the admissible indexing domain and would not arise as a separately admissible index in the restricted Hadamard framework. Hence the restricted indexed product would fail to realize the full reflected zero pair required by the canonical factorization.

Since the canonical Hadamard factorization is rigid [8], omission of any such zero changes the resulting function. Therefore, the restricted indexed product cannot coincide with F, contradicting Theorem 2. Hence no nontrivial zero can satisfy $\Re\varrho > \kappa/2$. Thus, every nontrivial zero satisfies $\Re\varrho = \kappa/2$. ■

Corollary 2 (Application to completed L-functions). *Let* Λ *be a completed* L*-function [4] that is entire of order one and satisfies a functional symmetry of the form*

$$\Lambda(w) = \varepsilon\,\Lambda(\kappa - w), \qquad |\varepsilon| = 1, \tag{55}$$

for some real constant κ. *Assume further that* Λ *admits a canonical Hadamard factorization, and that its reflected values are interpreted within the optimally restricted half-plane framework* $\Re z \geq \kappa/2$, *with values outside the admissible half-plane accessed only through the functional symmetry rewritten in admissible coordinates. Then every nontrivial zero* ϱ *of* Λ *satisfies* $\Re\varrho = \kappa/2$.

Proof. Since $\varepsilon \neq 0$, the factor ε does not affect the zero set of Λ. Thus, if $\Lambda(\varrho) = 0$, then the functional equation implies $\Lambda(\kappa - \varrho) = 0$, so the nontrivial zeros are symmetric under the reflection $w \mapsto \kappa - w$. Hence Λ satisfies the hypotheses of Theorem 3. Applying Theorem 3 gives $\Re\varrho = \kappa/2$ for every nontrivial zero ϱ. ■

6. Alleviating Potential Counterexamples

At this stage, a natural objection arises. There exist functions that resemble completed L-functions in certain structural respects, in that they satisfy a reflection-type functional symmetry, are entire of order one, and admit a Hadamard factorization [14–18]. Such functions can arise, for example, as a linear combinations of completed L-functions. These combinations may retain a formal symmetry and a canonical product representation, while differing from the classical L-function setting in one essential respect: they need not admit an Euler product.

At first glance, such functions may appear to provide counterexamples. They may possess zeros away from the symmetry line while still exhibiting formal features of reflection symmetry and Hadamard factorization. Since the restricted z-framework does not accommodate off-line zeros as admissible indexed objects, such examples might seem to challenge the conclusions of the preceding sections.

The purpose of the present section is to show that this appearance is misleading. These relaxed examples do not contradict the previous results, because they do not belong to the same structural class. Accordingly, they are not treated as genuine counterexamples, but as limiting cases that clarify the scope of the preceding theorems.

Definition 6 *(Symmetric linear combination). Let* F *and* G *be entire functions of order one, each admitting a Hadamard factorization, and suppose that*

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$$F(w) = F(\kappa - w), \qquad G(w) = G(\kappa - w), \tag{56}$$

for some real constant κ. *Define*

$$H(w) := F(w) + G(w). \tag{57}$$

The function H *is called a symmetric linear combination of* F *and* G *[14]. It satisfies the same functional symmetry*

$$H(w) = H(\kappa - w). \tag{58}$$

Remark 9 (Functional symmetry of linear combinations). *Since* F *and* G *are entire,* H *is entire. Moreover,*

$$H(\kappa - w) = F(\kappa - w) + G(\kappa - w) = F(w) + G(w) = H(w). \tag{59}$$

Thus, H *inherits the same reflection symmetry. Since the sum of two entire functions of order at most one is again entire of order at most one,* H *is entire of finite order at most one, and therefore admits a Hadamard factorization. Hence,* H *retains the formal features of entireness, reflection symmetry, and Hadamard factorization, even though it need not belong to the same structural class as the original functions.*

Remark 10 (Preservation of H under optimal restriction). *Define* H *as in Definition 6, and suppose that* H *is presented in an unrestricted variable* s *and in an optimally restricted variable* z*, with the two presentations agreeing on a nonempty open overlap region. Then, by Theorem 2, both presentations determine the same analytic object.*

Consequently, passage to the optimal restricted variable does not alter the entireness, reflection symmetry, or Hadamard factorization of H*. These structures survive the restriction unchanged. What the restriction changes is not the function itself, but the regime of direct admissibility through which those structures are accessed.*

Accordingly, if $H(w) = H(\kappa - w)$*, then under the optimal restriction,* $\Re z \geq \kappa/2$*, the symmetry axis is retained as the boundary of admissible access, while the underlying analytic object and its zeros remain unchanged.*

The zero structure of linear combinations is fundamentally different from the single-function structure considered in the previous sections. We now make this distinction explicit and show that it has no consequence for the restriction framework.

Lemma 6 (Cancellation zeros do not arise from the indexed mechanism). *Recall* H *from Definition 6. Then* $w \in \mathbb{C}$ *is a zero of* H *if and only if*

$$F(w) = -G(w). \tag{60}$$

Accordingly, zeros of H*, in general, arise through cancellation of the evaluated summands, and not through the admissible zero-indexing mechanism developed in the preceding sections for a single symmetric function.*

Proof. By Definition 6,

$$H(w) = F(w) + G(w).$$

Therefore

$$H(w) = 0 \tag{61}$$

if and only if

$$F(w) + G(w) = 0$$

which is equivalent to (60). Thus, the vanishing of H is determined by cancellation between the output values of the two summands.

This is structurally different from the admissible zero-indexing mechanism developed in the preceding sections. There, the restricted framework governs how a single symmetric function can be represented and how its zeros can appear as admissible indices in its Hadamard product. Here, the zero condition $F(w) = -G(w)$ arises only after the two functions have already been evaluated.

Thus, the restricted variable may govern the admissible presentation of F and G individually, but it does not govern the algebraic cancellation of their values. Therefore, the zeros of H are produced by cancellation between two evaluated summands, not by the admissible indexing mechanism established earlier. ■

Remark 11 (Hadamard factorization does not presuppose reflection symmetry). *Reflection symmetry is not a prerequisite for Hadamard factorization. Any entire function of finite order admits a Hadamard product*

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of the appropriate genus [8], whether or not it satisfies a relation of the form $F(w) = F(\kappa - w)$. *Accordingly, the existence of a Hadamard factorization alone does not place a function within the restricted zero-indexing framework developed earlier.*

Remark 12 (Symmetric linear combination does not fall under z-space indexing). *By Lemmas 6, the zeros of a symmetry linear combination arise through cancellation,*

$$H(w) = 0 \iff F(w) = -G(w), \tag{62}$$

rather than through the admissible indexed mechanism attached to a single symmetric function.

For this reason, the conclusion of Theorem 4 does not transfer to H *merely from the existence of a Hadamard product. The issue is not that* H *fails to admit a canonical product, but that its zeros are produced by a different structural mechanism. Thus, even when* H *retains formal analytic features such as entireness, symmetry, and Hadamard factorization, it need not lie within the same restricted zero-indexing framework as the individual summands.*

Theorem 4 (Linear combinations do not constitute counterexamples). *Let* H *be the symmetric linear combination defined by Definition 6, where* F, G, *and* H *are entire functions of order at most one, each admitting a Hadamard factorization and satisfying the common reflection symmetry (56) and (58).*

Suppose further that H *is considered under the same optimal restricted framework as in the preceding section. Then* H *does not constitute a counterexample to the restricted zero-indexing conclusion of Theorem 3, even if it possesses zeros away from the symmetry line* $\Re w = \kappa/2$. *Such zeros arise from cancellation between the analytic values of the summands by (62), and are therefore not governed by the restricted zero-indexing mechanism established there for an individual reflected function.*

Proof. By Definition 6 with Remark 9, the function H is entire, satisfies the same reflection symmetry as the summands, and admits a Hadamard factorization. By Theorem 2, these formal analytic structures survive passage to the optimal restricted variable unchanged, as noted in Remark 10.

However, through Lemma 6, the zeros of H are characterized by the cancellation relation (64). Such zeros do not arise through the admissible indexed mechanism governing the zeros of an individual reflected function in the restricted framework. Rather, they arise only after the analytic values of the summands have been formed and combined.

Therefore, any off-line zeros of H, if present, do not fall under the hypotheses of Theorem 3. The earlier conclusion applies only to functions whose zero structure is governed by that indexed framework. Since the zeros of H are generated by cancellation, H lies outside that structural class in the relevant sense. Consequently, H does not constitute a counterexample to the preceding result.

■

The discussion up to this point already covers classical examples such as the Davenport–Heilbronn function [15] and related L-function-like examples discussed in connection with the Selberg class [18], whether directly in this section or by extension of the preceding framework. Theorem 4 addresses the two-summand symmetric case, but does not yet cover more general linear combinations. The following results treat admissible Hadamard representations in this broader setting, including finite combinations with multiple summands and asymmetric linear combinations.

Remark 13 (Linear combination of admissible Hadamard representations). *Let* H *be defined by Definition 6, and suppose that each summand is considered within a restricted framework in which its Hadamard representation is admissible. Then* H *may be written as the sum of those admissible representations* $H(z) = F(z) + G(z)$, *with each summand expressed in its own justified Hadamard form.*

More generally, if

$$H_N(z) := \sum_{n=1}^{N} F_n(z), \tag{63}$$

where each F_n *is considered within a restricted framework in which its Hadamard representation is admissible, then* H_N *may be formed by summing those admissible representations term by term [14].*

This does not mean that the sum itself inherits the same restricted zero-indexing mechanism as an individual summand. Rather, it means only that the restricted framework permits the functions F_n *to be represented separately, after which their analytic values may be added. By Theorem 2, the resulting function is*

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still the same analytic object H_N. *However, its zeros arise from cancellation among those values, not from the zero-indexing law attached to any one summand separately.*

Remark 14 (Scope of the combination results). *An infinite series*

$$\sum_{n=1}^{\infty} F_n(z) \tag{64}$$

would require additional hypotheses to ensure convergence, preservation of holomorphy, control of order, and availability of Hadamard factorization. For that reason, the present section treats only finite combinations, since the infinite case lies outside the scope of the present argument.

Lemma 7 (Finite combinations and constant shifts do not recover indexed admissibility). *Define (63), where each value of where each* F_n *is entire of order at most one, admits a Hadamard factorization, and satisfies a reflection symmetry of the form*

$$F_n(z) = F_n(\kappa_n - z), \qquad \kappa_n \in \mathbb{R} \tag{65}$$

so that each F_n *has a symmetry axis*

$$\Re z = \kappa_n/2. \tag{66}$$

Assume that each F_n *satisfies Theorem 3 on its own restricted framework. Then Remark 13 applies term wise, where each* F_n *may be represented in its own justified restricted Hadamard form, and* H_N *is obtained by combining those analytic values.*

Further, for any constant $c \in \mathbb{C}$, *define*

$$\widetilde{H}_N(z) \coloneqq H_N(z) + c. \tag{67}$$

Then neither H_N *nor* $\widetilde{H}_N$ *acquires a single admissible restricted Hadamard indexing mechanism from the summands. The admissible restricted Hadamard structure remains attached to the individual functions* F_n, *regardless of the value of* c *or of the axes* $\kappa_n/2$.

Proof. By hypothesis, each F_n admits its own restricted Hadamard representation, so Remark 13 applies term wise. Hence the restricted framework justifies the separate representation of each summand, after which their analytic values may be added to form H_N.

This does not place the sum H_N under a single zero-indexing law of the type developed earlier for one reflected function. Rather, its zeros are determined by the combined cancellation condition

$$\sum_{n=1}^{N} F_n(z) = 0, \tag{68}$$

which arises only after evaluation of the summands.

The same remains true for (67). Since c is independent of z, it does not introduce any new reflected indexing structure; it only changes the output-level cancellation condition to

$$\sum_{n=1}^{N} F_n(z) = -c. \tag{69}$$

Thus, neither H_N nor $H_N + c$ acquires a single admissible restricted Hadamard indexing mechanism from the summands. The admissible restricted Hadamard structure remains attached to each F_n individually. ■

Thus far, this section excludes a broad class of apparent counterexamples arising from finite combinations and constant shifts. This does not classify every possible function lying outside the indexed Hadamard framework. What it does show is that the preceding restriction theorems are not challenged merely by exhibiting a function with formal symmetry and a Hadamard factorization. For a function to constitute a genuine counterexample, it must first be shown that its reflected access, zero indexing, and Hadamard realization are all governed by the same admissible operations of the restricted framework.

Proposition 3 (Scope of the restricted indexed-Hadamard framework). *The restriction theorems concerning indexed Hadamard factorization apply only to functions whose zero structure is governed by the admissible operations of the restricted framework. Accordingly, a function does not constitute a counterexample to those theorems unless it is first shown that its reflected access, zero indexing, and Hadamard realization are all controlled by that same restricted mechanism, and that it possesses a zero away from the relevant symmetry line.*

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Proof. The preceding restriction theorems depend on three structural facts: admissible reflected access, admissible zero indexing, and rigidity of the canonical Hadamard factorization within the restricted framework. If the zero structure of a function is not governed by those admissible operations, then the contradiction mechanism used in Theorems 1 and 3 cannot be formed. In that case, the function lies outside the hypotheses governing the restricted indexed-Hadamard theory and therefore does not constitute a counterexample to its conclusions. Even within that class, a genuine counterexample would still require the existence of a zero away from the relevant symmetry line. ■

Remark 15 (Euler-product structure as a structural indicator). *Proposition 3 shows that a genuine counterexample must first be shown to lie within the same restricted indexed-Hadamard framework as the preceding theorems. In this connection, the existence of an Euler product is a strong structural indicator. Although an Euler product may be expanded into other representations, such as a Dirichlet series, its factors belong to one and the same representation of a single analytic object. In this sense, the product presents one function through a unified multiplicative structure at the level of the complex variable.*

An Euler-product structure is strong evidence that reflected access, zero indexing, and Hadamard realization may be governed by one and the same admissible mechanism, as required in Proposition 3. By contrast, the absence of such a unified product structure leaves open the possibility that zeros are influenced by mechanism not directly governed by the restricted complex variable, such as cancellation between separately formed summands. Thus, while an Euler product is not itself a required assumption of Proposition 3, it provides a natural structural indicator that a function belongs to the relevant restricted class.

7. Conclusions

This paper has clarified and extended the restricted-variable framework introduced in Drake (2026). The central point is that restriction does not alter the underlying analytic object, provided the restricted presentation agrees with the original function on a nonempty open overlap domain. What restriction changes is the regime of admissible access through which that object is represented. In the model case of the completed zeta function, this permits a sharper attribution of the boundary mechanism: the geometric-series device acts at the level of the real exponent, while the structural consequences arise from the interaction of that restriction with functional symmetry and canonical Hadamard factorization.

On that basis, the paper refined the earlier treatment of reflected values and zero indexing. Functional symmetry acts on the argument of the function, but does not by itself induce a corresponding admissible indexing rule for zeros in the restricted Hadamard product. Accordingly, the restricted framework preserves functional symmetry while losing the ambient reflection principle at the level of admissible coordinates and indices. Thus, reflected zeros remain analytically present through symmetry, yet do not appear as separately admissible geometric indices unless they lie on the symmetry boundary itself. This yields the central dilemma in the zeta case that consistency between the canonical Hadamard factorization and the restricted admissible indexing mechanism forces the nontrivial zeros to lie on the critical line.

The paper then abstracted this mechanism beyond the specific derivation of Drake (2026). Under an optimal half-plane restriction aligned with a reflection symmetry $w \mapsto \kappa - w$, the admissible domain and its reflected image meet only on the symmetry axis. For an entire function of order one whose canonical Hadamard factorization is interpreted within such a restricted framework, the same boundary-alignment mechanism applies. In this way, the critical-line phenomenon is shown not to depend only on the original model derivation, but to reflect a more general structural interaction among symmetry, admissibility, and indexed factorization.

The paper also addressed apparent counterexamples arising from symmetric linear combinations. Although such functions may retain entireness, reflection symmetry, and Hadamard factorization, their zeros arise through cancellation between separately evaluated summands rather than through the restricted zero-indexing mechanism governing a single reflected function. For that reason, they do not belong to the same structural class and do not contradict the preceding theorems. A genuine counterexample must first be shown to lie within the same restricted indexed-Hadamard framework. Euler-product structure is a natural indicator of this, since it presents one analytic object through a unified internal multiplicative representation, whereas a linear combination combines analytically distinct objects only after their values have been formed. This distinction marks the scope boundary of the present framework.

Taken together, these results support the view that the symmetry line is not merely an externally imposed feature of a completed function. Under optimal restriction, it is the unique boundary at which admissible access, functional symmetry, and canonical zero indexing remain simultaneously compatible.

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