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Borislav Penev *

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A New Constructive Method for Approximating the Pontryagin Two-Input Linear Time-Optimal Control Problem by Decomposing It into Two Single-Input Problems and Recombining Their Optimal Trajectories

Borislav Penev [1,2]

[1] Department of Control Systems, Faculty of Electronics and Automatics, Technical University—Sofia, Branch Plovdiv, 4000 Plovdiv, Bulgaria; bpenev@tu-plovdiv.bg

[2] Center of Competence "Smart Mechatronic, Eco- and Energy-Saving Systems and Technologies", 25 Tsanko Dyustabanov St., 4000 Plovdiv, Bulgaria

Abstract

The proposed approach decomposes in a special way the original two input linear time-optimal control problem into two single-input linear time-optimal problems whose optimal solutions are subsequently recombined. A lemma and a theorem establish conditions under which the recombined control vector is a candidate for the optimal solution and provide a simple criterion, based on the Pontryagin Maximum Principle, to determine whether the obtained control is truly optimal or only near optimal. The method is illustrated on the canonical double integrator system, but with two independent inputs. The resulting control system preserves the bang-bang structure and switching sequence of the true optimal solution, while providing a transition time that exceeds the optimal value by only 0.55%. The proposed method offers a basis for developing a technique regarding the multi-input linear time-optimal control problems.

Keywords: time-optimal control; minimum time control; Pontryagin's maximum principle; synthesis of optimal systems; multi-input; two-input; single-input linear systems; switching surface

MSC: 49N35; 93B50; 49N05; 93B52; 93C05

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1. Introduction

The Mathematical theory of optimal processes with its basic result the Pontryagin Maximum Principle (PMP) [1] provides a great tool for researchers in the field of automatic control to resolve optimal control problems. For many engineering problems, it is crucial to take into account the limitations associated with the control of the system under consideration while striving for high performance indicators. A special case is the following so-called linear time-optimal control problem, to which special chapters and sections of the optimal control theory are devoted [1] (Chapter 3), [2] (Chapters 6, 7), [3] (Chapter II), [4–6] (Chapter 7). The system is described by a linear system of differential equations of order n, the initial state of the system at the initial time is known, the final state is also given, but the final time is not known in advance. The admissible controls are piecewise continuous functions taking its values from the admissible set U, representing a closed convex polyhedron, most often a symmetric rectangle in the r-dimensional space of admissible controls, where the space origin is the center of U. The problem consists of minimizing the transition time from the initial state to the final state of the system. The PMP represents a necessary condition of optimality, but in case the linear system is controllable on each input, i.e. the system satisfies the condition of normality, the PMP regarding the linear time-optimal control problem becomes necessary and sufficient condition of optimality. The time-optimal control stemmed from the PMP is a "bang-bang" control regarding each one of the components of the control vector function, the last takes its values at the vertices of the set U. The number of the so-called switchings is finite, it depends on the initial state. In case of existence of complex eigenvalues of the system matrix this number is not limited, but in case the eigenvalues of the system matrix are non-positive the number of switching regarding each input is limited by the order of the system. This is known as theorem about the number of switchings or the n intervals of constancy [1] (Chapter 3, §17, Theorem 10) [2] (Chapter 6, Theorem 6.8) [3] (Chapter II, §6, Theorem 2.11, pp. 115–118).

Although the existence of solid theoretical background studying the linear time-optimal control problem for more than 60 years since the first works on the topic the examples of application of the theory are most often limited to linear systems with one input and third order of the system. The attempts to obtain an analytical solution to the problem face serious difficulties, and researchers note, as in [7] (p. 1), that "… there is still no complete time-optimal analytical solution for systems higher than second order". The geometric representation of the so-called switching hyper-surfaces, where the switchings regarding the controls occur, is extremely difficult, even impossible, except some cases of single input systems of order at most three. In order to obtain a representation of the switching hyper-surfaces in [7] (p. 8) where "based on Bellman's principle of optimality, a series of switching surfaces and curves in phase space is generated using dynamic programming method".

The examples in the monographs and textbooks devoted to the cases of linear time-optimal control problems with more than one input are very rare. Although the PMP provides a theoretical foundation to solve such problems, the implementation is reduced to systems of second order with two inputs [1] (Chapter 3, §21. Examples, Example 1, Example 2), [2] (Chapter 7, §7.9), [3] (Chapter 2, §5. The Linear Time-Optimal Control Problem, Point 24. Example, pp. 102–108). The representations are in the phase plane of the system under consideration and show the possibility of applying the PMP not only to single-input systems. But the significant increase in the complexity of finding a solution for multi-input systems, accompanied by difficulties in geometric representation, probably became sufficient reason for the exhaustion of interest in further implementations.

The linear time-optimal control problem and the PMP is concerned in the recent study [8]. The authors apply the PMP and employ the parabolic state-space trajectories in the phase plane obtained as a result of "steering methods based on bang-bang time-optimal controls" of the double integrator system "to improve the performance of sampling-based kinodynamic planners" associated with Rapidly-exploring Random Tree (RRT) for path planning in robotics.

The paper [9] "addresses the time-optimal feedforward regulation problem with input and output constraints" for the "linear, multivariable (i.e. multi-input multi-output or MIMO) time-invariant systems". The study extends the so-called "generalized bang-bang control" [10] "for scalar (i.e. single-input single-output or SISO) systems when both input and output constraints are present" to the so-called "multivariable square (i.e. equal number of inputs and outputs) systems", where some results are based on the results regarding the PMP and time-optimal and linear time-optimal problems [11] (Chapter 10, Point 1. pp. 300–311). In [9] the authors consider an example of "TITO (two-inputs two-outputs) systems satisfying suitable controllability conditions". They mention [9] (p. 511) regarding the obtained results, that "by time discretization and linear programming, an approximation of the generalized bang-bang control can be computed". This once again assures the fact that increasing complexity imposes the application of the theoretical foundation along with powerful numerical optimization methods.

The recent studies [12,13] regarding the linear time-optimal control problem with non-positive eigenvalues of the system with one input but with no restriction on the system order, based on the Pontryagin Maximum Principle, unveil some new properties of the problem, which deepens and expands the geometric representation of system's behavior under time-optimal control. These properties lay the foundation of a new method for synthesizing the time-optimal control for systems of any order abandoning the need to construct the switching hyper-surface. This becomes attractive

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not only from the point of view of solving time-optimal control problems for high order systems with one input, but also as a basis for a first attempt to find a solution in the case of two inputs, which is the subject of this study. The core idea consists in a special decomposition of the two-input linear time-optimal control problem into two single-input linear time-optimal problems with their joint solution and subsequent recombination to obtain the solution of the original problem.

The paper is structured as follows. Section 2 presents the formulation of the linear time-optimal control problem with two inputs and a proposed novel approach to its solution, which leads to a lemma and a theorem derived there. Section 3 considers a two-input linear time-optimal control problem as an example, based on the canonical double integrator system but in case of two independent inputs. The resulting time-optimal control solution based on the PMP, a near time-optimal solution based on the proposed method, and a comparison of the time-optimal solution and the obtained near time-optimal one are presented successively. Section 4 is devoted to the discussion and concluding remarks on the results.

2. Formulation of the Problem and Solution

Let us consider the following continuous two-input linear time-optimal control problem of order $n,\ n \geq 2$. The system with a state-space vector

$$\boldsymbol{x} = (x_1 \quad \cdots \quad x_n)^T \tag{1}$$

is described as follows:

$$\dot{\boldsymbol{x}} = \mathrm{A}\boldsymbol{x} + B\boldsymbol{u}, \tag{2}$$

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix},$$

$$B = (B_1 \quad B_2),\ B_1 = \begin{pmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{n1} \end{pmatrix},\ B_2 = \begin{pmatrix} b_{12} \\ b_{22} \\ \vdots \\ b_{n2} \end{pmatrix}. \tag{3}$$

The admissible control vector is

$$\boldsymbol{u}(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}. \tag{4}$$

Let us suppose the considered system satisfies the condition of normality

$$rank(B_i\ AB_i\ A^2B_i\ \cdots\ A^{n-1}B_i) = 2 \text{ for } i = 1,2. \tag{5}$$

Let us assume that both $u_1(t)$ and $u_2(t)$ are independent piecewise continuous functions and each one takes its values from the range

$$-u_{1max} \leq u_1(t) \leq u_{1max}\,,\ u_{1max} = const > 0, \tag{6}$$

$$-u_{2max} \leq u_2(t) \leq u_{2max}\,,\ u_{2max} = const > 0, \tag{7}$$

and at the points of discontinuity τ_{1i} regarding $u_1(t)$ and τ_{2j} regarding $u_2(t)$ we have:

$$u_1(\tau_{1i}) = u_1(\tau_{1i} + 0), \tag{8}$$

$$u_2(\tau_{2j}) = u_2(\tau_{2j} + 0). \tag{9}$$

Hence, the allowable control set represents a symmetric rectangle U in the plane u_1u_2 where its origin O is an inner point and the center of U.

Let the initial state of the system at $t = 0$ be

$$\boldsymbol{x}(0) = \boldsymbol{x}_0 = (x_{10} \ \cdots \ x_{(n-1)0} \quad x_{n0})^T. \tag{10}$$

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The target state at the final time t_f, where t_f is unspecified, represents the origin of the system's state-space

$$\boldsymbol{x}(t_f) = \boldsymbol{x}_f = (0)_{n\times 1}. \tag{11}$$

The time-optimal control problem in terms here is to find an admissible control vector $\boldsymbol{u}(t)$ which transfers the system (2) from its initial state (10) to the final state (11) in minimum time, i.e. minimizing the performance index

$$J = t_f \to min. \tag{12}$$

Let us call the above problem "Problem P".

Let us now formulate two single-input linear time-optimal problems by decomposing Problem P in the following way. Let k be a positive number k, $0 < k < 1$. The system of the first one problem is composed of the matrices A and B_1 (3) and represents

$$\dot{\boldsymbol{x}}_1 = \mathrm{A}\boldsymbol{x}_1 + B_1 u_1. \tag{13}$$

The initial state of the system is

$$\boldsymbol{x}_1(0) = \boldsymbol{x}_{01} = (x_{10,1} \ \cdots \ x_{(n-1)0,1} \ \ x_{n0,1})^T, \tag{14}$$

and it represents

$$\boldsymbol{x}_{01} = k\boldsymbol{x}_0. \tag{15}$$

The admissible control u_1 is yet defined above piecewise continuous function $u_1(t)$, which takes its values form the range (6), and for whom at the points of discontinuity τ_{1i} we have (8).

The target state at the final time t_{1f}, where t_{1f} is unspecified, represents the origin of the system's state-space

$$\boldsymbol{x}_1(t_{1f}) = \boldsymbol{x}_{1f} = (0)_{n\times 1}. \tag{16}$$

The time-optimal problem, which we shall call "Problem P_1", is to find an admissible control $u_1(t)$ which transfers the system (13) from its initial state (14) to the final state (16) in minimum time, i.e. minimizing the performance index

$$J_1 = t_{1f} \to min. \tag{17}$$

Let us formulate in the same manner the second single-input linear time-optimal control problem. The system is composed of the matrices A and B_2 (3) and represents

$$\dot{\boldsymbol{x}}_2 = \mathrm{A}\boldsymbol{x}_2 + B_2 u_2. \tag{18}$$

The initial state of the system is

$$\boldsymbol{x}_2(0) = \boldsymbol{x}_{02} = (x_{10,2} \ \cdots \ x_{(n-1)0,2} \ \ x_{n0,2})^T, \tag{19}$$

and it represents

$$\boldsymbol{x}_{02} = (1-k)\boldsymbol{x}_0. \tag{20}$$

The admissible control u_2 is yet defined above piecewise continuous function $u_2(t)$. It takes its values form the range (7) and at the points of discontinuity τ_{2j} we have (9).

The target state at the final time t_{2f}, where t_{2f} is unspecified, represents the origin of the system's state-space

$$\boldsymbol{x}_2(t_{2f}) = \boldsymbol{x}_{2f} = (0)_{n\times 1}. \tag{21}$$

The time-optimal problem, which we shall call "Problem P_2", is to find an admissible control $u_2(t)$ which transfers the system (18) from its initial state (19) to the final state (21) in minimum time, i.e. minimizing the performance index

$$J_2 = t_{2f} \to min. \tag{22}$$

Let us keep in mind that both Problem P_1 and Problem P_2 satisfy the condition of normality since the initial Problem P satisfies it being controllable on each input. Therefore, aside from the fact that the PMP regarding both single-input linear time-optimal Problem P_1 and Problem P_2 is a necessary condition of optimality, the PMP also is a sufficient condition of optimality [3] (Chapter II,

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§5, Point 22, Theorem 2.8, p. 96). For both Problem P_1 and Problem P_2 the corresponding solutions exist [3] (Chapter II, §6, Point 29, Theorem 2.15, p. 129) and with respect to each one problem the solution is unique [3] (Chapter II, §6, Point 28, Theorem 2.14, p. 127).

Our goal is to try to find the solution of the initial Problem P based on the solutions of the single-input Problem P_1 and Problem P_2 and some new properties that would be discovered at studying the initial Problem P in this way. The idea is to lighten the load on solving the two-input Problem P by solving two single-input problems.

Let us notice the way the initial state $\boldsymbol{x}_{01}$ (14)–(15) of Problem P_1, and the initial state $\boldsymbol{x}_{02}$ (19)–(20) of Problem P_2 are chosen. Both $\boldsymbol{x}_{01}$ and $\boldsymbol{x}_{02}$ are two inner points lying on the segment with first end at the origin of the state-space of the system of Problem P and second end at the initial state $\boldsymbol{x}_0$ (10) of Problem P and $\boldsymbol{x}_{01}$ and $\boldsymbol{x}_{02}$ satisfy

$$\begin{aligned} \boldsymbol{x}_{01} + \boldsymbol{x}_{02} &= k\boldsymbol{x}_0 + (1-k)\boldsymbol{x}_0 \\ &= \boldsymbol{x}_0. \end{aligned} \tag{23}$$

Let us denote the solution of Problem P_1 – the minimum time and the optimal control function – as t_{1f}^o and $u_1^o(t), t \in [0, t_{1f}^o]$, the solution of Problem P_2 as t_{2f}^o and $u_2^o(t), t \in [0, t_{2f}^o]$, and the solution of the original Problem P as t_f^o and $\boldsymbol{u}^o(t), t \in [0, t_f^o]$.

Let us suppose that we have found such a positive number k, $0 < k < 1$, so that for the initial conditions $\boldsymbol{x}_{01}$ of Problem P_1 and $\boldsymbol{x}_{02}$ of Problem P_2 the minimum time t_{1f}^o of Problem P_1 and t_{2f}^o of Problem P_2 satisfy the equality

$$t_{1f}^o = t_{2f}^o. \tag{24}$$

The optimal trajectory in the n-dimensional state-space of the system of Problem P_1 with an initial state $\boldsymbol{x}_{01}$ under the optimal control $u_1^o(t), t \in [0,\ t_{1f}^o]$, of Problem P_1 is described as

$$\boldsymbol{x}_1(t) = e^{At}\boldsymbol{x}_{01} + \int_0^t e^{A\tau} B_1 u_1^o(t-\tau)\, d\tau \quad \text{for } t \in [0, t_{1f}^o]. \tag{25}$$

The final point of this trajectory represents the state-space origin of the system of Problem P_1

$$\boldsymbol{x}_1(t_{1f}^o) = e^{At_{1f}^o}\boldsymbol{x}_{01} + \int_0^{t_{1f}^o} e^{A\tau} B_1 u_1^o(t_{1f}^o-\tau)\, d\tau = (0)_{n\times 1}. \tag{26}$$

Analogically, the optimal trajectory in the n-dimensional state-space of the system of Problem P_2 with an initial state $\boldsymbol{x}_{02}$ under the optimal control $u_2^o(t), t \in [0, t_{2f}^o]$, of Problem P_2 is given by

$$\boldsymbol{x}_2(t) = e^{At}\boldsymbol{x}_{02} + \int_0^t e^{A\tau} B_2 u_2^o(t-\tau)\, d\tau \quad \text{for } t \in [0, t_{2f}^o]. \tag{27}$$

Its final point satisfies

$$\boldsymbol{x}_2(t_{2f}^o) = e^{At_{2f}^o}\boldsymbol{x}_{02} + \int_0^{t_{2f}^o} e^{A\tau} B_2 u_2^o(t_{2f}^o-\tau)\, d\tau = (0)_{n\times 1}. \tag{28}$$

Let us summarize both trajectories $\boldsymbol{x}_1(t)$ (25) and $\boldsymbol{x}_2(t)$ (27) in the state-space of Problem P having in mind that both trajectories are n-dimensional trajectories with identical length $t_{1f}^o = t_{2f}^o$ (24) in a state-space with the same system matrix A.

$$\boldsymbol{x}(t) = \boldsymbol{x}_1(t) + \boldsymbol{x}_2(t) \text{ for } t \in [0, t_{1f}^o]. \tag{29}$$

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$$\begin{aligned} \boldsymbol{x}(t) &= e^{At}\boldsymbol{x}_{01} + \int_0^t e^{A\tau}B_1 u_1^o(t-\tau)\,d\tau \\ &+ e^{At}\boldsymbol{x}_{02} + \int_0^t e^{A\tau}B_2 u_2^o(t-\tau)\,d\tau \\ &\text{for } t \in [0, t_{1f}^o]. \end{aligned} \tag{30}$$

After replacement of $\boldsymbol{x}_{01}$ and $\boldsymbol{x}_{02}$ with the expressions (15) and (20), we obtain

$$\begin{aligned} \boldsymbol{x}(t) &= e^{At}k\boldsymbol{x}_0 + \int_0^t e^{A\tau}B_1 u_1^o(t-\tau)\,d\tau \\ &+ e^{At}(1-k)\boldsymbol{x}_0 + \int_0^t e^{A\tau}B_2 u_2^o(t-\tau)\,d\tau \\ &\text{for } t \in [0, t_{1f}^o]. \end{aligned} \tag{31}$$

$$\begin{aligned} \boldsymbol{x}(t) &= e^{At}(k+1-k)\boldsymbol{x}_0 + \int_0^t e^{A\tau}B_1 u_1^o(t-\tau)\,d\tau + \int_0^t e^{A\tau}B_2 u_2^o(t-\tau)\,d\tau \\ &= e^{At}\boldsymbol{x}_0 + \int_0^t e^{A\tau}(B_1 \quad B_2)\begin{pmatrix} u_1^o(t-\tau) \\ u_2^o(t-\tau)\end{pmatrix} d\tau = e^{At}\boldsymbol{x}_0 + \int_0^t e^{A\tau}B\begin{pmatrix} u_1^o(t-\tau) \\ u_2^o(t-\tau)\end{pmatrix} d\tau \\ &\text{for } t \in [0, t_{1f}^o]. \end{aligned} \tag{32}$$

According to (24), (26) and (28), we have for $\boldsymbol{x}(t)$ at time $t_{1f}^o = t_{2f}^o$

$$\boldsymbol{x}(t_{1f}^o) = \boldsymbol{x}_1(t_{1f}^o) + \boldsymbol{x}_2(t_{1f}^o) = (0)_{n\times 1}. \tag{33}$$

Let the vector function $\boldsymbol{u}^*(t)$ be

$$\boldsymbol{u}^*(t) = \begin{pmatrix} u_1^o(t) \\ u_2^o(t)\end{pmatrix} \text{ for } t \in [0, t_{1f}^o]. \tag{34}$$

The Equations (32), (33), and (34) show that the control vector $\boldsymbol{u}^*(t)$ transfers the initial point $\boldsymbol{x}_0$ of Problem P to the final point – the state-space origin – in a transition time $t_{1f}^o = t_{2f}^o$.

Since the controls $u_1^o(t)$ and $u_2^o(t)$ are the time-optimal controls, the solutions, of Problem P_1 and Problem P_2, respectively – each of them, as the solution of a linear time-optimal control problem with a single input, according to the PMP, is a piecewise constant function with a unique finite number of switchings – from u_{1max} to $-u_{1max}$ or from $-u_{1max}$ to u_{1max} with regard to $u_1^o(t)$, and from u_{2max} to $-u_{2max}$ or from $-u_{2max}$ to u_{2max} with regard to $u_2^o(t)$ – occurring at a specific set of moments – and since both $u_1^o(t)$ and $u_2^o(t)$ have identical durations $t_{1f}^o = t_{2f}^o$, it follows that the control vector $\boldsymbol{u}^*(t)$ takes its values only from the following set:

$$S = \{(u_{1max}, u_{2max}), (u_{1max}, -u_{2max}), (-u_{1max}, u_{2max}), (-u_{1max}, -u_{2max})\}, \tag{35}$$

$$\boldsymbol{u}^*(t) \in S. \tag{36}$$

The last means that $\boldsymbol{u}^*(t)$ takes its values exclusively at the vertices of the allowable control set – the rectangle U. Hence, the vector function $\boldsymbol{u}^*(t)$ may represent the solution to the initial time-optimal control Problem P.

Let us now assume that the minimum time t_{1f}^o – the solution to Problem P_1 – differs from the minimum time t_{2f}^o – the solution to Problem P_2

$$t_{1f}^o \neq t_{2f}^o. \tag{37}$$

Let us suppose also for example

$$t_{1f}^o > t_{2f}^o. \tag{38}$$

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Let us repeat the derivations (25)–(26) regarding the optimal trajectory of Problem P_1 with an initial state $\boldsymbol{x}_{01}$ under the time-optimal control $u_1^o(t), t \in [0,\ t_{1f}^o]$, and the derivations (27)–(28) regarding the optimal trajectory of Problem P_2 with an initial state $\boldsymbol{x}_{02}$ under the optimal control $u_2^o(t), t \in [0, t_{2f}^o]$. Both trajectories $\boldsymbol{x}_1(t)$ and $\boldsymbol{x}_2(t)$ end at the origin of the state-space, but at different times. Let us construct the following control regarding Problem P_2

$$u_2'(t) = \begin{cases} u_2^o(t) & \text{for } t \in [0, t_{2f}^o], \\ 0 & \text{for } t \in (t_{2f}^o, t_{1f}^o]. \end{cases} \tag{39}$$

The prolongation of the control of Problem P_2 in the above way leads to staying still of the system of Problem P_2 at the state-space origin after reaching it at time t_{2f}^o until time t_{1f}^o and thus to an identical duration t_{1f}^o of both controls $u_1^o(t)$ and $u_2'(t)$ (39)

$$\boldsymbol{x}_2(t) = \begin{cases} e^{At}\boldsymbol{x}_{02} + \int_0^t e^{A\tau} B_2 u_2^o(t-\tau)\, d\tau & \text{for } t \in [0, t_{2f}^o], \\ (0)_{n\times 1} & \text{for } t \in (t_{2f}^o, t_{1f}^o]. \end{cases} \tag{40}$$

Thus, we obtain for the sum of both trajectories $\boldsymbol{x}_1(t)$ (25) and $\boldsymbol{x}_2(t)$ (40)

$$\boldsymbol{x}(t) = \boldsymbol{x}_1(t) + \boldsymbol{x}_2(t) \text{ for } t \in [0, t_{1f}^o]. \tag{41}$$

$$\begin{aligned} \boldsymbol{x}(t) \quad = \quad & e^{At}\boldsymbol{x}_{01} + \int_0^t e^{A\tau} B_1 u_1^o(t-\tau)\, d\tau \\ & + e^{At}\boldsymbol{x}_{02} + \int_0^t e^{A\tau} B_2 u_2'(t-\tau)\, d\tau \\ & \text{for } \ t \in [0, t_{1f}^o]. \end{aligned} \tag{42}$$

After replacement of $\boldsymbol{x}_{01}$ and $\boldsymbol{x}_{02}$ with the expressions (15) and (20), we obtain

$$\begin{aligned} \boldsymbol{x}(t) \quad = \quad & e^{At}k\boldsymbol{x}_0 + \int_0^t e^{A\tau} B_1 u_1^o(t-\tau)\, d\tau \\ & + e^{At}(1-k)\boldsymbol{x}_0 + \int_0^t e^{A\tau} B_2 u_2'(t-\tau)\, d\tau \\ & \text{for } \ t \in [0, t_{1f}^o]. \end{aligned} \tag{43}$$

$$\begin{aligned} \boldsymbol{x}(t) \quad & = \quad e^{At}(k+1-k)\boldsymbol{x}_0 + \int_0^t e^{A\tau} B_1 u_1^o(t-\tau)\, d\tau + \int_0^t e^{A\tau} B_2 u_2'(t-\tau)\, d\tau \\ & = \quad e^{At}\boldsymbol{x}_0 + \int_0^t e^{A\tau} [B_1 \quad B_2] \begin{pmatrix} u_1^o(t-\tau) \\ u_2'(t-\tau) \end{pmatrix} d\tau = e^{At}\boldsymbol{x}_0 + \int_0^t e^{A\tau} B \begin{pmatrix} u_1^o(t-\tau) \\ u_2'(t-\tau) \end{pmatrix} d\tau \\ & \text{for } \ t \in [0, t_{1f}^o]. \end{aligned} \tag{44}$$

According to (26) and (40), we have for $\boldsymbol{x}(t)$ at time t_{1f}^o

$$\boldsymbol{x}(t_{1f}^o) = \boldsymbol{x}_1(t_{1f}^o) + \boldsymbol{x}_2(t_{1f}^o) = (0)_{n\times 1}. \tag{45}$$

Let the vector function $\boldsymbol{u}^{*\prime}(t)$ be

$$\boldsymbol{u}^{*\prime}(t) = \begin{pmatrix} u_1^o(t) \\ u_2'(t) \end{pmatrix} \text{ for } t \in [0, t_{1f}^o] \tag{46}$$

We can conclude considering the function $u_2'(t)$ (39) that the control vector $\boldsymbol{u}^{*\prime}(t)$ for $t \in [0, t_{2f}^o]$ takes its values exclusively from the set S (35), but when $t \in (t_{2f}^o, t_{1f}^o]$ the vector function $\boldsymbol{u}^{*\prime}(t)$ takes its values exclusively from the set

$$S' = \{(u_{1max}, 0), (-u_{1max}, 0)\}. \tag{47}$$

$$\boldsymbol{u}^{*\prime}(t) \in \begin{cases} S & \text{for } t \in [0, t_{2f}^o], \\ S' & \text{for } t \in (t_{2f}^o, t_{1f}^o]. \end{cases} \tag{48}$$

All the points of the set S' (47) are inner points of the allowable control set - the rectangle U, i.e. are not vertices of U. Therefore, according to the PMP, the vector function $\boldsymbol{u}^{*\prime}(t)$ (46) is not definitely the solution to the initial Problem P.

The next step is the justification of existence of such a positive number k, $0 < k < 1$, so that at the initial conditions $\boldsymbol{x}_{01}$ (15) of Problem P_1 and $\boldsymbol{x}_{02}$ (20) of Problem P_2, the respective solutions t_{1f}^o of Problem P_1 and t_{2f}^o of Problem P_2 satisfy the equality (24).

Let us suppose first that the initial condition $\boldsymbol{x}_{01}$ of Problem P_1 represents the initial condition $\boldsymbol{x}_0$ of the original Problem P and we have found in this case the solution t_{1f}^o of Problem P_1. Let this time t_{1f}^o be T_1:

$$T_1 = t_{1f}^o. \tag{49}$$

Let us now form regarding the considered Problem P_1 the attainable set V_T the way [3] (Chapter II, §5, p. 85) (Pontryagin, Boltyanskii and colleagues use the term "sphere of attainability" instead). V_T represents here the set of all points for whom the solution t_{1f}^o of Problem P_1 having its initial state $\boldsymbol{x}_{01}$ at these points is equal or less T_1. Let V_T in this case be denoted also as $V_{T_1}^1$. The attainable set V_T is convex [3] (Chapter II, §5, Lemma 2.3, p. 85). The state-space origin as a final state of Problem P_1 is an inner point of the attainable set $V_{T_1}^1$. The initial condition $\boldsymbol{x}_0$ of the original Problem P. which now also represents the initial condition of Problem P_1, and for whom the obtained optimal time t_{1f}^o of Problem P_1 with an initial state at this point $\boldsymbol{x}_0$ is taken here as T_1, is a boundary point of $V_{T_1}^1$ [3] (Chapter II, §5, Consequence 2.6 of Lemma 2.5, p. 91). Let $\boldsymbol{x}_0$ be the point N_0.

$$N_0 \equiv \boldsymbol{x}_0. \tag{50}$$

Let k be a parameter and

$$k \geq 0. \tag{51}$$

If the point N_1 is defined as

$$N_1 \equiv k\boldsymbol{x}_0, \tag{52}$$

then N coincides with the origin O when $k = 0$, but when $0 < k < 1$ the point N_1 describes all the inner points of the segment ON_0. Both the points O and N_0 are points of $V_{T_1}^1$. Since the attainable set $V_{T_1}^1$ is convex, and N_1 is an inner point of the segment ON_0, then N_1 is also an inner point of $V_{T_1}^1$. Thus, the corresponding optimal time of Problem P_1 with an initial state at this point $N_1 \equiv k\boldsymbol{x}_0, 0 < k < 1$, is less the time T_1 according to [3] (Chapter II, §5, Lemma 2.4, p. 88). Moreover, it follows from Lemma 2.4 the continuity of the minimal time function with respect to the initial state. So, there exists some kind of continuous dependence of the time-optimal solution t_{1f}^o of Problem P_1 on the position of the point N_1 between the endpoints O and N_0 of the segment ON_0, which is a decreasing relation from T_1 to 0 when N_1 continuously describes all the points from N_0 to O on the segment ON_0 (the reachable set shrinks strictly when the initial state moves towards the state-space origin along the ray), i.e., when the parameter k continuously decreases on the interval $[0,1]$. So, when the parameter k continuously increases on the interval $[0,1]$ that function continuously increases from 0 to T_1 (the reachable set expands strictly when the initial state moves away from the state-space origin along the ray). Let that function be $f_1(k)$:

$$t_{1f}^o = f_1(k), \text{where } k \in [0,1], t_{1f}^o \in [0, T_1]. \tag{53}$$

Let us now consider Problem P_2 in the above-described manner. Let us suppose first that the initial condition $\boldsymbol{x}_{02}$ of Problem P_2 represents the initial condition $\boldsymbol{x}_0$ of the original Problem P and we have found in this case the solution t_{2f}^o of Problem P_2. Let this time t_{2f}^o be T_2:

$$T_2 = t_{2f}^o. \tag{54}$$

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Let us now form analogically regarding Problem P_2 the attainable set $V^2_{T_2}$. $V^2_{T_2}$ represents here the set of all points for whom the solution t^o_{2f} of Problem P_2 having its initial state $\boldsymbol{x}_{02}$ at these points is equal or less T_2. The attainable set $V^2_{T_2}$ is convex, the state-space origin O as a final state of Problem P_2 is an inner point of the attainable set $V^2_{T_2}$ while the point N_0 a boundary point of $V^2_{T_2}$.

If the point N_2 is defined as

$$N_2 \equiv (1-k)\boldsymbol{x}_0, \tag{55}$$

then N_2 coincides with N_0 when $k=0$, and coincides with the origin O when $k=1$, but when $0<k<1$ the point N_2 describes all the inner points of the segment ON_0 from with N_0 towards O. Both the points O and N_0 are points of $V^2_{T_2}$. Since the attainable set $V^2_{T_2}$ is convex, and N_2 is an inner point of the segment ON_0, then N_2 is also an inner point of $V^2_{T_2}$. Thus, the corresponding optimal time of Problem P_2 with an initial state at this point $N_2 \equiv (1-k)\boldsymbol{x}_0, 0<k<1$, is less than the time T_2 according to [3] (Chapter II, §5, Lemma 2.4, p. 88). Here, in contrast to the continuous increasing dependence of the time-optimal solution t^o_{1f} of Problem P_1 on the position of the point N_1 between the endpoints O and N_0 of the segment ON_0 when the parameter k continuously increases on the interval $[0,1]$, the continuous dependence of the time-optimal solution t^o_{2f} of Problem P_2 on the position of the point N_2 between the endpoints O and N_0 of the segment ON_0 is a decreasing relation from T_2 to 0 when N_2 continuously describes all the points from N_0 to O on the segment ON_0, i.e., when the parameter k continuously increases on the interval $[0,1]$. Let that function be $f_2(k)$.

$$t^o_{2f} = f_2(k), \text{where } k \in [0,1], t^o_{2f} \in [0, T_2]. \tag{56}$$

Thus, there are two continuous functions $f_1(k)$ and $f_2(k)$ defined on the same interval $k \in [0,1]$, $f_1(k)$ is a continuous increasing function while $f_2(k)$ is a continuous decreasing function, where:

$$\begin{aligned} f_1(0) &= 0\,, & f_1(1) &= T_1 > 0, \\ f_2(0) &= T_2 > 0\,, & f_2(1) &= 0. \end{aligned} \tag{57}$$

It is easy to see that there exists only one number k, $k \in (0,1)$, satisfying the equation

$$f_1(k) = f_2(k). \tag{58}$$

Let this k be k^o. Therefore,

$$t^o_{1f} = f_1(k^o) = f_2(k^o) = t^o_{2f}. \tag{59}$$

Thus, the following lemma has been proven.

Lemma 1. *Let Problem P_1 and Problem P_2 with initial states $\boldsymbol{x}_{01}$ (15) and $\boldsymbol{x}_{02}$ (20), respectively, where the parameter k varies from 0 to 1, be formulated based on Problem P. The dependence of the minimum time t^o_{1f} of the solution to Problem P_1 on the parameter k is a continuous increasing function starting from 0 at $k=0$, while the dependence of the minimum time t^o_{2f} of the solution to Problem P_2 on k is a continuous decreasing function becoming 0 at $k=1$. There exists only one number k, $k \in (0,1)$, named k^o so that when the initial states $\boldsymbol{x}_{01}$ of Problem P_1 and $\boldsymbol{x}_{02}$ of Problem P_2 are:*

$$\boldsymbol{x}_{01} = k^o \boldsymbol{x}_0, \tag{60}$$

$$\boldsymbol{x}_{02} = (1-k^o)\boldsymbol{x}_0, \tag{61}$$

the optimal – the minimum – time t^o_{1f} of Problem P_1 and the optimal time t^o_{2f} of Problem P_2 satisfy the equality

$$t^o_{1f} = t^o_{2f}. \tag{62}$$

In this case, the control vector $\boldsymbol{u}^*(t)$ (34), composed of the time-optimal control $u^o_1(t), t \in [0,\ t^o_{1f}]$, of Problem P_1 and the time-optimal control $u^o_2(t), t \in [0,\ t^o_{2f}]$, of Problem P_2, is a candidate for the solution to the original time-optimal control Problem P with an initial state $\boldsymbol{x}_0$.

First, let the time t^*_f be

$$t^*_f = t^o_{1f} = t^o_{2f}, \tag{63}$$

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and $\boldsymbol{x}^*(t)$ be the trajectory of the original Problem P obtained under the control vector $\boldsymbol{u}^*(t)$ (34) representing the sum of the time-optimal trajectories of Problem P_1 and Problem P_2:

$$\begin{aligned} &\boldsymbol{x}^*(t) = \boldsymbol{x}_1^o(t) + \boldsymbol{x}_2^o(t) \text{ for } t \in [0, t_f^o], \\ &\boldsymbol{x}_1^o(t) = e^{At}k^o\boldsymbol{x}_0 + \int_0^t e^{A\tau}B_1u_1^o(t-\tau)\,d\tau, \\ &\boldsymbol{x}_2^o(t) = e^{At}(1-k^o)\boldsymbol{x}_0 + \int_0^t e^{A\tau}B_2u_2^o(t-\tau). \end{aligned} \tag{64}$$

$$\begin{aligned} \boldsymbol{x}^*(t) &= e^{At}k^o\boldsymbol{x}_0 + \int_0^t e^{A\tau}B_1u_1^o(t-\tau)\,d\tau + e^{At}(1-k^o)\boldsymbol{x}_0 + \int_0^t e^{A\tau}B_2u_2^o(t-\tau)\,d\tau \\ &= e^{At}\boldsymbol{x}_0 + \int_0^t e^{A\tau}(B_1 \quad B_2)\begin{pmatrix} u_1^o(t-\tau) \\ u_2^o(t-\tau)\end{pmatrix}d\tau = e^{At}\boldsymbol{x}_0 + \int_0^t e^{A\tau}B\begin{pmatrix} u_1^o(t-\tau) \\ u_2^o(t-\tau)\end{pmatrix}d\tau \\ &= e^{At}\boldsymbol{x}_0 + \int_0^t e^{A\tau}B\boldsymbol{u}^*(t-\tau)\,d\tau \text{ for } t \in [0, t_f^*]. \end{aligned} \tag{65}$$

Let us now analyze the conditions which the control vector $\boldsymbol{u}^*(t)$ (34) must satisfy for being the time-optimal control solution to the initial Problem P. This means that the control vector $\boldsymbol{u}^*(t)$ (34) and the respective system trajectory $\boldsymbol{x}^*(t)$ must satisfy the PMP regarding the linear time-optimal control Problem P. In case of satisfying it as a necessary condition of optimality and considering that the system of Problem P satisfies the condition of normality (being controllable on each input), then, based on the fact that the PMP becomes also a sufficient condition of optimality [3] (Chapter II, §5, Point 22, Theorem 2.8, p. 96), $\boldsymbol{u}^*(t)$ (34) and the respective system trajectory $\boldsymbol{x}^*(t)$ become the only solution to the initial Problem P.

Pontryagin's function H (Hamilton function) regarding the linear time-optimal control problem [3] (Chapter II, §5, Point 19, Eqn. 2.24, p. 85) represents

$$H(\boldsymbol{x}(t), \boldsymbol{u}(t), \boldsymbol{\psi}(t)) = (\boldsymbol{\psi}(t))^T(A\boldsymbol{x}(t) + B\boldsymbol{u}(t)). \tag{66}$$

The vector function $\boldsymbol{\psi}(t)$ represents a solution of the adjoined system

$$\dot{\boldsymbol{\psi}} = -A^T\boldsymbol{\psi}. \tag{67}$$

Its solution is

$$\boldsymbol{\psi}(t) = e^{-A^Tt}\boldsymbol{\psi}_0. \tag{68}$$

Let the initial state regarding $\boldsymbol{\psi}$ be

$$\boldsymbol{\psi}_0^* = \boldsymbol{\psi}^*(0) = (\psi_{10}^* \ \cdots \ \psi_{(n-1)0}^* \ \ \psi_{n0}^*)^T. \tag{69}$$

and consider the following solution of the adjoined system called $\boldsymbol{\psi}^*(t)$:

$$\boldsymbol{\psi}^*(t) = e^{-A^Tt}\boldsymbol{\psi}_0^* \text{ for } t \in [0, t_f^*]. \tag{70}$$

We are going to solve the following problem. Provided that the control vector $\boldsymbol{u}^*(t)$ with its components $u_1^o(t)$ and $u_2^o(t)$ are known, let us consider the initial conditions of $\boldsymbol{\psi}_0^*$ (69) as unknown parameters, where the goal is to find out the conditions of existence a non-trivial solution regarding $\boldsymbol{\psi}_0^*$ so that the set $\boldsymbol{x}^*(t)$, $\boldsymbol{u}^*(t)$, and $\boldsymbol{\psi}^*(t)$ satisfies the PMP.

We obtain regarding H:

$$\begin{aligned} H(\boldsymbol{x}^*(t), \boldsymbol{u}(t), \boldsymbol{\psi}^*(t)) &= (\boldsymbol{\psi}^*(t))^T(A\boldsymbol{x}^*(t) + B\boldsymbol{u}(t)) \\ &= (\boldsymbol{\psi}^*(t))^TA\boldsymbol{x}^*(t) + (\boldsymbol{\psi}^*(t))^T(B_1 \quad B_2)\begin{pmatrix} u_1(t) \\ u_2(t)\end{pmatrix} \\ &= (\boldsymbol{\psi}^*(t))^TA\boldsymbol{x}^*(t) + (\boldsymbol{\psi}^*(t))^TB_1u_1(t) + (\boldsymbol{\psi}^*(t))^TB_2u_2(t). \end{aligned} \tag{71}$$

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$$\begin{aligned} \max_{\boldsymbol{u}\in U} H(\boldsymbol{x}^*(t),\boldsymbol{u}(t),\boldsymbol{\psi}^*(t)) &= \left(\boldsymbol{\psi}^*(t)\right)^T A\boldsymbol{x}^*(t) \\ &+ \max_{|u_1|\leq u_{1max}} \left(\boldsymbol{\psi}^*(t)\right)^T B_1\, u_1(t) \\ &+ \max_{|u_2|\leq u_{2max}} \left(\boldsymbol{\psi}^*(t)\right)^T B_2\, u_2(t). \end{aligned} \tag{72}$$

Let us consider the second component of the above sum (72). We obtain regarding $u_1(t)$

$$\max_{|u_1|\leq u_{1max}} \left(\boldsymbol{\psi}^*(t)\right)^T B_1\, u_1(t) = \left(\boldsymbol{\psi}^*(t)\right)^T B_1\, u_{1max}\, sign\left(\left(\boldsymbol{\psi}^*(t)\right)^T B_1\right) \tag{73}$$

Let us assume from the condition regarding the maximum of Hamilton function

$$\max_{\boldsymbol{u}\in U} H(\boldsymbol{x}^*(t),\boldsymbol{u}(t),\boldsymbol{\psi}^*(t)) = H(\boldsymbol{x}^*(t),\boldsymbol{u}^*(t),\boldsymbol{\psi}^*(t)), \tag{74}$$

with respect to the second component of (72), that in (73)

$$u_{1max}\, sign\left(\left(\boldsymbol{\psi}^*(t)\right)^T B_1\right) = u_1^o(t) \text{ for } t\in\left[0,t_f^*\right], \tag{75}$$

and the control $u_1^o(t)$ for $t\in\left[0,t_f^*\right]$ has k_1 switching times $t_{11},\dots,t_{1,k_1}$.

Then, we obtain regarding the function $\left(\boldsymbol{\psi}^*(t)\right)^T B_1$ that it must be equal to zero at the above set of times $t_{11},\dots,t_{1,k_1}$:

$$\left(\boldsymbol{\psi}^*(t)\right)^T B_1 = 0\ \forall t\in\left\{t_{11},\dots,t_{1,k_1}\right\}. \tag{76}$$

This means:

$${B_1}^T\boldsymbol{\psi}^*(t) = 0\ \forall t\in\left\{t_{11},\dots,t_{1,k_1}\right\}, \tag{77}$$

$${B_1}^T e^{-A^T t}\boldsymbol{\psi}_0^* = 0\ \forall t\in\left\{t_{11},\dots,t_{1,k_1}\right\}. \tag{78}$$

Thus, we have a homogeneous linear system of k_1 equations concerning all n unknown initial conditions $\psi_{10}^*,\cdots,\psi_{(n-1)0}^*,\psi_{n0}^*$ of $\boldsymbol{\psi}$.

Similarly, we obtain for the third component of the sum of (72),

$${B_2}^T e^{-A^T t}\boldsymbol{\psi}_0^* = 0\ \forall t\in\left\{t_{21},\dots,t_{2,k_2}\right\}, \tag{79}$$

where it is assumed that the control $u_2^o(t)$ for $t\in\left[0,t_f^*\right]$ has k_2 switching times $t_{21},\dots,t_{2,k_2}$.

Thus, we have in addition to (78) a second homogeneous linear system of k_2 equations concerning the same all n unknown initial states $\psi_{10}^*,\cdots,\psi_{(n-1)0}^*,\psi_{n0}^*$ of $\boldsymbol{\psi}$.

The PMP [3] (Chapter I, §2, Theorem 1.2, pp. 33–35) regarding the value of H at the end of the process

$$H\left(\boldsymbol{x}^*\left(t_f^*\right),\boldsymbol{u}^*\left(t_f^*\right),\boldsymbol{\psi}^*\left(t_f^*\right)\right) = 1. \tag{80}$$

Considering that

$$\begin{aligned} \boldsymbol{x}^*(t) &= (0)_{n\times 1}, \\ \boldsymbol{u}^*\left(t_f^*\right) &= \begin{pmatrix} u_1^o\left(t_f^*\right) \\ u_2^o\left(t_f^*\right) \end{pmatrix}, \end{aligned} \tag{81}$$

we obtain from (80) the following linear equation also referring to all n unknown initial conditions $\psi_{10}^*,\cdots,\psi_{(n-1)0}^*,\psi_{n0}^*$ of $\boldsymbol{\psi}$.

$$\left(u_1^o\left(t_f^*\right){B_1}^T e^{-A^T t_f^*} + u_2^o\left(t_f^*\right){B_2}^T e^{-A^T t_f^*}\right)\boldsymbol{\psi}_0^* = 1. \tag{82}$$

Summarizing the results of (78), (79), and (82), we can conclude that (78), (79), and (82) represent a linear system of

$$k_1 + k_2 + 1 \tag{83}$$

equations concerning all n unknown initial conditions $\psi_{10}^*,\cdots,\psi_{(n-1)0}^*,\psi_{n0}^*$ of $\boldsymbol{\psi}$.

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There are two options regarding the linear system (78), (79), and (82): be consistent or inconsistent one.

In case of consistent system, since it is not homogeneous, the solution is non-trivial. Thus, there exists a non-trivial solution (70) to the adjoined system (67) and the PMP is satisfied as a necessary condition of optimality. On the other hand, the system of Problem P satisfies the condition of normality (5). Since the PMP regarding the linear time-optimal control problem in case the system satisfies the condition of normality is not only a necessary but also a sufficient condition of optimality [3] (Chapter II, §5, Point 22, Theorem 2.8, p. 96), it follows that the control vector $\boldsymbol{u}^*(t)$ (34) and the trajectory $\boldsymbol{x}^*(t)$ (64), (65) are the only solution to the original Problem P.

In case of inconsistent system, it follows that there does not exist a non-trivial solution (70) to the adjoined system (67), which the PMP requires. Hence, the control vector $\boldsymbol{u}^*(t)$ (34) and the trajectory $\boldsymbol{x}^*(t)$ (64), (65) are not the time-optimal solution to the original Problem P. This happens even though the control vector $\boldsymbol{u}^*(t)$ takes it values exclusively from the set S (35), comprising only the vertices of the allowable control set – the symmetric rectangle U with a center at the origin O of the plane u_1u_2, and the system trajectory $\boldsymbol{x}^*(t)$ (64), (65) ends at the state-space origin.

Thus, the following theorem regarding the solution to the original Problem P has been proven.

Theorem 1. *Let the initial states $\boldsymbol{x}_{01}$ and $\boldsymbol{x}_{02}$ of Problem P_1 and Problem P_2 formulated based on Problem P be chosen according to Lemma 1:*

$$\boldsymbol{x}_{01} = k^o\boldsymbol{x}_0, \tag{84}$$

$$\boldsymbol{x}_{02} = (1 - k^o)\boldsymbol{x}_0. \tag{85}$$

The control vector $\boldsymbol{u}^*(t)$, composed of the time-optimal control $u_1^o(t)$ of Problem P_1 and the time-optimal control $u_2^o(t)$ of Problem P_2,

$$\boldsymbol{u}^*(t) = \begin{pmatrix} u_1^o(t) \\ u_2^o(t) \end{pmatrix} \text{ for } t \in [0, t_f^*] \tag{86}$$

and the trajectory $\boldsymbol{x}^*(t)$ of the original Problem P, obtained under $\boldsymbol{u}^*(t)$ and transferring the system to the state-space origin O at time t_f^*, are the only solution to the original Problem P according to the Pontryagin Maximum Principle if the linear system (78), (79), and (82) concerning all n unknown initial conditions $\psi_{10}^*, \cdots, \psi_{(n-1)0}^*, \psi_{n0}^*$ of the adjoined system (67) regarding $\boldsymbol{\psi}$ is consistent one.

In case the linear system (78), (79), and (82) is inconsistent one, then the vector $\boldsymbol{u}^*(t)$ and the trajectory $\boldsymbol{x}^*(t)$ are not the time-optimal control solution to the original Problem P.

The vector $\boldsymbol{u}^*(t)$ and the trajectory $\boldsymbol{x}^*(t)$ can only be considered as a somewhat near time-optimal control solution to Problem P since $\boldsymbol{x}^*(t)$, representing the sum of the time-optimal trajectories $\boldsymbol{x}_1^o(t)$ of Problem P_1 and $\boldsymbol{x}_2^o(t)$ of Problem P_2, is the result of $\boldsymbol{u}^*(t)$, which is a bang-bang control with values exclusively from the set S (35), comprising only the vertices of the allowable control set – the symmetric rectangle U with a center at the origin O of the plane u_1u_2.

3. Example

Let us formulate the following second order Problem P as an example. The system represents the well-known double integrator system but under a control vector consisting of two independent controls u_1 and u_2:

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} + B\boldsymbol{u}, \tag{87}$$

$$\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \qquad \boldsymbol{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \tag{88}$$

$$A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

$$B = (B_1 \quad B_2),\ B_1 = \begin{pmatrix} b_{11} \\ b_{21} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ B_2 = \begin{pmatrix} b_{12} \\ b_{22} \end{pmatrix} = \begin{pmatrix} 1 \\ 0.1612 \end{pmatrix}. \tag{89}$$

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Let us assume that both $u_1(t)$ and $u_2(t)$ are piecewise continuous functions and each one takes its values from the range:

$$-u_{1max} \le u_1(t) \le u_{1max}\,,\ u_{1max} = 1, \tag{90}$$

$$-u_{2max} \le u_2(t) \le u_{2max}\,,\ u_{2max} = 1. \tag{91}$$

Let us also assume, as we have already done at the initial formulation of Problem P, that at the points of discontinuity τ_{1i} regarding $u_1(t)$, and τ_{2j} regarding $u_2(t)$, we have (8) and (9). Let the admissible control vector composed by $u_1(t)$ and $u_2(t)$ be

$$\boldsymbol{u}(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}. \tag{92}$$

Let the initial state of the system at $t = 0$ be

$$\boldsymbol{x}(0) = \boldsymbol{x}_0 = \begin{pmatrix} x_{10} \\ x_{20} \end{pmatrix} = \begin{pmatrix} 1 \\ 0.44447 \end{pmatrix}. \tag{93}$$

The target state at the final time t_f, where t_f is unspecified, represents the state-space origin

$$\boldsymbol{x}(t_f) = \boldsymbol{x}_f = (0)_{2\times 1}. \tag{94}$$

The time-optimal Problem P in terms here and according to the originally formulated Problem P is to find an admissible control vector $\boldsymbol{u}(t)$ which transfers the system (88) from its initial state (93) to the final state (94) in minimum time, i.e. minimizing the performance index

$$J = t_f \to min. \tag{95}$$

The formulated time-optimal Problem P is controllable on each input, i.e. satisfies the condition of normality (5):

$$rank(B_1\ AB_1) = rank\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \middle| \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = rank\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 2, \tag{96}$$

$$\begin{aligned} rank(B_2\ AB_2) &= rank\left(\begin{pmatrix} 1 \\ 0.1612 \end{pmatrix} \middle| \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0.1612 \end{pmatrix}\right) \\ &= rank\begin{pmatrix} 1 & 0 \\ 0.1612 & 1 \end{pmatrix} = 2\,. \end{aligned} \tag{97}$$

The allowable control set U is convex since it represents a square with side 1 in the plane u_1u_2. It is also symmetric where the origin O of the plane u_1u_2 is an inner point and the center of U.

Following the deduced Lemma 1 and Theorem 1, the sequence of steps that we will perform consists of: Solving Problem P applying the PMP directly; Solving the pair of subproblems Problem P_1 and Problem P_2, arising from Problem P, and obtaining the optimal value k^o of the parameter k; Recombining the solutions of both Problem P_1 and Problem P_2, obtained at k_o, to obtain the control vector $\boldsymbol{u}^*(t)$ (86); Checking the linear system (78), (79), and (82) for consistency; Comparing the two solutions.

3.1. The Time-Optimal Control Solution of Problem P

Let us consider the following two piecewise constant functions $u_1^*(t)$ and $u_2^*(t)$ with identical duration of $t_f^*\ (s)$, each one having one switching from -1 to $+1$ at time $t_1^*\ (s)$ regarding $u_1^*(t)$, and at time $t_2^*\ (s)$ regarding $u_2^*(t)$:

$$u_1^*(t) = \begin{cases} -1 & \text{for } t \in [0, t_1^*) \\ +1 & \text{for } t \in [t_1^*, t_f^*], \end{cases} \tag{98}$$

$$u_2^*(t) = \begin{cases} -1 & \text{for } t \in [0, t_2^*) \\ +1 & \text{for } t \in [t_2^*, t_f^*], \end{cases} \tag{99}$$

$$t_1^* = 0.9675\ (s), \quad t_2^* = 1.1287\ (s), \quad t_f^* = 1.59625\ (s). \tag{100}$$

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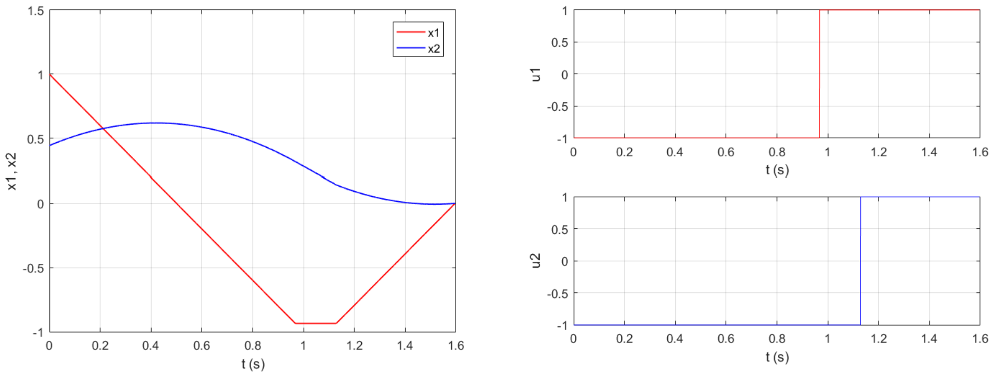
The control vector

$$\boldsymbol{u}^*(t) = \begin{pmatrix} u_1^*(t) \\ u_2^*(t) \end{pmatrix} \tag{101}$$

transfers the system from its initial state (93) to the state-space origin at time t_f^* as shown in Figure 1.



(a) (b)

Figure 1. The trajectory of the system (87)–(89) from its initial state (93) to the state-space origin under the control vector $\boldsymbol{u}^*(t)$ (98)–(101): (**a**) the process regarding the state-space variables x_1 and x_2; (**b**) the controls $u_1^*(t)$ and $u_2^*(t)$ of the control vector $\boldsymbol{u}^*(t)$.

Let us also call the above system trajectory with an initial state (93) obtained under the control vector $\boldsymbol{u}^*(t)$ (98)–(101) $\boldsymbol{x}^*(t)$:

$$\begin{aligned} \boldsymbol{x}^*(t) &= e^{At}\boldsymbol{x}_0 + \int_0^t e^{A(t-\tau)}B\boldsymbol{u}^*(\tau)\,d\tau \\ &= e^{At}\boldsymbol{x}_0 + \int_0^t e^{A(t-\tau)}(B_1 \quad B_2)\begin{pmatrix} u_1^*(\tau) \\ u_2^*(\tau) \end{pmatrix} d\tau \end{aligned} \tag{102}$$

$$\text{for } t \in [0, t_f^*],$$

$$\boldsymbol{x}^*(t) = e^{At}\boldsymbol{x}_0 + \int_0^t e^{A(t-\tau)}B_1 u_1^*(\tau)\,d\tau + \int_0^t e^{A(t-\tau)}B_2 u_2^*(\tau)\,d\tau \text{ for } t \in [0, t_f^*],$$
$$\text{where } e^{At} = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}, \tag{103}$$

$$\boldsymbol{x}^*(t_f^*) = \boldsymbol{x}^*(1.59625) = \begin{pmatrix} 1.1015 \times 10^{-16} \\ -9.7998 \times 10^{-13} \end{pmatrix} \cong \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{104}$$

We are going to show based on the PMP that $\boldsymbol{x}^*(t)$ and $\boldsymbol{u}^*(t)$ represent the solution of the considered time-optimal control Problem P. Pontryagin's function (Hamilton function) regarding the linear time-optimal control problem represents

$$H(\boldsymbol{x}(t), \boldsymbol{u}(t), \boldsymbol{\psi}(t)) = (\boldsymbol{\psi}(t))^T(A\boldsymbol{x}(t) + B\boldsymbol{u}(t)). \tag{105}$$

The vector $\boldsymbol{\psi}(t)$ represents a solution of the adjoined system

$$\dot{\boldsymbol{\psi}} = -A^T\boldsymbol{\psi}. \tag{106}$$

Its solution is

$$\begin{aligned} \boldsymbol{\psi}(t) &= e^{-A^Tt}\boldsymbol{\psi}_0 \\ &= \begin{pmatrix} 1 & -t \\ 0 & 1 \end{pmatrix}\boldsymbol{\psi}_0. \end{aligned} \tag{107}$$

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Let the initial state regarding $\boldsymbol{\psi}$ be

$$\boldsymbol{\psi}_0^* = \boldsymbol{\psi}^*(0) = \begin{pmatrix}\psi_{10}^* \\ \psi_{20}^*\end{pmatrix} = \begin{pmatrix}-882.5139 \times 10^{-3} \\ -912.1591 \times 10^{-3}\end{pmatrix}, \tag{108}$$

and consider the following solution of the adjoined system called $\boldsymbol{\psi}^*(t)$:

$$\begin{aligned}\boldsymbol{\psi}^*(t) &= e^{-A^T t}\boldsymbol{\psi}_0^* = \begin{pmatrix}1 & -t \\ 0 & 1\end{pmatrix}\begin{pmatrix}\psi_{10}^* \\ \psi_{20}^*\end{pmatrix} \\ &= \begin{pmatrix}\psi_{10}^* - \psi_{20}^* t \\ \psi_{20}^*\end{pmatrix} \text{ for } t \in [0, t_f^*].\end{aligned} \tag{109}$$

First, we will show that Pontryagin's function regarding the considered Problem P satisfies

$$\max_{\boldsymbol{u} \in U} H(\boldsymbol{x}^*(t), \boldsymbol{u}(t), \boldsymbol{\psi}^*(t)) = H(\boldsymbol{x}^*(t), \boldsymbol{u}^*(t), \boldsymbol{\psi}^*(t)). \tag{110}$$

We obtain regarding H:

$$\begin{aligned}H(\boldsymbol{x}^*(t), \boldsymbol{u}(t), \boldsymbol{\psi}^*(t)) &= (\boldsymbol{\psi}^*(t))^T (A\boldsymbol{x}^*(t) + B\boldsymbol{u}(t)) \\ &= (\boldsymbol{\psi}^*(t))^T A\boldsymbol{x}^*(t) + (\boldsymbol{\psi}^*(t))^T (B_1 \quad B_2)\begin{pmatrix}u_1(t) \\ u_2(t)\end{pmatrix} \\ &= (\boldsymbol{\psi}^*(t))^T A\boldsymbol{x}^*(t) + (\boldsymbol{\psi}^*(t))^T B_1 u_1(t) + (\boldsymbol{\psi}^*(t))^T B_2 u_2(t).\end{aligned} \tag{111}$$

$$\begin{aligned}H(\boldsymbol{x}^*(t), \boldsymbol{u}(t), \boldsymbol{\psi}^*(t)) &= \boldsymbol{\psi}^{*T}(t) A\boldsymbol{x}^*(t) \\ &+ \begin{pmatrix}\psi_{10}^* - \psi_{20}^* t \\ \psi_{20}^*\end{pmatrix}^T B_1 u_1(t) \\ &+ \begin{pmatrix}\psi_{10}^* - \psi_{20}^* t \\ \psi_{20}^*\end{pmatrix}^T B_2 u_2(t).\end{aligned} \tag{112}$$

Let us consider the second component of the sum of H (112)

$$\begin{aligned}\begin{pmatrix}\psi_{10}^* - \psi_{20}^* t \\ \psi_{20}^*\end{pmatrix}^T B_1 u_1(t) &= (\psi_{10}^* - \psi_{20}^* t \quad \psi_{20}^*)\begin{pmatrix}1 \\ 0\end{pmatrix} u_1(t) \\ &= (\psi_{10}^* - \psi_{20}^* t) u_1(t).\end{aligned} \tag{113}$$

The function $(\psi_{10}^* - \psi_{20}^* t)$ is a linear function on t:

$$(\psi_{10}^* - \psi_{20}^* t) = \begin{cases} 0 & \text{at } t = \dfrac{\psi_{10}^*}{\psi_{20}^*} = 0.9675 = t_1^*; \\ \psi_{10}^* = -882.5139 \times 10^{-3} < 0 & \text{at } t = 0; \\ (\psi_{10}^* - \psi_{20}^* t_f^*) = 0.57352 & \text{at } t = t_f^*. \end{cases} \tag{114}$$

Thus, the function $(\psi_{10}^* - \psi_{20}^* t)$ is a linear increasing function switching from negative to positive values at time t_1^*, which is the switching moment of $u_1^*(t)$ (98), (100) as it is also shown in Figure 1b. Therefore,

$$\begin{aligned}\max_{|u_1| \le 1} (\psi_{10}^* - \psi_{20}^* t) u_1(t) &= (\psi_{10}^* - \psi_{20}^* t)\, sign(\psi_{10}^* - \psi_{20}^* t) \\ &= (\psi_{10}^* - \psi_{20}^* t) u_1^*(t).\end{aligned} \tag{115}$$

Similarly, we obtain for the third component of the sum of H (112):

$$\begin{aligned}\begin{pmatrix}\psi_{10}^* - \psi_{20}^* t \\ \psi_{20}^*\end{pmatrix}^T B_2 u_2(t) &= (\psi_{10}^* - \psi_{20}^* t \quad \psi_{20}^*)\begin{pmatrix}1 \\ b_{22}\end{pmatrix} u_2(t) \\ &= (\psi_{10}^* - \psi_{20}^* t + \psi_{20}^* b_{22}) u_2(t).\end{aligned} \tag{116}$$

The function $(\psi_{10}^* + \psi_{20}^* b_{22} - \psi_{20}^* t)$ is a linear function on t:

$$(\psi_{10}^* + \psi_{20}^* b_{22} - \psi_{20}^* t) = \begin{cases} 0 & \text{at } t = \dfrac{\psi_{10}^* + \psi_{20}^* b_{22}}{\psi_{20}^*} = 1.1287 = t_2^*; \\ \psi_{10}^* + \psi_{20}^* b_{22} = -1.0296 < 0 & \text{at } t = 0; \\ \psi_{10}^* + \psi_{20}^* b_{22} - \psi_{20}^* t_f^* = 0.42648 & \text{at } t = t_f^*. \end{cases} \tag{117}$$

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Thus, the function $(\psi_{10}^* + \psi_{20}^* b_{22} - \psi_{20}^* t)$ is a linear increasing function switching from negative to positive values at time t_2^*, which is the switching moment of $u_2^*(t)$ (99), (100) as it is also shown in Figure 1b. Therefore,

$$\begin{aligned} &\max_{|u_2|\le 1} (\psi_{10}^* + \psi_{20}^* b_{22} - \psi_{20}^* t)\, u_2(t) \\ &= (\psi_{10}^* + \psi_{20}^* b_{22} - \psi_{20}^* t) sign(\psi_{10}^* + \psi_{20}^* b_{22} - \psi_{20}^* t) \\ &= (\psi_{10}^* + \psi_{20}^* b_{22} - \psi_{20}^* t) u_2^*(t). \end{aligned} \tag{118}$$

As a result, we obtain:

$$\begin{aligned} \max_{\boldsymbol{u}\in U} H(\boldsymbol{x}^*(t), \boldsymbol{u}(t), \boldsymbol{\psi}^*(t)) \; &= \left(\boldsymbol{\psi}^*(t)\right)^T A\boldsymbol{x}^*(t) \\ &+ \max_{|u_1|\le 1} \begin{pmatrix} \psi_{10}^* - \psi_{20}^* t \\ \psi_{20}^* \end{pmatrix}^T B_1 u_1(t) \\ &+ \max_{|u_2|\le 1} \begin{pmatrix} \psi_{10}^* - \psi_{20}^* t \\ \psi_{20}^* \end{pmatrix}^T B_2 u_2(t)\,; \end{aligned} \tag{119}$$

$$\begin{aligned} \max_{\boldsymbol{u}\in U} H(\boldsymbol{x}^*(t), \boldsymbol{u}(t), \boldsymbol{\psi}^*(t)) \; &= \left(\boldsymbol{\psi}^*(t)\right)^T A\boldsymbol{x}^*(t) \\ &+ \max_{|u_1|\le 1} (\psi_{10}^* - \psi_{20}^* t) u_1(t) \\ &+ \max_{|u_2|\le 1} (\psi_{10}^* + \psi_{20}^* b_{22} - \psi_{20}^* t)\, u_2(t)\,; \end{aligned} \tag{120}$$

$$\begin{aligned} \max_{\boldsymbol{u}\in U} H(\boldsymbol{x}^*(t), \boldsymbol{u}(t), \boldsymbol{\psi}^*(t)) \; &= \left(\boldsymbol{\psi}^*(t)\right)^T A\boldsymbol{x}^*(t) \\ &+ (\psi_{10}^* - \psi_{20}^* t) u_1^*(t) \\ &+ (\psi_{10}^* + \psi_{20}^* b_{22} - \psi_{20}^* t) u_2^*(t) \\ &= (\boldsymbol{\psi}^*(t))^T A\boldsymbol{x}^*(t) + \left(\boldsymbol{\psi}^*(t)\right)^T B\boldsymbol{u}^*(t) \\ &= H(\boldsymbol{x}^*(t), \boldsymbol{u}^*(t), \boldsymbol{\psi}^*(t)). \end{aligned} \tag{121}$$

We also obtain for $H(\boldsymbol{x}^*(t), \boldsymbol{u}^*(t), \boldsymbol{\psi}^*(t))$ at $t = t_f^*$ from (114) and (117):

$$\begin{aligned} H\left(\boldsymbol{x}^*(t_f^*), \boldsymbol{u}^*(t_f^*), \boldsymbol{\psi}^*(t_f^*)\right) \; &= \left(\boldsymbol{\psi}^*(t_f^*)\right)^T A\boldsymbol{x}^*(t_f^*) \\ &+ \left(\psi_{10}^* - \psi_{20}^* t_f^*\right) u_1^*(t_f^*) \\ &+ \left(\psi_{10}^* + \psi_{20}^* b_{22} - \psi_{20}^* t_f^*\right) u_2^*(t_f^*) \\ &= 0 + 0.57352 + 0.42648 \\ &= 1. \end{aligned} \tag{122}$$

Thus, we have just shown that the control vector $\boldsymbol{u}^*(t)$ (98)–(101) and the respective system trajectory $\boldsymbol{x}^*(t)$ (102) with an initial state (93) satisfy the PMP with respect to the considered linear time-optimal control Problem P. Taking into account that the PMP with respect to the linear time-optimal control problem in case the system satisfies the condition of normality is not only a necessary but also a sufficient condition for optimality, we can conclude that the control vector $\boldsymbol{u}^*(t)$ (98)–(101) and the system trajectory $\boldsymbol{x}^*(t)$ (102) reaching the state-space origin at time t_f^* (100) represent the only time-optimal solution to the considered Problem P.

3.2. A Near Time-Optimal Control Solution of Problem P Based on the Proposed Idea

Let us now try to solve the considered Problem P, based on the proposed idea, through the solutions of the two single-input problems – Problem P_1 and Problem P_2, obtained through the proposed decomposition of the initial Problem P. First, let us suppose k is a positive parameter $0 < k < 1$.

The system of the first Problem P_1 is composed of the matrices A and B_1 (89) and represents

$$\dot{\boldsymbol{x}}_1 = \mathrm{A}\boldsymbol{x}_1 + B_1 u_1. \tag{123}$$

The initial state of the system is

$$\boldsymbol{x}_1(0) = \boldsymbol{x}_{01} = (x_{10,1} \quad x_{10,2})^T, \tag{124}$$

$$\boldsymbol{x}_{01} = k\boldsymbol{x}_0. \tag{125}$$

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The admissible control u_1 is yet defined above piecewise continuous function $u_1(t)$, which takes its values form the range (90), and for whom at the points of discontinuity τ_{1i} we have (8).

The target state at the final time t_{1f}, where t_{1f} is unspecified, represents the origin of the system's state-space

$$\boldsymbol{x}_1(t_{1f}) = \boldsymbol{x}_{1f} = (0)_{n\times 1}. \tag{126}$$

The time-optimal Problem P_1 is to find an admissible control $u_1(t)$ which transfers the system (123) from its initial state (124)–(125) to the final state (126) in minimum time, i.e. minimizing the performance index

$$J_1 = t_{1f} \to min. \tag{127}$$

Let us formulate in the same manner the second single-input linear time-optimal control Problem P_2. The system is composed of the matrices A and B_2 (89) and represents the equations

$$\dot{\boldsymbol{x}}_2 = \mathrm{A}\boldsymbol{x}_2 + B_2 u_2. \tag{128}$$

The initial state of the system is

The admissible control u_2 is yet defined above piecewise continuous function $u_2(t)$. It takes its values form the range (91) and at the points of discontinuity τ_{2j} we have (9).

$$\boldsymbol{x}_2(0) = \boldsymbol{x}_{02} = (x_{20,1} \quad x_{20,2})^T, \tag{129}$$

$$\boldsymbol{x}_{02} = (1-k)\boldsymbol{x}_0. \tag{130}$$

The target state at the final time t_{2f}, where t_{2f} is unspecified, represents the origin of the system's state-space

$$\boldsymbol{x}_2(t_{2f}) = \boldsymbol{x}_{2f} = (0)_{n\times 1}. \tag{131}$$

The time-optimal Problem P_2 is to find an admissible control $u_2(t)$ which transfers the system (128) from its initial state (129)–(130) to the final state (131) in minimum time, i.e. minimizing the performance index

$$J_2 = t_{2f} \to min. \tag{132}$$

In order to fulfil the conditions of Lemma 1, several iterations on the positive number k, $0 < k < 1$, have been made solving Problem P_1 and Problem P_2 with respective initial conditions $\boldsymbol{x}_{01}$ of Problem P_1 and $\boldsymbol{x}_{02}$ of Problem P_2. The obtained final value of k with an accuracy of 1×10^{-6} (s) regarding the minimum times t_{1f}^o of Problem P_1 and t_{2f}^o of Problem P_2 is

$$k^o = 0.47167. \tag{133}$$

The respective initial condition $\boldsymbol{x}_{01}$ (125) of Problem P_1 and $\boldsymbol{x}_{02}$ (130) of Problem P_2 at $k^o = 0.47167$ are:

$$\boldsymbol{x}_{01} = k^o\boldsymbol{x}_0 = 0.47167\begin{pmatrix}1\\0.44447\end{pmatrix} = \begin{pmatrix}0.47167\\0.20964\end{pmatrix}, \tag{134}$$

$$\boldsymbol{x}_{02} = (1-k^o)\boldsymbol{x}_0 = (1-0.47167)\begin{pmatrix}1\\0.44447\end{pmatrix} = \begin{pmatrix}0.52833\\0.23483\end{pmatrix}, \tag{135}$$

and the obtained equal minimum times t_{1f}^o and t_{2f}^o are

$$t_{1f}^o = t_{2f}^o = 1.605\ (s). \tag{136}$$

Let t_f^* be

$$t_f^* = t_{1f}^o = t_{2f}^o = 1.605\ (s). \tag{137}$$

Problem P_1 and Problem P_2 have been solved numerically by the technique in [12,13], following [12] (Section 3.2 Synthesis Based on the New Property and the Method [14], pp. 9–13) and [13] (Section 4.1 Example 1, pp. 18–24). Both papers in [12,13] unveil new state-space properties of the linear time-optimal control problem with one input and real non-positive eigenvalues of the system following from the PMP. It is shown there that the positive semi-axis $0x_n$ of the system of order n is outside the switching hyper-surface of the considered problem, being wholly below or wholly above the switching hyper-surface of the problem, while the negative semi-axis $0x_n$ is outside the switching hyper-surface of the considered problem, being wholly below or wholly above the switching hyper-surface but in the opposite relation to the switching hyper-surface of the problem relative to the positive semi-axis $0x_n$. The determination of this relation regarding the considered problem is called "axes initialization". Here for both Problem P_1 and Problem P_2 the same result is obtained

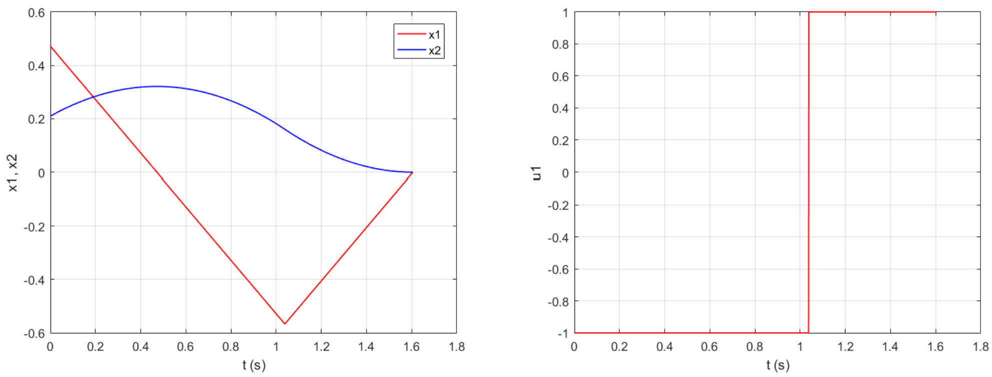
$$x_{2+} = -1, \tag{138}$$

which means that all the points of the negative semi-axis $0x_2$ of the system of Problem P_1 are above the switching curve, and the time-optimal control value for them is $+u_{1max}$. The same is true for the negative semi-axis $0x_2$ of the system of Problem P_2, and the time-optimal control value for all the points of this semi-axis is $+u_{2max}$. Based on the properties regarding the axes initialization and discovered in [12,13] other new properties, a foundation of a new method for synthesizing the time-optimal control without the need to describe the switching hyper-surfaces is obtained and illustrated by the examples mentioned above.

Table 1 represents in its first row the near optimal solution of Problem P_1 with an initial state $\boldsymbol{x}_{01}$ (134) and the obtained corresponding control function $u_1^{\tilde{o}}(t)$ associated with this problem. The near optimal solution of Problem P_2 with an initial state $\boldsymbol{x}_{02}$ (135) and the obtained corresponding control function $u_2^{\tilde{o}}(t)$ associated with this problem is presented in the second row of Table 1. The corresponding obtained processes with an accuracy of $\varepsilon_r = 0.0006$ regarding Problem P_1 under $u_1^{\tilde{o}}(t)$ and Problem P_2 under $u_2^{\tilde{o}}(t)$ are shown in Figure 2 and Figure 3.

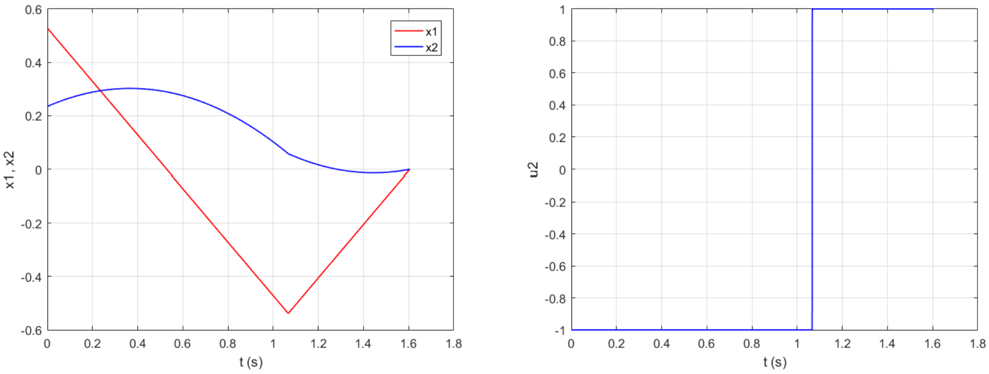
Table 1. Results of the solutions of Problem P_1 with an initial state $\boldsymbol{x}_{01}$ (134) and Problem P_2 with an initial state and $\boldsymbol{x}_{02}$ (135) obtained with an accuracy of $\varepsilon_r = 0.0006$ satisfying Lemma 1 at $k^o = 0.47167$ (133) with an accuracy of 1×10^{-6} (s).

Problem	**Control**	**Control in the first interval**	**Switching time t_{i1} (s)**	**Control in the second interval**	**Minimum time t_f^* (s)**
P_1	u_1	-1	1.038333	1	1.605
P_2	u_2	-1	1.066667	1	1.605



(**a**) (**b**)

Figure 2. Near time-optimal process of the system (123) of Problem P_1 from its initial state $\boldsymbol{x}_{01}$ (134) to the state-space origin obtained with an accuracy of $\varepsilon_r = 0.0006$: (**a**) the process regarding the state-space variables x_1 and x_2; (**b**) the control $u_1^{\tilde{o}}(t)$ presented in the first row of Table 1.



(a) (b)

Figure 3. Near time-optimal process of the system (128) of Problem P_2 from its initial state $\boldsymbol{x}_{02}$ (135) to the state-space origin obtained with an accuracy of $\varepsilon_r = 0.0006$: (**a**) the process regarding the state-space variables x_1 and x_2; (**b**) the control $u_2^{\tilde{o}}(t)$ presented in the second row of Table 1.

Let us now consider the initial Problem P (87)–(95) under the control vector composed of the solutions $u_1^o(t)$ of Problem P_1 and $u_2^o(t)$ of Problem P_2 as (86), bearing in mind that t_f^* represents (137),

$$\boldsymbol{u}^*(t) = \begin{pmatrix} u_1^o(t) \\ u_2^o(t) \end{pmatrix} \text{ for } t \in \left[0, t_f^*\right]. \tag{139}$$

Let the trajectory of the system of Problem P be $\boldsymbol{x}^*(t)$. It represents the sum of the obtained trajectories $\boldsymbol{x}_1^o(t)$ of Problem P_1 under $u_1^o(t)$ and $\boldsymbol{x}_2^o(t)$ of Problem P_2 under $u_2^o(t)$.

$$\begin{aligned} \boldsymbol{x}^*(t) &= e^{At}\boldsymbol{x}_0 + \int_0^t e^{A\tau} B \boldsymbol{u}^*(t-\tau)\, d\tau \\ &= e^{At}\boldsymbol{x}_0 + \int_0^t e^{A\tau} (B_1 \quad B_2) \begin{pmatrix} u_1^o(t-\tau) \\ u_2^o(t-\tau) \end{pmatrix} d\tau \\ &= e^{At}k^o\boldsymbol{x}_0 + e^{At}(1-k^o)\boldsymbol{x}_0 + \int_0^t e^{A\tau} B_1 u_1^o(t-\tau)\, d\tau + \int_0^t e^{A\tau} B_2 u_2^o(t-\tau)\, d\tau. \end{aligned} \tag{140}$$

$$\begin{aligned} \boldsymbol{x}^*(t) &= e^{At}k^o\boldsymbol{x}_0 + \int_0^t e^{A\tau} B_1 u_1^o(t-\tau)\, d\tau + e^{At}(1-k^o)\boldsymbol{x}_0 + \int_0^t e^{A\tau} B_2 u_2^o(t-\tau)\, d\tau \\ &\qquad \text{for } t \in \left[0, t_f^o\right]. \end{aligned} \tag{141}$$

$$\begin{aligned} \boldsymbol{x}^*(t) &= \boldsymbol{x}_1^o(t) + \boldsymbol{x}_2^o(t) \text{ for } t \in \left[0, t_f^o\right], \\ \boldsymbol{x}_1^o(t) &= e^{At}k^o\boldsymbol{x}_0 + \int_0^t e^{A\tau} B_1 u_1^o(t-\tau)\, d\tau, \\ \boldsymbol{x}_2^o(t) &= e^{At}(1-k^o)\boldsymbol{x}_0 + \int_0^t e^{A\tau} B_2 u_2^o(t-\tau). \end{aligned} \tag{142}$$

The trajectory $\boldsymbol{x}^*(t)$ and the control vector $\boldsymbol{u}^*(t)$ regarding the processes of the system of Problem P are shown in Figure 4. The parameters of the components $u_1^{\tilde{o}}(t)$ and $u_1^{\tilde{o}}(t)$ of $\boldsymbol{u}^*(t)$ (139) are presented in Table 1.

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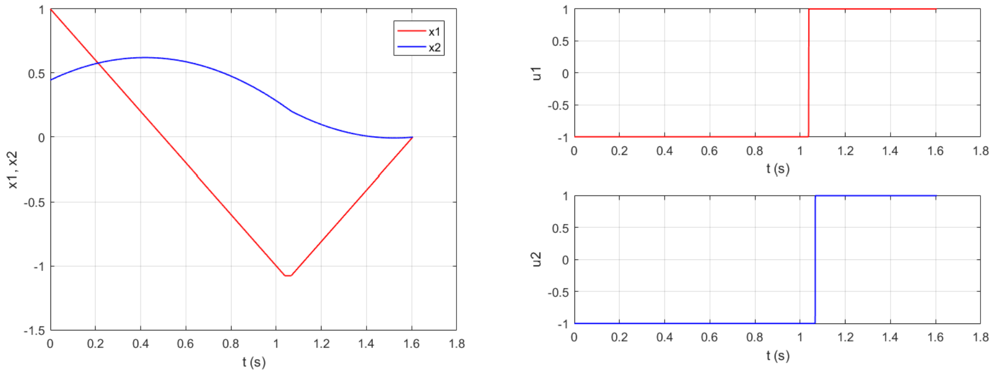

(a) (b)

Figure 4. The trajectory $\boldsymbol{x}^*(t)$ of the system (87)–(89) of Problem P from its initial state $\boldsymbol{x}_0$ (93) to the state-space origin obtained with an accuracy of $\varepsilon_r = 0.001$ under the control vector $\boldsymbol{u}^*(t)$ (139): (**a**) the process regarding the state-space variables x_1 and x_2; (**b**) the components $u_1^{\tilde{o}}(t)$ and $u_1^{\tilde{o}}(t)$ of the control vector $\boldsymbol{u}^*(t)$ (139) with parameters presented in Table 1.

Even though the only time-optimal solution of Problem P is found in the previous subsection 4.1, let us apply Theorem 1 regarding the control vector function (86), which is the obtained here regarding the considered example control vector function (139), and consider it as a candidate for the solution to the original time-optimal control Problem P with an initial state $\boldsymbol{x}_0$ (93) according to Lemma 1. We must check for consistency the linear system (78), (79), and (82). According to the data of Table 1 the number of switchings of $u_1^o(t)$ is $k_1 = 1$, the number of switchings of $u_2^o(t)$ is $k_2 = 1$. The Equations (77) represent

$$B_1{}^T e^{-A^T t_{11}} \boldsymbol{\psi}_0^* = 0. \quad (143)$$

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -t_{11} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \psi_{10}^* \\ \psi_{20}^* \end{pmatrix} = 0. \quad (144)$$

$$\begin{pmatrix} 1 & -t_{11} \end{pmatrix} \begin{pmatrix} \psi_{10}^* \\ \psi_{20}^* \end{pmatrix} = 0. \quad (145)$$

The Equations (79) represent

$$B_2{}^T e^{-A^T t_{21}} \boldsymbol{\psi}_0^* = 0. \quad (146)$$

$$\begin{pmatrix} 1 & 0.1612 \end{pmatrix} \begin{pmatrix} 1 & -t_{21} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \psi_{10}^* \\ \psi_{20}^* \end{pmatrix} = 0. \quad (147)$$

$$\begin{pmatrix} 1 & 0.1612 - t_{21} \end{pmatrix} \begin{pmatrix} \psi_{10}^* \\ \psi_{20}^* \end{pmatrix} = 0. \quad (148)$$

The Equation (82) represents

$$\left(u_1^o(t_f^*) B_1{}^T e^{-A^T t_f^*} + u_2^o(t_f^*) B_2{}^T e^{-A^T t_f^*} \right) \boldsymbol{\psi}_0^* = 1. \quad (149)$$

$$\left(\begin{pmatrix} 1 & -t_f^* \end{pmatrix} + \begin{pmatrix} 1 & 0.1612 - t_f^* \end{pmatrix} \right) \begin{pmatrix} \psi_{10}^* \\ \psi_{20}^* \end{pmatrix} = 1. \quad (150)$$

$$\begin{pmatrix} 2 & 0.1612 - 2t_f^* \end{pmatrix} \begin{pmatrix} \psi_{10}^* \\ \psi_{20}^* \end{pmatrix} = 1. \quad (151)$$

Thus, the linear system (78), (79), and (82), consisting here according to (83) of

$$k_1 + k_2 + 1 = 1 + 1 + 1 = 3 \quad (152)$$

equations, represents

$$\begin{pmatrix} 1 & -t_{11} \\ 1 & 0.1612 - t_{21} \\ 2 & 0.1612 - 2t_f^* \end{pmatrix} \begin{pmatrix} \psi_{10}^* \\ \psi_{20}^* \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \quad (153)$$

According to Table 1 $t_{11} = 1.038333$, $t_{21} = 1.066667$, and $t_f^* = 1.605$. Thus, the coefficient matrix and the augmented matrix of the above linear system are:

$$\begin{pmatrix} 1 & -1.0383 \\ 1 & -0.90547 \\ 2 & -3.0488 \end{pmatrix}, \tag{154}$$

$$\left(\begin{array}{cc|c} 1 & -1.0383 & 0 \\ 1 & -0.90547 & 0 \\ 2 & -3.0488 & 1 \end{array}\right). \tag{155}$$

The ranks of the above matrices (154) and (155) differ

$$rank\begin{pmatrix} 1 & -1.0383 \\ 1 & -0.90547 \\ 2 & -3.0488 \end{pmatrix} = 2 \neq rank\left(\begin{array}{cc|c} 1 & -1.0383 & 0 \\ 1 & -0.90547 & 0 \\ 2 & -3.0488 & 1 \end{array}\right) = 3, \tag{156}$$

and according to Kronecker-Capelli Theorem the considered linear system with unknowns ψ_{10}^* and ψ_{20}^* has no solution, i.e. is inconsistent. Thus, according to Theorem 1 the obtained by the proposed method control vector function (139) can only be considered as a somewhat near time-optimal control solution to the considered in this example Problem P.

3.3. A Comparison Between the Time-Optimal and the Obtained Near Time-Optimal Solution of Problem P

Let us first make a quality assessment of the obtained solutions of Problem P. The presented time-optimal solution in Subsection 4.1 fully complies with the PMP. According to the formulation of Problem P the two-input system satisfies the condition of normality being controllable on each input. The allowable control set U is convex since it represents a square with side 1 in the plane u_1u_2. It is also symmetric where the origin 0 of the plane u_1u_2 is an inner point and the center of U. Since that, regarding the linear time-optimal control problem the PMP is a necessary and sufficient condition of optimality. The minimum time is achieved through a bang-bang control regarding each of the two components u_1 and u_2 of the control vector, i.e. the values of the control vector function are taken only at the vertices of the square U. The sequence of the vertices of the square U that the time-optimal control vector $\boldsymbol{u}^*(t)$ (98)–(101) follows is

$$\{(-1,-1),(+1,-1),(+1,+1)\} \tag{157}$$

as shown in Figure 1b.

The obtained near time-optimal solution achieved also a trajectory with an end at the state-space O of the plane x_1x_2 but with an accuracy of $\varepsilon_r = 0.001$, obtained at the corresponding control vector (139) with values taken also exclusively at the vertices of the square U. The sequence of the vertices of the square U that the near time-optimal control vector $\boldsymbol{u}^*(t)$ (139) follows is the same time-optimal control sequence of the vertices of the square U (156) as shown in Table 1 and Figure 4b.

A quantitative comparison between the time-optimal and the obtained near time-optimal process is presented in Table 2. It is noticed that the switching times of the true time-optimal process regarding the optimal control functions $u_1(t)$ and $u_2(t)$ differ. Although this fact arises an initial bewilderment, the PMP does not require satisfying identical switching times regarding the allowable set of controls. The switching times are determined by the condition (110), which does not require exclusively identical switching times for all control functions composing the control vector function of the system. Even though the switching times of the obtained near time-optimal solution regarding the controls u_1 and u_2, compared to the corresponding time-optimal switching times of the controls u_1 and u_2, show light mismatches, the near time-optimal solution achieves a transition time only slightly longer than the time-optimal transition time with a relative error of less than 0.5%.

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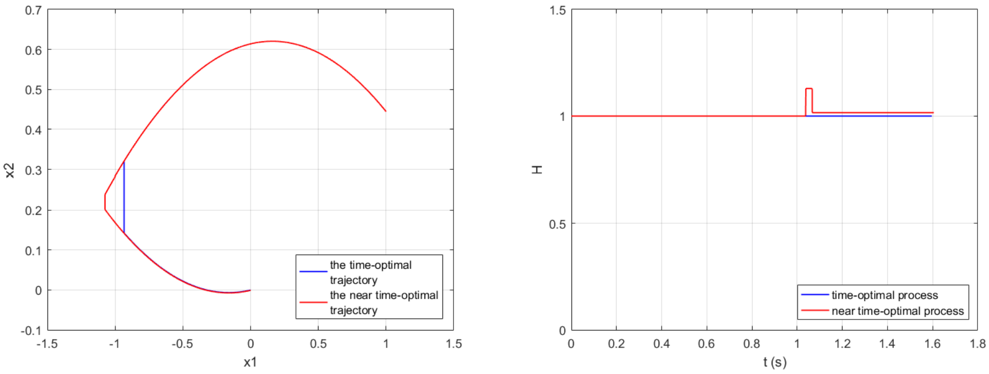
Table 2. A quantitative comparison between the true time-optimal and the near time-optimal solution of Problem P obtained by the proposed method.

Quantity (s)	True time-optimal control (s)	Proposed method (s)	Absolute error (s)	Relative error %
Final time	1.59625	1.605	+0.00875	0.54816
Switching time regarding u_1	0.9675	1.038333	+0.070833	7.3212
Switching time regarding u_2	1.1287	1.066667	−0.062033	5.496

The phase trajectories in the phase plane x_1x_2 regarding the time-optimal and the obtained near time-optimal solution are compared in Figure 5a. It can be noticed that the first and the last third part of the time-optimal and near time-optimal trajectory practically coincide. The first branch of each one of the two trajectories corresponds to the vertex $(-1,-1)$ of the square U, the middle branch corresponds to the vertex $(+1,-1)$, and the last third branch corresponds to the vertex $(+1,+1)$ of the square U. There is a small mismatch in the area where the switchings regarding the two controls occur, which corresponds to the middle part of the phase trajectory. The length of the middle branch of each of the two trajectories represents the time interval between the switching of u_1 and the switching of u_2 regarding the near time-optimal and the true time-optimal process, respectively. The duration of this branch is shorter at the near time-optimal process than the duration of the second branch of the true time-optimal process

$$(1.066667 - 1.038333) < (1.1287 - 0.9675), \tag{158}$$

while the duration of the first branch of the near time-optimal trajectory is longer than the true time-optimal one - $1.038333\ (s)$ against $0.9675\ (s)$, which may result in a longer transition time of the near time-optimal process. This is partly offset by an earlier jump to the third branch of the trajectory by the near time-optimal control. Thus, the transition time under the near time-optimal control is slightly longer than the time-optimal transition one. The relative error is only $0.55\ \%$.



(a) (b)

Figure 5. A comparison between the true time-optimal and the near time-optimal solution of Problem P obtained by the proposed method: (**a**) the trajectory of the system in the plane x_1x_2; (**b**) Pontryagin's function $H(\boldsymbol{x}^*(t),\boldsymbol{u}^*(t),\boldsymbol{\psi}^*(t))$ of the time-optimal process compared to $H(\boldsymbol{x}^*(t),\boldsymbol{u}^*(t),\boldsymbol{\psi}^*(t))$, where $\boldsymbol{x}^*(t)$ and $\boldsymbol{u}^*(t)$ are the obtained trajectory and control vector function of the near time-optimal solution, while $\boldsymbol{\psi}^*(t)$ is the solution of the adjoined system (106)–(109) regarding the time-optimal process.

Figure 5b shows Pontryagin's function

$$H(\boldsymbol{x}^*(t),\boldsymbol{u}^*(t),\boldsymbol{\psi}^*(t)) \text{ for } t \in [0, t_f^*], t_f^* = 1.59625\ (s), \tag{159}$$

of the time-optimal process, marked with blue, together with the function

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$$H(\boldsymbol{x}^*(t), \boldsymbol{u}^*(t), \boldsymbol{\psi}^*(t)) \text{ for } t \in [0, t_f^*], t_f^* = 1.605 \ (s), \tag{160}$$

marked with red, where $\boldsymbol{x}^*(t)$ and $\boldsymbol{u}^*(t)$ are the obtained trajectory and control vector function of the near time-optimal solution, while $\boldsymbol{\psi}^*(t)$ is the solution of the adjoined system (106)–(109) regarding the time-optimal process. From the point of view of the function H the replacement of $\boldsymbol{x}^*(t)$ and $\boldsymbol{u}^*(t)$ with the near time-optimal trajectory and the near time-optimal control vector function results in slightly less effectiveness, shown in the middle section of the process. After this short middle part of the process the function H remains constant, slightly shifted from the optimal one.

We can resume that:

- The obtained by the proposed method near time-optimal control is a bang-bang control with the same qualitative structure as the time-optimal one including the switching sequence of control vector at the vertices of the allowable set - the square U;
- The time-optimal trajectory in the phase plane x_1x_2 as well as the near time-optimal trajectory comprises three branches. The major parts of the first and the last third part of the time-optimal and near time-optimal trajectory practically coincide. There is a small mismatch concerning the middle part of the phase trajectory because of the small mismatch regarding the switching times of the two control functions. Both the time-optimal and the near-time optimal trajectories approach the state-space origin in practically the same way;
- The transition time under the near time-optimal control is slightly longer than the time-optimal transition one. The relative error is only 0.55 %.

This confirms the prediction of Theorem 1.

4. Discussion and Concluding Remarks

The experimental results convincingly show that the proposed method has potential. The novelty here is that the method decomposes the initial two-input problem into two tied together single-input problems. The first commitment is with the initial state of each one single-input problem, so that the first one part of the segment with one end at the state-space origin and second end at the initial state of two-input problem defines the initial state of the first single-input problem, while the rest part of the segment defines the initial state of the second single-input problem, along with the decomposition regarding the two controls. The second commitment is with the minimum times of the solutions of both single-input problems. Here it is shown the existence of such point of the considered segment so that both single-input problems achieve the same minimum time according to the PMP regarding each one problem which is the point of Lemma 1 and allowance to consider the recombination of both solutions of the single-input problems as a candidate for the time-optimal solution of the initial two-input problem.

This technique could be extended to multi-input systems through dividing the above segment in parts, whose number is equal to the number of system's inputs, while the second commitment remains the same – achievement of identical minimum times of the solutions of all single-input problems. Satisfying the second commitment looks predictable, having in mind the considerations at deducing Lemma 1.

The extension of Theorem 1 in the same manner looks also predictable and could provide an easy criterion to check the optimality of the recombination of the solutions of the single-input problems.

The decomposition of the initial two-input problem is provoked by the fact that the discovered new geometric state-space properties of the single-input linear time-optimal problem [12,13] following from the PMP provide a new time-optimal synthesis technique and obtaining the minimum time solution for high order systems [13] (Section 4.2. Example 2, pp. 25–30), while eliminating the need to construct switching hyper-surfaces. This advantage makes the idea of the decomposition of the two-input problem into two single-input problems attractive. The proposed separation is not only with respect to the controls, but also with respect to the initial state of the two-input problem dividing

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it into two initial states of each one single-input problem along the ray with beginning at the state-space origin and passing through the initial state of the two-input problem, on the one hand, and on the other hand, the existence of unique point at the ray that provides the same minimum times of both single-input time-optimal problems.

The combination of these theoretically backed novelties allows solving the canonical double integrator system with two independent inputs, as an example, and obtaining a near time-optimal solution with the same qualitative structure as the time-optimal one including the switching sequence of the control vector at the vertices of the allowable set - the square U, and minimum time slightly longer than the optimal time with relative error of only 0.55 %.

Further research could concern, in addition to the attempt to extend the obtained theoretical results to the general multi-input case, the decomposition technique regarding the separation of the initial state of the two-input problem between the initial states of the two single-input problems. This part could be further analyzed in order to configure a form of separation achieving a guaranteed true optimal solution at the recombination of the single-input solutions.

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Abbreviations

The following abbreviations are used in this manuscript:

PMP	Pontryagin Maximum Principle

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