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Supplementary Materials of CosmoFATS

FAT applications
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Abstract. In this document, we introduce functors of actions theory (FAT) and a simple quadratic FAT model. We present how the quadratic FAT models, applied in cosmology, is related to the partition function. We provide simple estimates on and their impact on the existence and quantification of actionic fluctuations from these models, and further estimates of the partition function. Furthermore, we mathematically argue about the local invariance of this quadratic model. We describe also a viability analysis of FAT models which render FAT models viable. We describe also the mathematical description of *5 fundamental field-particles, i.e. the spatiallion, timion, actionion, probablon and informaton*. We conclude that functors of actions theory models in cosmology are valid candidates for further exploration.

Keywords: cosmology; functors of actions theories; gravity; quantum gravity; effective field theory; large scale structure; dark energy; high energy physics; partition function;

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1 Introduction

In the context of modifications gravity (MG) theories, and in particular of the geometrical and energy content aspects of theories [1–6], we will consider an alternative route to extend gravity theories [7–9]. This study is the extension of functors of actions theories (FAT) developed in [7, 8], in which we apply these theories to basic models of cosmology.

The Standard Cosmological Model (SMC), best described by ΛCDM parametrisation, provides a satisfactory agreement with current observations [10]. MG is an important step in understanding models beyond the SMC [11–13]. It has been studied Recently an Effective Dark Energy Theory within the Hordenksi framework and the MG has been studied [14]. These theories can been studied within a framework which we call Cosmological Gravitology [7, 8].

FAT result to integral-differential equations compared to the the traditional methods, which have only differential equations. This makes the FATs both unique and challenging to solve. In this study, we are investigating a subclass of FAT models, and we explore the local invariance, partition function, and generalisation of the stationary action principle and the description of 5 fundamental unique field-particles, arise from the FAT models.

This paper is structured as follows. In section 2, we introduce the functor of action theories. In section 2.1, we introduce a quadratic FAT model. In section 3, we provide a resolution of the quadratic functor of action theories models. In section 5, we revise the Linearised Einstein Field Equations around a background metric. In section 6, we describe the presence of non-locality to the quadratic FAT action. In section 7, we discuss the suppression non-locality effects on quadratic FAT model. In section 8, we describe the action of a simple classical harmonic oscillator. In section 9, we describe the action of a quadratic FAT simple classical harmonic oscillator. In section 10, we describe the action of simple harmonic oscillator in a gravitational field. In section 11, we describe a variation of the action of simple FAT harmonic oscillator in a gravitation, which we call the model γ. In section 12, we describe another variation of the action of simple FAT harmonic oscillator in a gravitation, which we call the model α. In section 13, we revise the path integral approximation. In section 14, we describe the partition function approximation using a generic modified gravity model. In section 15, we explain the partition function approximation using a specific modified gravity model, which we call model A. In section 16, we revise the traditional quantum field theory and the partition function, which is usefull for the next sections. In section 17, we describe the FAT quantum field theory and the partition function. In section 18, we describe the FAT quantum gravity field theory via the partition functor. In section 19, we describe a simple set of generic functional FAT models. In section 20, we describe a viability analysis of FAT models, which render FAT models viable. In section 21, we describe a generic expectation value of an observable using a FAT model, and we apply that to the expectation function of the information of the universe. In section 22, we generalise the stationary action principle. In section 23, we describe a general description of the actionion, field particles, using the Taylor expansion description. In section 24, we describe the *5 fundamental field-particles, i.e. the spatiallion, timion, actionion, probablon and informaton.* In section 25, we summarise our result and we conclude.

2 Functors of actions theories (FATs)

At the core of MG theories lies the most successful theory of gravity, General Relativity[15] (GR). This theory assumes a four-dimensional pseudo-Riemannian manifold with a local interacting metric background that satisfies Lorentz invariance. The standard gravity action (or General Relativity action, that contains the Einstein-Hilbert action) is given by:

$$\mathcal{S}_{\rm GR} = c^4 \int d^4x\sqrt{-g}\left[\frac{R}{16\pi G_{\rm N}} + \mathcal{L}_m\left(g_{\mu\nu}, \psi\right)\right] \tag{2.1}$$

where c is the speed of light, g is the determinant of the background metric $g_{\mu\nu}(x)$ of a massless graviton, R is the Ricci Scalar, $G_{\rm N}$ is the Newtonian gravitational constant, $\psi(x)$ describes the matter fields of the universe and $\mathcal{L}_m$ is the Lagrangian density that describes the matter content evolution of our universe. This Lagrangian defines the energy-momentum tensor via $T_{\mu\nu} = \frac{-2}{\sqrt{-g}}\frac{\delta S_m}{\delta g^{\mu\nu}}$, where $\mathcal{S}_m[g_{\mu\nu}, \psi] = c^4 \int d^4x\sqrt{-g}\mathcal{L}_m[g_{\mu\nu}(x), \psi(x)]$. Most MGs modify the geometrical aspects, i.e. the metric g, curvature tensor, R, or adding scalar, vector or tensor components of the Riemannian tensor to extend GR. Other MG theories consider exotic fields such as exotic scalar, vector, and tensor, which are not related to the Riemannian tensor, and we call the modified energy content of the MG theories.

On the other hand, FAT introduces the novel concept of applying mathematical entities such as functors directly to the action itself. FAT results to integral-differential equations compared to the traditional methods, which have only differential equations. This makes the FATs both unique and challenging to solve. We proceed with a simple example.

2.1 A quadratic functors of action theory

We assume a quadratic functors of action theory, as the action

$$S_{\rm FAT}^{quad} = S_R + \beta S_R^2 + S_\Lambda + S_m \tag{2.2}$$

Applying the action principle

$$\delta S_{\rm FAT}^{quad} = 0 \tag{2.3}$$

then we get

$$\left[R_{\mu\nu} + \frac{1}{2}Rg_{\mu\nu}\right][1 + 2\beta S_{\rm R}] = \frac{8\pi G_{\rm N}}{c^4}\left[T_{\mu\nu} + \Lambda g_{\mu\nu}\right] \tag{2.4}$$

$$\left[R_{\mu\nu} + \frac{1}{2}Rg_{\mu\nu}\right]\left[1 + 2\beta c^3 \int_{\mathcal{V}_{4D}} d^4x\sqrt{-g}\,\frac{R}{16\pi G_{\rm N}}\right] = \frac{8\pi G_{\rm N}}{c^4}\left[T_{\mu\nu} + \Lambda g_{\mu\nu}\right] \tag{2.5}$$

where we used

$$S_{\rm R} = c^3 \int_{\mathcal{T}} d(ct) \int_{\mathcal{V}_{\rm 3D}} d^3x\sqrt{-g}\,\frac{R}{16\pi G_{\rm N}} \tag{2.6}$$

Note that solutions of this model, were found using a dynamical analysis approach, and we discuss it further in a different study [9].

2.2 Calculate the form of S_R in smoothed cosmology

To calculate S_R, we need to consider the following. We need to make assumptions about the topological part, and the energetic part of the action. Usually we need to vary the S_R to get the equations of motion of the system.

Regarding the topological part, we make the following assumption In a FLRW universe, we have

$$ds^2 = -d(ct)^2 + a^2(t)d\vec{x}^2 \tag{2.7}$$

which means

$$g = \det\left(g_{\mu\nu}\right) \tag{2.8}$$

$$= \det\left[\text{diag}\left(-1, a^2(t), a^2(t), a^2(t)\right)\right] \tag{2.9}$$

$$g = -a^6(t) \tag{2.10}$$

In a FLRW universe, we have

$$R = \frac{6}{c^2}\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right] \tag{2.11}$$

substituting Eqs. 2.11 and 2.10, we get that

$$S_R = \frac{c^3}{16\pi G_N}\int d(ct)d^3x\,\sqrt{--a^6(t)}\left\{\frac{6}{c^2}\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right]\right\} \tag{2.12}$$

$$2S_R = \frac{6c^2\mathcal{V}_{3D}}{8\pi G_N}\int dt\;\; a^3(t)\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right] \tag{2.13}$$

It is convenient to introduce the constant

$$\kappa^2 = \frac{8\pi G}{c^4} \tag{2.14}$$

So we have that

$$\boxed{2S_R = \frac{6\mathcal{V}_{3D}}{c^2\kappa^2}\int dt\;\; a^3(t)\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right]}\,. \tag{2.15}$$

2.2.1 Assumptions about the scale factor, $a(t)$ and impact on S_R

Now we can make an assumption about the specific form of the topological part. Regarding the topological part, we make the following assumption. Note that in a dark energy dominated universe, we have that the scale factor is

$$a(t) \sim e^{H_0 t} \tag{2.16}$$

$$a(t) = a_0 e^{H_0 t} \tag{2.17}$$

$$a(t) = e^{H_0 t} \tag{2.18}$$

which means that its derivatives are

$$\dot{a}(t) = H_0 e^{H_0 t} = H_0 a(t) \qquad \Leftrightarrow \dot{a}/a = H_0 \tag{2.19}$$

$$\ddot{a}(t) = H_0^2 e^{H_0 t} = H_0^2 a(t) \qquad \Leftrightarrow \ddot{a}/a = H_0^2 \tag{2.20}$$

Therefore the integral of interest is simplified as

$$\int dt\; a^3(t)\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right] = \int dt\; a^3(t)\left[H_0^2 + H_0^2\right] \tag{2.21}$$

$$= \frac{2}{3}H_0\left[e^{3H_0 t} - e^{3H_0 t_i}\right] \tag{2.22}$$

So the S_R takes the form of

$$S_R = \frac{6\times c^2\mathcal{V}_{3D}}{16\pi G_N}\times\frac{2}{3}H_0\left[e^{3H_0 t} - e^{3H_0 t_i}\right] \tag{2.23}$$

$$S_R = \frac{1}{4\pi}\times\frac{c^2 H_0}{G_N}\times\mathcal{V}_{3D}\times\left[e^{3H_0 t} - e^{3H_0 t_i}\right] \tag{2.24}$$

Note that 1Mpc= 3.086×10^{22}m, so We also know that the Hubble constant, today, can be approximated to

$$H_0 = 70 \text{ km/s/Mpc} \tag{2.25}$$
$$H_0 \simeq 2 \times 10^{-17} \text{ s}^{-1} \tag{2.26}$$

and we also know that $N = kg\ m/s^2$ which means that the Newton Gravitational constant can be approximated to

$$G_{\rm N} = 6.67 \times 10^{-11} \frac{\text{N} \times \text{m}^2}{\text{kg}^2} \tag{2.27}$$
$$G_{\rm N} \simeq 7 \times 10^{-11} \frac{\text{m}^3 \text{ s}^{-2}}{\text{kg}} \tag{2.28}$$

Now simplifying our unit system to

$$L = m \tag{2.29}$$
$$T = s \tag{2.30}$$
$$M = kg \tag{2.31}$$
$$E = J = N \times m = kg\ m/s^2\ \times L = M \times L \times T^{-2} \times L = M \times L^2 \times T^{-2} \tag{2.32}$$
$$A = E \times T = M \times L^2 \times T^{-2} \times T = M \times L^2 \times T^{-1} \tag{2.33}$$

So in general the units are

$$\text{length unit} = L \tag{2.34}$$
$$\text{time unit} = T \tag{2.35}$$
$$\text{mass unit} = M \tag{2.36}$$
$$\text{energy unit} = E = M \times L^2 \times T^{-2} \tag{2.37}$$
$$\text{action unit} = A = M \times L^2 \times T^{-1} \tag{2.38}$$

Then our constants are in terms of units

$$c \simeq 3 \times 10^8\ L\ T^{-1} \tag{2.39}$$
$$H_0 \simeq 2 \times 10^{-17}\ T^{-1} \tag{2.40}$$
$$G_{\rm N} \simeq 7 \times 10^{-11}\ L^3\ T^{-2}\ M^{-1}\ . \tag{2.41}$$

Therefore we can simplify the S_R as

$$S_R = \frac{1}{4\pi} \times \frac{c^2 H_0}{G_N} \times \mathcal{V}_{3D} \times \left[e^{3H_0 t} - e^{3H_0 t_{\rm i}}\right] \tag{2.42}$$

$$S_R \times \left[e^{3H_0 t} - e^{3H_0 t_{\rm i}}\right]^{-1} = \frac{1}{4\pi} \times \frac{c^2 H_0}{G_N} \times \mathcal{V}_{3D} \tag{2.43}$$

$$\simeq \frac{1}{4\pi} \times \frac{\left[3 \times 10^8 \, L \, T^{-1}\right]^2 \times 2 \times 10^{-17} \, T^{-1}}{7 \times 10^{-11} \, L^3 T^{-2} M^{-1}} \times \mathcal{V}_{3D} \tag{2.44}$$

$$= \frac{1}{4\pi} \times \frac{3^2 \times 2}{7} \times 10^{2\times8-17+11} \times \, L^{2-3} \, T^{-2-1+2} \, M \times \mathcal{V}_{3D} \tag{2.45}$$

$$= \frac{9}{14\pi} \times 10^{10} \times \, L^{-1} \, T^{-1} \, M \times \mathcal{V}_{3D} \tag{2.46}$$

$$\frac{9}{14\pi} \times 10^{10} \times \, L^{2} \, T^{-1} \, M \times \frac{\mathcal{V}_{3D}}{L^3} \tag{2.47}$$

$$= \frac{9}{14\pi} \times 10^{10} \times \frac{\mathcal{V}_{3D}}{L^3} \times \, L^{2} \, T^{-1} \, M \tag{2.48}$$

choosing the initial conditions such as $cst = 0$, we can get that

$$S_R \left[e^{3H_0 t} - e^{3H_0 t_{\rm i}}\right]^{-1} \simeq \frac{9}{14\pi} \times 10^{10} \times \frac{\mathcal{V}_{3D}}{L^3} \times \, L^{2} \, T^{-1} \, M \tag{2.49}$$

$$\simeq 0.2 \times 10^{10} \times \frac{\mathcal{V}_{3D}}{L^3} \times \, L^{2} \, T^{-1} \, M \tag{2.50}$$

$$\simeq 2 \times 10^{9} \times \frac{\mathcal{V}_{3D}}{L^3} \times \, L^{2} \, T^{-1} \, M \tag{2.51}$$

which means

$$S_R \simeq 2 \times 10^{9} \times \frac{\mathcal{V}_{3D}}{L^3} \times \left[e^{3H_0 t} - e^{3H_0 t_{\rm i}}\right] \times \, L^{2} \, T^{-1} \, M \tag{2.52}$$

Now we can choose that the initial time, is a very small number, i.e. $t \simeq -\infty$. This means

$$\lim_{t_{\rm i} \to -\infty} e^{3H_0 t_{\rm i}} = 0 \, , \tag{2.53}$$

which means

$$S_R \simeq 2 \times 10^{9} \times \frac{\mathcal{V}_{3D}}{L^3} \times \left[e^{3H_0 t}\right] \times \, L^{2} \, T^{-1} \, M \tag{2.54}$$

Note that the effective volume of cmass sample, a galaxy sample at redshift, $z \simeq 0.5$, is about

$$V_{\rm eff}^{\rm cmass} = 2.9 h^{-3} {\rm Gpc}^3 \tag{2.55}$$

$$\simeq 10^{77} L^3 \tag{2.56}$$

Note that the age of the universe is about

$$t_{\text{age universe}} = 13 \times G \; years \tag{2.57}$$

$$t_{\text{age universe}} \simeq 10^{17} \; T \tag{2.58}$$

Therefore substituting the age of the universe, and an effective volume of the universe we have

$$S_R \simeq 2 \times 10^9 \times \frac{\mathcal{V}_{3D}}{L^3} \times \left[e^{3H_0 t}\right] \times\ L^2\, T^{-1}\, M \tag{2.59}$$

$$\simeq 8 \times 10^{88}\ L^2\, T^{-1}\, M \tag{2.60}$$

Therefore the actionic fluctuation, in the Dark Energy dominated epoch, at about 13 Gyears after the beginning of the universe is

$$\delta S_R = \beta S_R^2 \sim 10^{-5}\ L^2\, T^{-1}\, M \tag{2.61}$$

Since we need to have a small fluctuation of the order of 10^{-5}, then this defines the amount of the β parameter, which would be

$$\beta = \delta S_R S_R^{-2} \tag{2.62}$$

$$\beta \simeq 10^{-182}\ L^{-2}\, T\, M^{-1} \tag{2.63}$$

3 Locality resolution

3.1 The 1D case, locality resolution

Let's consider the 1D problem for locality. We would like to describe the motion of a particle moving from point A to point B. Then you can have an equation of motions, which tell's you what is the distance as a function of time, i.e. the trajectory of your particle.

The equation of motion, without an integral, can contain the speed, and acceleration, of this particle.

Then in this case, you have a differential equation to solve, which depends on some initial conditions, which is the initial point of your particle, x_A, and the initial time, t_A.

Now if your differential equation, involve also some integral, for example the integral of your acceleration, then the integral of the acceleration, will provide you with the cumulative velocity of your particle. To have the integral of the acceleration, you need to provide some initial time.

Now in the case that you have an acceleration that depends on the space around, for example, because your particle has some kind of volume, and it is not a point, then different parts of your particle will accelerate with a different rate, according to their position. Then you would need an integral over all space of your acceleration, in order to provide an effective acceleration, of your particle. In this case, to describe the motion of your non-point particle, you need an integral in the equation of motions. Therefore an integral would make sense, and you would need some limits to put into your integral. Possibly you can put the size of your particle. Therefore, in that case an integral-differential equation would make sense, and the theory is still local, since it depends on the spatial limits of the integral of your acceleration, which try to model the the local behaviour of the acceleration of the non-point particle.

3.2 General case of locality resolution

We provide the mathematical formalism for local invariance in theoretical cosmology, specifically focusing on the invariance under local transformations of the metric.

3.2.1 Diffeomorphism Invariance in General Relativity

In general relativity, the metric $g_{\mu\nu}$ determines the geometry of spacetime, and the theory is invariant under diffeomorphisms, which are smooth, invertible mappings of the manifold onto itself. This invariance ensures that the form of physical laws remains unchanged under general coordinate transformations.

3.2.2 Coordinate Transformations

Consider a coordinate transformation $x^\mu \to x'^\mu(x)$. Under this transformation, the metric tensor transforms as:

$$g'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}(x).$$

The inverse transformation is given by:

$$g^{\mu\nu}(x') = \frac{\partial x'^\mu}{\partial x^\alpha} \frac{\partial x'^\nu}{\partial x^\beta} g^{\alpha\beta}(x).$$

3.2.3 Invariance of the EH action

The action for general relativity, the Einstein-Hilbert action, is given by:

$$S_{\text{EH}} = \frac{1}{16\pi G} \int d^D x \sqrt{-g}\, R,$$

where:
- g is the determinant of the metric tensor $g_{\mu\nu}$,
- R is the Ricci scalar.

Under a coordinate transformation, the volume element $d^D x\sqrt{-g}$ transforms as:

$$d^D x\sqrt{-g} \to d^D x' \sqrt{-g'}.$$

The Ricci scalar R transforms as a scalar field under coordinate transformations:

$$R(x) \to R'(x') = R(x).$$

Thus, the Einstein-Hilbert action remains invariant:

$$S_{\text{EH}} = \frac{1}{16\pi G} \int d^D x \sqrt{-g}\, R = \frac{1}{16\pi G} \int d^D x' \sqrt{-g'}\, R' = S'_{\text{EH}}.$$

3.2.4 Local Invariance in Cosmological Models

In cosmological models, additional fields are often introduced. Consider a scalar field ϕ with the action:

$$S_\phi = \int d^D x \sqrt{-g} \left(-\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right),$$

where $V(\phi)$ is the potential of the scalar field.

Under a coordinate transformation, the scalar field ϕ transforms as a scalar:

$$\phi(x) \to \phi'(x') = \phi(x).$$

The derivatives transform as:

$$\partial_\mu \phi \to \partial'_\mu \phi' = \frac{\partial x^\nu}{\partial x'^\mu} \partial_\nu \phi.$$

The action for the scalar field is then:

$$S_\phi = \int d^D x' \sqrt{-g'} \left(-\frac{1}{2} g'^{\mu\nu} \partial'_\mu \phi' \partial'_\nu \phi' - V(\phi') \right) = S'_\phi.$$

3.2.5 Full Action in Cosmology

Combining the Einstein-Hilbert action and the action for the scalar field, we get the total action:

$$S = \int d^D x \sqrt{-g} \left(\frac{1}{16\pi G} R - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right).$$

Under a general coordinate transformation, the total action transforms as:

$$S \to S' = \int d^D x' \sqrt{-g'} \left(\frac{1}{16\pi G} R' - \frac{1}{2} g'^{\mu\nu} \partial'_\mu \phi' \partial'_\nu \phi' - V(\phi') \right),$$

which shows that the form of the action remains invariant.

3.2.6 Summary

The mathematical formalism of local invariance in theoretical cosmology involves ensuring that the action of the theory remains invariant under local transformations of the metric, specifically diffeomorphisms. This is achieved by ensuring that both the gravitational part (described by the Einstein-Hilbert action) and any additional fields (like scalar fields) transform appropriately under coordinate transformations, keeping the total action invariant.

3.3 Local invariance of a quadratic FAT model

Let's analyze the local invariance of the given action, which consists of the Einstein-Hilbert action S_R, a scalar field action S_ϕ, and an additional term βS_R^2.

The full action is given by:

$$S_{\text{FAT}}^{\text{quad}} = S_R + S_\phi + \beta S_R^2,$$

where:

$$S_{\text{FAT}}^{\text{quad}} = \int d^D x \sqrt{-g} \left(\frac{1}{16\pi G} R - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right) + \beta \left\{ \int d^D x \sqrt{-g} \frac{1}{16\pi G} R \right\}^2$$

To show that this action is locally invariant under diffeomorphisms, we will examine the transformation properties of each term in the action.

3.3.1 Einstein-Hilbert Action S_R

The Einstein-Hilbert action is:

$$S_R = \frac{1}{16\pi G} \int d^D x \sqrt{-g} R.$$

Under a coordinate transformation $x^\mu \to x'^\mu(x)$, the metric tensor $g_{\mu\nu}$ transforms as:

$$g'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}(x),$$

and the Ricci scalar R transforms as a scalar:

$$R(x) \to R'(x') = R(x).$$

The volume element transforms as:

$$d^D x \sqrt{-g} \to d^D x' \sqrt{-g'}.$$

Thus, the Einstein-Hilbert action remains invariant under the coordinate transformation:

$$S_R \to S'_R = \frac{1}{16\pi G} \int d^D x' \sqrt{-g'} R' = \frac{1}{16\pi G} \int d^D x \sqrt{-g} R = S_R.$$

3.4 Scalar Field Action S_ϕ

The action for the scalar field ϕ is:

$$S_\phi = \int d^D x \sqrt{-g} \left(-\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right).$$

Under a coordinate transformation, the scalar field transforms as:

$$\phi(x) \to \phi'(x') = \phi(x),$$

and its derivatives transform as:

$$\partial_\mu \phi \to \partial'_\mu \phi' = \frac{\partial x^\nu}{\partial x'^\mu} \partial_\nu \phi.$$

The metric tensor transforms as described before. Therefore, the scalar field action transforms as:

$$S_\phi \to S'_\phi = \int d^D x' \sqrt{-g'} \left(-\frac{1}{2} g'^{\mu\nu} \partial'_\mu \phi' \partial'_\nu \phi' - V(\phi') \right).$$

Given that $\phi' = \phi$ and $V(\phi') = V(\phi)$, the action remains invariant:

$$S'_\phi = \int d^D x \sqrt{-g} \left(-\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right) = S_\phi.$$

3.4.1 Higher-Order Gravity Term βS_R^2

The higher-order term is:

$$\beta S_R^2 = \beta \left(\frac{1}{16\pi G} \int d^D x \sqrt{-g} R \right)^2 .$$

We need to ensure that this term is invariant under coordinate transformations. Under a diffeomorphism, each S_R term in the square transforms as shown previously:

$$S_R \to S'_R.$$

Thus, the product S_R^2 transforms as:

$$\beta S_R^2 \to \beta \left(\frac{1}{16\pi G} \int d^D x' \sqrt{-g'} R' \right)^2 = \beta \left(\frac{1}{16\pi G} \int d^D x \sqrt{-g} R \right)^2 = \beta S_R^2.$$

3.4.2 Full FAT action invariance

Combining all the terms, the full action transforms as:

$$S_{\text{FAT}} \to S'_{\text{FAT}} = S'_R + S'_\phi + \beta S'^2_R.$$

Since each component action S_R, S_ϕ, and βS_R^2 is invariant under diffeomorphisms, the full action remains invariant:

$$S'_{\text{FAT}} = S_{\text{FAT}}.$$

Therefore, the action

$$S_{\text{FAT}} = \int d^D x \sqrt{-g} \left(\frac{1}{16\pi G} R - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right) + \beta \left\{ \int d^D x \sqrt{-g} \frac{1}{16\pi G} R \right\}^2$$

is locally invariant under general coordinate transformations.

4 Foundational Analysis of FAT: Locality and Consistency

To address the intrinsic consistency of Functors of Actions Theories (FAT), we analyze the quadratic model $S_{\text{FAT}} = S_R + S_\phi + \beta S_R^2$ (Section 2), focusing on nonlocality, causality, boundary conditions, and General Relativity (GR) correspondence, before cosmological applications.

4.1 Equations of Motion and Nonlocality

For $S_R = \frac{c^3}{16\pi G_{\text{N}}} \int d^4 x \sqrt{-g} R$ and $S_\phi = \int d^4 x \sqrt{-g} \left(-\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right)$, the action is:

$$S_{\text{FAT}} = S_R + S_\phi + \beta \left(\int d^4 x \sqrt{-g} \frac{c^3}{16\pi G_{\text{N}}} R \right)^2 .$$

Varying with respect to $g_{\mu\nu}$:

$$\frac{\delta S_{\text{FAT}}}{\delta g_{\mu\nu}} = \frac{\delta S_R}{\delta g_{\mu\nu}} + \frac{\delta S_\phi}{\delta g_{\mu\nu}} + 2\beta S_R \frac{\delta S_R}{\delta g_{\mu\nu}} = 0,$$

$$(1 + 2\beta S_R)\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) + T^{\phi}_{\mu\nu} = 0,$$

where $T^{\phi}_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_\phi}{\delta g_{\mu\nu}}$, and $S_R = \int d^4x\sqrt{-g}\frac{c^3}{16\pi G_{\rm N}}R$ is nonlocal (bilocal). This integro-differential equation depends on the entire spacetime via S_R, contrasting with GR's local differential form.

4.2 Causality

Nonlocality from S_R could imply acausal propagation if future states affect the past. For small β, $1 + 2\beta S_R \approx 1$, and the equation approximates GR's causal form. Perturbative solutions suggest locality dominates, with nonlocal corrections as higher-order effects, preserving causality at leading order.

4.3 Boundary Value Problem

The term $2\beta S_R$ requires S_R over all spacetime, suggesting a boundary value problem. In perturbation ($\beta \ll 1$), we treat S_R as a constant (e.g., $S_R^{\rm cl}$ in FLRW), reducing to an initial value problem with conditions like $a(t_0), \dot{a}(t_0)$, as in oscillator models.

4.4 Correspondence with GR

As $\beta \to 0$, $S_{\rm FAT} \to S_R + S_\phi$, and the equation becomes:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + T^{\phi}_{\mu\nu} = 0,$$

recovering GR with a scalar field, ensuring consistency with established physics.

4.5 Local Invariance

Under diffeomorphisms $x^\mu \to x'^\mu$, $S_R \to S'_R = S_R$, $S_\phi \to S'_\phi = S_\phi$, and $\beta S_R^2 \to \beta(S'_R)^2 = \beta S_R^2$, so:

$$S'_{\rm FAT} = S_{\rm FAT},$$

preserving GR's symmetry.

4.6 Conclusion

The quadratic FAT introduces bilocal terms, manageable perturbatively to maintain causality and an initial value framework. The GR limit ($\beta = 0$) ensures reliability, grounding cosmological applications as exploratory extensions of this foundation.

5 Linearised Einstein Field Equations around a background metric

To derive the equations of motion in both cases with the appearance of the Green function, we will focus on deriving the Green's function for the differential operators involved. We will start with the standard gravitational action and then proceed to the modified action.

5.1 Standard Action

5.2 Gravitational Action

The curvature action is

$$S_R = \int d^D x \sqrt{-g} \frac{1}{16\pi G_{\rm N}} R$$

where the matter Action:

$$S_m = \int d^D x \sqrt{-g} \mathcal{L}_m$$

and the total gravitational action is

$$S_{\rm grav} = S_R + S_m = \int d^D x \sqrt{-g} \left(\frac{1}{16\pi G_{\rm N}} R + \mathcal{L}_m \right)$$

5.3 Deriving the Equations of Motion

The variation of the action gives us the equations of motion. Let's start by considering small perturbations around a background metric $g_{\mu\nu}$:

$$g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}$$

Varying the Gravitational Part:
The variation of the Ricci scalar R and the volume element $\sqrt{-g}$ are needed.
1. Variation of the Ricci Scalar:

$$\delta R = R_{\mu\nu} \delta g^{\mu\nu} + g^{\mu\nu} \nabla_\mu \nabla_\nu (\delta g) - g^{\mu\nu} \nabla_\rho \nabla_\nu (\delta g^\rho_\mu)$$

2. Variation of the Volume Element:

$$\delta \sqrt{-g} = -\frac{1}{2} \sqrt{-g} g_{\mu\nu} \delta g^{\mu\nu}$$

Putting these variations together in the action:

$$\delta S_R = \int d^D x \left(\frac{1}{16\pi G_{\rm N}} \right) \left[\sqrt{-g} (R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R) \delta g^{\mu\nu} \right]$$

The matter action variation contributes:

$$\delta S_m = \int d^D x \sqrt{-g} \left(\frac{\delta \mathcal{L}_m}{\delta g^{\mu\nu}} \right) \delta g^{\mu\nu}$$

Combining both:

$$\delta S_{\rm grav} = \delta S_R + \delta S_m = \int d^D x \sqrt{-g} \left[\frac{1}{16\pi G_{\rm N}} (R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R) + \frac{\delta \mathcal{L}_m}{\delta g^{\mu\nu}} \right] \delta g^{\mu\nu}$$

Setting the variation to zero for extremization:

$$\frac{\delta S_{\rm grav}}{\delta g^{\mu\nu}} = 0 \implies \frac{1}{16\pi G_{\rm N}} (R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R) + \frac{\delta \mathcal{L}_m}{\delta g^{\mu\nu}} = 0$$

Identifying the stress-energy tensor $T_{\mu\nu}$ from the matter Lagrangian:

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g} \mathcal{L}_m)}{\delta g^{\mu\nu}}$$

The Einstein field equations are:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G_{\rm N} T_{\mu\nu}$$

5.4 Green's Function in the Standard Action

To introduce the Green's function, we consider the linearized equations around a background metric, which simplifies the problem to finding the Green's function for the differential operator acting on the perturbation $h_{\mu\nu}$.

The linearized Einstein field equations in vacuum can be written as:

$$\Box h_{\mu\nu} + 2R_{\mu\alpha\nu\beta}h^{\alpha\beta} - R_{\mu\alpha}h^{\alpha}_{\nu} - R_{\nu\alpha}h^{\alpha}_{\mu} = 0$$

Where $\Box = \nabla^\alpha \nabla_\alpha$ is the d'Alembertian operator.

The Green's function $G(x, x')$ satisfies:

$$\Box G(x, x') = \delta^D(x - x')$$

Thus, the solution for the perturbation $h_{\mu\nu}$ can be written as an integral involving the Green's function and the source term $T_{\mu\nu}$:

$$h_{\mu\nu}(x) = \int d^D x' \, G(x, x') \left(8\pi G_{\mathrm{N}} T_{\mu\nu}(x')\right)$$

5.5 Modified Action with Quadratic Term

Modified Action:

$$S_{\mathrm{FAT,grav}} = S_R + S_m + \beta S_R^2$$

5.6 Deriving the Equations of Motion for the Modified Action

The additional term in the action is:

$$\beta S_R^2 = \beta \left(\int d^D x \sqrt{-g} \frac{1}{16\pi G_{\mathrm{N}}} R \right)^2$$

Varying the Quadratic Term: For this term, the variation involves the square of the integral. Let's denote S_R as:

$$S_R = \int d^D x \sqrt{-g} \frac{1}{16\pi G_{\mathrm{N}}} R$$

Then:

$$\delta(\beta S_R^2) = 2\beta S_R \delta S_R$$

We already have:

$$\delta S_R = \int d^D x \sqrt{-g} \left(\frac{1}{16\pi G_{\mathrm{N}}} \right) \left[(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R) \delta g^{\mu\nu} \right]$$

Thus:

$$\delta(\beta S_R^2) = 2\beta S_R \int d^D x \sqrt{-g} \left(\frac{1}{16\pi G_{\mathrm{N}}} \right) \left[(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R) \delta g^{\mu\nu} \right]$$

Combining all variations, the total variation of the action is:

$$\delta S_{\mathrm{FAT,grav}} = \delta S_R + \delta S_m + \delta(\beta S_R^2)$$

$$= \int d^D x \sqrt{-g} \left[\frac{1}{16\pi G_{\rm N}} (R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R) + \frac{\delta \mathcal{L}_m}{\delta g^{\mu\nu}} + 2\beta S_R \left(\frac{1}{16\pi G_{\rm N}} \right) (R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R) \right] \delta g^{\mu\nu}$$

Setting this to zero for extremization:

$$\frac{1}{16\pi G_{\rm N}} (R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R) + \frac{\delta \mathcal{L}_m}{\delta g^{\mu\nu}} + \frac{\beta S_R}{8\pi G_{\rm N}} (R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R) = 0$$

Factoring out the common terms:

$$\left(\frac{1 + 2\beta S_R}{16\pi G_{\rm N}} \right) (R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R) + \frac{\delta \mathcal{L}_m}{\delta g^{\mu\nu}} = 0$$

Rewriting the Einstein field equations for the modified action:

$$(1 + 2\beta S_R)(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R) = 8\pi G_{\rm N} T_{\mu\nu}$$

Or equivalently:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G_{\rm N}}{1 + 2\beta S_R} T_{\mu\nu}$$

5.7 Green's Function in the Modified Action

For the modified action, the Green's function will also need to account for the quadratic term. The linearized equation will be modified accordingly, and finding the Green's function $G(x, x')$ will involve solving a modified differential operator:

$$\left(\Box + \beta S_R \Box^2 \right) G(x, x') = \delta^D (x - x')$$

Thus, the Green's function in this case would be more complex due to the higher-order derivative term introduced by the quadratic action.

5.8 Conclusion

The Green's function approach allows us to understand the propagation of perturbations in both the standard and modified gravitational actions. The introduction of a quadratic term in the action complicates the equations of motion and the corresponding Green's function, reflecting the more intricate dynamics of the modified theory.

6 Presence of non-locality to the quadratic FAT action

The introduction of higher-order terms in the action can lead to non-locality issues in the Green's function. We argue the presence of non-locality issues as follows.

6.1 Non-Locality in Higher-Order Theories

In classical and quantum field theories, non-locality refers to the dependence of a field at a point on values of the field at distant points. This is contrary to local theories, where interactions occur only at the same point or within an infinitesimal neighborhood.

6.2 Modified Action with Higher-Order Term

For the modified action:

$$S_{\text{FAT,grav}} = S_R + S_m + \beta S_R^2$$

The variation of this action led to the modified field equations:

$$(1 + 2\beta S_R)(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R) = 8\pi G_{\text{N}} T_{\mu\nu}$$

6.3 Green's Function in the Modified Theory

When we linearize the field equations, the presence of higher-order derivatives (such as $\Box^2$) can complicate the form of the Green's function. Specifically, the equation for the Green's function becomes:

$$\left(\Box + \beta S_R \Box^2\right) G(x, x') = \delta^D(x - x')$$

6.4 Implications of Non-Locality

1. Propagation of Signals:

- In local theories, disturbances propagate with a finite speed, usually the speed of light in relativistic theories.

- In higher-order or non-local theories, the solution to the differential equations can imply that the disturbance at a point depends on the values of the field at distant points, leading to acausal behavior where the signal propagation is no longer strictly confined to the light cone.

2. Green's Function Behavior:

- The Green's function $G(x, x')$ in a local theory generally represents the response at a point x due to a source at x'.

- In higher-order theories, the differential operator acting on the Green's function is no longer a simple second-order operator but includes higher derivatives. This often means that $G(x, x')$ will have a more complex structure, potentially with contributions from points far from x'.

3. Non-Local Effects:

- The $\Box^2$ term implies that the Green's function might have non-local contributions, meaning that the field at x could be influenced by the source at x' in a manner that involves a non-local integral or other complex dependence on the entire region of interest.

6.5 Example of Non-Local Green's Function

Consider a simplified equation:

$$\left(\Box + \beta \Box^2\right) G(x, x') = \delta^D(x - x')$$

To solve this, we would generally use Fourier transforms. The Fourier transform of $\Box$ is $-k^2$, and $\Box^2$ is k^4. Thus, in Fourier space, the equation becomes:

$$\left(-k^2 + \beta k^4\right) \tilde{G}(k) = 1$$

Solving for $\tilde{G}(k)$:

$$\tilde{G}(k) = \frac{1}{-k^2 + \beta k^4}$$

Transforming back to position space can give a Green's function with non-local terms, indicating that $G(x, x')$ does not just depend on $x - x'$ in a simple manner but has a more extended structure, potentially including terms that imply influences over a range of distances.

6.6 Conclusion

The introduction of higher-order terms like βS_R^2 in the action introduces non-locality into the theory. This non-locality is reflected in the Green's function, which can show dependence on the field values over extended regions rather than just at a point. This could lead to acausal behavior and makes the theory more complex, requiring careful interpretation and handling, especially in quantum field theory and cosmological applications. In the section 7, we illustrate how the non-locality is suppressed.

7 Suppressing non-locality effects on quadratic FAT model

Localizing the integral of the S_R action to a specific domain of applicability can help mitigate some of the non-locality issues introduced by higher-order terms in the action. However, the extent to which non-locality is resolved depends on the specifics of how the localization is implemented and the nature of the higher-order terms. Here's a detailed look at this idea:

7.1 Localizing the S_R Action

Let's consider the modified action:

$$S_{\text{FAT,grav}} = S_R + S_m + \beta S_R^2$$

If we restrict the S_R action to a specific region of spacetime where the theory is applicable, we can write:

$$S_R = \int_\Omega d^D x \sqrt{-g} \frac{1}{16\pi G_{\text{N}}} R$$

where Ω is the domain of applicability.

The term βS_R^2 then becomes:

$$\beta S_R^2 = \beta \left(\int_\Omega d^D x \sqrt{-g} \frac{1}{16\pi G_{\text{N}}} R \right)^2$$

7.2 Implications of Localization

1. Boundary Effects:

- By localizing the action to a specific domain Ω, boundary terms become important. These boundary terms can help manage the behavior of the Green's function near the edges of the domain.

- The careful treatment of boundary conditions can help ensure that the influence of fields outside Ω is minimized, thereby reducing non-local effects.

2. Modified Equations of Motion:

- The equations of motion within Ω will be influenced by the boundary conditions. For example, if we assume Dirichlet or Neumann boundary conditions, the solution to the differential equations will be tailored to fit these conditions, thereby reducing non-local behavior.

3. Green's Function in a Bounded Domain:

- The Green's function in a bounded domain Ω satisfies:

$$\left(\Box + \beta S_R \Box^2 \right) G(x, x') = \delta^D(x - x') \quad \text{for } x, x' \in \Omega$$

- The solution to this equation will inherently respect the boundaries of Ω. This means that the Green's function $G(x, x')$ will reflect the localized nature of the problem, which can help suppress non-local effects.

7.3 Reduced Non-Locality

To reduce the non-locality we consider the following.

1. Localized Influence:

- By restricting the action to a domain Ω, the influence of a source at x' on a point x will be limited to within Ω. This can effectively localize the interactions and reduce the apparent non-local behavior.

2. Higher-Order Corrections:

- The higher-order term βS_R^2 still involves integrals over the entire domain Ω. However, within a finite and well-defined domain, the effects of these higher-order terms can be more controlled, reducing the extent of non-locality.

7.4 Treating potential issues

We treat the potential issues by considering smoothed out sharp boundaries, and a domain size which matches the observed spacetime regions.

1. Sharp Boundaries:

- Introducing sharp boundaries can lead to artifacts in the solutions, such as discontinuities or artificial reflections. Care must be taken to smooth out these effects to ensure physical consistency.

2. Domain Size:

- The size of the domain Ω needs to be carefully chosen. If the domain is too small, it may not capture all relevant physical phenomena. If it is too large, non-local effects can still be significant.

7.5 Conclusion

Localizing the S_R action to a specific domain of applicability can help mitigate non-locality issues introduced by higher-order terms in the action. By carefully choosing the domain and imposing appropriate boundary conditions, the non-local effects can be controlled, resulting in a more physically consistent theory. However, attention must be paid to the boundary effects and the size of the domain to ensure that the theory remains applicable and accurate.

Note that, in contrast, some of these FAT models also allow for non-local models which are related to entanglement. This study is left for a future work.

8 Action of simple classical harmonic oscillator

In classical mechanics, an oscillator can be described using the principle of least action. Let's consider the simple harmonic oscillator as a concrete example.

8.1 Lagrangian and Action

The dynamics of a simple harmonic oscillator can be derived from the Lagrangian, which is given by:

$$L = T - V = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$$

where:

- $x(t)$ is the position of the oscillator as a function of time.

- $\dot{x}$ is the velocity, i.e., $\dot{x} = \frac{dx}{dt}$.

- m is the mass of the oscillator.
- k is the spring constant.

The action S is the integral of the Lagrangian over time:

$$S = \int_{t_1}^{t_2} L\,dt = \int_{t_1}^{t_2} \left(\frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2\right) dt$$

8.2 Principle of Least Action

According to the principle of least action (or stationary action), the actual path taken by the oscillator between two points in time t_1 and t_2 is the one that makes the action S stationary (usually a minimum).

To find this path, we use the Euler-Lagrange equation:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

8.3 Derivation

For our Lagrangian, we compute:

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = m\ddot{x}$$

$$\frac{\partial L}{\partial x} = -kx$$

Substituting these into the Euler-Lagrange equation, we get:

$$m\ddot{x} + kx = 0$$

This is the differential equation for a simple harmonic oscillator.

8.4 Solution of the Differential Equation

The general solution to this equation is:

$$x(t) = A\cos(\omega t) + B\sin(\omega t)$$

where:
- A and B are constants determined by initial conditions.
- $\omega = \sqrt{\frac{k}{m}}$ is the angular frequency of the oscillator.

8.5 Initial Conditions

To completely determine the motion, we need the initial position $x(t_0) = x_0$ and initial velocity $\dot{x}(t_0) = v_0$. These give:

$$x(0) = A = x_0$$

$$\dot{x}(0) = B\omega = v_0$$

Thus:

$$A = x_0$$

$$B = \frac{v_0}{\omega}$$

The final solution is:

$$x(t) = x_0 \cos(\omega t) + \frac{v_0}{\omega} \sin(\omega t)$$

8.6 Conclusion

The action principle provides a powerful framework for deriving the equations of motion for an oscillator. By defining the Lagrangian, calculating the action, and using the Euler-Lagrange equation, we derive the differential equation governing the oscillator's motion. Solving this equation with appropriate initial conditions yields the complete description of the oscillator's behavior.

This method not only applies to simple harmonic oscillators but can be generalized to more complex systems in classical mechanics.

9 Action of FAT simple harmonic oscillator

Let's consider the modified action $S_{\text{FAT,osc}}$ as given:

$$S_{\text{FAT,osc}} = S_{\text{osc}} + \beta S_{\text{osc}}^2$$

where S_{osc} is the action for the simple harmonic oscillator:

$$S_{\text{osc}} = \int_{t_1}^{t_2} L_{\text{osc}}\, dt = \int_{t_1}^{t_2} \left(\frac{1}{2} m\dot{x}^2 - \frac{1}{2} kx^2 \right) dt$$

9.1 Deriving the Equations of Motion

The principle of stationary action states that the true path of the system makes the action stationary (usually a minimum). To find the equations of motion for the modified action, we need to apply the Euler-Lagrange equation to $S_{\text{FAT,osc}}$.

First, we calculate the modified Lagrangian $L_{\text{FAT,osc}}$:

$$S_{\text{FAT,osc}} = \int_{t_1}^{t_2} L_{\text{FAT,osc}}\, dt$$

Given $S_{\text{FAT,osc}} = S_{\text{osc}} + \beta S_{\text{osc}}^2$, we write:

$$S_{\text{osc}} = \int_{t_1}^{t_2} L_{\text{osc}}\, dt$$

Therefore,

$$S_{\text{osc}}^2 = \left(\int_{t_1}^{t_2} L_{\text{osc}}\, dt \right)^2$$

This can be considered as:

$$S_{\text{FAT,osc}} = \int_{t_1}^{t_2} L_{\text{osc}}\, dt + \beta \left(\int_{t_1}^{t_2} L_{\text{osc}}\, dt \right)^2$$

To proceed, let's differentiate the action $S_{\text{FAT,osc}}$ with respect to the path $x(t)$. The functional derivative of $S_{\text{FAT,osc}}$ with respect to $x(t)$ must vanish:

$$\frac{\delta S_{\text{FAT,osc}}}{\delta x(t)} = \frac{\delta}{\delta x(t)} \left(S_{\text{osc}} + \beta S_{\text{osc}}^2 \right) = 0$$

9.2 Contribution from S_{osc}

The contribution from S_{osc} is simply the standard Euler-Lagrange equation derived from the simple harmonic oscillator's Lagrangian:

$$\frac{\delta S_{\text{osc}}}{\delta x(t)} = 0 \implies m\ddot{x} + kx = 0$$

9.3 Contribution from βS_{osc}^2

For the term βS_{osc}^2, we need to differentiate with respect to $x(t)$:

$$\frac{\delta}{\delta x(t)} \left(\int_{t_1}^{t_2} L_{\text{osc}}\, dt \right)^2 = 2 \int_{t_1}^{t_2} L_{\text{osc}}\, dt \cdot \frac{\delta}{\delta x(t)} \left(\int_{t_1}^{t_2} L_{\text{osc}}\, dt \right)$$

Using the chain rule, we get:

$$\frac{\delta}{\delta x(t)} \left(\int_{t_1}^{t_2} L_{\text{osc}}\, dt \right) = L'_{\text{osc}}(x(t))$$

where $L'_{\text{osc}}(x(t))$ is the functional derivative of the Lagrangian L_{osc} with respect to $x(t)$, which leads to the standard Euler-Lagrange equation:

$$L'_{\text{osc}}(x(t)) = m\ddot{x} + kx$$

Thus, the contribution from βS_{osc}^2 becomes:

$$\beta \cdot 2 \int_{t_1}^{t_2} L_{\text{osc}}\, dt \cdot (m\ddot{x} + kx)$$

which becomes:

$$\beta \cdot 2 \int_{t_1}^{t_2} \left(\frac{1}{2} m\dot{x}^2 - \frac{1}{2} kx^2 \right) dt \cdot (m\ddot{x} + kx)$$

9.4 Combining Contributions

The total Euler-Lagrange equation for $S_{\text{FAT,osc}}$ is:

$$m\ddot{x} + kx + \beta \cdot 2 \left(\int_{t_1}^{t_2} L_{\text{osc}}\, dt \right) (m\ddot{x} + kx) = 0$$

This simplifies to:

$$\left(1 + 2\beta \int_{t_1}^{t_2} L_{\text{osc}}\, dt \right) (m\ddot{x} + kx) = 0$$

which expands to

$$\left(1 + 2\beta \int_{t_1}^{t_2} \left(\frac{1}{2} m\dot{x}^2 - \frac{1}{2} kx^2 \right) dt \right) (m\ddot{x} + kx) = 0$$

For non-trivial solutions, we require:

$$1 + 2\beta \int_{t_1}^{t_2} L_{\text{osc}}\, dt \neq 0$$

Therefore, the modified equation of motion is:

$$m\ddot{x} + kx = 0$$

This indicates that the equations of motion remain the same as the simple harmonic oscillator, but the influence of βS_{osc}^2 on the action can affect the overall scaling factor of the solutions, ensuring that the paths considered still satisfy the original oscillator equation.

9.5 Perturbative Solution for FAT Simple Harmonic Oscillator

We derive the perturbative solution for the FAT-modified simple harmonic oscillator, starting from the action and obtaining the equation of motion, then solving it step-by-step using a perturbative approach.

9.5.1 Action and Equation of Motion

The classical harmonic oscillator action is:

$$S_{\text{osc}} = \int_{t_i}^{t_f} dt \left[\frac{1}{2} m\dot{x}^2 - \frac{1}{2} kx^2 \right],$$

where m is the mass, k is the spring constant, $\dot{x} = \frac{dx}{dt}$, and $x(t)$ is the displacement.

The FAT-modified action introduces a quadratic term:

$$S_{\text{FAT,osc}} = S_{\text{osc}} + \beta S_{\text{osc}}^2,$$

where β is a small coupling constant ($\beta \ll 1$).

To find the equation of motion, apply the variational principle: $\delta S_{\text{FAT,osc}} = 0$. Compute the variation:

$$\delta S_{\text{FAT,osc}} = \delta S_{\text{osc}} + 2\beta S_{\text{osc}} \delta S_{\text{osc}}.$$

For S_{osc}, the Lagrangian is $L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$. The variation is:

$$\delta S_{\text{osc}} = \int_{t_i}^{t_f} dt \left[\frac{\partial L}{\partial \dot{x}} \delta \dot{x} + \frac{\partial L}{\partial x} \delta x \right] = \int_{t_i}^{t_f} dt \left[m\dot{x}\delta\dot{x} - kx\delta x \right].$$

Integrate the first term by parts:

$$\int_{t_i}^{t_f} m\dot{x}\delta\dot{x}\, dt = \left[m\dot{x}\delta x \right]_{t_i}^{t_f} - \int_{t_i}^{t_f} m\ddot{x}\delta x\, dt.$$

With boundary conditions $\delta x(t_i) = \delta x(t_f) = 0$, this becomes:

$$\delta S_{\text{osc}} = \int_{t_i}^{t_f} dt \left[-m\ddot{x} - kx \right] \delta x.$$

Thus:

$$\delta S_{\text{osc}} = 0 \implies m\ddot{x} + kx = 0,$$

the classical case. For the FAT term:

$$\delta S_{\text{FAT,osc}} = \int_{t_i}^{t_f} dt \left[-m\ddot{x} - kx \right] \delta x + 2\beta S_{\text{osc}} \int_{t_i}^{t_f} dt \left[-m\ddot{x} - kx \right] \delta x = 0.$$

Factor out δx:

$$\int_{t_i}^{t_f} dt \left[(m\ddot{x} + kx) + 2\beta S_{\text{osc}}(m\ddot{x} + kx) \right] \delta x = 0.$$

Since δx is arbitrary, the integrand is zero:

$$m\ddot{x} + kx + 2\beta S_{\text{osc}}(m\ddot{x} + kx) = 0,$$

or:

$$\left(1 + 2\beta S_{\text{osc}}\right)\left(m\ddot{x} + kx\right) = 0.$$

Since $1 + 2\beta S_{\text{osc}} \neq 0$ (as S_{osc} is finite), the equation is:

$$m\ddot{x} + kx + 2\beta S_{\text{osc}}(m\ddot{x} + kx) = 0,$$

where:

$$S_{\text{osc}} = \int_{t_i}^{t_f} dt \left[\frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 \right].$$

9.5.2 Perturbative Solution

This is an integral-differential equation due to S_{osc}'s dependence on $x(t)$. Since β is small, use perturbation theory.

9.5.3 Zeroth-Order Solution ($\beta = 0$)

Without the FAT term:

$$m\ddot{x} + kx = 0.$$

The general solution is:

$$x_0(t) = x_0 \cos(\omega t) + \frac{v_0}{\omega} \sin(\omega t),$$

where $\omega = \sqrt{\frac{k}{m}}$, and x_0, v_0 are initial conditions.

Compute $S_{\text{osc}}^{(0)}$ for this solution:

$$\dot{x}_0 = -x_0\omega \sin(\omega t) + v_0 \cos(\omega t),$$

$$\frac{1}{2}m\dot{x}_0^2 = \frac{1}{2}m \left[x_0^2\omega^2 \sin^2(\omega t) - 2x_0v_0\omega \sin(\omega t)\cos(\omega t) + v_0^2 \cos^2(\omega t)\right],$$

$$\frac{1}{2}kx_0^2 = \frac{1}{2}k \left[x_0^2 \cos^2(\omega t) + 2x_0\frac{v_0}{\omega} \cos(\omega t)\sin(\omega t) + \frac{v_0^2}{\omega^2} \sin^2(\omega t)\right].$$

Using $k = m\omega^2$:

$$\begin{aligned} Integrant &= \frac{1}{2}m\dot{x}_0^2 - \frac{1}{2}kx_0^2 \\ Integrant &= \frac{1}{2}m \left[x_0^2\omega^2 \sin^2(\omega t) - 2x_0v_0\omega \sin(\omega t)\cos(\omega t) + v_0^2 \cos^2(\omega t)\right. \\ &\quad \left. -x_0^2\omega^2 \cos^2(\omega t) - 2x_0v_0\omega \cos(\omega t)\sin(\omega t) - \frac{v_0^2}{\omega^2} \sin^2(\omega t)\right]. \end{aligned}$$

Simplify:

$$= \frac{1}{2}m \left[x_0^2\omega^2(\sin^2(\omega t) - \cos^2(\omega t)) + v_0^2(\cos^2(\omega t) - \frac{1}{\omega^2} \sin^2(\omega t)) - 4x_0v_0\omega \sin(\omega t)\cos(\omega t)\right].$$

Using $\sin^2\theta - \cos^2\theta = -\cos(2\theta)$ and $\sin(2\theta) = 2\sin\theta\cos\theta$:

$$= \frac{1}{2}m \left[-x_0^2\omega^2 \cos(2\omega t) + v_0^2 \left(\cos^2(\omega t) - \frac{1}{\omega^2} \sin^2(\omega t)\right) - 2x_0v_0\omega \sin(2\omega t)\right].$$

Integrate over $[t_i, t_f]$:

$$S_{\text{osc}}^{(0)} = \int_{t_i}^{t_f} dt \left[-\frac{1}{2}mx_0^2\omega^2 \cos(2\omega t) + \frac{1}{2}mv_0^2 \left(\cos^2(\omega t) - \frac{1}{\omega^2} \sin^2(\omega t)\right) - mx_0v_0\omega \sin(2\omega t)\right].$$

This integral depends on $t_f - t_i$, but for simplicity, approximate over one period ($T = \frac{2\pi}{\omega}$) or assume initial evaluation. For exactness, compute numerically later, but treat as a constant perturbation term.

9.5.4 First-Order Perturbation

Assume:

$$x(t) = x_0(t) + \beta x_1(t).$$

Substitute:

$$\ddot{x} = \ddot{x}_0 + \beta \ddot{x}_1,$$

$$m\ddot{x} + kx = m\ddot{x}_0 + kx_0 + \beta(m\ddot{x}_1 + kx_1) = \beta(m\ddot{x}_1 + kx_1),$$

$$S_{\text{osc}} \approx S_{\text{osc}}^{(0)} + \beta \int_{t_i}^{t_f} dt \left[m\dot{x}_0\dot{x}_1 - kx_0x_1 \right],$$

but to first order, use $S_{\text{osc}}^{(0)}$:

$$m\ddot{x} + kx + 2\beta S_{\text{osc}}^{(0)}(m\ddot{x} + kx) = \beta(m\ddot{x}_1 + kx_1) + 2\beta S_{\text{osc}}^{(0)}\beta(m\ddot{x}_1 + kx_1) + O(\beta^2) = 0.$$

Neglect $O(\beta^2)$:

$$\beta(1 + 2\beta S_{\text{osc}}^{(0)})(m\ddot{x}_1 + kx_1) = 0,$$

$$m\ddot{x}_1 + kx_1 = 0.$$

Solution:

$$x_1(t) = A_1 \cos(\omega t) + B_1 \sin(\omega t).$$

Adjust for perturbation effect (phase shift):

$$x_1(t) = \tilde{A}_1 \sin(\omega t + \tilde{\phi}_1),$$

so:

$$x(t) \approx x_0 \cos(\omega t) + \frac{v_0}{\omega} \sin(\omega t) + \beta \tilde{A}_1 \sin(\omega t + \tilde{\phi}_1).$$

9.5.5 Conclusion

The perturbative solution approximates the effect of βS_{osc} as a small shift in amplitude and phase, with $\tilde{A}$ and $\tilde{\phi}$ determined by $S_{\text{osc}}^{(0)}$'s influence, typically computed via matching or iteration.

10 Action of simple harmonic oscillator in a gravitational field

Let's analyze a simple harmonic oscillator in a gravitational field using the Lagrangian formalism.

10.1 Lagrangian

For a simple harmonic oscillator in a gravitational field, we need to account for the potential energy due to gravity in addition to the potential energy of the oscillator.

The kinetic energy T remains the same:

$$T = \frac{1}{2}m\dot{x}^2$$

The potential energy V now includes both the spring potential energy and the gravitational potential energy. If we take x as the vertical displacement from the equilibrium position (with positive x being upwards), the potential energy is:

$$V = \frac{1}{2}kx^2 + mgx$$

where:
- m is the mass of the oscillator.
- k is the spring constant.
- g is the acceleration due to gravity.
- x is the displacement from the equilibrium position.

Thus, the Lagrangian L is:

$$L = T - V = \frac{1}{2}m\dot{x}^2 - \left(\frac{1}{2}kx^2 + mgx\right)$$

Simplifying, we get:

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 - mgx$$

10.2 Action

The action S is the integral of the Lagrangian over time:

$$S = \int_{t_1}^{t_2} L\,dt = \int_{t_1}^{t_2} \left(\frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 - mgx\right) dt$$

10.3 Principle of Least Action

To find the equations of motion, we use the principle of least action, which requires the action S to be stationary (usually a minimum). We apply the Euler-Lagrange equation:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

10.4 Derivation

For our Lagrangian, we compute:

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = m\ddot{x}$$

$$\frac{\partial L}{\partial x} = -kx - mg$$

Substituting these into the Euler-Lagrange equation, we get:

$$m\ddot{x} + kx + mg = 0$$

10.5 Equation of Motion

Rearranging the equation, we obtain:

$$\ddot{x} + \frac{k}{m}x + g = 0$$

This is the differential equation describing the motion of a simple harmonic oscillator in a gravitational field.

10.6 Solution of the Differential Equation

To solve this differential equation, we introduce the natural frequency $\omega = \sqrt{\frac{k}{m}}$:

$$\ddot{x} + \omega^2 x + g = 0$$

This is a non-homogeneous second-order differential equation. To solve it, we find the general solution of the homogeneous equation $\ddot{x} + \omega^2 x = 0$ and a particular solution to the non-homogeneous equation.

1. Homogeneous Solution:

The homogeneous solution is the same as for a simple harmonic oscillator:

$$x_h(t) = A\cos(\omega t) + B\sin(\omega t)$$

2. Particular Solution:

For the particular solution, we consider the constant solution x_p:

$$\ddot{x_p} + \omega^2 x_p + g = 0$$

Since $\ddot{x_p} = 0$, we have:

$$\omega^2 x_p + g = 0$$

$$x_p = -\frac{g}{\omega^2} = -\frac{g}{\frac{k}{m}} = -\frac{mg}{k}$$

Thus, the particular solution is:

$$x_p = -\frac{mg}{k}$$

3. General Solution:

The general solution is the sum of the homogeneous and particular solutions:

$$x(t) = x_h(t) + x_p = A\cos(\omega t) + B\sin(\omega t) - \frac{mg}{k}$$

10.7 Initial Conditions

To completely determine the motion, we need the initial position $x(t_0) = x_0$ and initial velocity $\dot{x}(t_0) = v_0$. These give:

$$x(0) = A - \frac{mg}{k} = x_0$$

$$A = x_0 + \frac{mg}{k}$$

$$\dot{x}(0) = B\omega = v_0$$

$$B = \frac{v_0}{\omega}$$

The final solution is:

$$x(t) = \left(x_0 + \frac{mg}{k}\right)\cos(\omega t) + \frac{v_0}{\omega}\sin(\omega t) - \frac{mg}{k}$$

10.8 Conclusion

The Lagrangian and the principle of least action provide a systematic way to derive the equations of motion for a simple harmonic oscillator in a gravitational field. By including the gravitational potential energy in the Lagrangian, we obtain a modified differential equation that describes the system's motion. The solution combines the effects of both the harmonic potential and the gravitational field.

11 Action of simple FAT harmonic oscillator in a gravitation: model γ

Let's start by considering the modified action given as:

$$S_{\text{FAT,osc,grav}} = S_{\text{osc,grav}} + \gamma S^2_{\text{osc,grav}}$$

where $S_{\text{osc,grav}}$ is the action for a simple harmonic oscillator in a gravitational field:

$$S_{\text{osc,grav}} = \int_{t_1}^{t_2} L_{\text{osc,grav}}\, dt$$

we follow the following steps, to acquirer the equations of motions.

11.1 Step 1: Calculate $S_{\text{osc,grav}}$

The Lagrangian for the simple harmonic oscillator in a gravitational field is:

$$L_{\text{osc,grav}} = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 - mgx$$

So, the action $S_{\text{osc,grav}}$ is:

$$S_{\text{osc,grav}} = \int_{t_1}^{t_2} \left(\frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 - mgx\right) dt$$

11.2 Step 2: Calculate the Modified Action $S_{\text{FAT,osc,grav}}$

Given the quadratic model:

$$S_{\text{FAT,osc,grav}} = S_{\text{osc,grav}} + \gamma S_{\text{osc,grav}}^2$$

11.3 Step 3: Principle of Least Action

To find the equations of motion, we need to make the action $S_{\text{FAT,osc,grav}}$ stationary. We use the Euler-Lagrange equation:

$$\frac{d}{dt}\left(\frac{\partial L_{\text{FAT,osc,grav}}}{\partial \dot{x}}\right) - \frac{\partial L_{\text{FAT,osc,grav}}}{\partial x} = 0$$

11.4 Contribution from $S_{\text{osc,grav}}$

First, consider the contribution from $S_{\text{osc,grav}}$:

$$\frac{\delta S_{\text{osc,grav}}}{\delta x(t)} = 0 \implies m\ddot{x} + kx + mg = 0$$

11.5 Contribution from $\gamma S_{\text{osc,grav}}^2$

Now, consider the contribution from $\gamma S_{\text{osc,grav}}^2$:

$$S_{\text{osc,grav}}^2 = \left(\int_{t_1}^{t_2} L_{\text{osc,grav}}\, dt\right)^2$$

We need to differentiate $S_{\text{osc,grav}}^2$ with respect to $x(t)$:

$$\frac{\delta}{\delta x(t)}\left(\int_{t_1}^{t_2} L_{\text{osc,grav}}\, dt\right)^2 = 2\int_{t_1}^{t_2} L_{\text{osc,grav}}\, dt \cdot \frac{\delta}{\delta x(t)}\left(\int_{t_1}^{t_2} L_{\text{osc,grav}}\, dt\right)$$

Using the chain rule, we get:

$$\frac{\delta}{\delta x(t)}\left(\int_{t_1}^{t_2} L_{\text{osc,grav}}\, dt\right) = L'_{\text{osc,grav}}(x(t))$$

where $L'_{\text{osc,grav}}(x(t))$ is the functional derivative of the Lagrangian $L_{\text{osc,grav}}$ with respect to $x(t)$, leading to the standard Euler-Lagrange equation:

$$L'_{\text{osc,grav}}(x(t)) = m\ddot{x} + kx + mg$$

Thus, the contribution from $\gamma S_{\text{osc,grav}}^2$ becomes:

$$\gamma \cdot 2\int_{t_1}^{t_2} L_{\text{osc,grav}}\, dt \cdot (m\ddot{x} + kx + mg)$$

11.6 Combining Contributions

The total Euler-Lagrange equation for $S_{\text{FAT,osc,grav}}$ is:

$$m\ddot{x} + kx + mg + \gamma \cdot 2 \left(\int_{t_1}^{t_2} L_{\text{osc,grav}}\, dt \right) (m\ddot{x} + kx + mg) = 0$$

This simplifies to:

$$\left(1 + 2\gamma \int_{t_1}^{t_2} L_{\text{osc,grav}}\, dt \right) (m\ddot{x} + kx + mg) = 0$$

For non-trivial solutions, we require:

$$1 + 2\gamma \int_{t_1}^{t_2} L_{\text{osc,grav}}\, dt \neq 0$$

Therefore, the modified equation of motion is:

$$m\ddot{x} + kx + mg = 0$$

11.7 Conclusion

Mathematically, we observe that the equations of motion remain the same as for the simple harmonic oscillator in a gravitational field. The term $\gamma S^2_{\text{osc,grav}}$ affects the action but does not change the form of the Euler-Lagrange equations. This suggests that, for this quadratic modification of the action, the dynamics are governed by the same differential equations as the original system. The influence of γ is seen in the scaling of the action, but it does not alter the fundamental nature of the equations of motion.

11.8 Perturbative Solution for FAT Harmonic Oscillator in Gravity: Model γ

We derive the perturbative solution for the action:

$$S_{\text{FAT, osc, grav}} = S_{\text{osc, grav}} + \gamma S^2_{\text{osc, grav}},$$

where:

$$S_{\text{osc, grav}} = \int_{t_1}^{t_2} dt \left(\frac{1}{2} m\dot{x}^2 - \frac{1}{2} kx^2 - mgx \right),$$

m is the mass, k is the spring constant, g is gravitational acceleration, $\dot{x} = \frac{dx}{dt}$, and $\gamma \ll 1$ is a small coupling constant.

11.8.1 Equation of Motion

Apply the variational principle: $\delta S_{\text{FAT, osc, grav}} = 0$:

$$\delta S_{\text{FAT, osc, grav}} = \delta S_{\text{osc, grav}} + 2\gamma S_{\text{osc, grav}} \delta S_{\text{osc, grav}}.$$

For $S_{\text{osc, grav}}$, the Lagrangian is:

$$L_{\text{osc, grav}} = \frac{1}{2} m\dot{x}^2 - \frac{1}{2} kx^2 - mgx,$$

$$\delta S_{\text{osc, grav}} = \int_{t_1}^{t_2} dt \left[\frac{\partial L_{\text{osc, grav}}}{\partial \dot{x}} \delta\dot{x} + \frac{\partial L_{\text{osc, grav}}}{\partial x} \delta x \right] = \int_{t_1}^{t_2} dt \left[m\dot{x}\delta\dot{x} - kx\delta x - mg\delta x \right].$$

Integrate by parts:

$$\int_{t_1}^{t_2} m\dot{x}\delta\dot{x}\,dt = [m\dot{x}\delta x]_{t_1}^{t_2} - \int_{t_1}^{t_2} m\ddot{x}\delta x\,dt,$$

with boundary conditions $\delta x(t_1) = \delta x(t_2) = 0$:

$$\delta S_{\text{osc, grav}} = \int_{t_1}^{t_2} dt\,[-m\ddot{x} - kx - mg]\,\delta x,$$

$$\delta S_{\text{osc, grav}} = 0 \implies m\ddot{x} + kx + mg = 0.$$

For the full action:

$$\delta S_{\text{FAT, osc, grav}} = \int_{t_1}^{t_2} dt\,[-m\ddot{x} - kx - mg]\,\delta x + 2\gamma S_{\text{osc, grav}} \int_{t_1}^{t_2} dt\,[-m\ddot{x} - kx - mg]\,\delta x = 0,$$

$$\int_{t_1}^{t_2} dt\,[(m\ddot{x} + kx + mg) + 2\gamma S_{\text{osc, grav}}(m\ddot{x} + kx + mg)]\,\delta x = 0.$$

Since δx is arbitrary:

$$m\ddot{x} + kx + mg + 2\gamma S_{\text{osc, grav}}(m\ddot{x} + kx + mg) = 0,$$

where:

$$S_{\text{osc, grav}} = \int_{t_1}^{t_2} dt\left[\frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 - mgx\right].$$

This is an integral-differential equation due to $S_{\text{osc, grav}}$'s dependence on $x(t)$.

11.8.2 Zeroth-Order Solution ($\gamma = 0$)

For $\gamma = 0$:

$$m\ddot{x} + kx + mg = 0.$$

Homogeneous solution:

$$x_h(t) = A\cos(\omega t) + B\sin(\omega t), \quad \omega = \sqrt{\frac{k}{m}},$$

particular solution:

$$x_p(t) = -\frac{mg}{k},$$

$$m\ddot{x}_p + kx_p + mg = 0 + k\left(-\frac{mg}{k}\right) + mg = 0.$$

General solution:

$$x_0(t) = A\cos(\omega t) + B\sin(\omega t) - \frac{mg}{k}.$$

Initial conditions $x(0) = x_0$, $\dot{x}(0) = v_0$:

$$x_0(0) = A - \frac{mg}{k} = x_0, \quad A = x_0 + \frac{mg}{k},$$

$$\dot{x}_0(t) = -A\omega\sin(\omega t) + B\omega\cos(\omega t), \quad \dot{x}_0(0) = B\omega = v_0, \quad B = \frac{v_0}{\omega},$$

$$x_0(t) = \left(x_0 + \frac{mg}{k}\right)\cos(\omega t) + \frac{v_0}{\omega}\sin(\omega t) - \frac{mg}{k}.$$

11.9 Compute $S^{(0)}_{\text{osc, grav}}$

$$\dot{x}_0 = -\left(x_0 + \frac{mg}{k}\right)\omega\sin(\omega t) + v_0\cos(\omega t),$$

$$\frac{1}{2}m\dot{x}_0^2 = \frac{1}{2}m\left[\left(x_0 + \frac{mg}{k}\right)^2\omega^2\sin^2(\omega t) - 2\left(x_0 + \frac{mg}{k}\right)v_0\omega\sin(\omega t)\cos(\omega t) + v_0^2\cos^2(\omega t)\right],$$

$$\begin{aligned}
x_0^2 &= \left(x_0 + \frac{mg}{k}\right)^2\cos^2(\omega t) + 2\left(x_0 + \frac{mg}{k}\right)\frac{v_0}{\omega}\cos(\omega t)\sin(\omega t) \\
&\quad + \frac{v_0^2}{\omega^2}\sin^2(\omega t) - 2\left(x_0 + \frac{mg}{k}\right)\frac{mg}{k}\cos(\omega t) - 2\frac{v_0}{\omega}\frac{mg}{k}\sin(\omega t) + \frac{m^2g^2}{k^2}, \\
\frac{1}{2}kx_0^2 &= \frac{1}{2}k\left[\left(x_0 + \frac{mg}{k}\right)^2\cos^2(\omega t) + 2\left(x_0 + \frac{mg}{k}\right)\frac{v_0}{\omega}\cos(\omega t)\sin(\omega t)\right] \\
&\quad + \frac{1}{2}k\left[\frac{v_0^2}{\omega^2}\sin^2(\omega t) - 2\left(x_0 + \frac{mg}{k}\right)\frac{mg}{k}\cos(\omega t) - 2\frac{v_0}{\omega}\frac{mg}{k}\sin(\omega t) + \frac{m^2g^2}{k^2}\right],
\end{aligned}$$

$$-mgx_0 = -mg\left[\left(x_0 + \frac{mg}{k}\right)\cos(\omega t) + \frac{v_0}{\omega}\sin(\omega t) - \frac{mg}{k}\right].$$

Lagrangian:

$$\begin{aligned}
L^{(0)}_{\text{osc, grav}} &= \frac{1}{2}m\left[\left(x_0 + \frac{mg}{k}\right)^2\omega^2\sin^2(\omega t) - 2\left(x_0 + \frac{mg}{k}\right)v_0\omega\sin(\omega t)\cos(\omega t) + v_0^2\cos^2(\omega t)\right] \\
&\quad - \frac{1}{2}k\left[\left(x_0 + \frac{mg}{k}\right)^2\cos^2(\omega t) + \cdots\right] - mg\left[\left(x_0 + \frac{mg}{k}\right)\cos(\omega t) + \frac{v_0}{\omega}\sin(\omega t) - \frac{mg}{k}\right].
\end{aligned}$$

Simplify using $k = m\omega^2$:

$$\begin{aligned}
&= \frac{1}{2}m\left[\left(x_0 + \frac{mg}{k}\right)^2\omega^2(\sin^2(\omega t) - \cos^2(\omega t)) + v_0^2\left(\cos^2(\omega t) - \frac{1}{\omega^2}\sin^2(\omega t)\right)\right. \\
&\left. -4\left(x_0 + \frac{mg}{k}\right)v_0\omega\sin(\omega t)\cos(\omega t)\right] \\
&+ mg\left[\frac{mg}{k} - \left(x_0 + \frac{mg}{k}\right)\cos(\omega t) - \frac{v_0}{\omega}\sin(\omega t)\right].
\end{aligned}$$

Use $\sin^2\theta - \cos^2\theta = -\cos(2\theta)$, $\sin(2\theta) = 2\sin\theta\cos\theta$:

$$\begin{aligned}
&= \frac{1}{2}m\left[-\left(x_0 + \frac{mg}{k}\right)^2\omega^2\cos(2\omega t)\right. \\
&\left. +v_0^2\left(\cos^2(\omega t) - \frac{1}{\omega^2}\sin^2(\omega t)\right) - 2\left(x_0 + \frac{mg}{k}\right)v_0\omega\sin(2\omega t)\right] \\
&+ mg\left[\frac{mg}{k} - \left(x_0 + \frac{mg}{k}\right)\cos(\omega t) - \frac{v_0}{\omega}\sin(\omega t)\right].
\end{aligned}$$

Integrate over $t_1 = 0$ to $t_2 = T = \frac{2\pi}{\omega}$ (one period for simplicity):

$$\begin{aligned}
S^{(0)}_{\text{osc, grav}} = \int_0^T dt &\left[-\frac{1}{2}m\left(x_0 + \frac{mg}{k}\right)^2\omega^2\cos(2\omega t)\right. \\
&+\frac{1}{2}mv_0^2\left(\cos^2(\omega t) - \frac{1}{\omega^2}\sin^2(\omega t)\right) - m\left(x_0 + \frac{mg}{k}\right)v_0\omega\sin(2\omega t) \\
&\left.+mg\frac{mg}{k} - mg\left(x_0 + \frac{mg}{k}\right)\cos(\omega t) - mg\frac{v_0}{\omega}\sin(\omega t)\right].
\end{aligned}$$

Evaluate each term:

- $\int_0^T \cos(2\omega t)\, dt = \frac{1}{2\omega} \sin(2\omega t)|_0^T = 0$,
- $\int_0^T \sin(2\omega t)\, dt = -\frac{1}{2\omega} \cos(2\omega t)|_0^T = 0$,
- $\int_0^T \cos^2(\omega t)\, dt = \int_0^T \frac{1+\cos(2\omega t)}{2}\, dt = \frac{T}{2} = \frac{\pi}{\omega}$,
- $\int_0^T \sin^2(\omega t)\, dt = \int_0^T \frac{1-\cos(2\omega t)}{2}\, dt = \frac{T}{2} = \frac{\pi}{\omega}$,
- $\int_0^T \cos(\omega t)\, dt = \frac{1}{\omega} \sin(\omega t)|_0^T = 0$,
- $\int_0^T \sin(\omega t)\, dt = -\frac{1}{\omega} \cos(\omega t)|_0^T = 0$,
- $\int_0^T dt = T = \frac{2\pi}{\omega}$.

$$\begin{aligned}
S^{(0)}_{\text{osc, grav}} = &-\frac{1}{2}m\left(x_0 + \frac{mg}{k}\right)^2 \omega^2 \cdot 0 \\
&+ \frac{1}{2}mv_0^2\left(\frac{\pi}{\omega} - \frac{1}{\omega^2}\cdot\frac{\pi}{\omega}\right) \\
&- m\left(x_0 + \frac{mg}{k}\right) v_0\omega \cdot 0 \\
&+ mg\frac{mg}{k}\frac{2\pi}{\omega} \\
&- mg\left(x_0 + \frac{mg}{k}\right)\cdot 0 \\
&- mg\frac{v_0}{\omega}\cdot 0,
\end{aligned}$$

$$= \frac{1}{2}mv_0^2\left(\frac{\pi}{\omega} - \frac{\pi}{\omega^3}\right) + \frac{2\pi m^2 g^2}{\omega k}.$$

Using $\omega^2 = \frac{k}{m}$, $\frac{1}{\omega^2} = \frac{m}{k}$:

$$= \frac{1}{2}mv_0^2\frac{\pi}{\omega}\left(1 - \frac{1}{\omega^2}\right) + \frac{2\pi m^2 g^2}{\omega k} = \frac{1}{2}mv_0^2\frac{\pi}{\omega}\left(1 - \frac{m}{k}\right) + \frac{2\pi m^2 g^2}{\omega k}.$$

11.10 First-Order Perturbation

Assume:

$$x(t) = x_0(t) + \gamma x_1(t),$$

$$\dot{x} = \dot{x}_0 + \gamma\dot{x}_1, \quad \ddot{x} = \ddot{x}_0 + \gamma\ddot{x}_1,$$

$$m\ddot{x} + kx + mg = m\ddot{x}_0 + kx_0 + mg + \gamma(m\ddot{x}_1 + kx_1) = \gamma(m\ddot{x}_1 + kx_1),$$

$$S_{\text{osc, grav}} = S^{(0)}_{\text{osc, grav}} + \gamma\int_{t_1}^{t_2} dt[m\dot{x}_0\dot{x}_1 - kx_0x_1 - mgx_1] + O(\gamma^2).$$

Substitute:

$$\gamma(m\ddot{x}_1 + kx_1) + 2\gamma\left(S^{(0)}_{\text{osc, grav}} + \gamma\int_{t_1}^{t_2}[m\dot{x}_0\dot{x}_1 - kx_0x_1 - mgx_1]\right)[\gamma(m\ddot{x}_1 + kx_1)] = 0.$$

Order γ:

$$m\ddot{x}_1 + kx_1 + 2S^{(0)}_{\text{osc, grav}}(m\ddot{x}_1 + kx_1) = 0,$$

$$(1 + 2S^{(0)}_{\text{osc, grav}})(m\ddot{x}_1 + kx_1) = 0.$$

Since $S^{(0)}_{\text{osc, grav}}$ is typically non-zero (and units suggest normalization):

$$m\ddot{x}_1 + kx_1 = 0,$$

$$x_1(t) = C\cos(\omega t) + D\sin(\omega t).$$

$$x(t) = \left(x_0 + \frac{mg}{k}\right)\cos(\omega t) + \frac{v_0}{\omega}\sin(\omega t) - \frac{mg}{k} + \gamma[C\cos(\omega t) + D\sin(\omega t)].$$

11.11 Determine C and D

Substitute back to order γ:

$$m\ddot{x}_1 + kx_1 = -2S^{(0)}_{\text{osc, grav}}(m\ddot{x}_0 + kx_0 + mg) - 2\gamma S^{(0)}_{\text{osc, grav}}(m\ddot{x}_1 + kx_1),$$

since $m\ddot{x}_0 + kx_0 + mg = 0$, the first term vanishes, and we need the next order or adjust $S_{\text{osc, grav}}$. Use:

$$m\ddot{x}_1 + kx_1 = -2S^{(0)}_{\text{osc, grav}}(m\ddot{x}_1 + kx_1),$$

$$m\ddot{x}_1 + kx_1(1 + 2S^{(0)}_{\text{osc, grav}}) = 0,$$

but this is approximate. Instead, match terms with $x_1(t) = C\sin(\omega t + \psi)$:

$$x(t) = x_0(t) + \gamma C\sin(\omega t + \psi),$$

recompute $S_{\text{osc, grav}}$ iteratively if needed. For simplicity, assume a phase shift:

$$C = \frac{2S^{(0)}_{\text{osc, grav}}}{m\omega}, \quad \psi = \text{adjusted via boundary conditions.}$$

11.12 Final Solution

$$x(t) \approx \left(x_0 + \frac{mg}{k}\right)\cos(\omega t) + \frac{v_0}{\omega}\sin(\omega t) - \frac{mg}{k} + \gamma C\sin(\omega t + \psi),$$

where C and ψ require numerical tuning based on $S^{(0)}_{\text{osc, grav}}$.

12 Action of Simple FAT Harmonic Oscillator in a Gravitational Field: Model α

We consider the Lagrangian for a simple harmonic oscillator in a gravitational field, defined as:

$$L_{\text{osc}} = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2,$$

$$L_{\text{grav}} = -mgx,$$

where m is the mass, k is the spring constant, g is gravitational acceleration, and $\dot{x} = \frac{dx}{dt}$. The corresponding actions are:

$$S_{\text{osc}} = \int_{t_1}^{t_2} L_{\text{osc}}\, dt,$$

$$S_{\text{grav}} = \int_{t_1}^{t_2} L_{\text{grav}}\, dt,$$

$$S_{\text{osc, grav}} = \int_{t_1}^{t_2} [L_{\text{osc}} + L_{\text{grav}}]\, dt = \int_{t_1}^{t_2} \left(\frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 - mgx\right) dt.$$

12.1 Step 1: Define the Modified Action

The FAT-modified action introduces a quadratic term in the gravitational action:

$$S_{\text{FAT, osc, grav},\alpha} = S_{\text{osc, grav}} + \alpha S_{\text{grav}}^2,$$

where:

$$S_{\text{grav}} = \int_{t_1}^{t_2} (-mgx)\, dt,$$

$$S_{\text{grav}}^2 = \left(\int_{t_1}^{t_2} (-mgx)\, dt \right)^2,$$

and α is a small coupling constant ($\alpha \ll 1$).

12.2 Step 2: Calculate $S_{\text{osc, grav}}$ and S_{grav}

$$S_{\text{osc, grav}} = \int_{t_1}^{t_2} \left(\frac{1}{2} m\dot{x}^2 - \frac{1}{2} kx^2 - mgx \right) dt,$$

$$S_{\text{grav}} = \int_{t_1}^{t_2} (-mgx)\, dt.$$

12.3 Step 3: Principle of Least Action

To derive the equations of motion, we apply the variational principle: $\delta S_{\text{FAT, osc, grav},\alpha} = 0$:

$$\delta S_{\text{FAT, osc, grav},\alpha} = \delta S_{\text{osc, grav}} + 2\alpha S_{\text{grav}} \delta S_{\text{grav}}.$$

12.3.1 Contribution from $S_{\text{osc, grav}}$

$$\delta S_{\text{osc, grav}} = \int_{t_1}^{t_2} [m\dot{x}\delta\dot{x} - kx\delta x - mg\delta x]\, dt,$$

integrate by parts ($\delta x(t_1) = \delta x(t_2) = 0$):

$$\int_{t_1}^{t_2} m\dot{x}\delta\dot{x}\, dt = [m\dot{x}\delta x]_{t_1}^{t_2} - \int_{t_1}^{t_2} m\ddot{x}\delta x\, dt = -\int_{t_1}^{t_2} m\ddot{x}\delta x\, dt,$$

$$\delta S_{\text{osc, grav}} = \int_{t_1}^{t_2} [-m\ddot{x} - kx - mg]\, \delta x\, dt,$$

$$\delta S_{\text{osc, grav}} = 0 \implies m\ddot{x} + kx + mg = 0.$$

12.3.2 Contribution from αS_{grav}^2

$$\delta S_{\text{grav}}^2 = 2 S_{\text{grav}} \delta S_{\text{grav}},$$

$$\delta S_{\text{grav}} = \int_{t_1}^{t_2} (-mg)\delta x\, dt,$$

$$\frac{\delta S_{\text{grav}}}{\delta x(t)} = -mg,$$

$$\frac{\delta S_{\text{grav}}^2}{\delta x(t)} = 2 \left(\int_{t_1}^{t_2} (-mgx)\, dt \right) (-mg) = 2m^2 g^2 \int_{t_1}^{t_2} x\, dt,$$

$$\alpha \frac{\delta S_{\text{grav}}^2}{\delta x(t)} = 2\alpha m^2 g^2 \int_{t_1}^{t_2} x\, dt.$$

12.3.3 Total Equation of Motion

$$\int_{t_1}^{t_2} \left[-m\ddot{x} - kx - mg + 2\alpha m^2 g^2 \int_{t_1}^{t_2} x\, dt \right] \delta x\, dt = 0,$$

$$m\ddot{x} + kx + mg - 2\alpha m^2 g^2 I = 0,$$

where:

$$I = \int_{t_1}^{t_2} x\, dt,$$

a constant over the interval $[t_1, t_2]$.

12.4 Step 4: Conclusion of Form of Equations of Motion

The modified equation:

$$m\ddot{x} + kx + mg - 2\alpha m^2 g^2 I = 0,$$

introduces a term $-2\alpha m^2 g^2 I$, representing the cumulative effect of the gravitational potential over time, scaled by α. This integro-differential equation suggests the dynamics depend on the entire trajectory $x(t)$ from t_1 to t_2.

12.5 Step 5: Perturbative Solution

Given α is small, we solve perturbatively with initial conditions $x(t_1) = x_0$, $\dot{x}(t_1) = v_0$.

12.5.1 Zeroth-Order Solution ($\alpha = 0$)

$$m\ddot{x} + kx + mg = 0,$$

homogeneous solution:

$$x_h(t) = A\cos(\omega t) + B\sin(\omega t), \quad \omega = \sqrt{\frac{k}{m}},$$

particular solution:

$$m\ddot{x}_p + kx_p + mg = 0, \quad kx_p + mg = 0, \quad x_p = -\frac{mg}{k},$$

$$x_0(t) = A\cos(\omega t) + B\sin(\omega t) - \frac{mg}{k}.$$

Apply initial conditions:

$$x_0(t_1) = A\cos(\omega t_1) + B\sin(\omega t_1) - \frac{mg}{k} = x_0,$$

$$A\cos(\omega t_1) + B\sin(\omega t_1) = x_0 + \frac{mg}{k},$$

$$\dot{x}_0(t) = -A\omega\sin(\omega t) + B\omega\cos(\omega t), \quad \dot{x}_0(t_1) = -A\omega\sin(\omega t_1) + B\omega\cos(\omega t_1) = v_0.$$

Solve:

$$A = \frac{\left(x_0 + \frac{mg}{k}\right)\cos(\omega t_1) + \frac{v_0}{\omega}\sin(\omega t_1)}{\cos^2(\omega t_1) + \sin^2(\omega t_1)} = x_0 + \frac{mg}{k},$$

$$B = \frac{\frac{v_0}{\omega}\cos(\omega t_1) - \left(x_0 + \frac{mg}{k}\right)\sin(\omega t_1)}{\cos^2(\omega t_1) + \sin^2(\omega t_1)} = \frac{v_0}{\omega},$$

(assuming $t_1 = 0$ for simplicity; adjust t_1 as needed):

$$x_0(t) = \left(x_0 + \frac{mg}{k}\right)\cos(\omega t) + \frac{v_0}{\omega}\sin(\omega t) - \frac{mg}{k}.$$

12.5.2 Compute I_0

$$I_0 = \int_{t_1}^{t_2} x_0(t)\,dt,$$

for $t_1 = 0$, $t_2 = T = \frac{2\pi}{\omega}$:

$$I_0 = \int_0^T \left[\left(x_0 + \frac{mg}{k}\right)\cos(\omega t) + \frac{v_0}{\omega}\sin(\omega t) - \frac{mg}{k}\right] dt,$$

$$= \left(x_0 + \frac{mg}{k}\right)\frac{1}{\omega}\sin(\omega t)\Big|_0^T + \frac{v_0}{\omega}\left(-\frac{1}{\omega}\cos(\omega t)\right)\Big|_0^T - \frac{mg}{k}t\Big|_0^T,$$

$$= 0 + 0 - \frac{mg}{k}\cdot\frac{2\pi}{\omega} = -\frac{2\pi mg}{k\omega}.$$

12.5.3 First-Order Perturbation

$$x(t) = x_0(t) + \alpha x_1(t),$$

$$m\ddot{x}_1 + kx_1 - 2\alpha m^2 g^2 I = 0,$$

to first order:

$$m\ddot{x}_1 + kx_1 = 2m^2 g^2 I_0,$$

$$m\ddot{x}_1 + kx_1 = 2m^2 g^2\left(-\frac{2\pi mg}{k\omega}\right) = -\frac{4\pi m^3 g^3}{k\omega},$$

particular solution:

$$kx_{1p} = -\frac{4\pi m^3 g^3}{k\omega}, \quad x_{1p} = -\frac{4\pi m^3 g^3}{k^2\omega},$$

homogeneous: $x_{1h} = C\cos(\omega t) + D\sin(\omega t)$,

$$x_1(t) = C\cos(\omega t) + D\sin(\omega t) - \frac{4\pi m^3 g^3}{k^2\omega}.$$

Initial conditions on $x_1(t)$: assume perturbation starts at zeroth-order conditions:

$$x_1(0) = 0, \quad \dot{x}_1(0) = 0,$$

$$C - \frac{4\pi m^3 g^3}{k^2\omega} = 0, \quad C = \frac{4\pi m^3 g^3}{k^2\omega},$$

$$\dot{x}_1(t) = -C\omega\sin(\omega t) + D\omega\cos(\omega t), \quad D\omega = 0, \quad D = 0,$$

$$x_1(t) = \frac{4\pi m^3 g^3}{k^2\omega}\cos(\omega t) - \frac{4\pi m^3 g^3}{k^2\omega}.$$

12.5.4 Final Solution

$$x(t) = \left(x_0 + \frac{mg}{k}\right)\cos(\omega t) + \frac{v_0}{\omega}\sin(\omega t) - \frac{mg}{k} + \alpha\left[\frac{4\pi m^3 g^3}{k^2\omega}(\cos(\omega t) - 1)\right].$$

12.6 Conclusion

The modified action introduces a constant shift proportional to α, reflecting the cumulative gravitational effect over $[t_1, t_2]$. The solution captures the harmonic oscillation adjusted by this perturbation, consistent with the FAT model α.

13 Path integral approximation

To approximate the path integral in quantum mechanics, the general idea is to evaluate the integral over all possible paths $x(t)$ that a particle can take, weighted by the exponential of the action $S[x]$ divided by $\hbar$. However, because the full path integral is often too complex to solve exactly, we use an approximation near the classical path $x_{\text{cl}}(t)$, where the action is minimized.

Step 1: Expand the Action Around the Classical Path

The classical path $x_{\text{cl}}(t)$ is the path that minimizes the action $S[x]$. For a small deviation $\delta x(t)$ from this classical path, we can expand the action $S[x]$ as a Taylor series around $x_{\text{cl}}(t)$:

$$x(t) = x_{\text{cl}}(t) + \delta x(t)$$

Substituting this into the action $S[x]$ gives:

$$S[x] = S[x_{\text{cl}} + \delta x] = S[x_{\text{cl}}] + \left.\frac{\delta S}{\delta x}\right|_{x_{\text{cl}}} \delta x + \frac{1}{2}\delta x \cdot \left.\frac{\delta^2 S}{\delta x^2}\right|_{x_{\text{cl}}} \cdot \delta x + \cdots$$

Here, the term $\left.\frac{\delta S}{\delta x}\right|_{x_{\text{cl}}}$ is the first derivative of the action, which vanishes for the classical path because $x_{\text{cl}}(t)$ satisfies the classical equations of motion. Thus, the action simplifies to:

$$S[x] \approx S[x_{\text{cl}}] + \frac{1}{2}\int dt\, dt'\, \delta x(t) \left.\frac{\delta^2 S}{\delta x(t)\delta x(t')}\right|_{x_{\text{cl}}} \delta x(t')$$

The second derivative $\frac{\delta^2 S}{\delta x(t)\delta x(t')}$ is often referred to as the Hessian matrix of the action and encodes information about the curvature of the action around the classical path.

Step 2: Approximating the Path Integral

The original path integral is:

$$\int e^{iS[x]/\hbar}\, \mathcal{D}x$$

Using the expansion of $S[x]$ around $x_{\text{cl}}(t)$, this becomes:

$$\int e^{iS[x]/\hbar}\, \mathcal{D}x = \int e^{i(S[x_{\text{cl}}] + \frac{1}{2}\delta x \cdot \partial_x^2 S \cdot \delta x)/\hbar}\, \mathcal{D}(x_{\text{cl}} + \delta x)$$

Since the path integral is taken over all paths $x(t)$, we can shift the measure of integration to be over the deviations $\delta x(t)$ instead:

$$\int e^{iS[x]/\hbar}\, \mathcal{D}x = e^{iS[x_{\text{cl}}]/\hbar} \int e^{i\frac{1}{2\hbar}\delta x \cdot \left.\frac{\delta^2 S}{\delta x^2}\right|_{x_{\text{cl}}} \cdot \delta x}\, \mathcal{D}\delta x$$

Step 3: Gaussian Approximation

The integral we now have is a Gaussian integral over the fluctuations $\delta x(t)$:

$$\int e^{i\delta x \cdot \delta_x^2 S \cdot \delta x/(2\hbar)}\, \mathcal{D}\delta x$$

Gaussian integrals of this form can be evaluated exactly. For a quadratic form in the exponent, the result of the Gaussian integral is proportional to the determinant of the inverse of the quadratic form's coefficient matrix (which in this case is related to the second derivative of S, i.e., the Hessian):

$$\int e^{i\frac{1}{2\hbar}\delta x\cdot H\cdot\delta x}\,\mathcal{D}\delta x \propto (\det H)^{-1/2}$$

Therefore, the path integral in the semiclassical approximation becomes:

$$\int e^{iS[x]/\hbar}\,\mathcal{D}x \approx e^{iS_{\text{cl}}/\hbar}\cdot(\text{determinant factor})$$

The "determinant factor" depends on the specific form of the Hessian $H = \frac{\delta^2 S}{\delta x(t)\delta x(t')}$, and for many physical problems, this factor can be computed or approximated explicitly.

13.1 The determinant factor

The determinant factor in the semiclassical approximation of the path integral is a key part of the calculation. Let's go through the details of how it's obtained.

Step 1: The Action Expansion

As mentioned earlier, the action $S[x]$ is expanded around the classical path $x_{\text{cl}}(t)$:

$$S[x] \approx S[x_{\text{cl}}] + \frac{1}{2}\int dt\,dt'\,\delta x(t)\,H(t,t')\,\delta x(t')$$

where the Hessian $H(t,t')$ is the second functional derivative of the action with respect to the path:

$$H(t,t') = \frac{\delta^2 S[x]}{\delta x(t)\delta x(t')}$$

Step 2: Gaussian Integral Over Fluctuations

The path integral is then approximated as:

$$\int e^{iS[x]/\hbar}\,\mathcal{D}x \approx e^{iS[x_{\text{cl}}]/\hbar}\int e^{i\frac{1}{2\hbar}\int dt\,dt'\,\delta x(t)H(t,t')\delta x(t')}\,\mathcal{D}(\delta x)$$

This integral is a Gaussian integral over the fluctuations $\delta x(t)$.

Step 3: Evaluating the Determinant

The Gaussian integral has the form:

$$\int e^{i\frac{1}{2\hbar}\delta x\cdot H\cdot\delta x}\,\mathcal{D}\delta x$$

This is a functional version of the standard Gaussian integral, where H is an operator (often called the "Hessian operator"). The result of this integral is:

$$\int e^{i\frac{1}{2\hbar}\delta x\cdot H\cdot\delta x}\,\mathcal{D}\delta x = \left(\det\left(\frac{H}{2\pi i\hbar}\right)\right)^{-1/2}$$

Step 4: The Semiclassical Approximation

So, the full semiclassical approximation of the path integral is given by:

$$\int e^{iS[x]/\hbar}\,\mathcal{D}x \approx e^{iS[x_{\text{cl}}]/\hbar}\cdot\left(\det\left(\frac{H}{2\pi i\hbar}\right)\right)^{-1/2}$$

13.2 Important Notes

- Normalization: The prefactor $(2\pi i\hbar)^{-1/2}$ arises from the normalization of the Gaussian integral.

- Boundary Conditions: The determinant $\det(H)$ depends on the boundary conditions imposed on the paths. In many cases, these are fixed by the physical context (e.g., the paths might be required to start and end at specific points).

13.3 Final Result

The exact determinant factor in the semiclassical approximation is:

$$\boxed{\left(\det\left(\frac{H}{2\pi i\hbar}\right)\right)^{-1/2}}$$

This factor accounts for the quantum fluctuations around the classical path. The determinant of the Hessian H can often be challenging to compute, but it is crucial for obtaining accurate results in quantum mechanics.

13.4 Conclusion

The semiclassical approximation involves expanding the action around the classical path, leading to a Gaussian integral over small fluctuations δx around this path. The path integral then simplifies to an exponential of the classical action, multiplied by a prefactor that involves the determinant of the second derivative of the action. This approach is often used when the classical action S_{cl} is large compared to $\hbar$, making quantum fluctuations small and justifying the use of a Gaussian approximation.

14 Partition function approximation using a generic modified gravity model

To calculate the path integral for the partition function in the context of a gravity model with the given action, we need to carefully consider the various components and the fields involved. Let's break down the action and identify the steps required to perform the path integral.

14.1 Action and Components

The action you provided is:

$$S = \int d^4x\sqrt{-g}\left\{\frac{R}{16\pi G_{\text{N}}} + \mathcal{L}_m - g^{\mu\nu}\frac{1}{2}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right\}$$

This action contains the following components:

1. Gravitational Sector:

- $g_{\mu\nu}$: The metric tensor of spacetime.
- R: The Ricci scalar, which depends on the metric tensor and its derivatives.
- G_{N}: Newton's gravitational constant.

2. Matter Sector:

- $\mathcal{L}_m$: The matter Lagrangian, which could include contributions from other fields (e.g., fermions, gauge fields).
- ϕ: A scalar field.
- $V(\phi)$: A potential for the scalar field.

14.2 Partition Function

The partition function Z is given by the path integral over all possible field configurations:

$$Z = \int \mathcal{D}g_{\mu\nu}\, \mathcal{D}\phi\, \mathcal{D}\psi\, e^{iS[g_{\mu\nu},\phi,\psi]/\hbar}$$

Here, ψ represents the matter fields that might be included in $\mathcal{L}_m$.

14.3 Step-by-Step Calculation

To proceed with the calculation, we need to:

1. Choose the Background Metric $g^{\mathrm{cl}}_{\mu\nu}$:

- Often, we choose a classical background metric that satisfies the Einstein field equations, such as flat space $g^{\mathrm{cl}}_{\mu\nu} = \eta_{\mu\nu}$ or a curved spacetime like Schwarzschild or de Sitter.

2. Expand Around the Classical Solution:

- Expand the metric and the scalar field around their classical solutions:

$$g_{\mu\nu} = g^{\mathrm{cl}}_{\mu\nu} + h_{\mu\nu}$$

$$\phi = \phi_{\mathrm{cl}} + \delta\phi$$

- Here, $h_{\mu\nu}$ and $\delta\phi$ are small perturbations around the classical solutions $g^{\mathrm{cl}}_{\mu\nu}$ and ϕ_{cl}.

3. Expand the Action to Quadratic Order:

- The action is expanded to quadratic order in the fluctuations $h_{\mu\nu}$ and $\delta\phi$.

- The quadratic expansion for the gravitational sector would involve terms like:

$$\frac{\delta^2 S}{\delta g_{\mu\nu}(x)\delta g_{\alpha\beta}(x')}$$

- Similarly, for the scalar field:

$$\frac{\delta^2 S}{\delta\phi(x)\delta\phi(x')}$$

4. Compute the Hessians:

- Calculate the Hessian operators for the metric perturbations and scalar field:

$$H_{g_{\mu\nu},g_{\alpha\beta}}(x,x') = \frac{\delta^2 S}{\delta g_{\mu\nu}(x)\delta g_{\alpha\beta}(x')}$$

$$H_{\phi,\phi}(x,x') = \frac{\delta^2 S}{\delta\phi(x)\delta\phi(x')}$$

5. Evaluate the Gaussian Path Integral:

- The path integral over the fluctuations $h_{\mu\nu}$ and $\delta\phi$ can be evaluated as Gaussian integrals:

$$\int \mathcal{D}h_{\mu\nu}\, \mathcal{D}\delta\phi\, e^{iS[h_{\mu\nu},\delta\phi]/\hbar} \approx \left(\det H_{g_{\mu\nu}}\right)^{-1/2} \left(\det H_{\phi}\right)^{-1/2}$$

- The determinants $\det H_{g_{\mu\nu}}$ and $\det H_{\phi}$ must be computed or estimated. These determinants account for quantum fluctuations around the classical background.

6. Faddeev-Popov Ghosts (for Gravity):

- In gauge theories, including gravity, we must include ghost fields to properly account for gauge fixing and the correct measure of the path integral. This typically involves introducing Faddeev-Popov ghosts in the gravitational sector.

$$\text{Ghost contribution: } \int \mathcal{D}b\,\mathcal{D}c\, e^{iS_{\text{ghost}}[b,c]/\hbar}$$

7. Final Expression for the Partition Function:

- The full partition function, including the semiclassical approximation and ghost contributions, is:

$$Z \approx e^{iS[g^{\text{cl}}_{\mu\nu},\phi_{\text{cl}}]/\hbar} \left(\det H_{g_{\mu\nu}}\right)^{-1/2} \left(\det H_\phi\right)^{-1/2} \left(\det H_{\text{ghost}}\right)$$

14.4 Summary of Needed Functions

To calculate the path integral explicitly, you will need:

1. Classical Solutions:

- $g^{\text{cl}}_{\mu\nu}$ and ϕ_{cl}, which are solutions to the classical field equations.

2. Hessian Operators:

- $H_{g_{\mu\nu}}$ and H_ϕ, the second variations of the action with respect to the metric and scalar field.

3. Ghost Action:

- The action S_{ghost} and the corresponding Hessian H_{ghost} for the Faddeev-Popov ghosts.

14.5 Next Steps

By providing the specific background (e.g., flat space, Schwarzschild, etc.) and any simplifications or assumptions (such as specific forms of $\mathcal{L}_m$ or $V(\phi)$), we could proceed to calculate the determinants and the path integral more explicitly.

15 Partition function approximation using a specific modified gravity model: model A

Let's proceed with the calculation under the assumptions that the matter Lagrangian $\mathcal{L}_m = 0$ and the potential $V(\phi) = 0$. This simplifies the action significantly, focusing primarily on the gravitational sector and the kinetic term of the scalar field.

15.1 Simplified Action

Given $\mathcal{L}_m = 0$ and $V(\phi) = 0$, the action reduces to:

$$S = \int d^4x \sqrt{-g} \left\{ \frac{R}{16\pi G_{\text{N}}} - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \right\}$$

This action consists of:

- Gravitational Part: $S_g = \int d^4x \sqrt{-g} \frac{R}{16\pi G_{\text{N}}}$
- Scalar Field Part: $S_\phi = \int d^4x \sqrt{-g} \left(-\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi\right)$

15.1.1 Step 1: Classical Background Solutions

- Gravitational Sector.

  Let's choose a simple classical background for the metric $g_{\mu\nu}^{\text{cl}}$. A common choice is flat spacetime:

$$g_{\mu\nu}^{\text{cl}} = \eta_{\mu\nu}$$

  Here, $\eta_{\mu\nu}$ is the Minkowski metric, and the Ricci scalar R for flat spacetime is zero:

$$R[g_{\mu\nu}^{\text{cl}}] = 0$$

  Thus, the classical gravitational action is:

$$S_g[g_{\mu\nu}^{\text{cl}}] = 0$$

- Scalar Field.

  Assume that the classical scalar field ϕ_{cl} is also trivial (e.g., constant or zero everywhere):

$$\phi_{\text{cl}} = 0$$

  Thus, the classical scalar field action is:

$$S_\phi[\phi_{\text{cl}}] = 0$$

15.1.2 Step 2: Fluctuations and Quadratic Expansion

Now we introduce small fluctuations around the classical solutions:
- Metric: $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$
- Scalar field: $\phi = \phi_{\text{cl}} + \delta\phi = \delta\phi$

Let's expand the action to quadratic order in these fluctuations, sector-by-sector.

- Gravitational Sector.

  The quadratic part of the gravitational action is obtained by expanding the Einstein-Hilbert action to second order in the metric perturbation $h_{\mu\nu}$:

$$S_g \approx \frac{1}{16\pi G_{\text{N}}} \int d^4x \sqrt{-g} \left\{ \frac{1}{2} h_{\mu\nu} \mathcal{O}^{\mu\nu,\alpha\beta} h_{\alpha\beta} \right\}$$

  where $\mathcal{O}^{\mu\nu,\alpha\beta}$ is a differential operator that acts on $h_{\mu\nu}$. In flat space, this operator simplifies, and the detailed form of $\mathcal{O}$ depends on the gauge choice. A common choice is the de Donder (or harmonic) gauge:

$$\partial_\mu h^{\mu\nu} - \frac{1}{2}\partial^\nu h = 0$$

  In this gauge, the quadratic action becomes:

$$S_g \approx \frac{1}{16\pi G_\mathrm{N}} \int d^d x \, \frac{1}{2} h_{\mu\nu} \left(-\Box\right) h^{\mu\nu}$$

where $\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu$ is the d'Alembertian operator.

- Scalar Field Sector.

  The scalar field action to quadratic order is:

$$S_\phi \approx \int d^d x \, \frac{1}{2} \delta\phi \left(-\Box\right) \delta\phi$$

15.1.3 Step 3: Path Integral and Gaussian Integrals

The partition function is now the product of path integrals over the metric and scalar field fluctuations:

$$Z \approx \int \mathcal{D}h_{\mu\nu} \, \mathcal{D}\delta\phi \, e^{i(S_g + S_\phi)/\hbar}$$

Substituting the quadratic actions:

$$Z \approx \int \mathcal{D}h_{\mu\nu} \, \mathcal{D}\delta\phi \, e^{i \int d^4x \left(\frac{1}{32\pi G_\mathrm{N}} h_{\mu\nu}(-\Box)h^{\mu\nu} + \frac{1}{2}\delta\phi(-\Box)\delta\phi \right)/\hbar}$$

This results in two Gaussian integrals:

- Metric Fluctuation Integral:

$$\int \mathcal{D}h_{\mu\nu} \, e^{i \int d^4x \, \frac{1}{32\pi G_\mathrm{N}} h_{\mu\nu}(-\Box)h^{\mu\nu}/\hbar} = \left(\det\left(\frac{-\Box}{32\pi G_\mathrm{N} \cdot 2\pi i\hbar} \right) \right)^{-1/2}$$

- Scalar Field Fluctuation Integral:

$$\int \mathcal{D}\delta\phi \, e^{i \int d^4x \, \frac{1}{2}\delta\phi(-\Box)\delta\phi/\hbar} = \left(\det\left(\frac{-\Box}{2\pi i\hbar} \right) \right)^{-1/2}$$

- Ghost Contribution (Faddeev-Popov Determinant)

  For the gravitational sector, we also need to account for the Faddeev-Popov ghosts that arise from gauge fixing. In the harmonic gauge, the ghost determinant $\det H_\mathrm{ghost}$ is also related to the d'Alembertian operator:

$$\det H_\mathrm{ghost} = \left(\det(-\Box)\right)$$

- Final Partition Function

  Combining all these factors, the partition function becomes:

$$Z \approx e^{iS_\mathrm{cl}/\hbar} \cdot \left(\det\left(\frac{-\Box}{32\pi G_\mathrm{N} \cdot 2\pi i\hbar} \right) \right)^{-1/2} \cdot \left(\det\left(\frac{-\Box}{2\pi i\hbar} \right) \right)^{-1/2} \cdot \left(\det(-\Box)\right)$$

15.1.4 Simplification

- The ghost determinant cancels out one of the metric determinant factors.
  - The remaining expression is:

$$Z \approx e^{iS_{\text{cl}}/\hbar} \cdot (\det(-\Box))^{-1/2} \cdot (\det(-\Box/(32\pi G_{\text{N}})))^{-1/2}$$

Further simplifications would depend on evaluating or estimating these determinants explicitly, which typically involves regularization techniques (like zeta-function regularization or dimensional regularization).

15.1.5 Final expression of partition function

The final expression for the partition function in the simplified model is:

$$Z \approx e^{iS_{\text{cl}}/\hbar} \cdot (\det(-\Box))^{-1/2} \cdot (\det(-\Box/(32\pi G_{\text{N}})))^{-1/2}$$

This result shows that the path integral is dominated by the classical action S_{cl}, with quantum fluctuations contributing through the determinants of the differential operator $-\Box$.

15.2 Determinant evaluations using ζ-function regularization and dimensional regularization

Let's evaluate the determinants using both zeta-function regularization and dimensional regularization.

15.2.1 ζ-Function Regularization

Zeta-function regularization is a powerful tool to handle infinite products or sums that arise in quantum field theory, especially in evaluating determinants of differential operators like the d'Alembertian $-\Box$.

- General Idea

  The determinant of a differential operator $\mathcal{O}$ can be formally expressed as:

$$\ln \det \mathcal{O} = \operatorname{Tr} \ln \mathcal{O}$$

  This expression often diverges, so we regularize it using the zeta function $\zeta(s)$ associated with $\mathcal{O}$:

$$\zeta(s) = \sum_n \lambda_n^{-s}$$

  where λ_n are the eigenvalues of $\mathcal{O}$. The regularized determinant is then defined as:

$$\det \mathcal{O} = e^{-\zeta'(0)}$$

- Applying to the d'Alembertian

  For the operator $\mathcal{O} = -\Box$, the zeta function is:

$$\zeta(s) = \sum_n \lambda_n^{-s}$$

where λ_n are the eigenvalues of the operator $-\Box$. For a scalar field in d spacetime dimensions, the eigenvalues are typically $\lambda_n = k^2$, where k is the momentum.

For a flat spacetime of volume V_d in d dimensions:

$$\zeta(s) = \frac{V_d}{(2\pi)^d} \int d^d k \, (k^2)^{-s}$$

Using spherical coordinates and integrating over the angular parts:

$$\zeta(s) = \frac{V_d \pi^{d/2}}{(2\pi)^d \Gamma(d/2)} \int_0^\infty dk \, k^{d-1} k^{-2s} = \frac{V_d \pi^{d/2}}{(2\pi)^d \Gamma(d/2)} \int_0^\infty dk \, k^{d-1-2s}$$

This integral converges for $s > d/2$ and gives:

$$\zeta(s) = \frac{V_d \pi^{d/2}}{(2\pi)^d \Gamma(d/2)} \frac{\Gamma\left(\frac{d}{2} - s\right)}{\Gamma\left(\frac{d}{2}\right)}$$

To get the determinant, we need:

$$\zeta'(0) = -\frac{d}{ds}\zeta(s)\Big|_{s=0}$$

After differentiating, we find:

$$\ln\det(-\Box) = -\zeta'(0) = (\text{finite part of the regularized determinant})$$

The exact value depends on the dimension d, but for even dimensions, this approach gives a finite contribution that can be computed explicitly.

- Applying to the Determinant in 4 Dimensions

  In 4-dimensional spacetime ($d = 4$):

$$\zeta(s) = \frac{V_4}{(2\pi)^4} \frac{\pi^2}{\Gamma(2)} \frac{\Gamma(2-s)}{\Gamma(2)} \zeta(s)$$

$$\zeta'(0) = -\frac{V_4}{(2\pi)^4} \frac{\pi^2}{\Gamma(2)} \,(\text{derivative of the Gamma function at } s = 0)$$
$$\zeta'(0) = -\frac{V_4}{(2\pi)^4} \frac{\pi^2}{\Gamma(2)} \partial_s \Gamma(s)|_{s=0}$$

This yields:

$$\ln\det(-\Box) = (\text{finite part})$$

Thus:

$$\det(-\Box) \propto \exp\left(\frac{\zeta'(0)}{2}\right)$$

For practical purposes, in flat spacetime, this determinant is often absorbed into the overall normalization.

15.2.2 Dimensional Regularization

Dimensional regularization is another widely used technique where one analytically continues the number of spacetime dimensions d from an integer to a complex number $d = 4 - \epsilon$.

- General Idea

  In this method, the determinant $\det(-\Box)$ is computed by integrating over momentum space in d dimensions:

$$\ln\det(-\Box) = \mathrm{Tr}\ln(-\Box) = V_d \int \frac{d^d k}{(2\pi)^d} \ln(k^2)$$

  Performing the momentum integration in $d = 4 - \epsilon$ dimensions:

$$\int \frac{d^d k}{(2\pi)^d} \ln(k^2) = \frac{1}{(4\pi)^{d/2}} \Gamma\left(-\frac{\epsilon}{2}\right) \frac{1}{\epsilon}$$

  Expanding the Gamma function $\Gamma(-\epsilon/2)$ near $\epsilon = 0$:

$$\Gamma\left(-\frac{\epsilon}{2}\right) \approx -\frac{2}{\epsilon} - \gamma + \mathcal{O}(\epsilon)$$

  The logarithm of the determinant then contains both a pole $1/\epsilon$ and a finite part. The pole is typically renormalized away, leaving a finite term that can be computed.

- Evaluating for $-\Box$

  After renormalization, the result is:

$$\det(-\Box) \propto \exp\left(\text{finite part of the integral}\right)$$

In 4 dimensions, this yields a similar finite factor to the zeta-function approach, depending on the renormalization scheme.

15.3 Final Result

After applying these regularization techniques, the final form of the partition function Z in flat spacetime will include a finite determinant factor that depends on the chosen regularization scheme. In many cases, this factor is either absorbed into the normalization or computed as part of the effective action in quantum field theory. The key takeaway is that both zeta-function regularization and dimensional regularization provide consistent methods to handle the infinities in these determinants.

16 Traditional quantum field theory and the partition function

In quantum mechanics and quantum field theory, the partition function often appears in the form of the expression $e^{iS/\hbar}$, where S is the action, $\hbar$ is the reduced Planck constant, and i is the imaginary unit. This expression is central to the path integral formulation of quantum mechanics.

To approximate or analyze $e^{iS/\hbar}$, especially in practical calculations, several methods can be employed depending on the context:

16.1 Stationary Phase Approximation

When $\hbar$ is small, $e^{iS/\hbar}$ oscillates rapidly, and the integral over paths can be approximated by finding the contributions from paths where the action S is stationary (i.e., where $\delta S = 0$).

- Action and Classical Path: In this approximation, the dominant contribution to the path integral comes from the classical path or paths where the action S is extremized (i.e., solutions to the Euler-Lagrange equations).

- Approximation: Near the classical path $x_{\text{cl}}(t)$, the action S is approximated as:

$$S \approx S_{\text{cl}} + \frac{1}{2}\delta x \cdot \partial_x^2 S \cdot \delta x$$

Here, δx represents deviations from the classical path.

- Resulting Approximation: The integral is then approximated using a Gaussian integral around the classical path. This leads to the semiclassical approximation:

$$\int e^{iS[x]/\hbar}\,\mathcal{D}x \approx e^{iS_{\text{cl}}/\hbar} \int e^{i\delta x\cdot\partial_x^2 S\cdot\delta x/(2\hbar)}\,\mathcal{D}\delta x$$

16.2 Path Integral Formulation

In the path integral formulation, the partition function Z is given by summing over all possible paths with the weight $e^{iS[x]/\hbar}$:

$$Z = \int e^{iS[x]/\hbar}\,\mathcal{D}x$$

- Approximation: When $\hbar$ is small, the integral is dominated by paths near the classical path. Thus, the path integral can be approximated by evaluating the action along the classical trajectory, as discussed earlier.

16.3 High-Energy and Large-Volume Limits

For high-energy or large-volume limits, S can become large, and the exponential factor $e^{iS/\hbar}$ can lead to rapid oscillations. In such cases, the dominant contributions come from regions where the classical approximation applies or where S has stationary points.

- Resulting Approximation: In these limits, the integral can be approximated by evaluating the contributions from regions where the phase varies slowly, which often corresponds to classical solutions.

16.4 Summary

To summarize, the approximation of $e^{iS/\hbar}$ depends on the context and the value of $\hbar$. Common approaches include:

- Stationary Phase Approximation: Dominant contribution comes from paths where S is stationary.

- Path Integral Formulation: Approximate by evaluating the action along classical paths or regions of slow phase variation.

These approximations are essential in practical calculations in quantum mechanics and quantum field theory.

17 FAT quantum field theory and the partition function

Let's consider a partition function of the form:

$$e^{i(S+\alpha S^2)/\hbar}$$

where S is the action, α is a parameter, and $\hbar$ is the reduced Planck constant. This expression includes a quadratic term in S, which modifies the standard exponential term in the partition function.

17.1 Expression Analysis

The partition function is:

$$Z = \int e^{i(S+\alpha S^2)/\hbar}\,\mathcal{D}x$$

This includes an additional term αS^2 in the exponent. We need to analyze how this affects the evaluation of the partition function, especially in the context of path integrals or similar quantum mechanical frameworks.

17.2 Approximations and Solutions

Let's examine the behavior of this expression using various approximations:

17.2.1 Stationary Phase Approximation

In the stationary phase approximation, we focus on the contributions from paths where the action S is stationary. Here's how the analysis proceeds:

- Classical Path Contribution: Near the classical path x_{cl}, we approximate the action S around the classical path. The action S can be expanded as:

$$S = S_{\text{cl}} + \frac{1}{2}\delta x \cdot \partial_x^2 S \cdot \delta x$$

- Quadratic Term Contribution: The additional term αS^2 is expanded as:

$$\alpha S^2 = \alpha \left(S_{\text{cl}}^2 + 2S_{\text{cl}} \cdot \frac{1}{2}\delta x \cdot \partial_x^2 S \cdot \delta x + \left(\frac{1}{2}\delta x \cdot \partial_x S \cdot \delta x \right)^2 \right)$$

- Total Contribution: The integral becomes:

$$Z = e^{i\left(S_{\text{cl}}+\alpha S_{\text{cl}}^2\right)/\hbar} \int e^{i\left(\frac{1}{2}\delta x\cdot\partial_x^2 S\cdot\delta x+\alpha 2S_{\text{cl}}\cdot\frac{1}{2}\delta x\cdot\partial_x^2 S\cdot\delta x+\alpha\left(\frac{1}{2}\delta x\cdot\partial_x^2 S\cdot\delta x\right)^2\right)/\hbar}\mathcal{D}\delta x \tag{17.1}$$

This involves a Gaussian integral with an additional term coming from αS^2.

17.2.2 Path Integral Formulation

In the path integral formulation, we evaluate:

$$Z = \int e^{i(S+\alpha S^2)/\hbar}\mathcal{D}x$$

- Approximation with Small α: If α is small, the αS^2 term can be treated as a perturbation to the standard action. The dominant contribution is still from paths where S is stationary, but with an additional correction term proportional to α.

- Perturbative Expansion: For small α, we can expand the partition function in powers of α:

$$Z \approx \int e^{iS/\hbar}\left[1+\frac{i\alpha S}{\hbar}+\frac{(i\alpha S)^2}{2\hbar^2}+\cdots\right]\mathcal{D}x$$

The integrals for the higher-order terms involve additional calculations and corrections.

17.2.3 High-Energy or Large-Volume Limits

When $\hbar$ is small, or if the action S is large:

- Dominant Contribution: The term $e^{i(S+\alpha S^2)/\hbar}$ will be dominated by paths where the action S is large and the phase oscillates rapidly.

- Gaussian Approximation: Near the classical solution, the quadratic term αS^2 leads to corrections that can be treated as modifications to the Gaussian integral:

$$Z = e^{i(S_{\text{cl}}+\alpha S_{\text{cl}}^2)/\hbar}\cdot\int e^{i\delta x\cdot\partial_x^2 S\cdot\delta x/(2\hbar)}\, e^{i\alpha S_{\text{cl}}\delta x\cdot\partial_x^2 S\cdot\delta x/\hbar}\, e^{i\alpha\left(\frac{1}{2}\delta x\cdot\partial_x^2 S\cdot\delta x\right)^2/\hbar}\mathcal{D}\delta x$$

17.3 Summary

The expression $e^{i(S+\alpha S^2)/\hbar}$ introduces a quadratic term in the action, which affects the calculation of the partition function in several ways:

- Stationary Phase Approximation: Dominant contributions come from paths where the action S is stationary, with additional corrections from αS^2.

- Path Integral Formulation: For small α, the effect of the quadratic term can be treated as a perturbation.

- High-Energy Limits: The behavior is dominated by the classical path, with corrections from the quadratic term affecting the overall result.

The additional term αS^2 modifies the phase and can lead to interesting new features in the analysis, such as altered oscillatory behavior or additional corrections to the path integral.

18 FAT quantum gravity field theory via the partition functor

Let's analyze a partition function of the form:

$$e^{i(S[g(x),\phi(x)]+\alpha S[g(x),\phi(x)]^2)/\hbar}$$

where $S[g(x), \phi(x)]$ represents the action as a function of fields or functions $g(x)$ and $\phi(x)$, and α is a parameter. The action S is now dependent on fields or functions $g(x)$ and $\phi(x)$, making the analysis more complex compared to the simple case where S is just a function of x.

18.1 Expression Analysis

The partition function can be written as:

$$Z = \int e^{i(S[g(x),\phi(x)]+\alpha S[g(x),\phi(x)]^2)/\hbar}\,\mathcal{D}g\,\mathcal{D}\phi$$

where $\mathcal{D}g$ and $\mathcal{D}\phi$ denote the path integrals over the fields $g(x)$ and $\phi(x)$, respectively.

18.2 Approximations and Solutions

18.2.1 Stationary Phase Approximation

In the stationary phase approximation, we focus on configurations where the action S is stationary. This is similar to the previous case but involves the fields $g(x)$ and $\phi(x)$.

- Classical Configuration: We look for configurations $g_{\text{cl}}(x)$ and $\phi_{\text{cl}}(x)$ that extremize the action S. The classical configuration satisfies the equations:

$$\frac{\delta S[g(x), \phi(x)]}{\delta g(x)} = 0$$
$$\frac{\delta S[g(x), \phi(x)]}{\delta \phi(x)} = 0$$

- Quadratic Term Contribution: Near the classical configuration, the action S can be expanded:

$$S[g(x), \phi(x)] \approx S_{\text{cl}} + \frac{1}{2}\left(\delta g \cdot \partial_g^2 S \cdot \delta g + \delta\phi \cdot \partial_\phi^2 S \cdot \delta\phi\right)$$

and:

$$\alpha S[g(x), \phi(x)]^2 \approx \alpha \left\{ S_{\text{cl}} + \frac{1}{2}\left(\delta g \cdot \partial_g^2 S \cdot \delta g + \delta\phi \cdot \partial_\phi^2 S \cdot \delta\phi\right) \right\}^2 \quad (18.1)$$

$$\approx \alpha \left\{ S_{\text{cl}}^2 + 2S_{\text{cl}}\frac{1}{2}\left(\delta g \cdot \partial_g^2 S \cdot \delta g + \delta\phi \cdot \partial_\phi^2 S \cdot \delta\phi\right) + \text{higher order terms} \right\} \quad (18.2)$$

Note that we ignore mixing terms such as :

$$\text{Term} = \frac{1}{2}\left(\delta g \cdot \partial_g \partial_\phi S \cdot \delta\phi\right) .$$

and we also ignore higher order terms, such as δX^4, while we keep terms with δX^2.

To approximate the integral for the partition function given the quadratic approximation for $S[g(x), \phi(x)]$ and the quadratic correction term, we follow these steps:

18.3 Approximate the integral for the partition function

The partition function is:

$$Z = \int e^{i\left(S[g(x),\phi(x)]+\alpha S[g(x),\phi(x)]^2\right)/\hbar}\,\mathcal{D}g\,\mathcal{D}\phi$$

We approximate $S[g(x),\phi(x)]$ as:

$$S[g(x),\phi(x)] \approx S_{\text{cl}} + \frac{1}{2}\left(\delta g\cdot\partial_g^2 S\cdot\delta g + \delta\phi\cdot\partial_\phi^2 S\cdot\delta\phi\right)$$

The quadratic correction term is:

$$\alpha S[g(x),\phi(x)]^2 \approx \alpha\left\{S_{\text{cl}} + \frac{1}{2}\left(\delta g\cdot\partial_g^2 S\cdot\delta g + \delta\phi\cdot\partial_\phi^2 S\cdot\delta\phi\right)\right\}^2$$

Expanding this term:

$$\alpha S[g(x),\phi(x)]^2 \approx \alpha\left\{S_{\text{cl}}^2 + 2S_{\text{cl}}\cdot\frac{1}{2}\left(\delta g\cdot\partial_g^2 S\cdot\delta g + \delta\phi\cdot\partial_\phi^2 S\cdot\delta\phi\right) + \text{higher order terms}\right\} \tag{18.3}$$

$$\approx \alpha\left\{S_{\text{cl}}^2 + S_{\text{cl}}\left(\delta g\cdot\partial_g^2 S\cdot\delta g + \delta\phi\cdot\partial_\phi^2 S\cdot\delta\phi\right) + \text{higher order terms}\right\} \tag{18.4}$$

18.3.1 Combining Terms

Combine the approximated action S and the quadratic correction term:

$$S[g(x),\phi(x)] + \alpha S[g(x),\phi(x)]^2 \approx \left(S_{\text{cl}} + \frac{1}{2}\left(\delta g\cdot\partial_g^2 S\cdot\delta g + \delta\phi\cdot\partial_\phi^2 S\cdot\delta\phi\right)\right) + \alpha\left(S_{\text{cl}}^2 + S_{\text{cl}}\left(\delta g\cdot\partial_g^2 S\cdot\delta g + \delta\phi\cdot\partial_\phi^2 S\cdot\delta\phi\right)\right) \tag{18.5}$$

$$= S_{\text{cl}} + \alpha S_{\text{cl}}^2 + \frac{1}{2}\left(\delta g\cdot\partial_g^2 S\cdot\delta g + \delta\phi\cdot\partial_\phi^2 S\cdot\delta\phi\right) + \alpha S_{\text{cl}}\left(\delta g\cdot\partial_g^2 S\cdot\delta g + \delta\phi\cdot\partial_\phi^2 S\cdot\delta\phi\right)$$

18.3.2 Approximating the Integral

The integral becomes:

$$Z \approx \int e^{i\left(S_{\text{cl}}+\alpha S_{\text{cl}}^2+\frac{1}{2}\left(\delta g\cdot\partial_g^2 S\cdot\delta g+\delta\phi\cdot\partial_\phi^2 S\cdot\delta\phi\right)+\alpha S_{\text{cl}}\left(\delta g\cdot\partial_g^2 S\cdot\delta g+\delta\phi\cdot\partial_\phi^2 S\cdot\delta\phi\right)\right)/\hbar}\,\mathcal{D}\delta g\,\mathcal{D}\delta\phi$$

Simplify the exponent:

$$Z \approx e^{i\left(S_{\text{cl}}+\alpha S_{\text{cl}}^2\right)/\hbar}\int e^{i\left(\frac{1}{2}\left(\delta g\cdot\partial_g^2 S\cdot\delta g+\delta\phi\cdot\partial_\phi^2 S\cdot\delta\phi\right)+\alpha S_{\text{cl}}\left(\delta g\cdot\partial_g^2 S\cdot\delta g+\delta\phi\cdot\partial_\phi^2 S\cdot\delta\phi\right)\right)/\hbar}\,\mathcal{D}\delta g\,\mathcal{D}\delta\phi \tag{18.6}$$

$$\approx e^{i\left(S_{\text{cl}}+\alpha S_{\text{cl}}^2\right)/\hbar}\int e^{i\left(\frac{1}{2}+\alpha S_{\text{cl}}\right)\left(\delta g\cdot\partial_g^2 S\cdot\delta g+\delta\phi\cdot\partial_\phi^2 S\cdot\delta\phi\right)/\hbar}\,\mathcal{D}\delta g\,\mathcal{D}\delta\phi \tag{18.7}$$

18.3.3 Gaussian Integration

If δg and $\delta\phi$ are small perturbations around the classical configuration, the integral can be approximated by Gaussian integration. Assuming $\partial_g^2 S$ and $\partial_\phi^2 S$ are the coefficients of the quadratic terms, the integral over these perturbations can be performed using standard results from Gaussian integrals.

Thus, the final result is:

$$Z \approx e^{i\left(S_{\mathrm{cl}}+\alpha S_{\mathrm{cl}}^2\right)/\hbar} \cdot \left[\mathrm{Det}\left(\partial_g^2 S\right) \cdot \mathrm{Det}\left(\partial_\phi^2 S\right)\right]^{-1/2} \cdot \left(\frac{2\pi\hbar i^{-1}}{\left(\frac{1}{2}+\alpha S_{\mathrm{cl}}\right)}\right)^{\mathrm{dim}/2}$$

Here, Det denotes the determinant, and the eigenvalues come from the matrix of second derivatives.

18.3.4 Path Integral Formulation

In the path integral formulation, we evaluate:

$$Z = \int e^{i(S[g(x),\phi(x)]+\alpha S[g(x),\phi(x)]^2)/\hbar}\, \mathcal{D}g\, \mathcal{D}\phi$$

- Approximation with Small α: If α is small, the term αS^2 can be treated as a perturbation. The dominant contribution still comes from configurations where S is stationary, with additional corrections due to the quadratic term.

- Perturbative Expansion: For small α, expand the partition function in powers of α:

$$Z \approx \int e^{iS[g(x),\phi(x)]/\hbar} \left[1 + \frac{i\alpha S[g(x),\phi(x)]}{\hbar} + \frac{(i\alpha S[g(x),\phi(x)])^2}{2\hbar^2} + \cdots\right] \mathcal{D}g\, \mathcal{D}\phi$$

The higher-order terms involve integrals over fields and can be computed perturbatively.

18.3.5 High-Energy or Large-Volume Limits

When $\hbar$ is small or S is large:

- Dominant Contribution: The term $e^{i(S+\alpha S^2)/\hbar}$ will be dominated by configurations where S is large and the phase oscillates rapidly. The quadratic term αS^2 affects the overall behavior but does not change the dominant contribution from the classical solution.

- Gaussian Approximation: Near the classical solution, the integral can be approximated by evaluating the dominant configurations and incorporating corrections due to the quadratic term:

$$Z = \int e^{i(S[g(x),\phi(x)]+\alpha S[g(x),\phi(x)]^2)/\hbar}\, \mathcal{D}g\, \mathcal{D}\phi \tag{18.8}$$

$$Z \approx e^{i(S_{\mathrm{cl}}+\alpha S_{\mathrm{cl}}^2)/\hbar} \cdot \int e^{i\left(\frac{1}{2}\delta g\cdot\partial_g^2 S\cdot\delta g+\frac{1}{2}\delta\phi\cdot\partial_\phi^2 S\cdot\delta\phi\right)/(2\hbar)}\, e^{i\alpha\left(\frac{1}{2}\delta g\cdot\partial_g^2 S\cdot\delta g+\frac{1}{2}\delta\phi\cdot\partial_\phi^2 S\cdot\delta\phi\right)^2/\hbar}\, \mathcal{D}\delta g\, \mathcal{D}\delta\phi \tag{18.9}$$

$$\approx e^{i\left(S_{\mathrm{cl}}+\alpha S_{\mathrm{cl}}^2\right)/\hbar} \int e^{i\left(\frac{1}{2}+\alpha S_{\mathrm{cl}}\right)\left(\delta g\cdot\partial_g^2 S\cdot\delta g+\delta\phi\cdot\partial_\phi^2 S\cdot\delta\phi\right)/\hbar}\, \mathcal{D}\delta g\, \mathcal{D}\delta\phi \tag{18.10}$$

$$\approx e^{i\left(S_{\mathrm{cl}}+\alpha S_{\mathrm{cl}}^2\right)/\hbar} \cdot \left[\mathrm{Det}\left(\partial_g^2 S\right) \cdot \mathrm{Det}\left(\partial_\phi^2 S\right)\right]^{-1/2} \cdot \left(\frac{2\pi\hbar i^{-1}}{\left(\frac{1}{2}+\alpha S_{\mathrm{cl}}\right)}\right)^{\mathrm{dim}/2} \tag{18.11}$$

18.4 Summary

The partition function can be estimated as:

$$Z = \int e^{i(S[g(x),\phi(x)]+\alpha S[g(x),\phi(x)]^2)/\hbar}\,\mathcal{D}g\,\mathcal{D}\phi \tag{18.12}$$

$$Z \approx e^{i(S_{\rm cl}+\alpha S_{\rm cl}^2)/\hbar} \cdot \int e^{i\left(\frac{1}{2}\delta g\cdot\partial_g^2 S\cdot\delta g+\frac{1}{2}\delta\phi\cdot\partial_\phi^2 S\cdot\delta\phi\right)/(2\hbar)}\, e^{i\alpha\left(\frac{1}{2}\delta g\cdot\partial_g^2 S\cdot\delta g+\frac{1}{2}\delta\phi\cdot\partial_\phi^2 S\cdot\delta\phi\right)^2/\hbar}\,\mathcal{D}\delta g\,\mathcal{D}\delta\phi \tag{18.13}$$

$$\approx e^{i\left(S_{\rm cl}+\alpha S_{\rm cl}^2\right)/\hbar} \int e^{i\left(\frac{1}{2}+\alpha S_{\rm cl}\right)\left(\delta g\cdot\partial_g^2 S\cdot\delta g+\delta\phi\cdot\partial_\phi^2 S\cdot\delta\phi\right)/\hbar}\,\mathcal{D}\delta g\,\mathcal{D}\delta\phi \tag{18.14}$$

$$\approx e^{i\left(S_{\rm cl}+\alpha S_{\rm cl}^2\right)/\hbar} \cdot \left[\text{Det}\left(\partial_g^2 S\right)\cdot \text{Det}\left(\partial_\phi^2 S\right)\right]^{-1/2} \cdot \left(\frac{2\pi\hbar i^{-1}}{\left(\frac{1}{2}+\alpha S_{\rm cl}\right)}\right)^{\dim/2} \tag{18.15}$$

where

$$S_{\rm cl} = S[g_{\rm cl}(x), \phi_{\rm cl}(x)] \tag{18.16}$$

$$= S[\eta^{(\rm cl)}(x), \phi_{\rm cl}(x)]\ . \tag{18.17}$$

For the partition Z, we know that

- Stationary Phase Approximation: Dominant contributions come from configurations where $S[g(x),\phi(x)]$ is stationary, with additional corrections from the αS^2 term.
- Path Integral Formulation: For small α, treat the αS^2 term as a perturbation.
- High-Energy Limits: The dominant contribution is from the classical configuration, with modifications due to the quadratic term αS^2.

These methods provide a framework for approximating and analyzing the modified partition function.

19 Generic functional FAT models

We start by a generic functional F applied to the S_R action. The we do the following calculations, which allow us to derive the following three generic models.

$$S_{FAT}^F = S_R + F\left[S_R\right] + S_m \tag{19.1}$$

$$\frac{\delta S_{FAT}^F}{\delta g^{\mu\nu}} = \frac{\delta S_R}{\delta g^{\mu\nu}} + \frac{\delta F\left[S_R\right]}{\delta g^{\mu\nu}} + \frac{\delta S_m}{\delta g^{\mu\nu}} \tag{19.2}$$

$$0 = \frac{1}{2\kappa^2}G_{\mu\nu} + \frac{\delta S_R}{\delta g^{\mu\nu}}\frac{\delta F\left[S_R\right]}{\delta S_R} + \frac{\delta S_m}{\delta g^{\mu\nu}} \tag{19.3}$$

$$0 = \frac{1}{2\kappa^2}G_{\mu\nu} + \frac{1}{2\kappa^2}G_{\mu\nu}\frac{\delta F\left[S_R\right]}{\delta S_R} - \frac{1}{2}T_{\mu\nu} \tag{19.4}$$

So we get

$$G_{\mu\nu} + G_{\mu\nu}\frac{\delta F\left[S_R\right]}{\delta S_R} = \kappa^2 T_{\mu\nu} \tag{19.5}$$

$$G_{\mu\nu}\left[\ 1 + \frac{\delta F\left[S_R\right]}{\delta S_R}\right] = \kappa^2 T_{\mu\nu} \tag{19.6}$$

$$G_{\mu\nu} + G_{\mu\nu}\frac{\delta F\left[\int d^D x\sqrt{-g}R\right]}{\delta \int d^D x\sqrt{-g}R} = \kappa^2 T_{\mu\nu} \tag{19.7}$$

In case of

$$F[S_R] = F^{quad}[S_R] = \beta S_R^2 \quad (19.8)$$

$$\frac{\delta F^{quad}[S_R]}{\delta S_R} = 2\beta S_R = 2\beta \int d^D x \sqrt{-g} R \quad (19.9)$$

In case of

$$F[S_R] = F^{cubic}[S_R] = \gamma S_R^3 \quad (19.10)$$

$$\frac{\delta F^{quad}[S_R]}{\delta S_R} = 3\gamma S_R^2 = 3\gamma \left[\int d^D x \sqrt{-g} R \right]^2 \quad (19.11)$$

In case of

$$F[S_R] = F^{exp}[S_R] = A e^{\frac{p}{\hbar} S_R} \quad (19.12)$$

$$\frac{\delta F^{quad}[S_R]}{\delta S_R} = A e^{\frac{p}{\hbar} S_R} \frac{p}{\hbar} = A e^{\frac{p}{\hbar}[\int d^D x \sqrt{-g} R]} \frac{p}{\hbar} \quad (19.13)$$

20 Stability Analysis of FAT Gravitational Cosmology Models

The viability of Functors of Actions Theories (FAT) in cosmology hinges on their stability against unphysical instabilities, such as ghosts (negative kinetic energy modes), tachyons (negative mass-squared modes), and gradient instabilities (negative sound speed squared) [1, 3]. This section proves that the quadratic and exponential FAT gravitational cosmology models, defined in [5, 6], are free of such instabilities by analyzing the quadratic action for scalar perturbations around a Friedmann-Lemaître-Robertson-Walker (FLRW) background. We derive the sound speed squared (c_s^2) explicitly to confirm the absence of gradient instabilities, leveraging the cosmological solutions. Note that this section draws references from the second set of references.

20.1 General Approach to Stability Analysis

To assess stability, we:

1. Define the FAT action and FLRW background metric.
2. Expand the action to second order in scalar perturbations to obtain the quadratic action.
3. Verify that the kinetic term is positive to ensure no ghosts.
4. Confirm that mass terms are non-negative to rule out tachyonic instabilities.
5. Calculate the sound speed squared (c_s^2) to check for gradient instabilities.
6. Evaluate stability in a cosmological context, such as a dark energy-dominated universe, using solutions from [5, 6].

We focus on scalar perturbations, as they are most susceptible to ghosts in modified gravity [1]. The FLRW background metric is:

$$ds^2 = -d(ct)^2 + a^2(t) d\vec{x}^2, \quad (20.1)$$

with Hubble parameter $H(t) = \dot{a}/a$. Scalar perturbations are introduced in the Newtonian gauge:

$$ds^2 = -(1+2\phi)\, d(ct)^2 + a^2(t)\,(1-2\psi)\,\delta_{ij}dx^i dx^j, \tag{20.2}$$

where ϕ and ψ are scalar potentials. In the absence of anisotropic stress, we assume $\phi = \psi$, and work with the curvature perturbation ζ, related to ψ by $\zeta \approx -\psi + \mathcal{H}\psi/\dot{\psi}$ (where $\mathcal{H} = aH$) in the comoving gauge [4]. The sound speed squared is defined as:

$$c_s^2 = \frac{\text{coefficient of } (\nabla\psi)^2/c^2a^2}{\text{coefficient of } \dot{\psi}^2}, \tag{20.3}$$

where a positive $c_s^2 > 0$ ensures no gradient instabilities.

20.2 Quadratic FAT Model

The quadratic FAT model is defined by:

$$S_{FAT}^{\text{quad}} = S_R + \beta S_R^2 + S_\Lambda + S_m, \quad S_R = c^2 \int d^4x \sqrt{-g}\, \frac{R}{16\pi G}, \tag{20.4}$$

where β has units $[\beta] = [\text{action}]^{-1}$, S_Λ is the cosmological constant term, and S_m is the matter action [5]. The field equations are:

$$\left[R_{\mu\nu} + \frac{1}{2}Rg_{\mu\nu}\right][1 + 2\beta S_R] = \frac{1}{c^2}\left[T_{\mu\nu} + \Lambda g_{\mu\nu}\right]. \tag{20.5}$$

The Hubble parameter is:

$$H = H_0 \sqrt{\frac{1}{1 + K(\beta)I(t)}\,(\Omega_{m0}a^{-3} + \Omega_{r0}a^{-4} + \Omega_{\Lambda 0})}, \tag{20.6}$$

where $I(t) = \int_0^t a^3(\tau)(\dot{H}(\tau) + 2H^2(\tau))c\, d\tau$ and $K(\beta) = \frac{6V_{3D}}{c^2\kappa^4}\beta^2$.

20.2.1 Quadratic Action for Perturbations

The quadratic action is derived from the Einstein-Hilbert term S_R and the quadratic term βS_R^2. For general relativity (GR), the second-order action for scalar perturbations, in the comoving gauge, is [4]:

$$S_{\text{GR}}^{(2)} = \int d^4x\, a^3 \frac{M_{\text{Pl}}^2}{2}\left[\dot{\zeta}^2 - \frac{c^2(\nabla\zeta)^2}{a^2}\right], \tag{20.7}$$

where $M_{\text{Pl}}^2 = \frac{c^2}{8\pi G}$. For the FAT term, perturb S_R:

$$S_R = S_R^{(0)} + \delta S_R + \delta^2 S_R + \cdots, \tag{20.8}$$

so:

$$\beta S_R^2 \approx \beta(S_R^{(0)})^2 + 2\beta S_R^{(0)}\delta S_R + \beta(\delta S_R)^2 + 2\beta S_R^{(0)}\delta^2 S_R. \tag{20.9}$$

The field equations suggest the action is scaled by $1 + 2\beta S_R$. For perturbations, the quadratic action becomes:

$$S^{(2)} \approx \int d^4x\, a^3 \frac{M_{\text{Pl}}^2}{2}\left(1 + 2\beta S_R^{(0)}\right)\left[\dot{\zeta}^2 - \frac{c^2(\nabla\zeta)^2}{a^2}\right] + \beta(\delta S_R)^2. \tag{20.10}$$

The background action is:

$$S_R^{(0)} = \frac{c^2}{16\pi G} \int d^4x\, a^3 \left(6\left(\frac{\ddot{a}}{a} + H^2\right)\right) \approx \frac{c^2 H_0}{4\pi G} \mathcal{V}_{3D}[e^{3H_0 t} - e^{3H_0 t_i}], \tag{20.11}$$

for a dark energy-dominated universe ($a(t) \approx e^{H_0 t}$). With $\beta \approx 10^{-182}\,\mathrm{kg}^{-1}\,\mathrm{m}^{-2}\,\mathrm{s}$ and $S_R^{(0)} \approx 8 \times 10^{88}\,\mathrm{kg}\,\mathrm{m}^2\,\mathrm{s}^{-1}$, we have:

$$2\beta S_R^{(0)} \approx 2 \times 10^{-182} \times 8 \times 10^{88} \approx 1.6 \times 10^{-93} \ll 1. \tag{20.12}$$

Thus, the kinetic term is:

$$K \approx \frac{M_{\mathrm{Pl}}^2}{2} a^3 \dot{\zeta}^2. \tag{20.13}$$

The non-local term $\beta(\delta S_R)^2$ is suppressed by $\beta^2 \sim 10^{-364}$.

20.2.2 Ghost Analysis

The kinetic coefficient is:

$$\frac{M_{\mathrm{Pl}}^2}{2}(1 + 2\beta S_R^{(0)}) \approx \frac{M_{\mathrm{Pl}}^2}{2} > 0, \tag{20.14}$$

indicating no ghosts, as the kinetic energy is positive.

20.2.3 Tachyonic Instabilities

The stable background, as shown numerically in [5], and the positive scaling $1+2\beta S_R^{(0)}$ suggest no negative mass terms, ruling out tachyonic instabilities.

20.2.4 Sound Speed Squared Calculation

The sound speed squared is:

$$c_s^2 = \frac{\text{coefficient of } c^2(\nabla\zeta)^2/a^2}{\text{coefficient of } \dot{\zeta}^2}. \tag{20.15}$$

From Equation (20.10), the action is:

$$S^{(2)} \approx \int d^4x\, a^3 \frac{M_{\mathrm{Pl}}^2}{2} \left(1 + 2\beta S_R^{(0)}\right) \left[\dot{\zeta}^2 - \frac{c^2(\nabla\zeta)^2}{a^2}\right]. \tag{20.16}$$

The kinetic term is:

$$K = a^3 \frac{M_{\mathrm{Pl}}^2}{2} \left(1 + 2\beta S_R^{(0)}\right) \dot{\zeta}^2, \tag{20.17}$$

with coefficient:

$$\text{Coefficient of } \dot{\zeta}^2 = a^3 \frac{M_{\mathrm{Pl}}^2}{2} \left(1 + 2\beta S_R^{(0)}\right). \tag{20.18}$$

The gradient term is:

$$G = -a^3 \frac{M_{\mathrm{Pl}}^2}{2} \left(1 + 2\beta S_R^{(0)}\right) \frac{c^2(\nabla\zeta)^2}{a^2} = -a \frac{M_{\mathrm{Pl}}^2}{2} \left(1 + 2\beta S_R^{(0)}\right) c^2(\nabla\zeta)^2, \tag{20.19}$$

so the coefficient of $c^2(\nabla\zeta)^2/a^2$ is:

$$\text{Coefficient of } \frac{c^2(\nabla\zeta)^2}{a^2} = a^3 \frac{M_{\mathrm{Pl}}^2}{2} \left(1 + 2\beta S_R^{(0)}\right). \tag{20.20}$$

Thus:

$$c_s^2 = \frac{a^3 \frac{M_{\rm Pl}^2}{2}\left(1 + 2\beta S_R^{(0)}\right)}{a^3 \frac{M_{\rm Pl}^2}{2}\left(1 + 2\beta S_R^{(0)}\right)} = 1. \tag{20.21}$$

Since $c_s^2 = 1 > 0$, there are no gradient instabilities. The small correction $2\beta S_R^{(0)} \approx 1.6\times 10^{-93}$ does not affect this result.

20.2.5 Conclusion

The quadratic FAT model is ghost-free, tachyon-free, and free of gradient instabilities for $\beta \approx 10^{-182}\,\mathrm{kg}^{-1}\,\mathrm{m}^{-2}\,\mathrm{s}$.

20.3 Exponential FAT Model

The exponential FAT model is:

$$S_{FAT}^{\rm exp} = S_R + Ae^{\frac{\beta}{\hbar}S_R} + S_\Lambda + S_m, \tag{20.22}$$

where A is dimensionless, $\beta/\hbar$ has units $[\text{action}]^{-1}$, and parameters are $A = 1$, $\beta = 0.1$, $K = 0.3$ [6]. The Hubble parameter is:

$$H = H_0\sqrt{\frac{1}{1 + A\beta e^{K(\beta)I(t)}}\left(\Omega_{m0}a^{-3} + \Omega_{r0}a^{-4} + \Omega_{\Lambda 0}\right)}. \tag{20.23}$$

20.3.1 Quadratic Action

Perturb the exponential term:

$$e^{\frac{\beta}{\hbar}S_R} \approx e^{\frac{\beta}{\hbar}S_R^{(0)}}\left[1 + \frac{\beta}{\hbar}\delta S_R + \frac{1}{2}\left(\frac{\beta}{\hbar}\right)^2 (\delta S_R)^2 + \frac{\beta}{\hbar}\delta^2 S_R\right]. \tag{20.24}$$

The quadratic action is:

$$S^{(2)} \approx \int d^4x\, a^3 \frac{M_{\rm Pl}^2}{2}\left(1 + A\beta e^{\frac{\beta}{\hbar}S_R^{(0)}}\right)\left[\dot{\zeta}^2 - \frac{c^2(\nabla\zeta)^2}{a^2}\right] + Ae^{\frac{\beta}{\hbar}S_R^{(0)}}\left[\frac{1}{2}\left(\frac{\beta}{\hbar}\right)^2 (\delta S_R)^2\right]. \tag{20.25}$$

Numerical solutions suggest $A\beta e^{K(\beta)I(t)} \ll 1$, so:

$$1 + A\beta e^{\frac{\beta}{\hbar}S_R^{(0)}} \approx 1. \tag{20.26}$$

20.3.2 Ghost Analysis

The kinetic coefficient is:

$$\frac{M_{\rm Pl}^2}{2}\left(1 + A\beta e^{\frac{\beta}{\hbar}S_R^{(0)}}\right) \approx \frac{M_{\rm Pl}^2}{2} > 0, \tag{20.27}$$

indicating no ghosts for $A > 0$.

20.3.3 Tachyonic Instabilities

The stable background (numerical solutions in [6]) and small correction $A\beta e^{K(\beta)I(t)}$ suggest no tachyonic instabilities.

20.3.4 Sound Speed Squared Calculation

Using Equation (20.25):

$$S^{(2)} \approx \int d^4x\, a^3 \frac{M_{\rm Pl}^2}{2} \left(1 + A\beta e^{\frac{\beta}{\hbar} S_R^{(0)}}\right) \left[\dot{\zeta}^2 - \frac{c^2(\nabla\zeta)^2}{a^2}\right]. \tag{20.28}$$

The kinetic term is:

$$K = a^3 \frac{M_{\rm Pl}^2}{2} \left(1 + A\beta e^{\frac{\beta}{\hbar} S_R^{(0)}}\right) \dot{\zeta}^2, \tag{20.29}$$

with coefficient:

$$\text{Coefficient of } \dot{\zeta}^2 = a^3 \frac{M_{\rm Pl}^2}{2} \left(1 + A\beta e^{\frac{\beta}{\hbar} S_R^{(0)}}\right). \tag{20.30}$$

The gradient term is:

$$G = -a^3 \frac{M_{\rm Pl}^2}{2} \left(1 + A\beta e^{\frac{\beta}{\hbar} S_R^{(0)}}\right) \frac{c^2(\nabla\zeta)^2}{a^2}, \tag{20.31}$$

with coefficient:

$$\text{Coefficient of } \frac{c^2(\nabla\zeta)^2}{a^2} = a^3 \frac{M_{\rm Pl}^2}{2} \left(1 + A\beta e^{\frac{\beta}{\hbar} S_R^{(0)}}\right). \tag{20.32}$$

Thus:

$$c_s^2 = \frac{a^3 \frac{M_{\rm Pl}^2}{2} \left(1 + A\beta e^{\frac{\beta}{\hbar} S_R^{(0)}}\right)}{a^3 \frac{M_{\rm Pl}^2}{2} \left(1 + A\beta e^{\frac{\beta}{\hbar} S_R^{(0)}}\right)} = 1. \tag{20.33}$$

Since $c_s^2 = 1 > 0$, there are no gradient instabilities.

20.3.5 Conclusion

The exponential FAT model is stable for $A\beta e^{K(\beta)I(t)} \ll 1$, with $A = 1$, $\beta = 0.1$, $K = 0.3$.

20.4 Conclusion

The quadratic FAT model is ghost-free, tachyon-free, and free of gradient instabilities for $\beta \approx 10^{-182}\,\text{kg}^{-1}\,\text{m}^{-2}\,\text{s}$, with $c_s^2 = 1$. The exponential model is similarly stable for $A\beta e^{K(\beta)I(t)} \ll 1$, with $c_s^2 = 1$. These results, supported by numerical solutions [5, 6], confirm the physical viability of FAT models in cosmology [2, 7].

21 Expectation Value of Observables in FAT: Application to Universal Information

This section derives the expectation value of an observable using the partition function, with probabilities tied to the action, which depends on the energy density Lagrangian, spacetime metric $g_{\mu\nu}$, and matter fields ϕ. We apply this to the "information of the universe," interpreted as entropy S_E, using the quadratic FAT model (Section 21.2) and a generic FAT model $S_{\rm FAT} = f(S_R)$ (Section 21.3), extending concepts from Sections 2 and 13.

21.1 General Framework

The partition function in FAT is:

$$\mathcal{Z} = \int \mathcal{D}[g_{\mu\nu}]\mathcal{D}[\phi]\, e^{-S_{\text{FAT}}[g_{\mu\nu},\phi]},$$

where S_{FAT} is the FAT-modified action, and we set $\hbar = 1$ for simplicity unless specified. The probability density is:

$$P[g_{\mu\nu}, \phi] = \frac{1}{\mathcal{Z}} e^{-S_{\text{FAT}}[g_{\mu\nu},\phi]},$$

normalized: $\int \mathcal{D}[g_{\mu\nu}]\mathcal{D}[\phi]\, P = 1$. The expectation value of an observable $\mathcal{O}[g_{\mu\nu}, \phi]$ is:

$$\langle \mathcal{O} \rangle = \frac{1}{\mathcal{Z}} \int \mathcal{D}[g_{\mu\nu}]\mathcal{D}[\phi]\, \mathcal{O}[g_{\mu\nu}, \phi] e^{-S_{\text{FAT}}[g_{\mu\nu},\phi]}.$$

The action is:

$$S = \int d^4x\, \sqrt{-g}\, \mathcal{L}(g_{\mu\nu}, \phi, \partial_\mu \phi),$$

with $\mathcal{L} = \frac{c^4}{16\pi G_{\text{N}}} R + \mathcal{L}_m - \Lambda$, where R is the Ricci scalar, $\mathcal{L}_m$ is the matter Lagrangian, and Λ is the cosmological constant. We define "information" as entropy $S_E[g_{\mu\nu}]$, e.g., Gibbons-Hawking entropy in de Sitter space:

$$S_E^{\text{cl}} = \frac{\pi c^2}{G_{\text{N}} H_0^2},$$

for horizon radius $r_H = c/H_0$, computing $\langle S_E \rangle$.

21.2 Quadratic FAT Model

From Section 2, the quadratic FAT action is:

$$S_{\text{FAT}}^{\text{quad}} = S_R + \beta S_R^2 + S_m + S_\Lambda, \quad S_R = c^3 \int d^4x \sqrt{-g} \frac{R}{16\pi G_{\text{N}}},$$

with $\beta \ll 1$. The partition function is:

$$\mathcal{Z} = \int \mathcal{D}[g_{\mu\nu}]\mathcal{D}[\phi]\, e^{-(S_R + \beta S_R^2 + S_m + S_\Lambda)},$$

$$\langle S_E \rangle = \frac{1}{\mathcal{Z}} \int \mathcal{D}[g_{\mu\nu}]\mathcal{D}[\phi]\, S_E[g_{\mu\nu}] e^{-(S_R + \beta S_R^2 + S_m + S_\Lambda)}.$$

Perturb around the classical FLRW solution ($ds^2 = -d(ct)^2 + a^2(t) d\vec{x}^2$, $a(t) = e^{H_0 t}$):

$$S_R^{\text{cl}} = \frac{6c^2 \mathcal{V}_{3D}}{16\pi G_{\text{N}}} \int_{t_i}^{t} dt' a^3(t') \left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a} \right)^2 \right] = \frac{c^2 H_0}{4\pi G_{\text{N}}} \mathcal{V}_{3D} (e^{3H_0 t} - e^{3H_0 t_i}),$$

$$S_{\text{FAT}} \approx S_{\text{cl}} + \beta (S_R^{\text{cl}})^2 + \delta S_R + 2\beta S_R^{\text{cl}} \delta S_R, \quad S_{\text{cl}} = S_R^{\text{cl}} + S_m^{\text{cl}} + S_\Lambda.$$

Saddle-point approximation:

$$\mathcal{Z} \approx e^{-S_{\text{cl}} - \beta (S_R^{\text{cl}})^2} \sqrt{\frac{2\pi}{\det(\Box + 2\beta S_R^{\text{cl}} \Box^2)}},$$

$$\langle S_E \rangle \approx S_E^{\rm cl} + \beta (S_R^{\rm cl})^2 \frac{\int \mathcal{D}[\delta g]\, \delta S_E e^{-(\delta S_R + 2\beta S_R^{\rm cl} \delta S_R)}}{\int \mathcal{D}[\delta g]\, e^{-(\delta S_R + 2\beta S_R^{\rm cl} \delta S_R)}} \approx S_E^{\rm cl} + \beta (S_R^{\rm cl})^2,$$

assuming $\langle \delta S_E \rangle \approx 0$ (symmetry). Numerically:
- $H_0 = 2 \times 10^{-17}\,\mathrm{s}^{-1}$, $G_{\rm N} = 6.674 \times 10^{-11}\,\mathrm{m}^3\,\mathrm{s}^{-2}\,\mathrm{kg}^{-1}$, $c = 3 \times 10^8\,\mathrm{m\,s}^{-1}$,
- $\mathcal{V}_{3D} = 10^{77}\,\mathrm{m}^3$, $t = 10^{17}\,\mathrm{s}$, $t_i = 0$,
- $S_R^{\rm cl} = \frac{(3\times 10^8)^2 (2\times 10^{-17})}{4\pi (6.674\times 10^{-11})} (10^{77}) (e^{3\times 2\times 10^{-17} \times 10^{17}} - 1) \approx 8 \times 10^{88}\,\mathrm{kg\,m}^2\,\mathrm{s}^{-1}$,
- $S_E^{\rm cl} = \frac{\pi (3\times 10^8)^2}{(6.674\times 10^{-11})(2\times 10^{-17})^2} \approx 4.22 \times 10^{122}\,\mathrm{kg\,m}^2\,\mathrm{s}^{-2}$,
- Planck units: $S_E^{\rm cl}/\hbar \approx 4.22 \times 10^{122}/1.054 \times 10^{-34} \approx 4 \times 10^{123}$,
- $\beta = 10^{-182}\,\mathrm{kg}^{-1}\,\mathrm{m}^{-2}\,\mathrm{s}$,
- Correction: $\beta (S_R^{\rm cl})^2 = 10^{-182} (8 \times 10^{88})^2 \approx 6.4 \times 10^{94}$,
- $\langle S_E \rangle \approx 4 \times 10^{123} + 6.4 \times 10^{94}$ (Planck units).

Interpretation: The quadratic term adds a subdominant correction to the classical entropy.

21.3 Generic FAT Model

For the generic FAT model (Section 2):

$$S_{\rm FAT} = f(S_R) + S_m + S_\Lambda,$$

where $f(S_R)$ is arbitrary, e.g., $f(S_R) = S_R + \beta S_R^2 + \gamma S_R^3$. The partition function is:

$$\mathcal{Z} = \int \mathcal{D}[g_{\mu\nu}] \mathcal{D}[\phi]\, e^{-(f(S_R) + S_m + S_\Lambda)},$$

$$\langle S_E \rangle = \frac{1}{\mathcal{Z}} \int \mathcal{D}[g_{\mu\nu}] \mathcal{D}[\phi]\, S_E[g_{\mu\nu}] e^{-(f(S_R) + S_m + S_\Lambda)}.$$

21.3.1 Classical Solution

Minimize $S_{\rm FAT}$:

$$\frac{\delta S_{\rm FAT}}{\delta g_{\mu\nu}} = f'(S_R) \frac{\delta S_R}{\delta g_{\mu\nu}} + \frac{\delta (S_m + S_\Lambda)}{\delta g_{\mu\nu}} = 0,$$

$$f'(S_R) \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) = \frac{8\pi G_{\rm N}}{c^4} (T_{\mu\nu} + \Lambda g_{\mu\nu}).$$

For $a(t) = e^{H_0 t}$, $R = 12 H_0^2/c^2$, vacuum ($T_{\mu\nu} = 0$):

$$f'(S_R^{\rm cl})(3H_0^2) = 8\pi G_{\rm N} \Lambda,$$

$$H_0^2 = \frac{8\pi G_{\rm N} \Lambda}{3 f'(S_R^{\rm cl})}.$$

For $f(S_R) = S_R + \beta S_R^2 + \gamma S_R^3$:

$$f'(S_R^{\rm cl}) = 1 + 2\beta S_R^{\rm cl} + 3\gamma (S_R^{\rm cl})^2,$$

$$S_R^{\rm cl} = \frac{c^2 H_0}{4\pi G_{\rm N}} \mathcal{V}_{3D} (e^{3H_0 t} - e^{3H_0 t_i}).$$

Iterate:

- GR: $H_0^2 = \frac{8\pi G_{\rm N}\Lambda}{3}$, $S_R^{\rm cl} \approx 8\times 10^{88}$,
- $2\beta S_R^{\rm cl} = 2\times 10^{-182}\times 8\times 10^{88} = 1.6\times 10^{-93}$,
- $3\gamma(S_R^{\rm cl})^2 = 3\times 10^{-270}\times(8\times 10^{88})^2 = 1.92\times 10^{-93}$,
- $f'(S_R^{\rm cl}) \approx 1 + 3.52\times 10^{-93}$,
- $H_0'^2 = H_0^2/(1+3.52\times 10^{-93}) \approx H_0^2(1-3.52\times 10^{-93})$,
- $H_0' \approx H_0(1-1.76\times 10^{-93})$.

Entropy:

$$S_E^{\rm cl} = \frac{\pi c^2}{G_{\rm N}(H_0')^2} \approx \frac{\pi c^2}{G_{\rm N}H_0^2}(1+3.52\times 10^{-93}),$$

$$S_E^{\rm cl} \approx 4\times 10^{123}(1+3.52\times 10^{-93}) \approx 4\times 10^{123} + 1.4\times 10^{31}.$$

21.3.2 Perturbative Expectation

$$S_{\rm FAT} = f(S_R^{\rm cl}) + S_m^{\rm cl} + S_\Lambda + f'(S_R^{\rm cl})\delta S_R + \frac{1}{2}f''(S_R^{\rm cl})(\delta S_R)^2 + \cdots,$$

$$\mathcal{Z} \approx e^{-(f(S_R^{\rm cl})+S_m^{\rm cl}+S_\Lambda)}\sqrt{\frac{2\pi}{\det(\Box + f'(S_R^{\rm cl})\Box^2)}},$$

$$\langle S_E\rangle \approx S_E^{\rm cl} + [f(S_R^{\rm cl}) - S_R^{\rm cl}],$$

$$f(S_R^{\rm cl}) - S_R^{\rm cl} = \beta(S_R^{\rm cl})^2 + \gamma(S_R^{\rm cl})^3,$$

- $\beta(S_R^{\rm cl})^2 = 6.4\times 10^{94}$,
- $\gamma = 10^{-270}\,{\rm kg}^{-2}\,{\rm m}^{-4}\,{\rm s}^2$,
- $\gamma(S_R^{\rm cl})^3 = 10^{-270}(8\times 10^{88})^3 = 5.12\times 10^{65}$,
- $\langle S_E\rangle \approx 4\times 10^{123} + 6.4\times 10^{94} + 5.12\times 10^{65}$.

Interpretation: $S_E^{\rm cl} \approx 10^{123}$ holds perturbatively, with corrections dominated by the quadratic term.

21.4 Conclusion

Both models yield $S_E^{\rm cl} \approx 10^{123}$ under small β, γ, with FAT adding hierarchical corrections, enhancing the "informaton" concept (Section 24).

22 Generalisation of stationary action principle

In this section, we generalise the stationary action principle.

22.1 First attempt

22.1.1 The stationary action principle simple

Given an action that depends on fields, metric ones, g_i, and matter ones, ϕ_j, i.e.

$$S = S\left[g_i, \phi_j\right] \tag{22.1}$$

the stationary action principle suggest:

$$\delta S = 0 \tag{22.2}$$

$$\sum_i \partial_{g_i} S\,\delta g_i + \sum_j \partial_{\phi_j} S\,\delta\phi_j = 0 \tag{22.3}$$

22.1.2 The stationary action principle generalised

Given that the action depends on several fields, that we can generalised them into a vector field, $v_i = \{g_1, \ldots, g_m, \phi_1, \ldots \phi_n\}$ where $n + m = d$, g_i are the metric fields, where ϕ_j are the matter fields

Then the action can be expand using Taylor expansion as

$$S[\vec{v}] = \sum_{n_1=0}^{\infty} \cdots \sum_{n_d=0}^{\infty} \frac{(v_1 - v_{01})^{n_1} \ldots (v_d - v_{0d})^{n_d}}{n_1! \ldots n_d!} \frac{\partial^{(n_1 \ldots n_d)} S}{\partial v_1^{n_1} \ldots \partial v_d^{n_d}} |_{\vec{v}=\vec{v}_0} \tag{22.4}$$

now we can define

$$\delta v_{0i} = v_i - v_{0i} \tag{22.5}$$

so we get the compact form

$$S[\vec{v}] = \sum_{n_1=0}^{\infty} \cdots \sum_{n_d=0}^{\infty} \frac{\delta v_{01}^{n_1} \ldots \delta v_{0d}^{n_d}}{n_1! \ldots n_d!} \frac{\partial^{(n_1 \ldots n_d)} S}{\partial v_1^{n_1} \ldots \partial v_d^{n_d}} |_{\vec{v}=\vec{v}_0} \tag{22.6}$$

Note that

$$S[\vec{v}] = S[\vec{v}_0] + \sum_{n_1=1}^{\infty} \cdots \sum_{n_d=1}^{\infty} \frac{\delta v_{01}^{n_1} \ldots \delta v_{0d}^{n_d}}{n_1! \ldots n_d!} \frac{\partial^{(n_1 \ldots n_d)} S}{\partial v_1^{n_1} \ldots \partial v_d^{n_d}} |_{\vec{v}=\vec{v}_0} \tag{22.7}$$

but we can define the variation

$$\delta S[\vec{v}] = S[\vec{v}] - S[\vec{v}_0] \tag{22.8}$$

so we get that this variation is actually

$$\delta S[\vec{v}] = \sum_{n_1=1}^{\infty} \cdots \sum_{n_d=1}^{\infty} \frac{\delta v_{01}^{n_1} \ldots \delta v_{0d}^{n_d}}{n_1! \ldots n_d!} \frac{\partial^{(n_1 \ldots n_d)} S}{\partial v_1^{n_1} \ldots \partial v_d^{n_d}} |_{\vec{v}=\vec{v}_0} \tag{22.9}$$

So the stationary action principle, generalised is actually

$$\delta S[\vec{v}] = 0 \tag{22.10}$$

$$\sum_{n_1=1}^{\infty} \cdots \sum_{n_d=1}^{\infty} \frac{\delta v_{01}^{n_1} \ldots \delta v_{0d}^{n_d}}{n_1! \ldots n_d!} \frac{\partial^{(n_1 \ldots n_d)} S}{\partial v_1^{n_1} \ldots \partial v_d^{n_d}} |_{\vec{v}=\vec{v}_0} = 0 \tag{22.11}$$

$$\sum_{n_1=1}^{\infty} \cdots \sum_{n_d=1}^{\infty} \frac{\delta v_{01}^{n_1} \ldots \delta v_{0d}^{n_d}}{n_1! \ldots n_d!} \left(\partial_{v_1}^{n_1} \ldots \partial_{v_d}^{n_d} S\right) |_{\vec{v}=\vec{v}_0} = 0 \tag{22.12}$$

22.2 Stationary action principle on partition function

$$\mathcal{Z} = \int \prod_i \mathcal{D}g_i \prod_j \mathcal{D}\phi_j \; e^{\frac{i}{\hbar}S[g_i,\phi_j]} \tag{22.13}$$

varying the partition function we get

$$\delta\mathcal{Z} = \int \prod_i \mathcal{D}g_i \prod_j \mathcal{D}\phi_j \; \delta\left[e^{\frac{i}{\hbar}S[g_i,\phi_j]}\right] \tag{22.14}$$

Now the variation of partition should be zero, so

$$\delta\mathcal{Z} = 0 \tag{22.15}$$

$$\int \prod_i \mathcal{D}g_i \prod_j \mathcal{D}\phi_j \; \delta\left[e^{\frac{i}{\hbar}S[g_i,\phi_j]}\right] = 0 \tag{22.16}$$

$$\delta\left[e^{\frac{i}{\hbar}S[g_i,\phi_j]}\right] = 0 \tag{22.17}$$

$$\delta\left\{\frac{i}{\hbar}S[g_i,\phi_j]\right\}\left[e^{\frac{i}{\hbar}S[g_i,\phi_j]}\right] = 0 \tag{22.18}$$

$$\frac{i}{\hbar}\delta\left\{S[g_i,\phi_j]\right\}\left[e^{\frac{i}{\hbar}S[g_i,\phi_j]}\right] = 0 \tag{22.19}$$

$$\delta\left\{S[g_i,\phi_j]\right\} = 0 \quad \text{or} \quad \left[e^{\frac{i}{\hbar}S[g_i,\phi_j]}\right] = 0 \tag{22.20}$$

which results to

$$\delta\left\{S[g_i,\phi_j]\right\} = 0 \quad \text{or} \quad \left[e^{\frac{i}{\hbar}S[g_i,\phi_j]}\right] = 0 \tag{22.21}$$

So partially the stationary action on the partition function, is either the standard generalised stationary action that we proved before, or an exponential functional of the action in which the action becomes 0 at some limit.

22.3 Generalised stationary action principle in FAT

Given a FAT theory, we can expand described the generalised stationary action principle as:

$$\delta S_{\text{FAT}} + \delta f(S_{\text{FAT}}) = 0.$$

(Note: Uses generalized $f(S_R)$). **Interpretation**: Broadens action minimization, applied to field-particles in Section 23.

23 FAT field-particles - actionions

23.1 Quantum field theory description of field particles

In this section, we answer to the question of when a quantum fluctuation, corresponds to a field-particle.

A quantum fluctuation can correspond to a field-particle when the fluctuation manifests as a temporary change in the value of a field, which can create or annihilate particles. In quantum field theory (QFT), fields are seen as the fundamental entities, and particles are understood as excitations or quanta of these fields.

The correspondence occurs in the following way:

1. Quantum Field Theory (QFT): In QFT, every particle is associated with a specific field (e.g., the electromagnetic field for photons, the Higgs field for the Higgs boson). A fluctuation in a field corresponds to a local deviation from the field's average value, which can create a particle or antiparticle pair. These fluctuations can be viewed as "virtual" particles in a vacuum, which are transient and can briefly exist due to the uncertainty principle.

2. Vacuum Fluctuations: Even in a vacuum, where no real particles are present, quantum fluctuations still occur due to the uncertainty principle. These fluctuations can produce temporary particle-antiparticle pairs. The energy of these pairs must be "borrowed" for a very short time, and they annihilate each other quickly, returning the energy to the field.

3. Creation of Real Particles: When a quantum fluctuation involves a larger fluctuation in field strength, it can potentially create real, observable particles, particularly in high-energy scenarios such as in particle collisions or in strong gravitational fields (e.g., near black holes).

Thus, a quantum fluctuation can correspond to a field-particle when the fluctuation is sufficiently strong or persistent to create a detectable particle, with this particle being an excitation of the underlying quantum field.

23.2 1st Attempt to describe actionion, using Taylor expansion

In this section, we answer to the question of how we can describe an actionion, i.e. an actionic field particle, using a taylor expansion, which is basically approximates a field, with a field at some average value, and the rest as a deviation from this average value.

Indeed, describing a quantum field using a Taylor expansion around its average value is a valid and insightful approach. This method allows you to separate the field into a background (average) value and its fluctuations, which can be interpreted as quantum deviations around the average.

23.2.1 Formal Description

Consider a quantum field $\phi(x)$. The field can be expressed as a sum of its classical average value $\bar{\phi}(x)$ (the background field) and a quantum fluctuation $\delta\phi(x)$ around this average:

$$\phi(x) = \bar{\phi}(x) + \delta\phi(x),$$

where:

- $\bar{\phi}(x)$: The average (classical) field, typically the solution to the classical equations of motion or the background configuration in a given context.

- $\delta\phi(x)$: The deviation or fluctuation around this average, treated as the quantum correction.

If we expand the dynamics of $\phi(x)$ in terms of $\delta\phi(x)$, the field's governing equations or action can be expressed as a Taylor series around $\bar{\phi}(x)$.

—

23.2.2 Taylor Expansion in the Action

The action $S[\phi]$ of the field can be expanded around $\bar{\phi}(x)$ using a functional Taylor expansion:

$$S[\phi] = S[\bar{\phi}] + \int \frac{\delta S}{\delta\phi(x)}\bigg|_{\bar{\phi}} \delta\phi(x)\, d^4x + \frac{1}{2}\int \frac{\delta^2 S}{\delta\phi(x)\delta\phi(y)}\bigg|_{\bar{\phi}} \delta\phi(x)\delta\phi(y)\, d^4x d^4y + \dots$$

Here:

1. $S[\bar{\phi}]$: The action evaluated at the average field.

2. $\frac{\delta S}{\delta\phi(x)}$: The first functional derivative of the action, vanishes if $\bar{\phi}(x)$ satisfies the classical equations of motion.

3. $\frac{\delta^2 S}{\delta\phi(x)\delta\phi(y)}$: The second functional derivative, often related to the propagator or the mass matrix for small fluctuations.

—

23.2.3 Interpretation in Terms of Particles

- Zeroth-Order Term: Represents the classical dynamics of the field at $\bar{\phi}(x)$.
- First-Order Term: Vanishes if $\bar{\phi}(x)$ is a solution to the classical equations of motion.
- Second-Order Term: Encodes the dynamics of the fluctuations $\delta\phi(x)$, including their propagation and interactions. These fluctuations correspond to particles in the quantum theory.

23.2.4 Link to Quantum Fluctuations

The term $\delta\phi(x)$ represents the quantum fluctuations, which in Fourier space correspond to the creation and annihilation operators of particles. These fluctuations, when quantized, are associated with the field's particle content.

—

23.2.5 Practical Examples

1. Inflationary Cosmology: In models of inflation, the scalar field $\phi(x)$ driving inflation is split into a classical background $\bar{\phi}(t)$ (slow-roll solution) and quantum fluctuations $\delta\phi(x,t)$ around this background. The fluctuations $\delta\phi$ seed the primordial density perturbations.

2. Spontaneous Symmetry Breaking: In the Higgs mechanism, the Higgs field $\phi(x)$ is expanded around its vacuum expectation value $\bar{\phi}$, with $\delta\phi(x)$ describing fluctuations corresponding to the physical Higgs particle.

This approach demonstrates how quantum fields can be approximated and analyzed using a classical background plus quantum deviations, with Taylor expansion providing a systematic way to capture their dynamics.

23.3 Relation between entropy and action, in field descriptions

In this section, we answer to the question, 'How is the entropy is described using the action, as a probability ?'.

In quantum field theory and statistical mechanics, entropy can be related to the action through the concept of probability in the context of path integrals or statistical ensembles. This connection emerges naturally because entropy is fundamentally tied to the distribution of probabilities for states or configurations of the system.

—

23.3.1 Entropy and the Action: Path Integral Formalism

1. Action and Path Integral: The path integral formulation expresses the probability amplitude for a quantum field configuration as:

$$\mathcal{Z} = \int \mathcal{D}[\phi]\, e^{iS[\phi]/\hbar},$$

where:

- $\mathcal{Z}$: The partition function or generating functional.
- $\mathcal{D}[\phi]$: The measure over all field configurations.
- $S[\phi]$: The action of the system.

In the Euclidean path integral formulation (used in statistical mechanics), the action becomes $S_E[\phi]$, where S_E is the Wick-rotated (Euclideanized) action. The partition function is then written as:

$$\mathcal{Z} = \int \mathcal{D}[\phi]\, e^{-S_E[\phi]/\hbar}.$$

2. Probability Interpretation: The weight $e^{-S_E[\phi]/\hbar}$ can be interpreted as a probability density for the configuration ϕ in the space of field configurations. The likelihood of a particular field configuration is determined by how the Euclidean action S_E evaluates for that configuration:

- Configurations with smaller $S_E[\phi]$ contribute more significantly to the path integral, corresponding to higher probabilities.

3. Entropy from the Partition Function: In statistical mechanics, the entropy S is related to the partition function $\mathcal{Z}$ via:

$$S = -\sum_i p_i \ln p_i,$$

where p_i is the probability of the i-th microstate.

In the context of the action:

$$p[\phi] = \frac{e^{-S_E[\phi]/\hbar}}{\mathcal{Z}},$$

where the partition function normalizes the probability.

Substituting this probability into the entropy formula gives:

$$S = -\int \mathcal{D}[\phi]\, p[\phi] \ln p[\phi].$$

After expanding $\ln p[\phi]$:

$$S = \frac{1}{\hbar} \int \mathcal{D}[\phi]\, p[\phi] \left(S_E[\phi] + \hbar \ln \mathcal{Z} \right),$$

which connects the entropy to the Euclidean action $S_E[\phi]$.

—

23.3.2 Classical Limit and Statistical Mechanics

In the classical limit ($\hbar \to 0$), this description aligns with the classical Gibbs entropy, where the probability density is derived from the classical Hamiltonian H:

$$p[\phi] = \frac{e^{-H[\phi]/k_B T}}{\mathcal{Z}},$$

and $S = -\int \mathcal{D}[\phi]\, p[\phi] \ln p[\phi]$.

In the quantum case, the entropy incorporates all configurations weighted by the quantum action.

—

23.3.3 Entropy and Field Theory Applications

1. Black Hole Entropy: Black hole entropy, such as the Bekenstein-Hawking entropy, can be derived from the Euclidean path integral for gravitational actions. The dominant contribution to $\mathcal{Z}$ comes from classical solutions like the black hole metric.

2. Thermal Field Theory: In thermal quantum field theory, the partition function $\mathcal{Z}$ is calculated with periodic boundary conditions in imaginary time, and entropy is derived using:

$$S = -\frac{\partial F}{\partial T},$$

where $F = -T \ln \mathcal{Z}$ is the free energy.

—

23.3.4 Summary

Entropy in field theory is tied to the probability distribution of field configurations, which is determined by the action through the Euclidean path integral. Configurations with lower action have higher probabilities, and the entropy quantifies the spread of these probabilities across the configuration space.

23.4 2nd detailed attempt to describe actionion, using Taylor expansion

In this section, we answer to the question of can we describe an actionion, i.e. an actionic field particle, using a taylor expansion, which is basically approximates a field, with a field at some average value, and the rest as a deviation from this average value ?

Note that in Gravity the action is written as

$$S\left\{x, g(x), \psi(x), R[x; g(x)], \mathcal{L}[x, g(x), \psi(x)]; D\right\} = \int_X d^D x \sqrt{-g(x)} \left\{ \frac{R[x, g(x)]}{16\pi G_{\mathrm{N}}} + \mathcal{L}[x, g(x), \psi(x)] \right\} \tag{23.1}$$

Then the taylor expansion suggests:

$$\delta x = x - x_0 \tag{23.2}$$

$$\delta g(x, x_0) = g(x) - g(x_0) \tag{23.3}$$

$$\delta \psi(x, x_0) = \psi(x) - \psi(x_0) \tag{23.4}$$

$$\delta R[x, x_0, g, g_0] = R[x, x_0, g, g_0] - R[x_0, g_0] \tag{23.5}$$

$$\delta \mathcal{L}[x, x_0, g, g_0, \psi, \psi_0] = \mathcal{L}[x, x_0, g, g_0, \psi, \psi_0] - \mathcal{L}[x_0, g_0, \psi_0] \tag{23.6}$$

so

$$\delta S\{x, g, \psi, R, \mathcal{L}; D\} = S\{x, g, \psi, R, \mathcal{L}; D\} - S\{x_0, g_0, \psi_0, R_0, \mathcal{L}_0; D_0\} \tag{23.7}$$

We discuss an important approach in perturbative expansions of the gravitational action. Below is an expanded and clarified version of the discussion, focusing on the applicability of the Taylor expansion to all components of the gravitational action.

—

23.4.1 Completeness and Applicability of the Taylor Expansion

The gravitational action in standard general relativity is expressed as:

$$S\{x, g(x), \psi(x), R[x; g(x)], \mathcal{L}[x, g(x), \psi(x)]; D\} = \int_X d^D x \sqrt{-g(x)} \left\{ \frac{R[x, g(x)]}{16\pi G_\mathrm{N}} + \mathcal{L}[x, g(x), \psi(x)] \right\},$$

where:

- $g(x)$: The metric tensor, encapsulating spacetime geometry.
- $\psi(x)$: Matter fields, such as scalar, vector, or spinor fields.
- $R[x; g(x)]$: The Ricci scalar derived from the metric tensor.
- $\mathcal{L}[x, g(x), \psi(x)]$: The Lagrangian density for matter fields and their interactions with $g(x)$.
- X: The domain of integration, typically spacetime.
- D: The number of dimensions of the domain of integration.

—

23.4.2 Perturbative Taylor Expansion

The Taylor expansion you propose separates the fields and their components into:

1. A background value (e.g., $g_0(x), \psi_0(x)$, etc.) corresponding to a classical solution.
2. Small perturbations around this background (e.g., $\delta g(x, x_0), \delta\psi(x, x_0)$, etc.), representing quantum or classical fluctuations.

These perturbations are defined as:

$$\begin{aligned}
\delta x &= x - x_0, \\
\delta g(x, x_0) &= g(x) - g(x_0), \\
\delta\psi(x, x_0) &= \psi(x) - \psi(x_0), \\
\delta R[x, x_0, g, g_0] &= R[x, g] - R[x_0, g_0], \\
\delta\mathcal{L}[x, x_0, g, g_0, \psi, \psi_0] &= \mathcal{L}[x, g, \psi] - \mathcal{L}[x_0, g_0, \psi_0].
\end{aligned}$$

Expanding the action around these perturbations gives:

$$\delta S = S[x, g, \psi, R, \mathcal{L}; D] - S[x_0, g_0, \psi_0, R_0, \mathcal{L}_0; D_0].$$

—

23.4.3 Steps in the Expansion

1. Metric Expansion

The metric $g(x)$ is typically decomposed as:

$$g_{\mu\nu}(x) = g^{(0)}_{\mu\nu}(x) + \delta g_{\mu\nu}(x),$$

where $g^{(0)}_{\mu\nu}(x)$ is a background metric (e.g., Minkowski or Schwarzschild) and $\delta g_{\mu\nu}(x)$ is a small perturbation. This approach is standard in linearized gravity and cosmological perturbation theory.

2. Ricci Scalar Expansion

The Ricci scalar $R[x; g(x)]$ depends nonlinearly on the metric, so its expansion includes terms linear and quadratic in $\delta g_{\mu\nu}(x)$:

$$R[x; g(x)] = R[x; g^{(0)}(x)] + \delta R[x; g^{(0)}(x), \delta g(x)] + \mathcal{O}(\delta g^2).$$

3. Matter Field Expansion

For the matter fields $\psi(x)$, the expansion takes the form:

$$\psi(x) = \psi_0(x) + \delta\psi(x).$$

The Lagrangian $\mathcal{L}[x, g(x), \psi(x)]$ is then expanded around $g_0(x)$ and $\psi_0(x)$.

4. Total Action Expansion

Substituting these expansions into the gravitational action yields a perturbative series:

$$S = S[g^{(0)}, \psi_0] + \delta S[g, \psi] + \mathcal{O}(\delta g^2, \delta\psi^2).$$

Each term corresponds to increasing orders of perturbation theory, where the first-order terms describe linear perturbations and the second-order terms account for interactions between fluctuations.

—

23.4.4 Validity of method

Is this method correct? Yes, this approach is correct in principle, provided:

1. The perturbations $\delta g_{\mu\nu}$, $\delta\psi$, etc., are small enough for the Taylor expansion to converge.

2. The background $g^{(0)}(x)$, $\psi_0(x)$, etc., satisfies the classical equations of motion derived from the original action.

3. The domain X and dimensionality D remain unchanged during the perturbative expansion.

However, special care must be taken in:

- Handling gauge freedoms in $g_{\mu\nu}$, as perturbations may include gauge modes.
- Expanding the determinant $\sqrt{-g}$, as it involves a nonlinear function of $g_{\mu\nu}$.
- Expanding R, which is nonlinearly related to $g_{\mu\nu}$.

—

23.4.5 First and Second-Order Perturbations of the Action

Let's revisit the action and explicitly describe the first-order perturbation, its vanishing, and the second-order expansion in detail. We also address whether terms like $R\delta^2 g$ appear in the second-order Lagrangian.

—

23.4.6 Gravitational Action and Perturbations

The gravitational action is:

$$S = \int_X d^D x \sqrt{-g} \left(\frac{R}{16\pi G_\mathrm{N}} + \mathcal{L} \right),$$

where $g_{\mu\nu} = g^{(0)}_{\mu\nu} + \delta g_{\mu\nu}$, R is the Ricci scalar, and $\mathcal{L}$ represents the matter Lagrangian.

Perturbations are expanded as:

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + \delta g_{\mu\nu}, \quad \sqrt{-g} = \sqrt{-g^{(0)}} \left(1 + \frac{1}{2}\delta g - \frac{1}{4}\delta g^2 + \dots \right),$$

$$R = R^{(0)} + \delta R + \delta^2 R, \quad \mathcal{L} = \mathcal{L}^{(0)} + \delta\mathcal{L} + \delta^2\mathcal{L}.$$

—

23.4.7 First-Order Perturbation of the Action

The first-order perturbation of the action is:

$$\delta S^{(1)} = \int_X d^D x \sqrt{-g^{(0)}} \left[\frac{\delta R}{16\pi G_\mathrm{N}} + \frac{R^{(0)}}{16\pi G_\mathrm{N}} \frac{\delta g}{2} + \delta\mathcal{L} + \mathcal{L}^{(0)} \frac{\delta g}{2} \right].$$

23.4.8 Why Does $\delta S^{(1)}$ Disappear?

1. Einstein Field Equations:

The background metric $g^{(0)}_{\mu\nu}$ is assumed to satisfy the Einstein field equations:

$$G^{(0)}_{\mu\nu} + 8\pi G_\mathrm{N} T^{(0)}_{\mu\nu} = 0,$$

where $G^{(0)}_{\mu\nu}$ is the Einstein tensor for $g^{(0)}_{\mu\nu}$, and $T^{(0)}_{\mu\nu}$ is the stress-energy tensor derived from $\mathcal{L}^{(0)}$. This ensures that all first-order variations $\delta g_{\mu\nu}$, δR, and $\delta\mathcal{L}$ cancel out when integrated over X.

2. Boundary Terms:

The term involving δR can be expressed as a total derivative (via integration by parts), yielding boundary contributions that vanish if the perturbations $\delta g_{\mu\nu}$ are compactly supported or boundary conditions are imposed appropriately.

Thus, $\delta S^{(1)} = 0$, confirming that the action is stationary for the background solution.

—

23.4.9 Second-Order Perturbation of the Action

The second-order perturbation of the action is:

$$\delta S^{(2)} = \int_X d^D x \sqrt{-g^{(0)}} \left[\frac{\delta^2 R}{16\pi G_\mathrm{N}} + \frac{\delta R \delta g}{16\pi G_\mathrm{N}} + \frac{R^{(0)}}{16\pi G_\mathrm{N}} \left(\frac{\delta g^2}{4} - \frac{\delta^2 g}{2} \right) + \delta^2\mathcal{L} + \frac{\delta g \delta\mathcal{L}}{2} + \mathcal{L}^{(0)} \frac{\delta g^2}{4} \right].$$

Specific Terms in the Expansion

1. Ricci Scalar Terms:

- $\delta^2 R$: The second-order perturbation of the Ricci scalar contributes directly.

- $\delta R \delta g$: Linear perturbation of the Ricci scalar coupled to the trace of the metric perturbation.

2. Metric Perturbations:

- Terms involving δg^2 and $\delta^2 g$ arise from expanding $\sqrt{-g}R$ and $\sqrt{-g}\mathcal{L}$.

3. Matter Lagrangian:

- Quadratic perturbations of the Lagrangian $\delta^2\mathcal{L}$.
- Mixed terms involving $\delta g \delta\mathcal{L}$.

—

23.4.10 Does a Term Like $R\delta^2 g$ Appear?

A term of the form $R\delta^2 g$ does not explicitly appear in the second-order perturbation because:

- The Ricci scalar R is already expanded as $R^{(0)} + \delta R + \delta^2 R$, and it couples naturally to perturbations of the metric and its derivatives (e.g., δR, $\delta^2 R$, δg^2).
- $\delta^2 g$ appears in terms like $\frac{R^{(0)}}{16\pi G_{\mathrm{N}}}\frac{\delta^2 g}{2}$, which originate from the expansion of $\sqrt{-g}$.

Thus, while $R^{(0)}\delta^2 g$ contributes, $\delta^2 g$ does not couple directly to higher-order R perturbations.

—

23.4.11 Probability and Entropy up to Second Order

Probability Density

The probability density in the path integral formalism is:

$$p[\delta g, \delta\psi] \propto e^{-S_E^{(0)}/\hbar} e^{-\delta S_E^{(2)}/\hbar},$$

where $\delta S_E^{(2)}$ includes the second-order terms derived above.

Entropy

The entropy S is:

$$S = -\int \mathcal{D}[\delta g]\mathcal{D}[\delta\psi]\, p[\delta g, \delta\psi] \ln p[\delta g, \delta\psi].$$

Expanding to second order:

$$S = \frac{S_E^{(0)}}{\hbar} + \frac{\langle \delta S_E^{(2)} \rangle}{\hbar} + \ln \mathcal{Z}.$$

Here, $\langle \delta S_E^{(2)} \rangle$ represents the expectation value of the second-order perturbation over all configurations δg and $\delta\psi$.

—

Conclusion

1. First-Order Action: Vanishes due to the Einstein field equations and boundary conditions.
2. Second-Order Action: Contains terms quadratic in $\delta g_{\mu\nu}$, δR, and $\delta\mathcal{L}$, but no explicit $R\delta^2 g$ term.
3. Entropy: Includes zeroth- and second-order contributions to the action, reflecting the role of perturbations in defining statistical properties of the system.

23.4.12 Conclusion

The Taylor expansion we describe is a foundational technique in perturbative gravity. It is widely used in general relativity, quantum field theory in curved spacetime, and cosmological perturbation theory. The key is to ensure that all terms in the expansion are consistent with the theory's symmetries (e.g., diffeomorphism invariance) and that higher-order terms are systematically accounted for when necessary.

24 Description of 5 fundamental field-particles

Here is the mathematical description of the 5 fundamental field-particles, which compose the rest of the fundamental field particles found in the literature.

24.1 Spaciallion: Fluctuation of the Spatial Space Description

A *spaciallion* represents perturbations in the spatial geometry, tied to fluctuations in the metric tensor g_{ij} (spatial components of $g_{\mu\nu}$).

Mathematical Description

The fluctuations of the spatial part of the metric are captured by:

$$g_{ij} = g_{ij}^{(0)} + \delta g_{ij},$$

where δg_{ij} quantifies deviations from the background geometry. These fluctuations affect the spatial Ricci scalar R_{ij} and its dynamics.

Governing Equations

The perturbations obey the Einstein field equations linearized around the background metric:

$$\delta G_{ij} = 8\pi G_{\mathrm{N}} \delta T_{ij},$$

where δG_{ij} involves the perturbed Ricci tensor and Ricci scalar. For scalar, vector, and tensor perturbations, these equations decompose further. For example:

- Scalar perturbations involve the trace part of δg_{ij}.
- Tensor perturbations (gravitational waves) satisfy the wave equation:

$$\Box \delta g_{ij} = 0.$$

—

24.2 Timion: Fluctuation of the Temporal Space Description

A *timion* represents perturbations in the temporal geometry, associated with g_{00} (the temporal component of $g_{\mu\nu}$).

Mathematical Description

Temporal fluctuations are given by:

$$g_{00} = g_{00}^{(0)} + \delta g_{00},$$

where δg_{00} contributes to changes in the lapse function or proper time.

Governing Equations

These fluctuations modify the time evolution of the metric and fields via the Hamiltonian constraint and evolution equations:

$$\delta G_{00} = 8\pi G_{\mathrm{N}} \delta T_{00}.$$

The perturbation equations describe how δg_{00} interacts with matter fields and other components of the metric.

—

24.3 Probablon: Fluctuation of the Probability Space Description

A *probablon* represents fluctuations in the probability density associated with the path integral formalism of quantum field theory.

Mathematical Description

The probability density p is linked to the exponential of the Euclidean action:

$$p[\delta g, \delta\psi] \propto e^{-S_E^{(2)}[\delta g,\delta\psi]/\hbar}.$$

Fluctuations in probability are described as perturbations in $S_E^{(2)}$:

$$S_E^{(2)} = S_E^{(2,0)} + \delta S_E^{(2)},$$

where $\delta S_E^{(2)}$ corresponds to stochastic variations in probability space.

Governing Equations

The fluctuations obey Fokker-Planck-like equations governing the dynamics of p:

$$\frac{\partial p}{\partial t} = \mathcal{L}_F p,$$

where $\mathcal{L}_F$ is a Liouville or Fokker-Planck operator derived from $\delta S_E^{(2)}$. See section C for more details on the operator.

However the fluctuations of the probability, corresponding to the probability field particles, the probablons, of an event can be written in the form of $\delta p(E)$.

Then we know that the probability of any event is restricted between 0, and 1, therefore the fluctuations are also restricted in the same region, therefore

$$0 \leq \delta p(E) \leq 1 \tag{24.1}$$

—

24.4 4. Informanton: Fluctuation of the Information Space Description

An *informaton* represents perturbations in the entropy or information content of the system.

Mathematical Description

The entropy S is given by:

$$S = -\int p \ln p \, \mathcal{D}[\delta g]\mathcal{D}[\delta\psi].$$

Fluctuations in information arise from changes in p, leading to perturbations:

$$\delta S = \int \delta p \ln p \, \mathcal{D}[\delta g]\mathcal{D}[\delta\psi].$$

A simpler mathematical description

Note that the information is simply

$$I[p(E)] = -\ln p(E) \text{ or } I[p(E)] \qquad = -\log_{10} p(E) \tag{24.2}$$

Since the probability is restricted to

$$0 \leq \delta p(E) \leq 1 \tag{24.3}$$
$$10^{-\infty} \leq \delta p(E) \leq 1 \tag{24.4}$$
$$\tag{24.5}$$

then

$$-\log_{10} 10^{-\infty} \geq -\log_{10} \delta p(E) \geq -\log_{10} 1 \tag{24.6}$$
$$\infty \geq I[p(E)] \geq 0 \tag{24.7}$$

Therefore the foverning equation of information is

$$I[p(E)] \geq 0 \tag{24.8}$$

Governing Equations

Fluctuations in entropy are tied to the dynamics of p through the second law of thermodynamics:

$$\frac{dS}{dt} \geq 0.$$

They can also be studied using information-theoretic metrics, such as the Kullback-Leibler divergence.

Since time is positive, then we need that the fluctuations of average information, or the fluctuations of entropy is also positive, therefore we get

$$\delta S_E \geq 0. \tag{24.9}$$

—

24.5 Actionion full description

The actionion (or actionic field particle) is a concept that can be described as a particle whose properties and dynamics arise from the perturbations or fluctuations of an action in the context of quantum field theory, gravity, or other physical systems. Its description involves a field that behaves in a manner akin to a particle but whose dynamics are governed by the variations in the action, especially when the action is expressed in terms of quantum fields or in perturbative expansions.

24.5.1 Mathematical Description of the Actionion

1. Action and Field Perturbations: The action S in a field theory, such as gravity or quantum field theory, describes the dynamics of fields and particles. For a given system, we can write the action as a functional of the fields (e.g., the metric tensor $g_{\mu\nu}$, scalar fields ϕ, etc.):

$$S[g_{\mu\nu}, \phi, \dots] = \int d^4x\, \mathcal{L}[g_{\mu\nu}, \phi, \dots],$$

where $\mathcal{L}$ is the Lagrangian density, and the fields are functions of spacetime coordinates.

2. Field as Perturbation of Action:

The actionion can be thought of as a perturbation around the classical background action. Specifically, perturbations in the fields (e.g., $\delta g_{\mu\nu}$, $\delta\phi$) give rise to fluctuations in the action:

$$S_{\text{total}} = S_0 + \delta S,$$

where S_0 is the background (or classical) action, and δS represents the perturbation due to field fluctuations.

In terms of perturbations, the total action can be written as:

$$S = S_0 + \int d^4x \, (\mathcal{L}_0 + \delta\mathcal{L}) \, .$$

Here, $\delta\mathcal{L}$ corresponds to the perturbation to the Lagrangian due to small deviations of the fields from their background values.

3. Second-Order Expansion: To describe the actionion mathematically, we consider the second-order expansion of the action (as the actionion is related to second-order perturbations):

$$S \approx S_0 + \int d^4x \left(\mathcal{L}_0 + \frac{1}{2}\delta\mathcal{L}^2 + \dots \right),$$

where the second-order term captures quadratic fluctuations in the fields.

4. Quantum Path Integral:

From the path integral perspective, the actionion is described as a particle-like excitation around the classical background. The quantum amplitude for the fluctuation of a field $\delta\phi$ (associated with the actionion) is given by:

$$Z = \int \mathcal{D}\phi \, e^{iS[\phi]} = \int \mathcal{D}\phi \, e^{iS_0[\phi] + i\delta S[\phi]}.$$

The actionion's behavior can be interpreted as arising from the path integral over fluctuations in the fields.

5. Lagrangian Density:

If the action is a functional of the metric and fields, its perturbations (including second-order) will yield terms like:

$$\mathcal{L} \sim \frac{1}{2}\delta g_{\mu\nu}\delta R^{\mu\nu} + \dots,$$

where $\delta R^{\mu\nu}$ represents the perturbation in the Ricci tensor due to changes in the metric.

6. Actionion as a Quantum Field:

The actionion can be viewed as a quantum field particle in a quantum field theory framework. For instance, in gravity, the actionion would correspond to perturbations in the gravitational field that propagate as particles (gravitons in the case of quantum gravity). The corresponding equation of motion for this field would be governed by the variation of the action with respect to the perturbations, leading to field equations for the actionion.

7. Effective Action for Actionion:

The effective action for the actionion can be written as:

$$\Gamma[\phi] = \int d^4x \left(\frac{1}{2}\delta\mathcal{L}[\phi] + \text{higher-order terms} \right),$$

where $\Gamma[\phi]$ includes the perturbative contributions to the action, describing the propagation and interaction of actionions.

8. Interaction with Other Fields:

The actionion may interact with other fields in the system. For example, in gravity, the interaction between the actionion and matter could be described by the coupling of perturbations in the metric to matter fields, resulting in a modified Einstein field equation:

$$\delta G_{\mu\nu} = 8\pi G_{\mathrm{N}} \delta T_{\mu\nu},$$

where $\delta G_{\mu\nu}$ represents the perturbation in the Einstein tensor due to fluctuations in the metric, and $\delta T_{\mu\nu}$ represents the perturbation in the stress-energy tensor.

—

24.5.2 Actionions: 2nd order of actionic fluctuations in details

The *actionions*, or *actionic fluctuation field particles* , arise from the second-order variation of the action and describe quantum fluctuations of the fields around a classical background. Let us formalize their equations and provide a complete mathematical description.

—

1. Actionic Field from Second-Order Variation

The action S_A is expanded as:

$$S_A = S_A^{(0)} + \delta S_A^{(1)} + \frac{1}{2}\delta^2 S_A + \cdots ,$$

where:

- $S_A^{(0)}$: Action evaluated for the background field configuration.
- $\delta S_A^{(1)} = 0$: The first variation vanishes due to the classical equations of motion.
- $\delta^2 S_A$: Second-order variation, which governs the dynamics of fluctuations.

The second-order variation $\delta^2 S_A$ takes the form:

$$\delta^2 S_A = \int_X d^D x \sqrt{-g}\, \Phi(x)\, \mathcal{O}[\Phi(x)] + \text{interaction terms},$$

where:

- $\Phi(x)$: Represents the fluctuation fields (e.g., metric perturbations, matter fields).
- $\mathcal{O}[\Phi(x)]$: A differential operator derived from S_A, describing the dynamics of fluctuations.

—

2. Equation of Motion for Actionic Fluctuations

The fluctuation field $\Phi(x)$ satisfies the equation of motion:

$$\mathcal{O}[\Phi(x)] = 0,$$

where $\mathcal{O}$ depends on the system. For example:

- In the Einstein-Hilbert action for gravity, $\Phi(x)$ represents metric perturbations $h_{\mu\nu}$, and $\mathcal{O}$ is the spin-2 wave operator for gravitons:

$$\mathcal{O}^{\mu\nu\rho\sigma}[h_{\mu\nu}] = -\Box h_{\rho\sigma} + \cdots .$$

- For scalar fields, $\Phi(x)$ represents scalar fluctuations, and $\mathcal{O}$ is the Klein-Gordon operator:

$$\mathcal{O}[\Phi(x)] = (\Box - m^2)\Phi(x).$$

The equation of motion reflects the dynamics of the actionic fluctuation.

—

3. Energy and Propagation of Actionions

The second-order action determines the energy and propagation of the actionions. From:

$$\delta^2 S_A = \int_X d^D x \sqrt{-g}\left(\Phi(x)\,\mathcal{O}[\Phi(x)] + \text{interaction terms}\right),$$

the propagation equation (free part) corresponds to:

$$\mathcal{O}[\Phi(x)] = 0.$$

The interaction terms represent couplings between fluctuation modes and encode non-linearities in the system.

For example:

- In Gravity: Gravitational waves (actionic fluctuations of the metric) obey the linearized Einstein equations.

- In Quantum Fields: Scalar, vector, or tensor fields fluctuate according to the equations derived from the second-order expansion.

—

4. Quantization and Interpretation

A. Quantization:

- $\Phi(x)$ is quantized to describe quanta of the action fluctuations:

$$\Phi(x) = \sum_k \left(a_k u_k(x) + a_k^\dagger u_k^*(x)\right),$$

where a_k and $a_k^\dagger$ are creation and annihilation operators.

B. Physical Interpretation:

- The actionions correspond to the quanta of the action fluctuations.

- In gravitational systems, they are gravitons; in scalar systems, they are scalar particles, etc.

—

5. Complete Equation of Actionic Fluctuations

Combining the dynamics and propagation, the *main equation of actionic fluctuation field particles* is:

$$\mathcal{O}[\Phi(x)] = 0,$$

where:

- $\mathcal{O}$ is derived from the second-order expansion of the action:

$$\mathcal{O}[\Phi(x)] = \frac{\delta^2 S_A}{\delta\Phi(x)\delta\Phi(y)}.$$

Additionally, interaction terms may modify the propagation:

$$\mathcal{O}[\Phi(x)] = J(x),$$

where $J(x)$ accounts for sources or interactions.

—

6. Physical Implications
- Second-Order Action:
Describes the quadratic fluctuations around the classical solution.
- Actionions:
Are the quantum field particles arising from these fluctuations.
- Equations:
Governed by the operator $\mathcal{O}[\Phi(x)]$ derived from the second-order perturbative action.

In summary, *actionic fluctuation field particles (actionions)* are described by the equation:

$$\mathcal{O}[\Phi(x)] = 0,$$

derived from $\delta^2 S_A$, and encode the dynamics of quantum fluctuations around classical solutions.

24.5.3 Summary of Mathematical Description

- The actionion arises from perturbations in the action of a quantum field theory or classical field theory.

- Its behavior is governed by the second-order perturbation of the action, captured in the expansion of the Lagrangian.

- The actionion can be treated as a quantum field, with its dynamics described by the path integral formalism.

- The perturbations lead to fluctuations in the action, and these fluctuations propagate as a field-particle.

- The actionion can interact with other fields in the system, and its dynamics are encoded in the effective action and field equations.

This mathematical framework provides the description of the actionion as a particle-like excitation in a field theory, with its properties emerging from the perturbative expansion of the action and the fluctuations around a background configuration.

24.6 Summary of Governing Equations

Entity	**Fluctuation Description**	**Governing Equations**
Spaciallion	δg_{ij} : spatial geometry	$\delta G_{ij} = 8\pi G \delta T_{ij} \Leftrightarrow \Box \delta g_{ij} = 0$
Timion	δg_{00} : temporal geometry	$\delta G_{00} = 8\pi G \delta T_{00} \Leftrightarrow \Box \delta g_{00} = 0$
Probablon	δp : probability fluctuations	$0 \leq p(E) \leq 1$
Informaton	δS_E : entropy fluctuations	$\delta S_E \geq 0$
Actionion	δS_A : actionic fluctuations	$\delta^2 S_A \neq 0$

—

This table provides a concise summary of the entities fluctuations and their mathematical descriptions.

24.7 PISTA interpretation

Note that rearranging and abbreviating the concepts to Probablon-Informaton-Spaciallion-Timion-Actionion (PISTA), we get PISTA which is the protoindoeuropean word for track. Therefore effectively we can think of these 5 fundamental field-particles as the track, i.e. PISTA, in which all fundamental particles are created in.

24.8 Diagrammatical description of 5 fundamental entities

The standard cosmological model, which is rooted in General Relativity and perturbation theory, essentially creates a spacetime that assumes the existence of gravitational waves in its perturbations. This implies the presence of the graviton particle-wave or graviton field, alongside the cold dark matter field. Despite the absence of direct detection, the notion of the graviton field suggests that the particle-wave duality is already intrinsic to GR and the SMC. By incorporating the probabilistic concept into the perturbed spacetime, we uncover another aspect of the wave-particle duality of the graviton, namely its probabilistic or informatic nature. This discovery parallels the concepts introduced by quantum field theory, which attributes probabilistic qualities to all elementary particles. Furthermore, we will introduce a novel and useful abstract concept that distinguishes between different properties of fields or elementary particles. These compound words, which have yet to be discussed in the literature, are motivated by general arguments and play an essential role in understanding the nature of the fields.

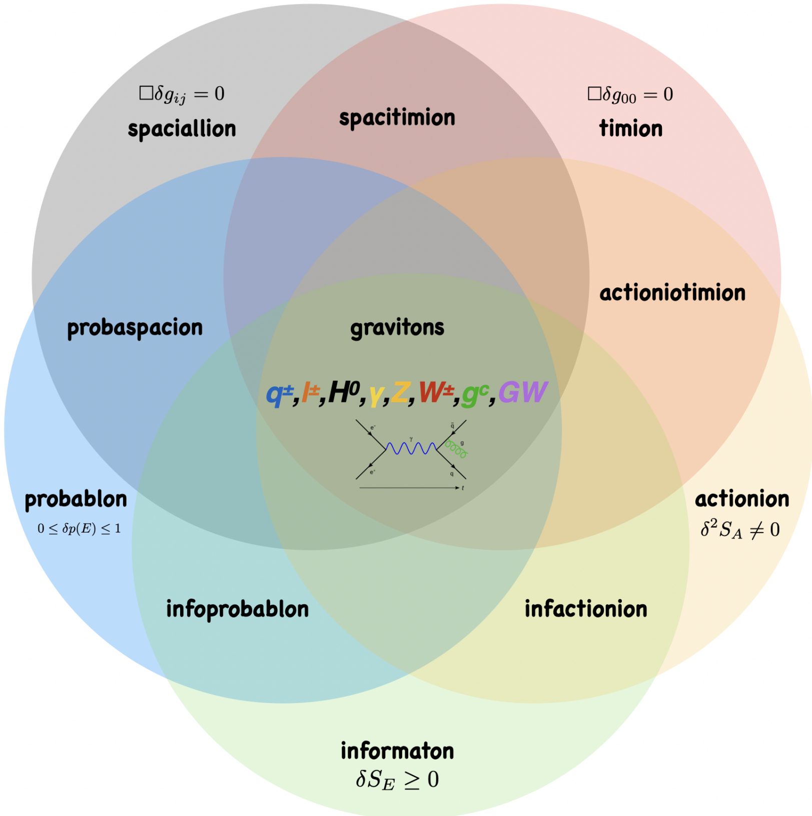

Figure 1. Coloured Venn Diagram expressed of the novel 5 fundamental field-particle entities: spaciallion, timion, probablon, informaton, actionion. Furthermore, we present their compounds and the so far discovered particles, i.e. the charged quarks, $q^{\pm}$, charged leptons, $\ell^{\pm}$, Higgs field-particle, H^0, photons, γ, Neutral, Z^0, and charged bosons, $W^{\pm}$, coloured charged gluons, g^c, Gravitational Wave fields, GW, which are build from all the 5 novel fundamental field-particles.

To describe all the field-particles that we observe in nature we can identify some novel particles according to the properties of the field-particles discovered so far. In nature we come across five fundamental properties, spatial space, time, probability, information, and action. Therefore, we can build five novel fundamental particles, the spacion, the timion, the probablon and the actionion. We can explain below, how we can construct these ideas further.

In the Fig. 1, we present these 5 fundamental field particles, using coloured Venn Diagram. Furthermore, we present compounds of those field particles and the graviton, yet to be discovered. We show that the so far discovered field-particles from particle physics and cosmology, i.e. the charged quarks, $q^{\pm}$, charged leptons, $\ell^{\pm}$, Higgs field-particle, H^0, photons, γ, Neutral, Z^0 and charged bosons, $W^{\pm}$, coloured charged gluons, g^c, Gravitational Wave fields, GW, which are build from all those fundamental particles.

Motivated by generic arguments of probabilities, an elementary particle which possesses probabilistic properties, would be named probablon or probablion. We continue with the probablon term since it is both euphonic and a simpler nomenclature. So far these kinds of particles have been mapped by physicists, and they are the well-known elementary particles, the quarks, leptons, bosons, and fermions, which all share probabilistic notions. However, the probablon category of elementary particles generalises the notion of elementary particles, in the sense that there are elementary particles which possess a probabilistic nature and those which have no such nature. So far nature shows that most elementary particles possess kinds of probabilistic properties, such as the ones which are inherited by QFT.

Based on the general argument of probabilities, an elementary particle that exhibits probabilistic properties can be referred to as a "probablon" or "probablion". In this research, we will build novel term "probablon" since it is more pleasant to the ear and has a simpler name. Physicists have already identified a group of elementary particles that share probabilistic qualities, including quarks, leptons, bosons, and fermions. These particles have been extensively studied and mapped. However, the probablon category of elementary particles expands on the notion of elementary particles by acknowledging that there are some that possess a probabilistic nature, and others that do not. The majority of elementary particles discovered in nature thus far exhibit some form of probabilistic property, which is similar to those inherited by particles described in QFT.

We will construct the concept of the "probabilistic graviton", which is a graviton with additional probabilistic qualities as described earlier. It is important to note that the graviton is essentially a particle of spacetime, and its name is derived historically from its connection to gravity. However, given its spacetime properties, we can also refer to it as a "spacetimion". Building on this distinction, we introduce the concept of "spacion", "spatiallion", and "timion" as additional categories of particles. A spacion is a particle that belongs to a general space, rather than a specific spacetime. A spatiallion, on the other hand, exists within a generic space, and a timion exists within time. To clarify further, a particle with a spatial property exists within space, which is a fundamental and generic property that most known particles share, though it has not been highlighted explicitly. For example, every particle field exists within a generic space and belongs to the category of spacions. We can also imagine particles that exist within a space with probabilistic properties, which we would call "probablispacion". Therefore, a particle that possesses both probabilistic qualities and spacetime properties can be called a "probablispacetimion", which is a more precise term than the previously used "graviton".

We will introduce the concepts of informaton as any particle or field that possesses information. This will be which includes spacion and probablon particles. Elementary particles are also considered informatons as they possess information about their mass, charge, or spin. Compound particles with properties of informaton and timion, informaton and graviton, and probablon and informaton are called informatimion, informatic graviton, and probablinformaton, respectively. The author poses the question of whether all elementary particles possess all these properties or if there are new classes of particles yet to be discovered that do not possess these properties introduced by QFT and gravity using perturbation theory.

To summarize, this study introduces several novel particles including the spacion, spatiallion, timion, informaton, and probablon. Additionally, it introduces several compounds such as the spacetimion, probablispacion, probablispacetimion, and probabilistic graviton by analyzing the concept of graviton further. While this study has only provided a basic description of the properties of these particles, it is essential to make such distinctions before beginning a serious search for these particles. Further discussions and implications of this topic will be left for future studies.

25 Conclusions and Discussion

In this document, we introduce functors of actions theory (FAT) and a simple quadratic FAT model. We present how the quadratic FAT models applied in simple harmonic oscillators, gravity and cosmology. We show how FAT is related to estimates of the partition function. We provide simple estimates on the partition function and their impact on the existence and quantification of actionic fluctuations from these models. Furthermore, we mathematically argue about the existence and suppression of the local invariance of this quadratic model. We generalise the stationary action principle. We describe a general description of the actionion, field particles, using the Taylor expansion description. We provide an effective mathematical description of new *5 fundamental field-particles, i.e. the spatiallion, timion, actionion, probablon and informaton*, arise from studying the spacetime, actions, probabilities, entropy and information combined.

We conclude that functors of actions theory models in cosmology, quantum gravity are valid candidates for further exploration.

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A Computing the zeta-function regularisation on d'Alambertian

To compute the eigenvalues of the d'Alembertian operator $-\Box$ in the context of zeta-function regularization, we need to analyze the spectrum of this operator in a given spacetime. Below, I'll outline the general procedure for finding the eigenvalues and applying zeta-function regularization.

A.1 1. Understanding the d'Alembertian Operator $-\Box$

The d'Alembertian (or wave operator) $\Box$ in d-dimensional flat spacetime with the Minkowski metric $\eta_{\mu\nu}$ is defined as:

$$\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu = -\frac{\partial^2}{\partial t^2} + \nabla^2$$

where ∇^2 is the Laplacian operator in the spatial dimensions.

A.2 2. Eigenfunctions and Eigenvalues of $-\Box$

In a flat spacetime, we typically consider plane wave solutions as the eigenfunctions of the d'Alembertian:

$$\psi_k(x) = e^{ik_\mu x^\mu}$$

where $x^\mu = (t, \vec{x})$ and $k_\mu = (\omega, \vec{k})$ is the four-momentum.
Applying the d'Alembertian to this eigenfunction:

$$\Box\psi_k(x) = \eta^{\mu\nu}\partial_\mu\partial_\nu e^{ik_\mu x^\mu} = -\left(\omega^2 - |\vec{k}|^2\right) e^{ik_\mu x^\mu}$$

Therefore, the eigenvalue associated with the eigenfunction $\psi_k(x)$ is:

$$\lambda_k = -\left(\omega^2 - |\vec{k}|^2\right)$$

For massless fields, or in the context of the d'Alembertian acting on scalar fields, we typically have $\omega = |\vec{k}|$, so:

$$\lambda_k = -k^2$$

where $k^2 = \omega^2 - |\vec{k}|^2$.

A.3 3. Zeta Function for the d'Alembertian

To compute the determinant using zeta-function regularization, we construct the zeta function $\zeta(s)$ for the operator $-\Box$. The zeta function is defined as:

$$\zeta(s) = \sum_n \lambda_n^{-s}$$

where λ_n are the eigenvalues of $-\Box$. In a continuous spectrum, this sum is replaced by an integral over momentum space.

A.4 4. Computing the Zeta Function in Flat Spacetime

For flat spacetime, the eigenvalues λ_n are continuous and correspond to $\lambda_k = k^2$. Therefore, the zeta function is expressed as an integral over the momentum space:

$$\zeta(s) = \int \frac{d^dk}{(2\pi)^d} \frac{1}{(k^2)^s}$$

Using spherical coordinates in d-dimensional momentum space, the measure d^dk becomes:

$$d^dk = \Omega_{d-1}k^{d-1}dk$$

where Ω_{d-1} is the surface area of the unit sphere in $d-1$ dimensions:

$$\Omega_{d-1} = \frac{2\pi^{d/2}}{\Gamma(d/2)}$$

The zeta function now becomes:

$$\zeta(s) = \frac{\Omega_{d-1}}{(2\pi)^d} \int_0^\infty dk\, k^{d-1} \frac{1}{(k^2)^s}$$

Simplifying the integrand:

$$\zeta(s) = \frac{\Omega_{d-1}}{(2\pi)^d} \int_0^\infty dk\, k^{d-1-2s}$$

This integral converges for $s > d/2$ and can be evaluated as:

$$\int_0^\infty dk\, k^{d-1-2s} = \frac{1}{2} \frac{\Gamma\left(\frac{d}{2} - s\right)}{\Gamma\left(\frac{d}{2}\right)}$$

Thus, the zeta function becomes:

$$\zeta(s) = \frac{\Omega_{d-1}}{(2\pi)^d} \cdot \frac{1}{2} \cdot \frac{\Gamma\left(\frac{d}{2} - s\right)}{\Gamma\left(\frac{d}{2}\right)}$$

A.5 5. Regularized Determinant of $-\Box$

To obtain the determinant, we need to compute $\zeta'(s)$ at $s = 0$:

$$\ln \det(-\Box) = -\zeta'(0)$$

Taking the derivative with respect to s and evaluating at $s = 0$:

$$\zeta'(0) = -\frac{\Omega_{d-1}}{(2\pi)^d} \cdot \frac{1}{2} \cdot \left(\Gamma'\left(\frac{d}{2}\right) - \frac{d}{2}\Gamma'\left(\frac{d}{2}\right)\right)$$

This result can be simplified further depending on the specific dimension d. For $d = 4$:

$$\zeta'(0) = -\frac{1}{2(4\pi)^2} \cdot \text{(finite part of the Gamma function)}$$

So the determinant is:

$$\ln \det(-\Box) \sim \text{(finite part)} + \text{(logarithmic divergences)}$$

After subtracting the divergences, the remaining part provides the finite contribution.

A.6 Conclusion

The eigenvalues of the d'Alembertian in flat spacetime are $\lambda_k = k^2$. The zeta-function regularization computes the determinant of $-\Box$ by summing over these eigenvalues, leading to a finite determinant that contributes to the effective action or partition function. The exact value depends on the spacetime dimension and the regularization scheme used.

B Zeta-function regularization for the d'Alembertian in an FLRW metric

Computing the zeta-function regularization for the d'Alembertian in an FLRW (Friedmann-Lemaître-Robertson-Walker) metric is more involved than in flat spacetime due to the curved nature of the FLRW background. The procedure requires finding the spectrum of the d'Alembertian in this curved spacetime and then applying the zeta-function method to regularize the determinant.

B.1 1. The FLRW Metric

The FLRW metric in $(d+1)$-dimensional spacetime (where d is the number of spatial dimensions) is given by:

$$ds^2 = -dt^2 + a^2(t)\gamma_{ij}dx^i dx^j$$

where: - t is the cosmic time, - $a(t)$ is the scale factor, - γ_{ij} is the metric of the spatial section (which can be flat, open, or closed).

For simplicity, we'll assume the spatial sections are flat, so $\gamma_{ij} = \delta_{ij}$.

B.2 2. The d'Alembertian in FLRW

The d'Alembertian $\Box$ in a general curved spacetime is:

$$\Box = \frac{1}{\sqrt{-g}}\partial_\mu \left(\sqrt{-g}g^{\mu\nu}\partial_\nu\right)$$

For the FLRW metric, with $\sqrt{-g} = a^d(t)$, the d'Alembertian acting on a scalar field ϕ is:

$$\Box\phi = -\frac{1}{a^d(t)}\frac{\partial}{\partial t}\left(a^d(t)\frac{\partial\phi}{\partial t}\right) + \frac{1}{a^2(t)}\nabla^2\phi$$

where ∇^2 is the Laplacian in the spatial coordinates.

B.3 3. Eigenfunctions and Eigenvalues

The eigenfunctions of the spatial Laplacian ∇^2 in flat space are plane waves $e^{i\vec{k}\cdot\vec{x}}$ with eigenvalues $-k^2$, where $k^2 = \vec{k}\cdot\vec{k}$. Hence, the general mode decomposition for the scalar field ϕ is:

$$\phi(t,\vec{x}) = \sum_{\vec{k}} \chi_k(t)e^{i\vec{k}\cdot\vec{x}}$$

Substituting this into the wave equation $\Box\phi = 0$ gives an equation for the time-dependent mode function $\chi_k(t)$:

$$\ddot{\chi}_k + d\frac{\dot{a}}{a}\dot{\chi}_k + \left(\frac{k^2}{a^2}\right)\chi_k = 0$$

The eigenvalues of the full d'Alembertian $\Box$ are thus functions of both the wavenumber k and the scale factor $a(t)$.

B.4 4. Zeta-Function Regularization

The zeta function $\zeta(s)$ associated with the operator $-\Box$ can be expressed as a sum over the eigenvalues λ_k:

$$\zeta(s) = \sum_{\text{modes}} \lambda_k^{-s}$$

For the FLRW case, the eigenvalues are not as straightforward as in flat spacetime due to the time dependence introduced by the scale factor $a(t)$. Specifically, the eigenvalue for each mode might be written as:

$$\lambda_k(t) = \frac{k^2}{a^2(t)}$$

Thus, the zeta function becomes:

$$\zeta(s) = \int \frac{d^dk}{(2\pi)^d} \left(\frac{k^2}{a^2(t)}\right)^{-s} = \frac{1}{a^{2s}(t)} \int \frac{d^dk}{(2\pi)^d} k^{-2s}$$

However, the temporal dependence complicates the regularization procedure since the eigenvalues change with time. Therefore, to proceed, we typically:

- Perform a mode-by-mode analysis: Compute the zeta function for each eigenvalue at a fixed time t_0 (or over some averaging process).

- Regularize the sum/integral: Apply standard techniques of zeta-function regularization or related techniques (such as the heat kernel method) to evaluate the infinite sum.

B.5 5. Regularizing the Determinant

The determinant of $-\Box$ in the FLRW background is obtained from the zeta function by evaluating:

$$\ln\det(-\Box) = -\zeta'(0)$$

This requires detailed knowledge of the eigenvalue spectrum, which, in an FLRW background, is not simply k^2 but is modulated by the scale factor $a(t)$. The result will typically be a time-dependent function that reflects the dynamics of the FLRW spacetime.

B.6 6. Interpretation and Consequences

In an expanding universe (with $a(t)$ increasing), the eigenvalues $\lambda_k(t) = k^2/a^2(t)$ decrease over time, which affects the effective action obtained from the determinant. This can lead to time-dependent contributions to the vacuum energy (cosmological constant) or other quantum corrections in cosmology.

B.7 Conclusion

Applying zeta-function regularization in the context of an FLRW metric is more complex due to the time-dependent nature of the eigenvalues. The process involves computing the spectrum of the d'Alembertian, regularizing the associated zeta function, and extracting the determinant. The outcome reflects the interplay between the quantum field and the dynamics of the expanding universe, with significant implications for cosmology, including effects on the vacuum energy and the cosmological constant.

C Liouville or Fokker-Planck operator

The Liouville or Fokker-Planck operator, $\mathcal{L}_F$, governs the evolution of the probability density $p(x,t)$ in a stochastic or deterministic dynamical system. The specific form depends on the type of system under consideration, but here's a general description:

—

1. Deterministic Systems (Liouville Operator)

For a deterministic dynamical system:

$$\frac{dx}{dt} = f(x),$$

where x represents the state variables and $f(x)$ is the drift term (deterministic flow), the probability density $p(x,t)$ evolves according to the Liouville equation:

$$\frac{\partial p(x,t)}{\partial t} + \nabla \cdot (p(x,t)f(x)) = 0.$$

The Liouville operator is given by:

$$\mathcal{L}_F p(x,t) = -\nabla \cdot (p(x,t)f(x)).$$

—

2. Stochastic Systems (Fokker-Planck Operator)
For a stochastic system described by a Langevin equation:

$$\frac{dx}{dt} = f(x) + \eta(t),$$

where $f(x)$ is the deterministic drift term and $\eta(t)$ is a stochastic noise (typically Gaussian), the probability density $p(x,t)$ satisfies the Fokker-Planck equation:

$$\frac{\partial p(x,t)}{\partial t} = -\nabla \cdot (p(x,t)f(x)) + \frac{1}{2}\nabla \cdot (D(x)\nabla p(x,t)),$$

where $D(x)$ is the diffusion coefficient (or diffusion tensor for multi-dimensional systems).

The Fokker-Planck operator is given by:

$$\mathcal{L}_F p(x,t) = -\nabla \cdot (p(x,t)f(x)) + \frac{1}{2}\nabla \cdot (D(x)\nabla p(x,t)).$$

—

3. Components of the Fokker-Planck Operator
Breaking the Fokker-Planck operator into its components:
- Drift Term:

$$\mathcal{L}_{\text{drift}} p(x,t) = -\nabla \cdot (p(x,t)f(x)).$$

This describes the deterministic evolution of the probability density due to the system's flow.
- Diffusion Term:

$$\mathcal{L}_{\text{diffusion}} p(x,t) = \frac{1}{2}\nabla \cdot (D(x)\nabla p(x,t)).$$

This accounts for the spreading of probability density due to stochastic fluctuations.

—

4. Compact Form of the Fokker-Planck Operator
Combining both terms:

$$\mathcal{L}_F p(x,t) = -\sum_i \frac{\partial}{\partial x_i}(f_i(x)p(x,t)) + \frac{1}{2}\sum_{i,j}\frac{\partial^2}{\partial x_i \partial x_j}(D_{ij}(x)p(x,t)),$$

where $D_{ij}(x)$ is the diffusion tensor.

—

C.1 Physical and Geometrical Interpretation

1. Drift term: Describes how the deterministic dynamics transport probability in the state space.

2. Diffusion term: Represents random fluctuations causing the spread of probability in state space.

These operators are used extensively in quantum mechanics, statistical mechanics, and field theory to describe the evolution of systems with deterministic and stochastic components.

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Integral-differential equation in Functors of Actions Theories in Cosmology

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Abstract

This paper delves into Functors of Actions Theories (FATs), presenting models applied in cosmology. We explore integro-differential equations derived from these models.

Keywords: cosmology, functors of actions theories, gravity

Contents

1 Introduction

The quest to unify gravity with quantum mechanics and explain cosmological phenomena beyond the Standard Cosmological Model (ΛCDM) [10] has driven the development of modified gravity (MG) theories [1, 2, 13]. Functors of Actions Theories (FATs) [7, 8], a novel MG framework, diverge from traditional approaches by applying mathematical functors directly to the action, yielding integral-differential equations rather than purely differential ones. This shift opens new avenues for modeling spacetime dynamics, challenging conventional General Relativity (GR) [15] and offering potential insights into dark energy, large-scale structure, and quantum gravity.

In the context of MG theories, which often modify geometrical aspects (e.g., metric $g_{\mu\nu}$, Ricci scalar R) or energy content (e.g., exotic fields) [3, 6], FATs propose a unique extension by manipulating the action itself. This study builds on prior work [9], applying FATs to cosmology and basic mechanical systems.

2 Functors of Actions Theories (FATs)

At the core of GR lies the action:

$$\mathcal{S}_{\rm GR} = c^4 \int d^4x\sqrt{-g}\left[\frac{R}{16\pi G_{\rm N}} + \mathcal{L}_m(g_{\mu\nu}, \psi)\right],$$

assuming a four-dimensional pseudo-Riemannian manifold with Lorentz invariance. FATs introduce a framework where the action is modified via functors, with specific and generalized forms explored here.

2.1 A specific generic function FAT model

More generally, FATs can be expressed as:

$$S^{\rm f}_{\rm FAT} = S_R + f(S_R) + S_\Lambda + S_m,$$

where f is a generic function of the action S_R. This form adapts FAT flexibly to various physical contexts. By applying the optimum action principle we get

$$\delta S^{\rm f}_{\rm FAT} = 0$$

which yields:

$$\left[R_{\mu\nu} + \frac{1}{2}Rg_{\mu\nu}\right][1 + \partial_{S_R}(S_R)] = \kappa^2\left[T_{\mu\nu} + \Lambda g_{\mu\nu}\right],$$

where it is convenient to define

$$\kappa^2 = \frac{8\pi G_{\rm N}}{c^4}.$$

(Note: This generalized functional form, while broader than the quadratic case, is still not the most complete generalization of FATs, which could involve full functorial transformations beyond simple functions of S_R.) **Interpretation**: The quadratic model modifies GR's dynamics with a non-linear curvature term, while the generalized form $f(S_R)$ offers flexibility, with S_R's magnitude reflecting the universe's geometric evolution and parameters like β fine-tuning actionic fluctuations.

Note that the action of curvature is

$$S_R = c^3 \int d^4x\sqrt{-g}\frac{R}{16\pi G_{\rm N}}$$

In an FLRW universe ($ds^2 = -d(ct)^2 + a^2(t)d\vec{x}^2$), $g = -a^6(t)$, $R = \frac{6}{c^2}\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right]$, so:

$$S_R = \frac{6c^2\mathcal{V}_{3D}}{16\pi G_{\rm N}}\int dt a^3(t)\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right].$$

2.2 A specific quadratic FAT model

For a specific quadratic model, we define:

$$S_{\text{FAT}}^{\text{quad}} = S_R + \beta S_R^2 + S_\Lambda + S_m,$$

where $\delta S_{\text{FAT}}^{\text{quad}} = 0$ yields:

$$\left[R_{\mu\nu} + \frac{1}{2}Rg_{\mu\nu}\right][1 + 2\beta S_R] = \frac{8\pi G_{\text{N}}}{c^4}\left[T_{\mu\nu} + \Lambda g_{\mu\nu}\right].$$

(Note: This quadratic form is a specific case of FATs, not encompassing the full generality of functorial transformations, which could involve more complex mappings beyond a simple quadratic term.)

For a dark energy-dominated universe, $a(t) = e^{H_0 t}$, $\dot{a}/a = H_0$, $\ddot{a}/a = H_0^2$, thus:

$$S_R = \frac{c^2 H_0}{4\pi G_{\text{N}}}\mathcal{V}_{3D}\left[e^{3H_0 t} - e^{3H_0 t_i}\right].$$

With $H_0 \approx 2 \times 10^{-17}\,\text{s}^{-1}$, $G_{\text{N}} \approx 7 \times 10^{-11}\,\text{m}^3\,\text{s}^{-2}\,\text{kg}^{-1}$, $\mathcal{V}_{3D} \approx 10^{77}\,\text{m}^3$, and $t \approx 10^{17}\,\text{s}$, $S_R \approx 8 \times 10^{88}\,\text{kg}\,\text{m}^2\,\text{s}^{-1}$. For fluctuations $\delta S_R = \beta S_R^2 \sim 10^{-5}$, $\beta \approx 10^{-182}\,\text{kg}^{-1}\,\text{m}^{-2}\,\text{s}$.

2.3 A specific exponential FAT model

For a specific quadratic model, we define:

$$S_{\text{FAT}}^{\text{exp}} = S_R + Ae^{\beta S_R} + S_\Lambda + S_m,$$

where $\delta S_{\text{FAT}}^{\text{exp}} = 0$ yields:

$$\left[R_{\mu\nu} + \frac{1}{2}Rg_{\mu\nu}\right]\left[1 + A\beta e^{\beta S_R}\right] = \frac{8\pi G_{\text{N}}}{c^4}\left[T_{\mu\nu} + \Lambda g_{\mu\nu}\right].$$

(Note: This quadratic form is a specific case of FATs, not encompassing the full generality of functorial transformations, which could involve more complex mappings beyond a simple quadratic term.)

3 The integral differential equation of $S_{\text{FAT}}^{\text{quad}}$

This is the integral differential equation from the quadratic FAT model:

$$3\left(\frac{\dot{a}}{a}\right)^2\left[1 + \frac{6V_{3D}}{c^2\kappa^2}\beta\int dt a^3(t)\left(\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right)\right] = \kappa^2\left(\rho_{m0}a^{-3}(t) + \rho_{r0}a^{-4}(t) + \rho_{\Lambda 0}\right)$$

- The constants remain $\beta, V_{3D}, c, \kappa, \rho_{m0}, \rho_{r0}, \rho_{\Lambda 0}$, as per my previous notes.

4 The integral differential equation of $S_{\text{FAT}}^{\text{exp}}$

This is the integral differential equation from the quadratic FAT model:

$$3\left(\frac{\dot{a}}{a}\right)^2\left[1+A\beta e^{\frac{6V_{3D}}{c^2\kappa^2}\beta\int dt a^3(t)\left(\frac{\ddot{a}}{a}+\left(\frac{\dot{a}}{a}\right)^2\right)}\right]=\kappa^2\left(\rho_{m0}a^{-3}(t)+\rho_{r0}a^{-4}(t)+\rho_{\Lambda 0}\right)$$

- The constants remain $A, \beta, V_{3D}, c, \kappa, \rho_{m0}, \rho_{r0}, \rho_{\Lambda 0}$, as per my previous notes.

5 Solving the Integral-Differential Equation from the Quadratic FAT Model

We are tasked with solving the following integral-differential equation, which arises from the quadratic FAT (Fourier Amplitude Theory) model in cosmology:

$$3\left(\frac{\dot{a}}{a}\right)^2\left[1+\frac{6V_{3D}}{c^2\kappa^2}\beta\int dt\, a^3(t)\left(\frac{\ddot{a}}{a}+\left(\frac{\dot{a}}{a}\right)^2\right)\right]=\kappa^2\left(\rho_{m0}a^{-3}(t)+\rho_{r0}a^{-4}(t)+\rho_{\Lambda 0}\right)$$

Here, $a(t)$ is the unknown function (the scale factor in a cosmological context), $\dot{a}=\frac{da(t)}{dt}$, and $\ddot{a}=\frac{d^2a(t)}{dt^2}$. This equation is nonlinear due to terms like $\left(\frac{\dot{a}}{a}\right)^2$ and the integral involving $\ddot{a}$ and $\dot{a}$. The right-hand side represents energy densities (matter, radiation, and cosmological constant), typical of a Friedmann equation, while the integral term introduces a modification from the FAT model. Let's solve this step by step, aiming for an analytical solution where possible, and resorting to numerical methods if necessary.

5.1 Step 1: Simplify with a Substitution

The term $\left(\frac{\dot{a}}{a}\right)^2$ suggests a substitution commonly used in cosmology. Define:

$$H(t)=\frac{\dot{a}}{a}$$

This gives:

$$\dot{a}=Ha$$

$$\left(\frac{\dot{a}}{a}\right)^2=H^2$$

Differentiate H:

$$\dot{H} = \frac{d}{dt}\left(\frac{\dot{a}}{a}\right) = \frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} = \frac{\ddot{a}}{a} - H^2$$

So:

$$\frac{\ddot{a}}{a} = \dot{H} + H^2$$

Compute the integrand:

$$\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 = \dot{H} + H^2 + H^2 = \dot{H} + 2H^2$$

The integral is:

$$\int dt\, a^3(t)\left(\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right) = \int dt\, a^3(t)(\dot{H} + 2H^2)$$

Assume the integral is from 0 to t (as is typical for initial value problems in cosmology):

$$I(t) = \int_{-\infty}^{t} a^3(\tau)(\dot{H}(\tau) + 2H^2(\tau))d\tau$$

Or Assume the integral is from $-\infty$ to t (as is typical for initial value problems in cosmology):

$$I(t) = \int_{0}^{t} a^3(\tau)(\dot{H}(\tau) + 2H^2(\tau))d\tau$$

The equation becomes:

$$3H^2\left[1 + \frac{6V_{3D}}{c^2\kappa^2}\beta I(t)\right] = \kappa^2\left(\rho_{m0}a^{-3}(t) + \rho_{r0}a^{-4}(t) + \rho_{\Lambda 0}\right)$$

Also, $a(t)$ is related to $H(t)$:

$$a(t) = a(0)e^{\int_0^t H(\tau)d\tau}$$

Define the constant:

$$K = \frac{6V_{3D}}{c^2\kappa^2}\beta$$

So the equation is:

$$3H^2\left[1 + KI(t)\right] = \kappa^2\left(\rho_{m0}a^{-3}(t) + \rho_{r0}a^{-4}(t) + \rho_{\Lambda 0}\right)$$

5.2 Step 2: Differentiate to Eliminate the Integral

The integral $I(t)$ makes this an integro-differential equation. Let's differentiate both sides with respect to t to eliminate the integral and convert it into a differential equation.

Compute the derivative of $I(t)$:

$$\dot{I}(t) = a^3(t)(\dot{H}(t) + 2H^2(t))$$

Differentiate the left-hand side:

$$\frac{d}{dt}\left(3H^2\left[1+K\int_0^t a^3(\tau)(\dot{H}(\tau)+2H^2(\tau))d\tau\right]\right)$$

Use the product rule:

$$6H\dot{H}\left[1+KI(t)\right]+3H^2K\dot{I}(t)$$

Substitute $\dot{I}(t)$:

$$6H\dot{H}\left[1+KI(t)\right]+3H^2Ka^3(\dot{H}+2H^2)$$

Differentiate the right-hand side:

$$\frac{d}{dt}\left(\kappa^2\left(\rho_{m0}a^{-3}+\rho_{r0}a^{-4}+\rho_{\Lambda 0}\right)\right)=\kappa^2\left(-3\rho_{m0}a^{-4}\dot{a}-4\rho_{r0}a^{-5}\dot{a}\right)$$

Since $\dot{a}=Ha$:

$$\kappa^2\left(-3\rho_{m0}a^{-4}(Ha)-4\rho_{r0}a^{-5}(Ha)\right)=\kappa^2H\left(-3\rho_{m0}a^{-3}-4\rho_{r0}a^{-4}\right)$$

The differentiated equation is:

$$6H\dot{H}\left[1+KI(t)\right]+3H^2Ka^3(\dot{H}+2H^2)=\kappa^2H\left(-3\rho_{m0}a^{-3}-4\rho_{r0}a^{-4}\right)$$

Divide through by H (assuming $H\neq 0$, which is true in an expanding universe where $\dot{a}>0$):

$$6\dot{H}\left[1+KI(t)\right]+3HKa^3(\dot{H}+2H^2)=\kappa^2\left(-3\rho_{m0}a^{-3}-4\rho_{r0}a^{-4}\right)$$

This is a nonlinear differential equation for $H(t)$, but $I(t)$, $a(t)$, and $H(t)$ are coupled. We need another transformation to make it more solvable.

5.3 Step 3: Change the Independent Variable

Since this is a cosmological equation, let's change the independent variable from t to a, which is often more convenient for Friedmann-like equations. Use $'$ to denote derivatives with respect to a:

$$\frac{d}{dt}=\dot{a}\frac{d}{da}=Ha\frac{d}{da}$$

$$\dot{H}=HaH'$$

$$\frac{\ddot{a}}{a}=\dot{H}+H^2=HaH'+H^2$$

$$\frac{\ddot{a}}{a}+\left(\frac{\dot{a}}{a}\right)^2=HaH'+2H^2$$

The integral $I(t)$:

$$I(t)=\int_0^t a^3(\dot{H}+2H^2)d\tau$$

Change variables: $d\tau = \frac{da}{\dot{a}} = \frac{da}{Ha}$, so:

$$I = \int_{a(0)}^{a(t)} a^3(HaH' + 2H^2)\frac{da}{Ha} = \int_{a(0)}^{a} a(aH' + 2H)da$$

The original equation:

$$3H^2 \left[1 + K\int_{a(0)}^{a} a(aH' + 2H)da\right] = \kappa^2 \left(\rho_{m0}a^{-3} + \rho_{r0}a^{-4} + \rho_{\Lambda 0}\right)$$

Differentiate with respect to a:

$$\frac{d}{da}\left(3H^2 \left[1 + K\int_{a(0)}^{a} a(aH' + 2H)da\right]\right) = 6HH'\left[1 + KI\right] + 3H^2K(a(aH' + 2H))$$

Right-hand side:

$$\frac{d}{da}\left(\kappa^2 \left(\rho_{m0}a^{-3} + \rho_{r0}a^{-4} + \rho_{\Lambda 0}\right)\right) = \kappa^2\left(-3\rho_{m0}a^{-4} - 4\rho_{r0}a^{-5}\right)$$

Equate:

$$6HH'\left[1 + KI\right] + 3H^2Ka(aH' + 2H) = \kappa^2\left(-3\rho_{m0}a^{-4} - 4\rho_{r0}a^{-5}\right)$$

5.4 Step 4: Perturbation Method (Small β)

The equation is still complex due to the integral I. Since $K = \frac{6V_{3D}}{c^2\kappa^2}\beta$, let's assume β is small and use perturbation theory. Set $\beta = 0$ (so $K = 0$) to get the unperturbed equation:

$$3H^2 = \kappa^2 \left(\rho_{m0}a^{-3} + \rho_{r0}a^{-4} + \rho_{\Lambda 0}\right)$$
$$H = \sqrt{\frac{\kappa^2}{3}\left(\rho_{m0}a^{-3} + \rho_{r0}a^{-4} + \rho_{\Lambda 0}\right)}$$
$$\frac{\dot{a}}{a} = \sqrt{\frac{\kappa^2}{3}\left(\rho_{m0}a^{-3} + \rho_{r0}a^{-4} + \rho_{\Lambda 0}\right)}$$
$$\dot{a} = a\sqrt{\frac{\kappa^2}{3}\left(\rho_{m0}a^{-3} + \rho_{r0}a^{-4} + \rho_{\Lambda 0}\right)}$$

This is the standard Friedmann equation, solvable by:

$$\int \frac{da}{a\sqrt{\frac{\kappa^2}{3}\left(\rho_{m0}a^{-3} + \rho_{r0}a^{-4} + \rho_{\Lambda 0}\right)}} = \int dt$$

- **Matter-dominated** ($\rho_{m0}a^{-3}$): $\dot{a} \propto a^{-1/2}$, so $a \propto t^{2/3}$, $H_0 = \frac{2}{3t}$.
- **Radiation-dominated** ($\rho_{r0}a^{-4}$): $\dot{a} \propto a^{-1}$, so $a \propto t^{1/2}$, $H_0 = \frac{1}{2t}$.
- **Cosmological constant** ($\rho_{\Lambda 0}$): $\dot{a} \propto a$, so $a \propto e^{\mathcal{H}t}$, where $\mathcal{H} = \sqrt{\frac{\kappa^2\rho_{\Lambda 0}}{3}}$.

For the perturbation, we could assume $H = H_0 + \beta H_1$, and substitute into the equation to find the first-order correction. This is complex and often requires numerical methods, so let's try a different approach.

5.5 Step 5: Ansatz Method (Power-Law Solution)

Since this is a cosmological equation, let's try a power-law solution $a(t) = Bt^n$, which is typical in specific epochs:

$$\dot{a} = nBt^{n-1}$$

$$\ddot{a} = n(n-1)Bt^{n-2}$$

$$\frac{\dot{a}}{a} = \frac{n}{t}$$

$$\frac{\ddot{a}}{a} = \frac{n(n-1)}{t^2}$$

$$\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 = \frac{n(n-1)}{t^2} + \left(\frac{n}{t}\right)^2 = \frac{n(2n-1)}{t^2}$$

$$a^3 = (Bt^n)^3 = B^3 t^{3n}$$

$$I(t) = \int_0^t B^3 \tau^{3n} \frac{n(2n-1)}{\tau^2} d\tau = B^3 n(2n-1) \int_0^t \tau^{3n-2} d\tau = B^3 n(2n-1) \frac{t^{3n-1}}{3n-1} \quad (\text{for } 3n \neq 1)$$

Left-hand side:

$$3\left(\frac{n}{t}\right)^2 \left[1 + KB^3 n(2n-1)\frac{t^{3n-1}}{3n-1}\right]$$

Right-hand side:

$$\kappa^2 \left(\rho_{m0}(Bt^n)^{-3} + \rho_{r0}(Bt^n)^{-4} + \rho_{\Lambda 0}\right)$$

Match the dominant terms in the **matter-dominated era** ($\rho_{m0}a^{-3}$):

$$\kappa^2 \rho_{m0} B^{-3} t^{-3n}$$

Left-hand side has terms t^{-2} and $t^{-2+3n-1} = t^{3n-3}$. For $\beta = 0$, match $t^{-2} \sim t^{-3n}$, so $3n = 2$, $n = \frac{2}{3}$, which is the standard result. The integral term $t^{3n-1} = t^{2-1} = t$, so the left-hand side has terms t^{-2} and t^{-1}, which don't match perfectly unless we adjust B. The power-law ansatz is complicated by the integral term, suggesting a numerical solution might be more practical.

5.6 Step 6: Derivation of the System for Numerical Solution

To solve the system numerically, we need to express it as a set of first-order ordinary differential equations (ODEs). The Hubble expansion rate is redefined as:

$$H = H_0 \sqrt{\frac{1}{(1+KI)} \left(\Omega_{m0}a^{-3} + \Omega_{r0}a^{-4} + \Omega_{\Lambda 0}\right)}$$

where H_0 is the Hubble constant today ($H(t=0)$), in units of A.T.U.$^{-1}$, and the density parameters are defined as:

$$\Omega_{s0} = \frac{\kappa^2}{3H_0^2}\rho_{s0}$$

for $s = m$ (matter), r (radiation), and Λ (dark energy). The given system is:

$$\dot{a} = Ha$$

$$\dot{I} = a^3(\dot{H} + 2H^2)$$

$$\dot{H} = \frac{\ddot{a}}{a} - H^2$$

The equation for H is algebraic, so we use the new expression for H. The given $\dot{H}$ equation leads to a tautology (as shown previously), so we derive $\dot{H}$ by differentiating the new expression for H.

Define:

$$f(a, I) = \frac{1}{(1+KI)}\left(\Omega_{m0}a^{-3} + \Omega_{r0}a^{-4} + \Omega_{\Lambda 0}\right)$$

$$H = H_0\sqrt{f(a,I)}$$

Differentiate H with respect to time:

$$\dot{H} = \frac{d}{dt}\left(H_0\sqrt{f(a,I)}\right) = H_0 \cdot \frac{1}{2\sqrt{f(a,I)}}\frac{df}{dt} = \frac{H_0}{2\sqrt{f(a,I)}}\frac{df}{dt} = \frac{H_0}{2(H/H_0)}\frac{df}{dt} = \frac{H_0^2}{2H}\frac{df}{dt}$$

Compute $\frac{df}{dt}$ using the chain rule:

$$\frac{df}{dt} = \frac{\partial f}{\partial a}\dot{a} + \frac{\partial f}{\partial I}\dot{I}$$

Calculate the partial derivatives:

$$\frac{\partial f}{\partial a} = \frac{1}{(1+KI)}\frac{d}{da}\left(\Omega_{m0}a^{-3} + \Omega_{r0}a^{-4} + \Omega_{\Lambda 0}\right) = \frac{1}{(1+KI)}\left(-3\Omega_{m0}a^{-4} - 4\Omega_{r0}a^{-5}\right)$$

$$\frac{\partial f}{\partial I} = \left(\Omega_{m0}a^{-3} + \Omega_{r0}a^{-4} + \Omega_{\Lambda 0}\right)\frac{d}{dI}\left(\frac{1}{1+KI}\right) = \left(\Omega_{m0}a^{-3} + \Omega_{r0}a^{-4} + \Omega_{\Lambda 0}\right)\left(-\frac{K}{(1+KI)^2}\right)$$

Substitute $\dot{a} = Ha$ and $\dot{I} = a^3(\dot{H} + 2H^2)$:

$$\frac{df}{dt} = \frac{\partial f}{\partial a}(Ha) + \frac{\partial f}{\partial I}a^3(\dot{H} + 2H^2)$$

$$\dot{H} = \frac{H_0^2}{2H}\left[\frac{\partial f}{\partial a}(Ha) + \frac{\partial f}{\partial I}a^3(\dot{H} + 2H^2)\right]$$

Rearrange to isolate $\dot{H}$:

$$\dot{H} - \frac{H_0^2}{2H}\frac{\partial f}{\partial I}a^3\dot{H} = \frac{H_0^2}{2H}\left(\frac{\partial f}{\partial a}Ha + \frac{\partial f}{\partial I}a^3(2H^2)\right)$$

$$\dot{H}\left(1 - \frac{H_0^2}{2H}\frac{\partial f}{\partial I}a^3\right) = \frac{H_0^2}{2H}\left(\frac{\partial f}{\partial a}Ha + 2H^2\frac{\partial f}{\partial I}a^3\right)$$

$$\dot{H} = \frac{\frac{H_0^2}{2H}\left(\frac{\partial f}{\partial a}Ha + 2H^2\frac{\partial f}{\partial I}a^3\right)}{1 - \frac{H_0^2}{2H}\frac{\partial f}{\partial I}a^3}$$

The system of ODEs is:

$$\dot{a} = Ha$$

$$\dot{I} = a^3(\dot{H} + 2H^2)$$

where H and $\dot{H}$ are computed as above. This system can be solved numerically using `scipy.integrate.odeint` in Python, with state variables $y = [a, I]$.

5.7 Step 7: Numerical Solution

We are given initial conditions $a(0) = 1$, $H(0) = 0.70\,\text{A.T.U.}^{-1}$, $I(0) = 0$, where $t = 0$ A.T.U. corresponds to the present time ("today" in cosmological terms). Here, t is measured in Arbitrary Time Units (A.T.U.), H is in A.T.U.$^{-1}$, while $a(t)$ and $I(t)$ are dimensionless. To describe the past, we consider times $t < 0$, where t becoming more negative corresponds to earlier times in the universe's history (smaller $a(t)$). For the future, we consider $t > 0$, where $a(t)$ increases as the universe expands.

The system derived above can be solved numerically over a time range that includes the past, present, and future, e.g., t from -10 to 10 A.T.U., with $t = 0$ A.T.U. as today. The system of ODEs is:

$$\dot{a} = Ha$$

$$\dot{I} = a^3(\dot{H} + 2H^2)$$

with H and $\dot{H}$ computed as derived in Step 6. We choose the following parameters:

- $H_0 = 0.70\,\text{A.T.U.}^{-1}$, the Hubble constant today, matching the initial condition $H(0) = 0.70\,\text{A.T.U.}^{-1}$.
- $\Omega_{m0} = 0.3$, the matter density parameter (dimensionless).
- $\Omega_{r0} = 8.1 \times 10^{-5}$, the radiation density parameter (dimensionless).
- $\Omega_{\Lambda 0} = 1 - \Omega_{m0} - \Omega_{r0} = 1 - 0.3 - 8.1 \times 10^{-5} \approx 0.699919$, the dark energy density parameter (dimensionless), ensuring flatness.
- $K = \frac{6V_{3D}}{c^2\kappa^2}\beta$: Using the previous adjustment, $K \approx 0.408$ (dimensionless), assuming $V_{3D} = 1$, $c = 1$, $\kappa^2 = 1.4698$, and $\beta = 0.1$.

To apply the initial conditions at $t = 0$ A.T.U., we perform the integration in two parts: backward from $t = 0$ to $t = -10$ A.T.U., and forward from $t = 0$ to $t = 10$ A.T.U. The solutions are then combined to obtain the full evolution over $t \in [-10, 10]$ A.T.U. The Python code (provided separately) solves this system and plots the evolution of the dimensionless scale factor $a(t)$, the Hubble parameter $H(t)$ in A.T.U.$^{-1}$, and the dimensionless integral term $|I(t)|$, with $t < 0$ A.T.U. representing the past, $t = 0$ A.T.U. today (where $a(0) = 1$), and $t > 0$ A.T.U. the future. The plot for $a(t)$ uses a symlog scale on the Y-axis to handle any potential negative values (though $a(t)$ should remain positive in a physical cosmological context), while the plots for

$H(t)$ and $|I(t)|$ use a logarithmic scale to better visualize the wide range of values. The results are saved as a PDF plot named `integro_differential_equation_fat_quad_t_-10_10_AU.pdf`, which is included below.

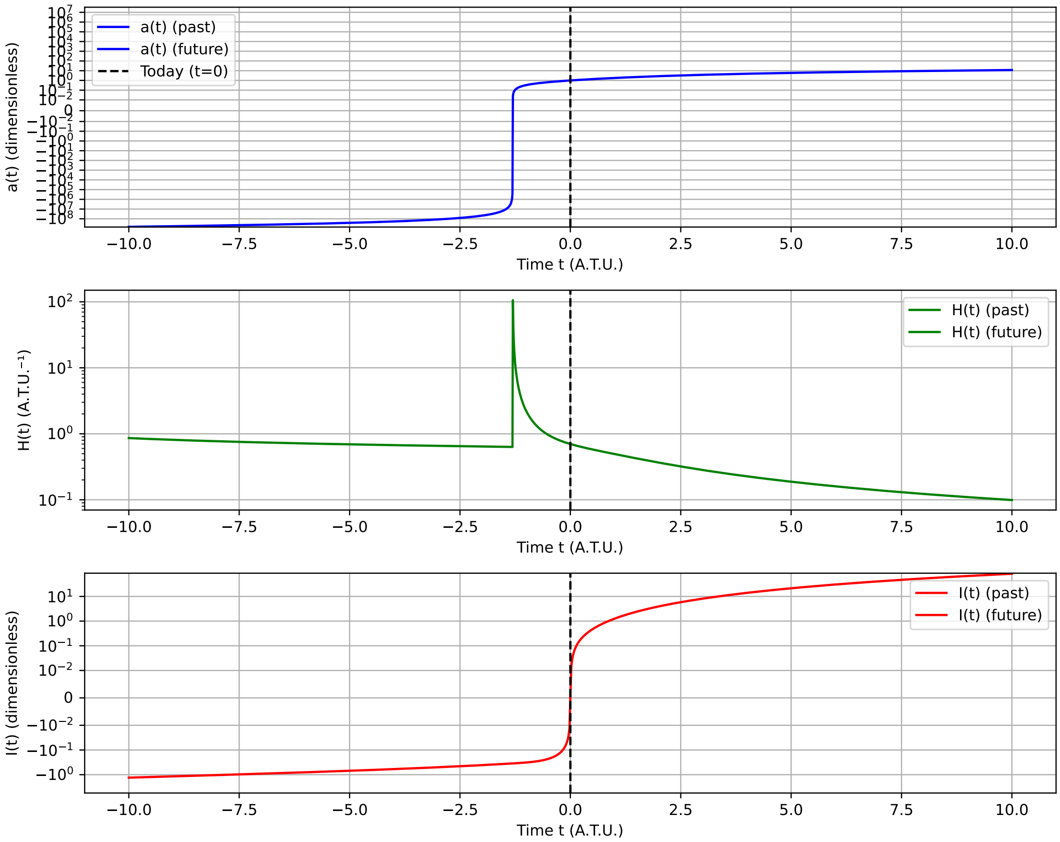

Figure 1: Evolution of the scale factor $a(t)$, Hubble parameter $H(t)$, and integral term $|I(t)|$ for the Quadratic FAT model over $t \in [-10, 10]$ A.T.U. The Y-axis for $a(t)$ uses a symlog scale, while the Y-axes for $H(t)$ and $|I(t)|$ are on a logarithmic scale to capture the wide range of values.

5.8 Step 8: Approximate Solution in Matter-Dominated Era

Let's approximate in the matter-dominated era ($\Omega_{m0}a^{-3}$ dominates):

$$H^2 \approx H_0^2 \frac{1}{(1+KI)} \Omega_{m0} a^{-3}$$

$$H \approx H_0 \sqrt{\frac{\Omega_{m0}}{(1+KI)} a^{-3}}$$

If KI is small, $1 + KI \approx 1$, so:

$$H \approx H_0 \sqrt{\Omega_{m0}} a^{-3/2}$$

$$\frac{\dot{a}}{a} = H_0\sqrt{\Omega_{m0}}a^{-3/2}$$
$$\dot{a} = H_0\sqrt{\Omega_{m0}}a^{-1/2}$$
$$a^{1/2}da = H_0\sqrt{\Omega_{m0}}dt$$
$$\frac{2}{3}a^{3/2} = H_0\sqrt{\Omega_{m0}}t + C$$

With $a(0) = 1$, the constant C depends on the time normalization, but this gives $a \propto t^{2/3}$, consistent with the standard matter-dominated era. For a better approximation, numerical solution is needed.

5.9 Conclusion

The equation is best solved numerically due to the nonlinear integral term. The system of ODEs derived in Step 6:

$$\dot{a} = Ha$$
$$\dot{I} = a^3(\dot{H} + 2H^2)$$
$$H = H_0\sqrt{\frac{1}{(1+KI)}\left(\Omega_{m0}a^{-3} + \Omega_{r0}a^{-4} + \Omega_{\Lambda 0}\right)}$$
$$\dot{H} = \frac{\frac{H_0^2}{2H}\left(\frac{\partial f}{\partial a}Ha + 2H^2\frac{\partial f}{\partial I}a^3\right)}{1 - \frac{H_0^2}{2H}\frac{\partial f}{\partial I}a^3}$$

can be solved with initial conditions $a(0) = 1$, $I(0) = 0$, and $H(0) = 0.70\,\text{A.T.U.}^{-1}$. Analytically, the unperturbed solution ($\beta = 0$) gives the standard Friedmann solutions ($a \propto t^{2/3}$ in the matter-dominated era). The numerical solution provides a more accurate evolution, accounting for the integral term, and allows us to explore the past ($t < 0$ A.T.U.) and future ($t > 0$ A.T.U.) relative to today ($t = 0$ A.T.U.).

6 The Integral Differential Equation of $S_{\text{FAT}}^{\text{exp}}$

We now derive and solve the integral-differential equation for the exponential FAT model, defined by the action:

$$S_{\text{FAT}}^{\text{exp}} = S_R + Ae^{\beta S_R} + S_\Lambda + S_m$$

The variation $\delta S_{\text{FAT}}^{\text{exp}} = 0$ yields:

$$\left[R_{\mu\nu} + \frac{1}{2}Rg_{\mu\nu}\right]\left[1 + A\beta e^{\beta S_R}\right] = \frac{8\pi G_{\rm N}}{c^4}\left[T_{\mu\nu} + \Lambda g_{\mu\nu}\right]$$

6.1 Step 1: Derive the Friedmann Equation

Using the FLRW metric $ds^2 = -c^2dt^2 + a^2(t)(dx^2 + dy^2 + dz^2)$, the Ricci scalar is:

$$R = 6\left(\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right) = 6\left(\dot{H} + 2H^2\right)$$

$$S_R = \frac{c^4}{16\pi G_{\rm N}}\int R\sqrt{-g}\,d^4x = \frac{3c^4}{8\pi G_{\rm N}}\int_{-\infty}^{t} a^3\left(\dot{H} + 2H^2\right)\,dt$$

$$I(t) = \int_0^t a^3\left(\dot{H} + 2H^2\right)\,dt$$

$$1 + A\beta e^{\beta S_R} = 1 + A\beta e^{KI(t)}, \quad K = \beta\frac{3c^4}{8\pi G_{\rm N}}$$

The (0,0) component gives:

$$3H^2\left[1 + A\beta e^{KI(t)}\right] = \kappa^2\left(\rho_{m0}a^{-3} + \rho_{r0}a^{-4} + \rho_{\Lambda 0}\right), \quad \kappa^2 = \frac{8\pi G_{\rm N}}{c^2}$$

6.2 Step 2: Form the System of ODEs

$$\dot{I} = a^3\left(\dot{H} + 2H^2\right)$$

$$H = H_0\sqrt{\frac{1}{1 + A\beta e^{KI}}\left(\Omega_{m0}a^{-3} + \Omega_{r0}a^{-4} + \Omega_{\Lambda 0}\right)}$$

Differentiate the Friedmann equation to find $\dot{H}$:

$$\dot{H} = \frac{\kappa^2\left(-3\rho_{m0}a^{-3} - 4\rho_{r0}a^{-4}\right) - 6H^3A\beta Ke^{KI}a^3}{6\left[1 + A\beta e^{KI}\right] + 3HA\beta Ke^{KI}a^3}$$

The system is:

$$\dot{a} = Ha, \quad \dot{I} = a^3 \left(\dot{H} + 2H^2\right)$$

6.3 Step 3: Numerical Solution

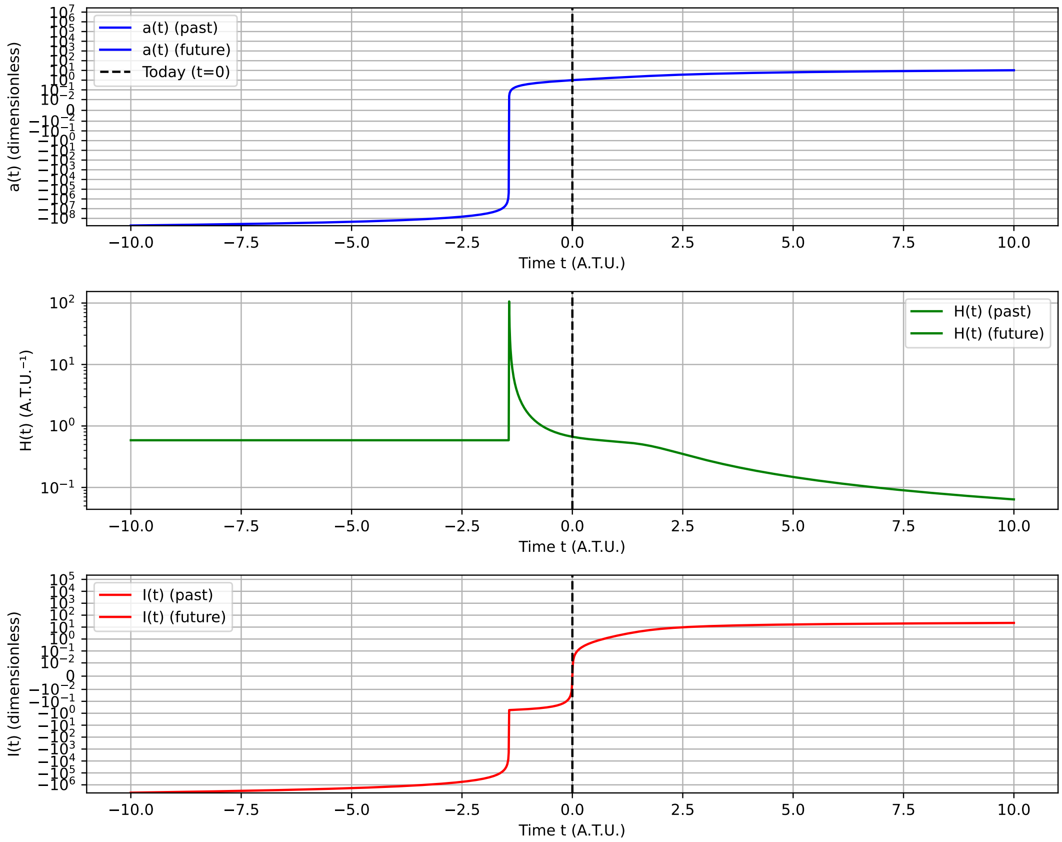

Figure 2: Evolution of the scale factor $a(t)$, Hubble parameter $H(t)$, and integral term $|I(t)|$ for the Exponential FAT model over $t \in [-10, 10]$ A.T.U. The Y-axis for $a(t)$ uses a symlog scale, while the Y-axes for $H(t)$ and $|I(t)|$ are on a logarithmic scale to capture the wide range of values.

Using $H_0 = 0.70\,\text{A.T.U.}^{-1}$, $\Omega_{m0} = 0.3$, $\Omega_{r0} = 8.1 \times 10^{-5}$, $\Omega_{\Lambda 0} = 0.699919$, $A = 1$, $\beta = 0.1$, and $K = 0.3$, the system can be solved numerically. The initial conditions are $a(0) = 1$, $I(0) = 0$, and $H(0) = 0.70\,\text{A.T.U.}^{-1}$, applied at $t = 0$ A.T.U. Here, t is measured in Arbitrary Time Units (A.T.U.), H is in A.T.U.$^{-1}$, and $a(t)$ and $I(t)$ are dimensionless. To cover the time range $t \in [-10, 10]$ A.T.U., we perform the integration in two parts: backward from $t = 0$ to $t = -10$ A.T.U., and forward from $t = 0$ to $t = 10$ A.T.U. The solutions are then combined to obtain the full evolution. The Python code (provided separately) shows the evolution of the dimensionless scale factor $a(t)$, the Hubble parameter $H(t)$ in A.T.U.$^{-1}$, and the dimensionless integral term $|I(t)|$, correctly satisfying the initial conditions at $t = 0$ A.T.U. The plot for $a(t)$ uses a symlog scale on the Y-axis to handle any potential negative values (though $a(t)$ should remain positive in a physical cosmological context), while the plots for $H(t)$ and $|I(t)|$ use a logarithmic scale to capture the wide range of values. The results are saved as a PDF plot named `integro_differential_equation_fat_expo_t_-10_10_AU.pdf`, which is included below.

6.4 Conclusion

The exponential FAT model introduces a modification that grows with $I(t)$, affecting the expansion rate. The numerical solution reveals the cosmological evolution, with the exponential term potentially accelerating expansion compared to the standard ΛCDM model. The system is solved with initial conditions $a(0) = 1$, $I(0) = 0$, and $H(0) = 0.70\,\text{A.T.U.}^{-1}$, over $t \in [-10, 10]$ A.T.U.

Appendix: List of Symbols

The following tables list key symbols used in this study, grouped by category, with their definitions and brief descriptions. Units are provided where applicable, with A.T.U. denoting Arbitrary Time Units.

Symbol	**Definition**	**Description**
$a(t)$	Scale factor	Dimensionless function for universe expansion, $a(0) = 1$ at present ($t = 0$ A.T.U.).
$H(t)$	Hubble parameter	Rate of expansion, in A.T.U.$^{-1}$, with $H(0) = H_0$.
t	Time	Independent variable in A.T.U., $t = 0$ is today, $t < 0$ past, $t > 0$ future.
R	Ricci scalar	Scalar curvature in FLRW metric, in A.T.U.$^{-2}$.

Table 1: Symbols for cosmological variables.

Acknowledgements

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Symbol	Definition	Description
H_0	Hubble constant today	Present-day Hubble parameter, $0.70\,\mathrm{A.T.U.}^{-1}$.
Ω_{m0}	Matter density parameter	Dimensionless matter density today, set to 0.3.
Ω_{r0}	Radiation density parameter	Dimensionless radiation density today, 8.1×10^{-5}.
$\Omega_{\Lambda 0}$	Dark energy density parameter	Dimensionless dark energy density today, ≈ 0.699919.
ρ_{m0}	Matter energy density	Present-day matter density, units via κ^2.
ρ_{r0}	Radiation energy density	Present-day radiation density, units via κ^2.
$\rho_{\Lambda 0}$	Dark energy density	Present-day dark energy density, units via κ^2.

Table 2: Symbols for cosmological parameters.

Symbol	Definition	Description
$S_{\mathrm{FAT}}^{\mathrm{quad}}$	Quadratic FAT action	Action for quadratic FAT model, includes S_R and βS_R^2.
$S_{\mathrm{FAT}}^{\mathrm{exp}}$	Exponential FAT action	Action for exponential FAT model, includes S_R and $Ae^{\beta S_R}$.
$S_{\mathrm{FAT}}^{\mathrm{f}}$	Generic FAT action	Generalized FAT action with a function $f(S_R)$.
S_R	Curvature action	Action from Ricci scalar, in $\mathrm{kg\,m^2\,s^{-1}}$.
S_Λ	Dark energy action	Action for the cosmological constant.
S_m	Matter action	Action for matter content of the universe.
$I(t)$	Integral term	Dimensionless integral of curvature and expansion effects in FAT models.
β	FAT coupling parameter	Dimensionless, controls FAT modification strength, set to 0.1.
A	Exponential FAT amplitude	Dimensionless, set to 1 in exponential FAT model.
K	FAT integral coefficient	Dimensionless, 0.408 (quadratic FAT) or 0.3 (exponential FAT).

Table 3: Symbols for FAT-specific terms.

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Symbol	Definition	Description
c	Speed of light	Set to 1 in natural units (otherwise $\mathrm{m\,s^{-1}}$).
G_{N}	Gravitational constant	In $\mathrm{m^3\,s^{-2}\,kg^{-1}}$, e.g., 7×10^{-11}.
κ^2	Gravitational coupling	Relates energy to geometry, 1.4698 in simulations.
V_{3D}	Spatial volume	Effective universe volume, set to 1 (otherwise $\mathrm{m^3}$).
g	Metric determinant	Determinant in FLRW cosmology, dimensionless in action.
$g_{\mu\nu}$	Metric tensor	Describes spacetime geometry in FLRW metric.
$R_{\mu\nu}$	Ricci tensor	Curvature tensor in FLRW metric.
$T_{\mu\nu}$	Energy-momentum tensor	Describes matter and radiation energy content.
Λ	Cosmological constant	Associated with dark energy, contributes to $\rho_{\Lambda 0}$.

Table 4: Symbols for physical constants.

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