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CosmoFATs: Cosmological Functors of Actions Theories

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Abstract: This paper delves into Functors of Actions Theories (FAT) which applied in cosmology, presenting models applied across cosmology, harmonic oscillators, and quantum gravity. We explore partition functions, local invariance, non-locality suppression, and actionic fluctuations, generalizing the stationary action principle and introducing five fundamental field-particles: spatiallion, timion, probablon, informaton, and actionion. We apply a viability analysis to ensure that FAT cosmology models are viable. We provide physical solutions to the integro-differential equations created by FAT gravitational cosmology models. Retaining detailed discussions from the original manuscript, with comprehensive interpretations, all sections, references, and symbol explanations, FATs emerge as a promising framework for advancing cosmological and quantum gravity research. The FAT framework might provide alternative aspects to modified gravity, dark energy and explore further the cosmological model with the manifestation of FAT fluctuations, the so called actionions.

Keywords: cosmology; functors of actions theories; modified gravity; quantum gravity; dark energy; partition function; universe.

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1. Introduction

The quest to unify gravity with quantum mechanics and explain cosmological phenomena beyond the Standard Cosmological Model (ΛCDM) [1] has driven the development of modified gravity (MG) theories [2,3,18,30]. Several efforts are made in order to improve this model from the observational [4–7,9–12,14] and theoretical [2,3,15,16,18,25,26,28,30] points of view.

The special issue focuses on cosmological anomalies and tensions that challenge the ΛCDM model, such as the Hubble parameter, H_0 discrepancy between early- and late-time probes, the S_8 tension in large-scale structure clustering, and CMB anomalies suggesting possible isotropy violations [4]. These discrepancies hint at new physics beyond the standard model, motivating alternative theoretical frameworks like FATs [25,26]. By applying mathematical functors to the gravitational action, FATs introduce integro-differential equations that offer novel insights into dark energy, structure formation, and quantum gravity [2], potentially addressing these tensions. This work explores cosmological FATs (CosmoFATs) to test their ability to resolve observational conflicts and provide a complementary path to existing MG theories [3], aligning with the special issue's aim to confront cosmology with innovative theoretical models.

Functors of Actions Theories (FATs) [25,26], a novel MG framework, diverge from traditional approaches by applying mathematical functors directly to the action, yielding integro-differential equations rather than purely differential ones. This shift opens new avenues for modeling spacetime dynamics, challenging conventional General Relativity (GR) [32] as well as standard MG theory models [3,28] and offering potential insights into dark energy, large-scale structure, and quantum gravity. In this work, we consider a path in formulating quantum gravity, and possibly a unified theory of everything, which is complementary to current efforts discussed in Nordita Program [2].

FATs can be reanalyzed through advanced generalized tensor theories by leveraging the unified framework of generalized tensor indices, their transformation properties, offering new topological geometrical insights of fractional manifolds [33] into FAT principles [25,26]. We leave this interesting observation to a future study.

In the context of MG theories, which often modify geometrical aspects (e.g., metric $g_{\mu\nu}$, Ricci scalar R) or energy content (e.g., exotic fields) [19,24], FATs propose a unique extension by manipulating the action itself. This study builds on prior work [20,21,27], applying FATs to cosmology in Section 2 and to basic mechanical systems in Section 3. In Section 4, we study foundational aspects of FATs. We assess local invariance (Section 5), non-locality in quadratic FATs (Section 6), and non-locality suppression (Section 7). In Section 8, we provide a viability analysis, to ensure the FAT gravitational cosmology models are viable. We provide physical solutions to the integro-differential equations created by FAT gravity models in Section 9. We aim to quantify its impact on partition functions (Sections 10–11.3). We describe the expectation value of the Universal information from FAT models in Section 12. We generalize the stationary action principle (Section 13), and define five fundamental field-particles (Section 14) which go beyond the particle-fields described by the standard model of particle physics ([35]). Detailed discussions from the original manuscript are retained in supplementary material, with sections linked narratively and enriched by appendices (Appendix B–A), positioning FATs as a robust candidate for future research.

2. Functors of Actions Theories (FAT)

At the core of GR lies the action:

$$\mathcal{S}_{\rm GR} = c^4 \int d^4x \sqrt{-g}\left[\frac{R}{16\pi G_{\rm N}} + \mathcal{L}_m(g_{\mu\nu}, \psi)\right], \tag{1}$$

assuming a four-dimensional pseudo-Riemannian manifold with Lorentz invariance. FATs introduce a framework where the action is modified via functors, with specific and generalized forms explored here.

2.1. A Specific Quadratic FAT Model

For a specific quadratic model, we define:

$$S_{\rm FAT}^{\rm quad} = S_R + \beta S_R^2 + S_\Lambda + S_m, \quad S_R = c^3 \int d^4x \sqrt{-g}\frac{R}{16\pi G_{\rm N}}, \tag{2}$$

where $\delta S_{\rm FAT}^{\rm quad} = 0$ yields:

$$\left[R_{\mu\nu} + \frac{1}{2}R g_{\mu\nu}\right][1 + 2\beta S_R] = \frac{8\pi G_{\rm N}}{c^4}\left[T_{\mu\nu} + \Lambda g_{\mu\nu}\right]. \tag{3}$$

(Note: This quadratic form is a specific case of FATs, not encompassing the full generality of functorial transformations, which could involve more complex mappings beyond a simple quadratic term.) In an FLRW universe ($ds^2 = -d(ct)^2 + a^2(t)d\vec{x}^2$), $g = -a^6(t)$, $R = \frac{6}{c^2}\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right]$, so:

$$S_R = \frac{6c^2\mathcal{V}_{3D}}{16\pi G_{\rm N}} \int dt a^3(t)\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right]. \tag{4}$$

For a dark energy-dominated universe, $a(t) = e^{H_0 t}$, $\dot{a}/a = H_0$, $\ddot{a}/a = H_0^2$, thus:

$$S_R = \frac{c^2 H_0}{4\pi G_{\rm N}}\mathcal{V}_{3D}\left[e^{3H_0 t} - e^{3H_0 t_i}\right]. \tag{5}$$

With $H_0 \approx 2\times10^{-17}\,{\rm s}^{-1}$, $G_{\rm N} \approx 7\times10^{-11}\,{\rm m^3\,s^{-2}\,kg^{-1}}$, $\mathcal{V}_{3D} \approx 10^{77}\,{\rm m}^3$, and $t \approx 10^{17}\,{\rm s}$, $S_R \approx 8\times 10^{88}\,{\rm kg\,m^2\,s^{-1}}$, the FAT fluctuations are $\delta S_R = \beta S_R^2 \sim 10^{-5}$, in which $\beta \approx 10^{-182}\,{\rm kg^{-1}\,m^{-2}\,s}$.

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2.2. A Specific Generic Function FAT Model

More generally, FATs can be expressed as:

$$S_{\text{FAT}} = f(S_R), \tag{6}$$

where f is a generic function of the action S_R. This form adapts FAT flexibly to various physical contexts, beyond the quadratic case $f_{\text{quad}}(S_R) = S_R + \beta S_R^2 + S_\Lambda + S_m$ we can consider the exponential case, $f_{\text{expo}}(S_R) = S_R + Ae^{\frac{\beta}{\hbar}S_R} + S_\Lambda + S_m$. (Note: This generalized functional form, while broader than the quadratic case, is still not the most complete generalization of FATs, which could involve full functorial transformations beyond simple functions of S_R.) **Interpretation**: The quadratic model modifies GR's dynamics with a non-linear curvature term, while the generalized form $f(S_R)$ offers flexibility, with S_R's magnitude reflecting the universe's geometric evolution and parameters like β fine-tuning actionic fluctuations, setting the stage for Foundation analysis in Section 4, and invariance analysis in Section 5.

3. Oscillator Modelling

Shifting from cosmology in Section 2, we explore FAT's adaptability in oscillatory mechanics.

3.1. Classical Harmonic Oscillator

Action:

$$S_{\text{osc}} = \int dt \left[\frac{1}{2} m\dot{x}^2 - \frac{1}{2} kx^2 \right], \tag{7}$$

yields:

$$m\ddot{x} + kx = 0. \tag{8}$$

Solution (standard differential method, initial conditions $x(0) = x_0$, $\dot{x}(0) = v_0$):

$$x(t) = x_0 \cos(\omega t) + \frac{v_0}{\omega} \sin(\omega t), \quad \omega = \sqrt{\frac{k}{m}}. \tag{9}$$

Interpretation: This describes harmonic motion with frequency ω, a baseline for FAT extensions.

3.2. FAT Simple Harmonic Oscillator

FAT extends Section 3.1 with the quadratic form:

$$S_{\text{FAT,osc}} = S_{\text{osc}} + \beta S_{\text{osc}}^2, \tag{10}$$

giving:

$$m\ddot{x} + kx + 2\beta S_{\text{osc}}(m\ddot{x} + kx) = 0. \tag{11}$$

With $S_{\text{osc}} = \int_{t_i}^{t_f} dt \left[\frac{1}{2} m\dot{x}^2 - \frac{1}{2} kx^2 \right]$, this integro-differential equation is solved perturbatively (integro-differential method):

$$x(t) \approx x_0 \cos(\omega t) + \frac{v_0}{\omega} \sin(\omega t) + \beta \tilde{A} \sin(\omega t + \tilde{\phi}), \tag{12}$$

where $\tilde{A}$ and $\tilde{\phi}$ are small corrections from S_{osc}. **Interpretation**: The non-linear quadratic term shifts oscillation dynamics, introducing perturbations.

3.3. FAT Oscillator in Gravity: Model γ

For gravitational effects beyond Section 3.2:

$$S_{\text{FAT,osc,grav}} = S_{\text{osc,grav}} + \gamma S_{\text{osc,grav}}^2, \quad S_{\text{osc,grav}} = \int dt \left[\frac{1}{2} m\dot{x}^2 - mgx \right], \tag{13}$$

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(note: quadratic form used here), yielding:

$$m\ddot{x} + mg + 2\gamma S_{\text{osc,grav}}(m\ddot{x} + mg) = 0. \tag{14}$$

Solved via the integro-differential method (perturbing $x_0(t) = x_0 + v_0 t - \frac{1}{2}gt^2$):

$$x(t) \approx x_0 + v_0 t - \frac{1}{2}gt^2 + \gamma B t^3, \tag{15}$$

B being a correction. Classical solution:

$$m\ddot{x} + mg = 0, \quad x(t) = x_0 + v_0 t - \frac{1}{2}gt^2. \tag{16}$$

Interpretation: The γ-term adds a cubic deviation to gravitational motion.

3.4. FAT Oscillator in Gravity: Model α

Extending Section 3.3, we define:

$$S_{\text{FAT,osc,grav}} = S_{\text{osc,grav}} + \alpha S_{\text{grav}}^2, \quad S_{\text{grav}} = \int_{t_1}^{t_2} (-mgx)\,dt, \tag{17}$$

where $S_{\text{osc,grav}} = \int_{t_1}^{t_2} \left(\frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 - mgx\right)dt$, and $\alpha \ll 1$. This yields:

$$m\ddot{x} + kx + mg - 2\alpha m^2 g^2 \int_{t_1}^{t_2} x\,dt = 0. \tag{18}$$

Solved perturbatively (integro-differential method) with $x(t_1) = x_0$, $\dot{x}(t_1) = v_0$:

$$x(t) \approx \left(x_0 + \frac{mg}{k}\right)\cos(\omega t) + \frac{v_0}{\omega}\sin(\omega t) - \frac{mg}{k} + \alpha C(\cos(\omega t) - 1), \tag{19}$$

where $\omega = \sqrt{\frac{k}{m}}$, $C = \frac{4\pi m^3 g^3}{k^2 \omega}$ (for $t_2 - t_1 = \frac{2\pi}{\omega}$). **Interpretation**: The α-term introduces a gravitational coupling shift, enhancing oscillator dynamics, bridging to quantum explorations in Section 10.

4. Foundational Analysis of FAT: Locality and Consistency

To address the intrinsic consistency of FATs, we analyze the quadratic model $S_{\text{FAT}} = S_R + S_\phi + \beta S_R^2$ (Section 2), focusing on nonlocality, causality, boundary conditions, and General Relativity (GR) correspondence, before cosmological applications.

4.1. Equations of Motion and Nonlocality

For $S_R = \frac{c^3}{16\pi G_N}\int d^4x\sqrt{-g}R$ and $S_\phi = \int d^4x\sqrt{-g}\left(-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right)$, the action is:

$$S_{\text{FAT}} = S_R + S_\phi + \beta\left(\int d^4x\sqrt{-g}\frac{c^3}{16\pi G_N}R\right)^2. \tag{20}$$

Varying with respect to $g_{\mu\nu}$:

$$\frac{\delta S_{\text{FAT}}}{\delta g_{\mu\nu}} = \frac{\delta S_R}{\delta g_{\mu\nu}} + \frac{\delta S_\phi}{\delta g_{\mu\nu}} + 2\beta S_R \frac{\delta S_R}{\delta g_{\mu\nu}} = 0, \tag{21}$$

$$(1 + 2\beta S_R)\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) + T_{\mu\nu}^\phi = 0, \tag{22}$$

where $T_{\mu\nu}^\phi = -\frac{2}{\sqrt{-g}}\frac{\delta S_\phi}{\delta g_{\mu\nu}}$, and $S_R = \int d^4x\sqrt{-g}\frac{c^3}{16\pi G_N}R$ is nonlocal (bilocal). This integro-differential equation depends on the entire spacetime via S_R, contrasting with GR's local differential form.

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4.2. Causality

Nonlocality from S_R could imply acausal propagation if future states affect the past. For small β, $1 + 2\beta S_R \approx 1$, and the equation approximates GR's causal form. Perturbative solutions (e.g., Section 3.4) suggest locality dominates, with nonlocal corrections as higher-order effects, preserving causality at leading order.

4.3. Boundary Value Problem

The term $2\beta S_R$ requires S_R over all spacetime, suggesting a boundary value problem. In perturbation ($\beta \ll 1$), we treat S_R as a constant (e.g., S_R^{cl} in FLRW), reducing to an initial value problem with conditions like $a(t_0), \dot{a}(t_0)$, as in oscillator models (Section 3).

4.4. Correspondence with GR

As $\beta \to 0$, $S_{\mathrm{FAT}} \to S_R + S_\phi$, and the equation becomes:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + T^{\phi}_{\mu\nu} = 0, \tag{23}$$

recovering GR with a scalar field, ensuring consistency with established physics.

4.5. Local Invariance

Under diffeomorphisms $x^\mu \to x'^\mu$, $S_R \to S'_R = S_R$, $S_\phi \to S'_\phi = S_\phi$, and $\beta S_R^2 \to \beta (S'_R)^2 = \beta S_R^2$, so:

$$S'_{\mathrm{FAT}} = S_{\mathrm{FAT}}, \tag{24}$$

preserving GR's symmetry.

4.6. Linearized Einstein Field Equations

From Section 2, standard GR action $S_{\mathrm{grav}} = S_R + S_m$ linearizes around $g_{\mu\nu} + \delta g_{\mu\nu}$:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G_{\mathrm{N}} T_{\mu\nu}, \tag{25}$$

with Green's function:

$$\Box G(x, x') = \delta^D (x - x'). \tag{26}$$

Using the quadratic FAT model:

$$S_{\mathrm{FAT,grav}} = S_R + S_m + \beta S_R^2, \tag{27}$$

(note: this is the quadratic case), we obtain:

$$(1 + 2\beta S_R)\left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R\right) = 8\pi G_{\mathrm{N}} T_{\mu\nu}, \tag{28}$$

and:

$$\left(\Box + \beta S_R \Box^2\right) G(x, x') = \delta^D (x - x'). \tag{29}$$

Perturbations $\delta g_{\mu\nu}$ adjust curvature and volume terms, complicating dynamics. **Interpretation**: FAT introduces a non-linear gravitational response, with the Green's function reflecting higher-order propagation effects, leading to non-locality discussions in Section **??**.

4.7. Conclusion

The quadratic FAT introduces bilocal terms, manageable perturbatively to maintain causality and an initial value framework. The GR limit ($\beta = 0$) ensures reliability, grounding cosmological applications (e.g., Section 3.4) as exploratory extensions of this foundation.

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5. Local Invariance Resolution

Extending FAT's framework from Section 2, we analyze the quadratic case:

$$S_{\text{FAT}}^{\text{quad}} = S_R + S_\phi + \beta S_R^2, \tag{30}$$

where $S_\phi = \int d^D x \sqrt{-g}\left(-\frac{1}{2} g^{\mu\nu}\partial_\mu \phi \partial_\nu \phi - V(\phi)\right)$ (note: here we use the quadratic form, a specific instance of FATs). Under diffeomorphisms $g'_{\mu\nu} = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}$, R transforms as a scalar, and $\sqrt{-g}$ adjusts, ensuring:

$$S_{\text{FAT}}^{\text{quad}} = S_{\text{FAT}}^{\text{quad}'}. \tag{31}$$

Furthermore, in 1D, locality holds if integrals model local behavior (e.g., particle size limits). **Interpretation**: This invariance aligns FAT with GR's diffeomorphism symmetry, ensuring physical consistency, with locality preserved by appropriate integral bounds, tested further in Section 4.6.

6. Non-Locality in Quadratic FAT

Following linearized dynamics in Section 4.6, the quadratic βS_R^2 term (a specific FAT instance) induces non-locality:

$$\left(\Box + \beta S_R \Box^2\right) G(x, x') = \delta^D(x - x'), \tag{32}$$

with Fourier transform:

$$\tilde{G}(k) \approx \frac{1}{-k^2 + \beta S_R k^4}. \tag{33}$$

Higher-order derivatives suggest acausal signal propagation beyond the light cone, unlike GR's locality. **Interpretation**: This non-locality implies field responses depend on distant points, challenging causality but offering new dynamical insights, addressed in Section 7.

7. Suppressing Non-Locality

To address non-locality from Section 6 in the quadratic model, localizing S_R approximates:

$$\left(\Box + \beta S_R \Box^2\right) G(x, x') \approx \Box G(x, x'). \tag{34}$$

This reduces higher-order terms, aligning with local theories. **Interpretation**: Localization mitigates acausal effects, making FAT more compatible with observed physics, paving the way for mechanical applications in Section 3.

8. Viability summary for FAT gravitational cosmology models

The quadratic and exponential Functors of Actions Theories (FAT) gravitational cosmology models, introduced in [25,26], are demonstrated to be stable against unphysical instabilities, as detailed in supplementary material. Analysis of the quadratic and exponential FAT actions for scalar perturbations around a Friedmann-Lemaître-Robertson-Walker (FLRW) background confirms the absence of ghosts (negative kinetic energy modes), tachyons (negative mass-squared modes), and gradient instabilities (negative sound speed squared). Both models exhibit a sound speed squared of $c_s^2 \approx 1$, ensuring stable perturbation propagation [28,30]. The quadratic model is stable for $\beta \approx 10^{-182}\,\text{kg}^{-1}\,\text{m}^{-2}\,\text{s}$, while the exponential model requires $A\beta e^{K(\beta)I(t)} \ll 1$, achieved with parameters $A = 1$, $\beta = 0.1$, and $K = 0.3$. These results, supported by numerical solutions and consistent with observational [4,5] and theoretical [15,16] constraints, affirm the physical viability of FAT models in gravitational cosmology.

9. Integro-differential equations solutions of FAT cosmology

In this section, we present two integro-differential equations of FAT gravitational cosmology models, and we provide their solutions. Let $a(t), A, \beta, V_{3D}, c, \kappa, \rho_{m0}, \rho_{r0}, \rho_{\Lambda 0}$, be the scale factor, the exponential gravity FAT term amplitude parameter, the exponential (or quadratic) gravity FAT term

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coupling parameter, the 3D observable volume, the light speed constant, the reduced Gravitational and light speed constant, the matter energy density, the radiation energy density, and the dark energy density, respectively.

9.1. The Integral and the Constant of the FAT Term

The integral of the FAT term which appears to the gravity models is:

$$I(t) = \int dt\, a^3(t)\left(\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right) = \int dt\, a^3(t)(\dot{H} + 2H^2), \tag{35}$$

We assume the integral is from 0 to t (as is typical for initial value problems in cosmology), so we get

$$I(t) = \int_0^t a^3(\tau)(\dot{H}(\tau) + 2H^2(\tau))d\tau. \tag{36}$$

The constant which appears in the FAT term is

$$K(\beta) = \frac{6V_{3D}}{c^2\kappa^2}\beta. \tag{37}$$

This term is constant in the observed known volume, V_{3D}.

9.2. The Integro-Differential Equation of Quadratic FAT Gravity $S^{\text{quad}}_{\text{FAT}}$

The integro-differential equation from the quadratic FAT model is

$$3\left(\frac{\dot{a}}{a}\right)^2[1 + K(\beta)I(t)] = \kappa^2\left(\rho_{m0}a^{-3}(t) + \rho_{r0}a^{-4}(t) + \rho_{\Lambda 0}\right), \tag{38}$$

which is rewritten in the form of the Hubble rate as

$$H = H_0\sqrt{\frac{1}{1 + K(\beta)I(t)}(\Omega_{m0}a^{-3} + \Omega_{r0}a^{-4} + \Omega_{\Lambda 0})}. \tag{39}$$

The integro-differential equation is reduced to a system of ordinary differential equations which is:

$$\dot{a} = Ha, \tag{40}$$
$$\dot{I} = a^3(\dot{H} + 2H^2). \tag{41}$$

This system can be solved numerically using `scipy.integrate.odeint` in Python, with state variables $y = [a, I]$.

9.3. Numerical Solution of Quadratic FAT Gravity $S^{\text{quad}}_{\text{FAT}}$

We are given initial conditions $a(0) = 1$, $H(0) = 0.70\,\text{A.T.U.}^{-1}$, $I(0) = 0$, where $t = 0$ A.T.U. corresponds to the present time ("today" in cosmological terms). Here, t is measured in Arbitrary Time Units (A.T.U.), H is in A.T.U.$^{-1}$, while $a(t)$ and $I(t)$ are dimensionless. To describe the past, we consider times $t < 0$, where t becoming more negative corresponds to earlier times in the universe's history (smaller $a(t)$). For the future, we consider $t > 0$, where $a(t)$ increases as the universe expands.

The system derived above can be solved numerically over a time range that includes the past, present, and future, e.g., t from -10 to 10 A.T.U., with $t = 0$ A.T.U. as today. We choose the following parameters:

- $H_0 = 0.70\,\text{A.T.U.}^{-1}$, the Hubble constant today, matching the initial condition $H(0) = 0.70\,\text{A.T.U.}^{-1}$.
- $\Omega_{m0} = 0.3$, the matter density parameter (dimensionless).
- $\Omega_{r0} = 8.1 \times 10^{-5}$, the radiation density parameter (dimensionless).

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- $\Omega_{\Lambda 0} = 1 - \Omega_{m0} - \Omega_{r0} = 1 - 0.3 - 8.1 \times 10^{-5} \approx 0.699919$, the dark energy density parameter (dimensionless), ensuring flatness.
- $K = \frac{6V_{3D}}{c^2\kappa^2}\beta$: Using the previous adjustment, $K \approx 0.408$ (dimensionless), assuming $V_{3D} = 1$, $c = 1$, $\kappa^2 = 1.4698$, and $\beta = 0.1$.

To apply the initial conditions at $t = 0$ A.T.U., we perform the integration in two parts: backward from $t = 0$ to $t = -10$ A.T.U., and forward from $t = 0$ to $t = 10$ A.T.U. The solutions are then combined to obtain the full evolution over $t \in [-10, 10]$ A.T.U. The Python code (provided separately) solves this system and plots the evolution of the dimensionless scale factor $a(t)$, the Hubble parameter $H(t)$ in A.T.U.$^{-1}$, and the dimensionless integral term $|I(t)|$, with $t < 0$ A.T.U. representing the past, $t = 0$ A.T.U. today (where $a(0) = 1$), and $t > 0$ A.T.U. the future. The plot for $a(t)$ uses a symlog scale on the Y-axis to handle any potential negative values (though $a(t)$ should remain positive in a physical cosmological context), while the plots for $H(t)$ and $|I(t)|$ use a logarithmic scale to better visualize the wide range of values. We present the results of the temporal evolution of scale factor, Hubble expansion rate, and the FAT term integral, in Figure 1. This evolution is similar and comparable to the standard temporal evolution of the corresponding quantities. Deviation of the quadratic FAT temporal evolution of scale factor, and Hubble expansion rate in respect of the standard temporal evolution may signify the existence of actionions.

Figure 1. Evolution of the scale factor $a(t)$, Hubble parameter $H(t)$, and integral term $|I(t)|$ for the Quadratic FAT model over $t \in [-10, 10]$ A.T.U. The Y-axis for $a(t)$ uses a symlog scale, while the Y-axes for $H(t)$ and $|I(t)|$ are on a logarithmic scale to capture the wide range of values.

9.4. The Integro-Differential Equation of Exponential FAT Gravitational Cosmology $S_{\mathrm{FAT}}^{\exp}$

The integro-differential equation from the exponential FAT model is

$$3\left(\frac{\dot{a}}{a}\right)^2\left[1 + A\beta e^{K(\beta)I(t)}\right] = \kappa^2\left(\rho_{m0}a^{-3}(t) + \rho_{r0}a^{-4}(t) + \rho_{\Lambda 0}\right), \tag{42}$$

which is rewritten in the form of the Hubble rate as

$$H = H_0\sqrt{\frac{1}{1 + A\beta e^{K(\beta)I(t)}}(\Omega_{m0}a^{-3} + \Omega_{r0}a^{-4} + \Omega_{\Lambda 0})}. \tag{43}$$

The integro-differential equation is reduced to a system of ordinary differential equations which is:

$$\dot{a} = Ha, \tag{44}$$
$$\dot{I} = a^3(\dot{H} + 2H^2). \tag{45}$$

This system can be solved numerically using `scipy.integrate.odeint` in Python, with state variables $y = [a, I]$.

9.5. Numerical Solution of Exponential FAT Gravitational Cosmology $S_{\mathrm{FAT}}^{\exp}$

Using $H_0 = 0.70$ A.T.U.$^{-1}$, $\Omega_{m0} = 0.3$, $\Omega_{r0} = 8.1 \times 10^{-5}$, $\Omega_{\Lambda 0} = 0.699919$, $A = 1$, $\beta = 0.1$, and $K = 0.3$, the system can be solved numerically. The initial conditions are $a(0) = 1$, $I(0) = 0$, and $H(0) = 0.70$ A.T.U.$^{-1}$, applied at $t = 0$ A.T.U. Here, t is measured in Arbitrary Time Units (A.T.U.), H is in A.T.U.$^{-1}$, and $a(t)$ and $I(t)$ are dimensionless. To cover the time range $t \in [-10, 10]$ A.T.U., we perform the integration in two parts: backward from $t = 0$ to $t = -10$ A.T.U., and forward from $t = 0$ to $t = 10$ A.T.U. The solutions are then combined to obtain the full evolution. The Python code (provided separately) shows the evolution of the dimensionless scale factor $a(t)$, the Hubble parameter $H(t)$ in A.T.U.$^{-1}$, and the dimensionless integral term $|I(t)|$, correctly satisfying the initial conditions at $t = 0$ A.T.U. The plot for $a(t)$ uses a symlog scale on the Y-axis to handle any potential negative values (though $a(t)$ should remain positive in a physical cosmological context), while the plots for

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$H(t)$ and $|I(t)|$ use a logarithmic scale to capture the wide range of values. We present the results of the temporal evolution of scale factor, Hubble expansion rate, and the FAT term integral, in Figure 2. This evolution is similar and comparable to the standard temporal evolution of the corresponding quantities. Deviation of the exponential FAT temporal evolution of scale factor, and Hubble expansion rate in respect of the standard temporal evolution may signify the existence of actionions.

Figure 2. Evolution of the scale factor $a(t)$, Hubble parameter $H(t)$, and integral term $|I(t)|$ for the Exponential FAT model over $t \in [-10, 10]$ A.T.U. The Y-axis for $a(t)$ uses a symlog scale, while the Y-axes for $H(t)$ and $|I(t)|$ are on a logarithmic scale to capture the wide range of values.

10. Path Integral Approximations

Transitioning from Section 3:

$$\mathcal{Z} = \int \mathcal{D}[\phi] e^{-S_E[\phi]/\hbar}, \tag{46}$$

with:

$$\mathcal{Z} \propto \det{}^{-1/2}\left(\frac{\delta^2 S}{\delta\phi^2}\right). \tag{47}$$

Interpretation: Weights quantum configurations, standard in QFT [29], leading to Section 10.1.

10.1. Partition Function in Modified Gravity

Extending Section 10:

$$\mathcal{Z} \propto \exp\left(-\frac{S_{\text{cl}}}{\hbar}\right) \det{}^{-1/2}\left(\frac{\delta^2 S}{\delta\phi^2}\right). \tag{48}$$

(Note: Applies to generic $S_{\text{FAT}} = f(S_R)$ from Section 2). **Interpretation**: Combines classical and quantum contributions, refined in Section 10.2.

10.2. Partition Function: Model A

For a simplified FAT from Section 10.1:

$$\mathcal{Z} = \int \mathcal{D}[\phi] e^{-S_{\text{cl}}/\hbar} e^{-\delta^2 S/\hbar}, \tag{49}$$

using ζ-function regularization. **Interpretation**: Separates classical paths and fluctuations, contrasting with Section 11.1.

11. Field Theory Expansion

In this section, we are expanding QFT and QG.

11.1. Traditional Quantum Field Theory

Building on Section 10.2:

$$\mathcal{Z} \approx e^{-S_{\text{cl}}/\hbar} \sqrt{\frac{2\pi\hbar}{\det(-\Box)}}. \tag{50}$$

Interpretation: Emphasizes classical dominance, adapted by FAT in Section 11.2.

11.2. FAT Quantum Field Theory

FAT modifies Section 11.1 with the quadratic form:

$$\mathcal{Z} \approx e^{-(S_R + \beta S_R^2)/\hbar} \sqrt{\frac{2\pi\hbar}{\det(-\Box + \beta S_R \Box^2)}}. \tag{51}$$

(Note: Quadratic case used here). **Interpretation**: Incorporates non-linearity, extended to gravity in Section 11.3.

11.3. *FAT Quantum Gravity via Partition Functor*

From Section 11.2:

$$\mathcal{Z} \approx e^{-S_{\text{FAT}}/\hbar} \sqrt{\frac{2\pi\hbar}{\det(\mathcal{O})}}. \tag{52}$$

(Note: Applies to $S_{\text{FAT}} = f(S_R)$). **Interpretation**: Encapsulates quantum gravity corrections, leading to Section 13.

12. Expectation Value of Universal Information in FAT

To probe FAT's foundational implications, we compute the expectation value of an observable—here, the universe's "information" as entropy S_E (cf. Section 14)—using the partition function. The action depends on the Lagrangian density $\mathcal{L} = \frac{c^4}{16\pi G_N} R + \mathcal{L}_m - \Lambda$, metric $g_{\mu\nu}$, and matter fields ϕ.

For the quadratic model (Section 3.4):

$$\mathcal{Z} = \int \mathcal{D}[g_{\mu\nu}]\mathcal{D}[\phi]\, e^{-(S_{\text{osc,grav}} + \alpha S^2_{\text{grav}})}, \tag{53}$$

$$\langle S_E \rangle = \frac{1}{\mathcal{Z}} \int \mathcal{D}[g_{\mu\nu}]\mathcal{D}[\phi]\, S_E e^{-(S_{\text{osc,grav}} + \alpha S^2_{\text{grav}})}, \tag{54}$$

generalized to cosmology as $S_{\text{FAT}} = S_R + \beta S_R^2 + S_m + S_\Lambda$, with $S_R = c^3 \int d^4x \sqrt{-g} \frac{R}{16\pi G_N}$. In a de Sitter universe ($a(t) = e^{H_0 t}$), $S_E^{\text{cl}} = \frac{\pi c^2}{G_N H_0^2} \approx 10^{123}$ (Planck units), and perturbatively:

$$\langle S_E \rangle \approx 10^{123} + \beta(8 \times 10^{88})^2 \approx 10^{123} + 6.4 \times 10^{94}, \tag{55}$$

for $\beta \approx 10^{-182}\,\text{kg}^{-1}\,\text{m}^{-2}\,\text{s}$.

For a generic model, $S_{\text{FAT}} = f(S_R) + S_m + S_\Lambda$, e.g., $f(S_R) = S_R + \beta S_R^2 + \gamma S_R^3$:

$$\langle S_E \rangle \approx S_E^{\text{cl}} + \left\{ f(S_R^{\text{cl}}) - S_R^{\text{cl}} \right\} \approx 10^{123} + 6.4 \times 10^{94} + 5.1 \times 10^{65}, \tag{56}$$

with $\gamma \approx 10^{-270}\,\text{kg}^{-2}\,\text{m}^{-4}\,\text{s}^2$, where $f'(S_R^{\text{cl}}) \approx 1$ preserves $S_E^{\text{cl}} \approx 10^{123}$.

Interpretation: Both models yield a classical entropy consistent with de Sitter, with FAT corrections enhancing dynamics, supporting quantum explorations (Section 10) while grounding the theory's consistency.

13. Generalized Stationary Action Principle

Given that the action depends on several fields, that we can generalised them into a vector field, $v_i = \{g_1, \ldots, g_m, \phi_1, \ldots \phi_n\}$ where $n + m = d$, g_i are the metric fields, where ϕ_j are the matter fields. Then the action can be expanded using Taylor expansion as:

$$S[\vec{v}] = \sum_{n_1=0}^{\infty} \cdots \sum_{n_d=0}^{\infty} \frac{(v_1 - v_{01})^{n_1} \ldots (v_d - v_{0d})^{n_d}}{n_1! \ldots n_d!} \frac{\partial^{(n_1 \ldots n_d)} S}{\partial v_1^{n_1} \ldots \partial v_d^{n_d}} \Big|_{\vec{v}=\vec{v}_0}, \tag{57}$$

and by defining the perturbations as

$$\delta v_{0i} = v_i - v_{0i}, \quad \delta S[\vec{v}] = S[\vec{v}] - S[\vec{v}_0], \tag{58}$$

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we get the compact form of the generalised stationary action principle as:

$$\delta S[\vec{v}] = 0, \tag{59}$$

$$\sum_{n_1=1}^{\infty} \cdots \sum_{n_d=1}^{\infty} \frac{\delta v_{01}^{n_1} \ldots \delta v_{0d}^{n_d}}{n_1! \ldots n_d!} \frac{\partial^{(n_1 \ldots n_d)} S}{\partial v_1^{n_1} \ldots \partial v_d^{n_d}} |_{\vec{v}=\vec{v}_0} = 0, \tag{60}$$

$$\sum_{n_1=1}^{\infty} \cdots \sum_{n_d=1}^{\infty} \frac{\delta v_{01}^{n_1} \ldots \delta v_{0d}^{n_d}}{n_1! \ldots n_d!} (\partial_{v_1}^{n_1} \ldots \partial_{v_d}^{n_d} S) |_{\vec{v}=\vec{v}_0} = 0. \tag{61}$$

From Section 2, we can expand even further the generalised stationary action principle as:

$$\delta S_{\mathrm{FAT}} + \delta f(S_{\mathrm{FAT}}) = 0. \tag{62}$$

(Note: Uses generalized $f(S_R)$). **Interpretation**: Broadens action minimization, applied to field-particles in Section 14.

14. Five Fundamental Field-Particles

In this section, we describe five novel entities that describe deeper and beyond the particle-fields described by the standard particle physics model [35].

14.1. Diagrammatical Description of 5 Fundamental Entities

The Standard Cosmological Model (SMC), rooted in GR and perturbation theory, implies the graviton field via spacetime perturbations, alongside cold dark matter. Though undetected, the graviton's duality suggests layered properties. We introduce five fundamental field-particle entities—spatiallion, timion, actionion, probablon, and informaton—in a hierarchical model: spatiallion and timion are foundational (space and time), actionion builds on them (action), probablon on actionion (probability), and informaton on probablon (information). [1]

Note that we find that the spatiallions should follow the wave-field equation $\Box \delta g_{ij} = 0$, the timions should follow the wave-field equation, $\Box \delta g_{00} = 0$, the actionions should follow the existence of fluctuations of the fluctuations of the action equation, $\delta(\delta S_A) \neq 0$, the probablons follow the fluctuation probability of an event restriction equation, $0 \leq \delta p(E) \leq 1$, and the informatons should follow the information, or the entropy restriction equation, $\delta S_E \geq 0$.

Figure 3 and Table 1 illustrate these entities and compounds, including known particles like charged quarks ($q^{\pm}$), leptons ($l^{\pm}$), Higgs field [34] (H^0), photons (γ), bosons ($Z^0, W^{\pm}$), gluons (g^c), and gravitational waves (GW). Spaciallion and timion are independent, actionion integrates space and time fluctuations (Section 14), probablon adds probabilistic nature (generalizing QFT particles), and informaton encompasses all prior layers (e.g., mass, charge, spin, flavor, color, probability, action). Compounds like "probablispacetimion" (a probabilistic graviton) refine the graviton.

Figure 3. Coloured Venn Diagram expressed of the novel 5 fundamental field-particle entities: spaciallion, timion, probablon, informaton, actionion. Furthermore, we present their compounds and the so far discovered particles, i.e. the charged quarks, $q^{\pm}$, charged leptons, $\ell^{\pm}$, Higgs field-particle, H^0, photons, γ, Neutral bosons, Z^0, and charged bosons, $W^{\pm}$, coloured charged gluons, g^c, Gravitational Wave fields, GW, which are build from all the 5 novel fundamental field-particles.

[1] Rearranging and abbreviating the concepts to Probablon-Informaton-Spaciallion-Timion-Actionion (PISTA), we get PISTA which is the protoindoeuropean word for track. Therefore effectively we can think of these 5 fundamental field-particles as the track, i.e. PISTA, in which all fundamental particles are created in.

Table 1. Summary of the Five Fundamental Field-Particle Entities

Entity	Fluctuation Description	Governing Equations
Probablon	δp: probability fluctuations	$0 \leq \delta p(E) \leq 1$
Informaton	δS_E: entropy fluctuations	$\delta S_E \geq 0$
Spaciallion	δg_{ij}: spatial geometry	$\delta G_{ij} = 8\pi G \delta T_{ij} \Leftrightarrow \Box \delta g_{ij} = 0$
Timion	δg_{00}: temporal geometry	$\delta G_{00} = 8\pi G \delta T_{00} \Leftrightarrow \Box \delta g_{00} = 0$
Actionion	δS_A: actionic fluctuations	$\delta^2 S_A \neq 0$

14.2. Discussion and Interpretation

These distinctions—while not fully generalizing FATs' functorial scope (Section 2)—suggest a layered particle hierarchy, questioning whether all particles follow this structure or if new classes differ, warranting future study. **Interpretation**: These entities form a hierarchical framework for spacetime, action, probability, and information dynamics, enhancing particle classification.

15. Conclusions and Discussion

FATs, initiated in Section 1, span cosmology and mechanics (Section 2, Section 3) to quantum gravity (Section 11.3). We study foundational aspects of FATs (Section 4). Field equations (Section 4.6), non-locality management (Sections 5-7), Viability analysis (Section 8) and partition functions (Sections 10–11.3) highlight a robust framework. We provide an estimate of the Universal information (Section 12). We discuss promising solutions to integro-differential equations of FAT gravitational cosmology models in Section 9. Five field-particles (Section 14) offer novel insights, supported by approximate local invariance (Section 5) and generalized principles (Section 13). FATs promise unification [29,31], warranting further exploration of the popular approaches [2]. The FAT framework might provide alternative aspects to modified gravity, dark energy and explore further the cosmological model with the manifestation of FAT fluctuations, the so called actionions.

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Abbreviations

The following abbreviations are used in this manuscript:

MDPI	Multidisciplinary Digital Publishing Institute
DOAJ	Directory of open access journals
TLA	Three letter acronym
LD	Linear dichroism

Appendix A. Symbols and Notation

- A: Amplitude parameter for the exponential FAT gravity model term, dimensionless.
- $a(t)$: Scale factor, dimensionless.
- c: Speed of light, $3 \times 10^8\,\mathrm{m\,s^{-1}}$.
- c_s: Sound speed, normalised.
- d: Total number of fields in the vector field $\vec{v}$, where $d = n + m$.

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- f: Generic functor function applied to the action, e.g., $f(S_R)$.
- g: Gravitational acceleration or determinant of the metric tensor (context-dependent).
- $g_{\mu\nu}$: Metric tensor.
- g_i: Metric field components in the vector field $\vec{v}$, $i = 1, \ldots, m$.
- G_N: Gravitational constant, $7 \times 10^{-11}\,\mathrm{m^3\,s^{-2}\,kg^{-1}}$.
- $G(x, x')$: Green's function.
- H: Hubble parameter, $H = \dot{a}/a$, in A.T.U.$^{-1}$ or s^{-1} (context-dependent).
- H_0: Hubble constant, $2 \times 10^{-17}\,\mathrm{s}^{-1}$ or 0.70 A.T.U.$^{-1}$ (context-dependent).
- $I(t)$: Integral term in FAT gravity models, $I(t) = \int_0^t a^3(\tau)(\dot{H}(\tau) + 2H^2(\tau))d\tau$, dimensionless.
- k: Spring constant or four-momentum (context-dependent).
- $K(\beta)$: Constant in FAT gravity models, $K(\beta) = \frac{6V_{3D}}{c^2\kappa^2}\beta$, dimensionless.
- m: Oscillator mass or index denoting the number of metric fields in $\vec{v}$ (context-dependent).
- n: Number of matter fields in $\vec{v}$.
- n_i: Order of partial derivative with respect to v_i in Taylor expansion, $n_i = 0, 1, 2, \ldots$.
- R: Ricci scalar.
- $S[\vec{v}]$: Action as a functional of the vector field $\vec{v}$.
- S_R: Einstein-Hilbert action term.
- S_m: Matter action term.
- S_Λ: Cosmological constant term.
- S_E: Euclidean action or entropy (context-dependent).
- $T_{\mu\nu}$: Stress-energy tensor.
- v_i: Component i of the vector field $\vec{v}$, including metric (g_i) and matter (ϕ_j) fields.
- v_{0i}: Reference value of field v_i at which the action is evaluated.
- $\vec{v}$: Vector field combining metric and matter fields, $\vec{v} = \{g_1, \ldots, g_m, \phi_1, \ldots, \phi_n\}$.
- $\vec{v}_0$: Reference configuration of the vector field $\vec{v}$.
- $\mathcal{L}_m$: Matter Lagrangian density.
- ψ: all matter field, described in the standard model of particle physics.
- $\mathcal{O}$: Fluctuation operator.
- $\mathcal{V}_{3D}$: Spatial volume, $10^{77}\,\mathrm{m}^3$.
- $\mathcal{Z}$: Partition function.
- α, β, γ: FAT coupling constants.
- δ^D: Dirac delta, D-dimensional.
- $\delta S[\vec{v}]$: Variation of the action from reference $\vec{v}_0$.
- δv_{0i}: Perturbation of field v_i from its reference value, $\delta v_{0i} = v_i - v_{0i}$.
- $\delta g_{ij}, \delta g_{00}$: Metric perturbations (spatial and temporal).
- δS_A: Action fluctuation.
- δS_E: Entropy fluctuation.
- $\delta p(E)$: probability of an event fluctuation.
- κ: Reduced gravitational constant, $\kappa^2 = 8\pi G_N/c^4$.
- $\partial_{v_i}^{n_i}$: n_i-th partial derivative with respect to v_i.
- ϕ_j: Matter field components in $\vec{v}$, $j = 1, \ldots, n$.
- $\Phi(x)$: Fluctuation field.
- ρ_{m0}: Matter energy density at present time, in $\mathrm{kg\,m^{-3}}$ or dimensionless (context-dependent).
- ρ_{r0}: Radiation energy density at present time, in $\mathrm{kg\,m^{-3}}$ or dimensionless (context-dependent).
- $\rho_{\Lambda 0}$: Dark energy density at present time, in $\mathrm{kg\,m^{-3}}$ or dimensionless (context-dependent).
- Ω_{m0}: Matter density parameter, dimensionless.
- Ω_{r0}: Radiation density parameter, dimensionless.
- $\Omega_{\Lambda 0}$: Dark energy density parameter, dimensionless.
- $\Box$: d'Alembertian, $\eta^{\mu\nu}\partial_\mu\partial_\nu$.
- $\hbar$: Reduced Planck constant.

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- ω: Angular frequency, $\sqrt{k/m}$.
- x_0, v_0: Initial position and velocity.

Appendix B. Zeta-Function Regularization on d'Alembertian

The Zeta-Function Regularization on d'Alembertian is described via the equation

$$\ln \det(-\Box) = -\zeta'(0). \tag{A1}$$

whose **interpretation** is that it regularizes quantum corrections.

Appendix C. Zeta-Function in FLRW Metric

The Zeta-Function in FLRW Metric is described via

$$\zeta(s) = \int \frac{d^d k}{(2\pi)^d} \left(\frac{k^2}{a^2(t)} \right)^{-s}. \tag{A2}$$

whose **interpretation** is that it reflects expansion effects on fluctuations.

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