Article

Not peer-reviewed version

On the Complete Analogy of Complex Analysis and Real Analysis in the Field of Vectors V_2

Branko Saric *

Posted Date: 20 February 2025

doi: 10.20944/preprints202502.1614.v1

Keywords: geometric product; the field of vectors

Article

On the Complete Analogy of Complex Analysis and Real Analysis in the Field of Vectors $\mathbf{V}_2$

Branko Sarić

Faculty of Technical Sciences, University of Kragujevac, Čačak 32000, Serbia; saric.b@mts.rs

Abstract: On the basis of the isomorphic algebraic structures of the field of complex numbers $\mathbb{C}$ and the 2-dimensional *Euclidean* field of vectors $\mathbf{V}_2$, in terms of identical geometric products of elements, in this paper vector integral identities have been derived for scalar and vector fields in $\mathbf{V}_2$, which are vector analogues of the well-known integral identities of complex analysis. In doing so, special attention is given to the vector analogue in $\mathbf{V}_2$ of *Cauchy's* calculus of residues.

Keywords: geometric product; the field of vectors

MSC: Primary 14A05, 26B12; Secondary 14A25, 26B20

---

1. Introduction

A geometric algebra (*Clifford* algebra) is an extension of elementary algebra to work with geometrical objects such as vectors. It is built out of two fundamental operations addition and geometric product, [2]. The multiplication of vectors alone results in objects called multivectors, among which are bivectors, the name applied in this paper to the objects of the bivector field $\mathbb{R}^2$, corresponding to the field of vectors $\mathbf{V}_2$. Compared with other formalisms for manipulating geometric objects, geometric algebra supports dividing by a vector. The geometric product was first mentioned by *Grassmann*, who founded the so-called external algebra, [3]. After that, *Clifford* himself greatly expanded upon *Grassmann's* work, to form geometric algebra, named after him *Clifford* algebra [2], by unifying both *Grassmann's* algebra and *Hamilton's* quaternion algebra. In the middle of the 20th century, *Hestenes* repopularized the term geometric algebra [4,5].

On the other hand, although rarely used explicitly, a geometric representation of complex numbers is implicitly based on its structure of the *Euclidean* 2-dimensional vector space. If the binary operation of the product $\bar{w}z$, of two complex numbers $\bar{w}$ and z, is considered as the sum of the inner product $w \circ z = (\bar{w}z + w\bar{z})/2 = \mathbf{1}\ \mathrm{Re}(\bar{w}z)$ and outer product $w \wedge z = (\bar{w}z - w\bar{z})/2 = \boldsymbol{i}\ \mathrm{Im}(\bar{w}z)$, where $\mathbf{1} = (1, i0)$ and $\boldsymbol{i} = (0, i)$, and i is an imaginary unit, it can be said that $\bar{w}z$ is in the form of a geometric product of two $i$vectors (two complex numbers), as two geometric objects belonging to the $i$vector field $\mathbb{C}$ (to the field of complex numbers $\mathbb{C}$). For any complex number z, its absolute value $|z|$ is its *Euclidean* norm denoted by r, and the argument $\arg z$ is the polar angle φ. Since ordered pairs represent both complex numbers and vectors, the binary operation of the product of two complex numbers (two ordered pairs), in the form of the geometric product wz ($wz = \bar{w} \circ z + \bar{w} \wedge z$), will be the basis for modifying *Grassmann's* geometric product of vectors, which is defined as the sum of the inner (scalar) and outer (vector) products of two vectors. By this modification, the geometric product of two vectors becomes commutative, similar to the product of complex numbers themselves, which still supports vector division. In this manner, a complete analogy is established between the algebra of complex numbers and the modified *Clifford* algebra in the *Euclidean* 2-dimensional field of vectors $\mathbf{V}_2$. On the basis of that analogy, the paper presents the most important vector integral identities, in the real *Euclidean* field of vectors $\mathbf{V}_2$, which are vector analogies to the well-known integral identities of complex analysis.

 doi:10.20944/preprints202502.1614.v1

1.1. Realireal Vector Space $V_{\mathbb{C}}$

The ordered pairs $\mathbf{1} = (1, i0)$ and $\boldsymbol{i} = (0, i)$ are the basis of the 2-dimensional real*i*real vector space $V_{\mathbb{C}}$ [6], which is the *Cartesian* product $V_{\mathbb{R}} \times iV_{\mathbb{R}}$ of a 1-dimensional real vector space $V_{\mathbb{R}}$ and a 1-dimensional *i*real vector space $iV_{\mathbb{R}}$, and as such, $V_{\mathbb{C}}$ is an additive *Abelian* (commutative) group of elements $(x, iy) = \mathbf{1}x + \boldsymbol{i}y$. As both of these 1-dimensional vector spaces are defined over the field of real numbers $\mathbb{R}$, the real vector space $V_{\mathbb{R}}$ can be said to be a field of real numbers $\mathbb{R}$, whereas the *i*real vector space $iV_{\mathbb{R}}$ cannot be a field, and therefore not a field of imaginary numbers iy. On the other hand, if the vector space $V_{\mathbb{C}}$ is complemented by a binary operation of the product of two elements (a, ib) and (c, id), which corresponds to the matrix product, in such a manner that

$$(a, ib)(c, id) \eqqcolon \begin{bmatrix} a & ib \\ ib & a \end{bmatrix} \begin{bmatrix} c & id \\ id & c \end{bmatrix} = \tag{1}$$

$$= \begin{bmatrix} ac - bd & i(ad + bc) \\ i(ad + bc) & ac - bd \end{bmatrix} \eqqcolon (ac - bd, i(ad + bc)),$$

where both the commutative axiom of multiplication and the associative axiom of multiplication are satisfied, as well as the distributive axiom, and in addition the element $(x, -iy)/(x^2 + y^2)$, which corresponds to the inverse matrix $\begin{bmatrix} x & iy \\ iy & x \end{bmatrix}^{-1}$, is the inverse element of the element $(x, iy) \neq (0, i0)$, then the 2-dimensional real*i*real vector space $V_{\mathbb{C}}$ can be said to be defined over the field of complex numbers $\mathbb{C}$, that is, $V_{\mathbb{C}}$ is the field of complex numbers $\mathbb{C}$. More precisely, on the one hand, the ordered pair $(V_{\mathbb{C}}, \mathbb{R})$ is the vector space $V_{\mathbb{C}}$ over the scalar field of real numbers $\mathbb{R}$, and on the other hand, after complementation with the binary operation of the product of the elements, the ordered pair $(V_{\mathbb{C}}, \mathbb{C})$ is the vector space over the *i*vector field $\mathbb{C}$, that is, $V_{\mathbb{C}}$ is the *i*vector field $\mathbb{C}$. Accordingly, complex numbers z can be said to be *i*vectors $(x, iy) = \mathbf{1}x + \boldsymbol{i}y$, the elements of the vector space $V_{\mathbb{C}}$, that is, of the *i*vector field $\mathbb{C}$ and as such can be multiplied either by real numbers as scalars or by complex numbers z as *i*vectors. When the *i*vector (a, ib) is multiplying by the imaginary unit i, then the order of of the elements, in the resulting ordered pair, is changed, so the resulting ordered pair is not an element of the *i*vector field $\mathbb{C}$. Therefore, from that perspective, multiplying complex numbers by imaginary numbers is absolutely unacceptable. A complex number can be multiplied by the *i*vector $\boldsymbol{i}$, so that $(x, iy)\boldsymbol{i} = \boldsymbol{i}(x, iy) = -y\mathbf{1} + x\boldsymbol{i} = (-y, ix)$ and $\boldsymbol{i}^2 = \boldsymbol{i}\boldsymbol{i} = -\mathbf{1}$.

How the operator $\mathbf{c}\boldsymbol{i}\mathbf{s}\cdot = (\cos\cdot, i\sin\cdot) = \mathbf{1}\cos\cdot + \boldsymbol{i}\sin\cdot$ has the most important properties of an exponential function, since $\mathbf{c}\boldsymbol{i}\mathbf{s}\cdot\mathbf{c}\boldsymbol{i}\mathbf{s}\cdot = \mathbf{c}\boldsymbol{i}\mathbf{s}(\cdot + \cdot)$, $(\mathbf{c}\boldsymbol{i}\mathbf{s}\cdot)^{-1} = \mathbf{c}\boldsymbol{i}\mathbf{s}(-\cdot)$ and $d(\mathbf{c}\boldsymbol{i}\mathbf{s}\cdot) = \boldsymbol{i}\mathbf{c}\boldsymbol{i}\mathbf{s}\cdot d\cdot$, the operator $\exp(\boldsymbol{i}\cdot)$ can be said to be the exponential form of the operator $\mathbf{c}\boldsymbol{i}\mathbf{s}\cdot$. For above reason, multiplication of the operator $\mathbf{c}\boldsymbol{i}\mathbf{s}\cdot = \exp(\boldsymbol{i}\cdot)$ by imaginary numbers is unacceptable, but multiplication by either a real number or an *i*vector is acceptable.

1.2. The Field of Vectors $\mathbf{V}_2$

The basis of the 2-dimensional real vector space $V_{\mathbb{R}^2}$ (the *Cartesian* square of the 1-dimensional real vector space $V_{\mathbb{R}}$), as an additive *Abelian* group of elements (x, y), consists of ordered pairs $\mathbf{1} = (1, 0)$ and $\hat{\mathbf{1}} = (0, 1)$, such that $(x, y) = x\mathbf{1} + y\hat{\mathbf{1}}$. Analogous to the binary operation of the product of the elements, which we used to complement the real*i*real vector space $V_{\mathbb{C}}$, it is also possible to complement the vector space $V_{\mathbb{R}^2}$, with the binary operation of the product of two elements (a, b) and (c, d), which corresponds to the matrix product, in such a manner that

$$(a, b)(c, d) \eqqcolon \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \begin{bmatrix} c & d \\ -d & c \end{bmatrix} = \tag{2}$$

$$= \begin{bmatrix} ac - bd & ad + bc \\ -(ad + bc) & ac - bd \end{bmatrix} \eqqcolon (ac - bd, ad + bc),$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 20 February 2025 doi:10.20944/preprints202502.1614.v1

where the element $(x/(x^2+y^2), -y/(x^2+y^2))$, which corresponds to the inverse matrix $\begin{bmatrix} x & y \\ -y & x \end{bmatrix}^{-1}$, is the inverse element of the vector space $V^*_{\mathbb{R}^2} = V_{\mathbb{R}^2} \backslash \{(0,0)\}$. In this case both the commutative axiom of multiplication and the associative axiom of multiplication are satisfied, as well as the distributive axiom. Therefore, on the one hand, the ordered pair $(V_{\mathbb{R}^2}, \mathbb{R})$ is the vector space $V_{\mathbb{R}^2}$ over the scalar field of real numbers $\mathbb{R}$, whereas on the other hand, the ordered pair $(V_{\mathbb{R}^2}, \mathbb{R}^2)$ is a vector space over the bivector field $\mathbb{R}^2$, that is, $V_{\mathbb{R}^2}$ is the bivector field $\mathbb{R}^2$, whose elements are the bivectors $(x,y) = x\mathbf{1} + y\hat{\mathbf{1}}$. Accordingly, the bireal numbers w are bivectors (x,y), elements of the vector space $V_{\mathbb{R}^2}$, that is, of the bivector field $\mathbb{R}^2$ and as such can be multiplied either by real numbers as scalars or by bireal numbers as bivectors.

The field of vectors $\mathbf{V}_2$ corresponds to the bivector field $\mathbb{R}^2$, in the sense of the correspondence: $\mathbf{1} \risingdotseq \mathbf{e}$ and $\hat{\mathbf{1}} \risingdotseq \hat{\mathbf{e}}$, where the unit vectors $\mathbf{e}$ and $\hat{\mathbf{e}}$ are orthogonal basis vectors of the field of vectors $\mathbf{V}_2$. The vectors $\mathbf{r} = x\mathbf{e} + y\hat{\mathbf{e}}$, as elements of the field $\mathbf{V}_2$, correspond to the bireal numbers $w \in \mathbb{R}^2$. In other words, there is a one to one correspondence between the field of vectors $\mathbf{V}_2$ and the set of bireal numbers $\mathbb{R}^2$. If $(x,-y) \risingdotseq x\mathbf{e} - y\hat{\mathbf{e}} = \bar{\mathbf{r}}$, then $r = \|\bar{\mathbf{r}}\| = \|(x,-y)\| = \left|\sqrt[2]{x^2+y^2}\right|$ is the norm over the field of vectors $\mathbf{V}_2$ and over the bivector field $\mathbb{R}^2$, simultaneously. In addition, $\bar{\mathbf{r}}\mathbf{r} \fallingdotseq (x,-y)(x,y) = r^2\mathbf{1} \risingdotseq r^2\mathbf{e}$ and $r^2\mathbf{r}^{-1} = \bar{\mathbf{r}}$. It is quite clear that the inverse element $\mathbf{r}^{-1}$ allows division by a vector in the field of vectors $\mathbf{V}_2$.

The binary operation of the product of two vectors $\bar{\mathbf{r}}_1 = a\mathbf{e} - b\hat{\mathbf{e}} \in \mathbf{V}_2$ and $\mathbf{r}_2 = c\mathbf{e} + d\hat{\mathbf{e}} \in \mathbf{V}_2$, as follows

$$\bar{\mathbf{r}}_1\mathbf{r}_2 = \frac{1}{2}(\bar{\mathbf{r}}_1\mathbf{r}_2 + \mathbf{r}_1\bar{\mathbf{r}}_2) + \frac{1}{2}(\bar{\mathbf{r}}_1\mathbf{r}_2 - \mathbf{r}_1\bar{\mathbf{r}}_2) := (\bar{\mathbf{r}}_1 \cdot \bar{\mathbf{r}}_2)\mathbf{e} - (\bar{\mathbf{r}}_1 \times \bar{\mathbf{r}}_2) \times \mathbf{e} = \tag{3}$$

$$= (\mathbf{r}_1 \cdot \mathbf{r}_2)\mathbf{e} + (\mathbf{r}_1 \times \mathbf{r}_2) \times \mathbf{e} = [(a\mathbf{e} + b\hat{\mathbf{e}}) \cdot (c\mathbf{e} + d\hat{\mathbf{e}})]\mathbf{e} + [(a\mathbf{e} + b\hat{\mathbf{e}}) \times (c\mathbf{e} + d\hat{\mathbf{e}})] \times \mathbf{e},$$

where $\mathbf{r}_1 \cdot \mathbf{r}_2 = \|\mathbf{r}_1\|\|\mathbf{r}_2\|\cos\theta$ and $(\mathbf{r}_1 \times \mathbf{r}_2) \times \mathbf{e} = \|\mathbf{r}_1\|\|\mathbf{r}_2\|\sin\theta\hat{\mathbf{e}}$, which is obviously commutative and corresponds to the product $(a,-b)(c,d) = (ac+bd, ad-bc)$ of two bivectors $(a,-b)$ and (c,d), is the geometric product of these two vectors. Here,

$$\frac{1}{2}(\bar{\mathbf{r}}_1\mathbf{r}_2 + \mathbf{r}_1\bar{\mathbf{r}}_2) := (\mathbf{r}_1 \cdot \mathbf{r}_2)\mathbf{e} = \mathbf{r}_1 \circ \mathbf{r}_2 \fallingdotseq (a,b) \circ (c,d) \text{ and} \tag{4}$$

$$\frac{1}{2}(\bar{\mathbf{r}}_1\mathbf{r}_2 - \mathbf{r}_1\bar{\mathbf{r}}_2) := (\mathbf{r}_1 \times \mathbf{r}_2) \times \mathbf{e} = \mathbf{r}_1 \wedge \mathbf{r}_2 \fallingdotseq (a,b) \wedge (c,d). \tag{5}$$

In addition, $(a,-b)(c,d) = (a,b) \circ (c,d) + (a,b) \wedge (c,d)$ and

$$[(a,b) \circ (c,d)]^2 - [(a,b) \wedge (c,d)]^2 = (a,-b)(c,d)(c,-d)(a,b) = \tag{6}$$

$$= (a,-b)(c,d)(a,b)(c,-d) = \|(a,b)\|^2\|(c,d)\|^2\mathbf{1}.$$

The previously introduced concepts of the geometric product and bivector are closely related to the same concepts in *Clifford* algebra [2]. However, there is evidently a crucial difference between *Clifford* algebra and the algebra of the field of vectors $\mathbf{V}_2$, which is reflected in the fact that the geometric product of the elements of the field of vectors $\mathbf{V}_2$, on the one hand, is the element of the field of vectors $\mathbf{V}_2$, corresponding to the bivector, the element of the bivector field $\mathbb{R}^2$, and on the other hand, it is also commutative, which means that the field of vectors $\mathbf{V}_2$, in addition to being an additive *Abelian* group, is also a multiplicative *Abelian* group.

If $\mathbf{r}_0$ is the unit vector of the vector $\mathbf{r}$ ($\mathbf{r} = r\mathbf{r}_0$), then $\mathbf{r}_0^{-1} = \bar{\mathbf{r}}_0$ and

$$\frac{\mathbf{r}_{0_1}}{\mathbf{r}_{0_2}} = \mathbf{r}_{0_1}\bar{\mathbf{r}}_{0_2} = (\mathbf{r}_{0_1} \cdot \mathbf{r}_{0_2})\mathbf{e} - (\mathbf{r}_{0_1} \times \mathbf{r}_{0_2}) \times \mathbf{e}, \tag{7}$$

that is,

$$\mathbf{r}_{0_1} \circ \mathbf{r}_{0_2} = \frac{1}{2}\left(\frac{\mathbf{r}_{0_2}}{\mathbf{r}_{0_1}} + \frac{\mathbf{r}_{0_1}}{\mathbf{r}_{0_2}}\right) \text{ i } \mathbf{r}_{0_1} \wedge \mathbf{r}_{0_2} = \frac{1}{2}\left(\frac{\mathbf{r}_{0_2}}{\mathbf{r}_{0_1}} - \frac{\mathbf{r}_{0_1}}{\mathbf{r}_{0_2}}\right), \tag{8}$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 20 February 2025 doi:10.20944/preprints202502.1614.v1

where $\mathbf{r}_{0_1} \circ \mathbf{r}_{0_2} = (\mathbf{r}_{0_1} \cdot \mathbf{r}_{0_2})\mathbf{e}$ i $\mathbf{r}_{0_1} \wedge \mathbf{r}_{0_2} = (\mathbf{r}_{0_1} \times \mathbf{r}_{0_2}) \times \mathbf{e}$ are the symmetric and antisymmetric parts of the geometric product $\bar{\mathbf{r}}_{0_1}\mathbf{r}_{0_2}$, respectively. Therefore, to emphasize once again, the geometric product of two vectors is a vector in $\mathbf{V}_2$. Accordingly, $\mathbf{re} = (\bar{\mathbf{r}} \cdot \mathbf{e})\mathbf{e} + (\bar{\mathbf{r}} \times \mathbf{e}) \times \mathbf{e} = x\mathbf{e} + y\hat{\mathbf{e}} = \mathbf{r}$ and $\mathbf{r}\hat{\mathbf{e}} = (\bar{\mathbf{r}} \cdot \hat{\mathbf{e}})\mathbf{e} + (\bar{\mathbf{r}} \times \hat{\mathbf{e}}) \times \mathbf{e} = \mathbf{r}_\perp$. The vector $\mathbf{r}_\perp$ is orthogonal to the vector $\mathbf{r}$, that is, $\mathbf{r}_\perp$ is the vector obtained by rotating the vector $\mathbf{r} = r\mathbf{r}_0 = r(\mathbf{e}\cos\varphi + \hat{\mathbf{e}}\sin\varphi)$, where φ is the angle between the vector $\mathbf{r}$ and the vector $\mathbf{e}$, by $\pi/2$ radians in the positive mathematical direction. In addition, the unit vector $\bar{\mathbf{r}}_0 = \cos\varphi\mathbf{e} - \sin\varphi\hat{\mathbf{e}}$ is the inverse vector $\mathbf{r}_0^{-1}$ of the unit vector $\mathbf{r}_0$ of the vector $\mathbf{r}$, and therefore also the unit vector of the inverse vector $\mathbf{r}^{-1}$ of the vector $\mathbf{r}$, so that $\mathbf{r}^{-1} = \bar{\mathbf{r}}_0/r$ and $\bar{\mathbf{r}} = r\bar{\mathbf{r}}_0$.

The main purpose of the paper is to derive vector integral identities, in the field of vectors $\mathbf{V}_2$, which are analogous to the known integral identities of complex analysis, on the basis of the analogy of the *i*vector field $\mathbb{C}$ and the bivector field $\mathbb{R}^2$, which corresponds to the field of vectors $\mathbf{V}_2$.

2. The Main Results

As $\mathbf{c\hat{e}s}\cdot\mathbf{c\hat{e}s}\cdot = \mathbf{c\hat{e}s}(\cdot+\cdot)$, $(\mathbf{c\hat{e}s}\cdot)^{-1} = \mathbf{c\hat{e}s}(-\cdot)$ i $d(\mathbf{c\hat{e}s}\cdot) = \hat{\mathbf{e}}\mathbf{c\hat{e}s}\cdot d\cdot$, the bivector operator $\mathbf{c\hat{e}s}\cdot = \mathbf{e}\cos\cdot + \hat{\mathbf{e}}\sin\cdot$ also has the properties of an exponential function, similar to the *i*vector operator $\mathbf{c\dot{i}s}\cdot$. The operator $\exp(\hat{\mathbf{e}}\cdot)$ is the exponential form of the operator $\mathbf{c\hat{e}s}\cdot$. Since $\mathbf{r}_0 = \mathbf{c\hat{e}s}\varphi = \exp(\hat{\mathbf{e}}\varphi)$, one more analogy with complex analysis is the notion of the so-called vector logarithmic function $\log\mathbf{r} = \ln r\mathbf{e} + \varphi\hat{\mathbf{e}}$, where $2\varphi\hat{\mathbf{e}} = 2\log\mathbf{r}_0 = \log(\mathbf{r}_0/\bar{\mathbf{r}}_0)$. In addition, $\mathrm{Log}\,\mathbf{r} = \ln r\mathbf{e} + (\varphi \pm 2\pi n)\hat{\mathbf{e}}$, $n \in \mathbb{N}$. Let $\hat{\mathbf{r}}_0 = -\mathbf{r}_{0\perp}^{-1} = \mathbf{e}\sin\varphi + \hat{\mathbf{e}}\cos\varphi = \mathbf{s\hat{e}c}\varphi = \hat{\mathbf{e}}\bar{\mathbf{r}}_0$. The ordered pair of vectors $(\bar{\mathbf{r}}_0, \hat{\mathbf{r}}_0)$ is the inverse orthonormal basis with respect to the orthonormal basis $(\mathbf{r}_0, \mathbf{r}_{0\perp})$ of the field of vectors $\mathbf{V}_2$. For an arbitrary vector $\boldsymbol{\varrho} \in \mathbf{V}_2$, the vector $\boldsymbol{\varrho}\boldsymbol{r}_0$ is the rotated vector $\boldsymbol{\varrho}$, in the positive mathematical direction, by the angle φ, and the vector $\boldsymbol{\varrho}\boldsymbol{r}_{0\perp}$ by the angle $\pi/2+\varphi$. The geometric products of the vector $\boldsymbol{\varrho}$ with the inverse basis vectors $\bar{\mathbf{r}}_0$ and $\hat{\mathbf{r}}_0$ rotate the vector $\boldsymbol{\varrho}$ by the angles $-\varphi$ and $\pi/2-\varphi$, respectively, in the positive mathematical direction. On the basis of the geometric products $\mathbf{ee} = \mathbf{e}$, $\mathbf{e}\hat{\mathbf{e}} = \hat{\mathbf{e}}$ and $\hat{\mathbf{e}}\hat{\mathbf{e}} = -\mathbf{e}$, that is,

$$\mathbf{r}_0\mathbf{r}_0 = (\bar{\mathbf{r}}_0 \cdot \mathbf{r}_0)\mathbf{e} + (\bar{\mathbf{r}}_0 \times \mathbf{r}_0) \times \mathbf{e} = (\cos^2\varphi - \sin^2\varphi)\mathbf{e} + 2(\cos\varphi\sin\varphi)\hat{\mathbf{e}}, \tag{9}$$

$\mathbf{r}_0\bar{\mathbf{r}}_0 = \mathbf{e}$ and $\mathbf{r}_0\hat{\mathbf{r}}_0 = \hat{\mathbf{e}}$ ($\hat{\mathbf{r}} = \bar{\mathbf{r}}\hat{\mathbf{e}}$), all other combinations of geometric products of the basis vectors $\mathbf{r}_0$, $\mathbf{r}_{0\perp}$, $\bar{\mathbf{r}}_0$ and $\hat{\mathbf{r}}_0$ can also be obtained.

If we introduce the differential operator $d = dr\partial_r + d\varphi\partial_\varphi$, then $d\mathbf{r} = dr\mathbf{r}_0 + d\varphi\mathbf{r}_\perp$. Hence, $d\mathbf{r}_\perp = \hat{\mathbf{e}}d\mathbf{r} = dr\mathbf{r}_{0\perp} - d\varphi\mathbf{r}$ and $d\hat{\mathbf{r}} = \hat{\mathbf{e}}d\bar{\mathbf{r}} = \hat{\mathbf{e}}(dr\bar{\mathbf{r}}_0 - d\varphi\hat{\mathbf{r}}) = dr\hat{\mathbf{r}}_0 + d\varphi\bar{\mathbf{r}}$. Since $2r\cos\varphi\mathbf{e} = \mathbf{r} + \bar{\mathbf{r}}$ and $2r\sin\varphi\hat{\mathbf{e}} = \mathbf{r} - \bar{\mathbf{r}}$, the vector operators of partial derivatives are introduced, as a vector analogue of the *Virtinger* operators [16],

$$\eth_{\mathbf{r}} = \partial_{\mathbf{r}}r\partial_r + \partial_{\mathbf{r}}\varphi\partial_\varphi = \frac{1}{2}(\bar{\mathbf{r}}_0\partial_r - \frac{\hat{\mathbf{r}}_0}{r}\partial_\varphi) \text{ and } \eth_{\bar{\mathbf{r}}} = \bar{\eth}_{\mathbf{r}} = \frac{1}{2}(\mathbf{r}_0\partial_r + \frac{\mathbf{r}_{0\perp}}{r}\partial_\varphi). \tag{10}$$

Here,

$$\partial_{\mathbf{r}}r = \frac{1}{2r}\partial_{\mathbf{r}}r^2 = \frac{\bar{\mathbf{r}}_0}{2} \text{ and } \partial_{\mathbf{r}}\varphi = \cos^2\varphi\partial_{\mathbf{r}}\tan\varphi = (\frac{\mathbf{r}+\bar{\mathbf{r}}}{2r})^2\frac{-2\hat{\mathbf{e}}\bar{\mathbf{r}}}{(\mathbf{r}+\bar{\mathbf{r}})^2} = -\frac{\hat{\mathbf{r}}_0}{2r}. \tag{11}$$

It is important to emphasize that when geometric products and geometric quotients are differentiated, the same rules apply as when ordinary products and quotients are differentiated. Namely,

$$d\frac{\mathbf{r}}{\bar{\mathbf{r}}} = (d\frac{1}{r^2})\mathbf{r}^2 + \frac{1}{r^2}d\mathbf{r}^2 = -2(\frac{\mathbf{r}^2}{r^3}dr - \frac{1}{r^2}\mathbf{r}d\mathbf{r}) = \tag{12}$$

$$= -2[\frac{\mathbf{r}^2}{2r^4}(\bar{\mathbf{r}}d\mathbf{r} + \mathbf{r}d\bar{\mathbf{r}}) - \frac{1}{r^2}\mathbf{r}d\mathbf{r}] = -2[\frac{1}{2\bar{\mathbf{r}}^2}(\bar{\mathbf{r}}d\mathbf{r} + \mathbf{r}d\bar{\mathbf{r}}) - \frac{1}{\bar{\mathbf{r}}^2}\bar{\mathbf{r}}d\mathbf{r}] = \frac{\bar{\mathbf{r}}d\mathbf{r} - \mathbf{r}d\bar{\mathbf{r}}}{\bar{\mathbf{r}}^2}.$$

The vector orerator

$$\mathbf{d} = \mathbf{r}_0(d\mathbf{r}\eth_{\mathbf{r}} + d\bar{\mathbf{r}}\eth_{\bar{\mathbf{r}}}) = 2\mathbf{r}_0(d\mathbf{r} \circ \eth_{\bar{\mathbf{r}}}) = \mathbf{r}_0(dr\partial_r + d\varphi\partial_\varphi) = \mathbf{r}_0 d, \tag{13}$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 20 February 2025 doi:10.20944/preprints202502.1614.v1

as a symmetrical part of the geometric product $2d\mathbf{r}\eth_{\mathbf{r}}$, is a radial vector differential operator. The antisymmetric part

$$\mathbf{d}_{\perp} = \mathbf{r}_0(d\mathbf{r}\eth_{\mathbf{r}} - d\bar{\mathbf{r}}\eth_{\bar{\mathbf{r}}}) = -2\mathbf{r}_0(d\mathbf{r} \wedge \eth_{\bar{\mathbf{r}}}) = \mathbf{r}_{0\perp}(rd\varphi\partial_r - \frac{1}{r}dr\partial_{\varphi}) = \mathbf{r}_{0\perp}\hat{d} \tag{14}$$

is a transverse vector differential operator. It is obvious that the vector operator $2\eth_{\bar{\mathbf{r}}}$ is a gradient operator in polar coordinates. The symmetric part of the geometric product $2\eth_{\mathbf{r}}\mathbf{F}$ is the divergence vector (div) of the vector field $\mathbf{F} = F\mathbf{r}_0 + F_{\perp}\mathbf{r}_{0\perp}$, and the antisymmetric part is the curl of the vector field $\mathbf{F}$, since

$$\bar{\mathbf{r}}_0\eth_{\bar{\mathbf{r}}} \circ \bar{\mathbf{r}}_0\mathbf{F} + \bar{\mathbf{r}}_0\eth_{\bar{\mathbf{r}}} \wedge \bar{\mathbf{r}}_0\mathbf{F} = \mathbf{r}_0\eth_{\mathbf{r}}(\bar{\mathbf{r}}_0\mathbf{F}) = -\frac{\bar{\mathbf{r}}_0}{2r}\mathbf{F} + \eth_{\mathbf{r}}\mathbf{F} \text{ and} \tag{15}$$

$$\eth_{\mathbf{r}}\mathbf{F} = \frac{1}{r^2}(\bar{\mathbf{r}}\eth_{\bar{\mathbf{r}}} \circ \bar{\mathbf{r}}\mathbf{F}) + \frac{1}{r^2}(\bar{\mathbf{r}}\eth_{\bar{\mathbf{r}}} \wedge \bar{\mathbf{r}}\mathbf{F}), \tag{16}$$

so that

$$2\eth_{\mathbf{r}}\mathbf{F} = \frac{\bar{\mathbf{r}}_0\mathbf{r}_0}{r}[\partial_r(rF) + \partial_{\varphi}F_{\perp}] + \frac{\bar{\mathbf{r}}_0\mathbf{r}_{0\perp}}{r}[\partial_r(rF_{\perp}) - \partial_{\varphi}F] = \tag{17}$$

$$= \bar{\mathbf{r}}_0\mathbf{r}_0 \,\mathrm{div}\mathbf{F} + \,\mathrm{curl}\mathbf{F} \times \bar{\mathbf{r}}_0\mathbf{r}_0,$$

that is,

$$2\eth_{\bar{\mathbf{r}}}\mathbf{F} = \frac{\mathbf{r}_0\mathbf{r}_0}{r}(r\partial_r F - \partial_{\varphi}F_{\perp}) + \frac{\mathbf{r}_0\mathbf{r}_{0\perp}}{r}(r\partial_r F_{\perp} + \partial_{\varphi}F) - \frac{\mathbf{F}}{\bar{\mathbf{r}}} = \mathrm{grad}\mathbf{F}, \tag{18}$$

where

$$\mathbf{r}_0\eth_{\mathbf{r}} \circ \bar{\mathbf{r}}_0\mathbf{F} + \mathbf{r}_0\eth_{\mathbf{r}} \wedge \bar{\mathbf{r}}_0\mathbf{F} = \bar{\mathbf{r}}_0\eth_{\bar{\mathbf{r}}}(\bar{\mathbf{r}}_0\mathbf{F}) = \bar{\mathbf{r}}_0(\frac{1}{2r}\mathbf{F} + \bar{\mathbf{r}}_0\eth_{\bar{\mathbf{r}}}\mathbf{F}), \tag{19}$$

$$\mathrm{div}\mathbf{F} = \frac{2}{r^2}(\bar{\mathbf{r}}\eth_{\bar{\mathbf{r}}} \cdot \bar{\mathbf{r}}\mathbf{F}) \text{ and } \mathrm{curl}\mathbf{F} = \frac{2}{r^2}(\bar{\mathbf{r}}\eth_{\bar{\mathbf{r}}} \times \bar{\mathbf{r}}\mathbf{F}).$$

On the other hand, as $\mathbf{r} = r\mathbf{c\hat{e}s}\varphi = r\exp(\varphi\hat{\mathbf{e}})$, it follows that $2\varphi\hat{\mathbf{e}} = \log(\mathbf{r}/\bar{\mathbf{r}})$,

$$\mathbf{d}\varphi = \mathbf{r}_0 d\varphi = \mathbf{r}_0(\eth_{\mathbf{r}}\varphi d\mathbf{r} + \eth_{\bar{\mathbf{r}}}\varphi d\bar{\mathbf{r}}) = -\frac{\mathbf{r}_{0\perp}}{2}d\log(\frac{\mathbf{r}}{\bar{\mathbf{r}}}) \text{ and} \tag{20}$$

$$\mathbf{d}r = \mathbf{r}_0 dr = \mathbf{r}_0(\eth_{\mathbf{r}}rd\mathbf{r} + \eth_{\bar{\mathbf{r}}}rd\bar{\mathbf{r}}) = \mathbf{r}_0 d(\mathbf{r}\bar{\mathbf{r}})^{\frac{1}{2}} = \frac{\mathbf{r}}{2}d\log(\mathbf{r}\bar{\mathbf{r}}). \tag{21}$$

Therefore,

$$\mathbf{r}_{0\perp}drd\varphi = \frac{\mathbf{r}}{4}d\log(\mathbf{r}\bar{\mathbf{r}})d\log(\frac{\mathbf{r}}{\bar{\mathbf{r}}}). \tag{22}$$

In addition,

$$\frac{\mathbf{r}^2}{4}d\log(\mathbf{r}\bar{\mathbf{r}})d\log(\frac{\mathbf{r}}{\bar{\mathbf{r}}}) = \frac{\mathbf{r}_0^2}{4r^2}(\bar{\mathbf{r}}d\mathbf{r} + \mathbf{r}d\bar{\mathbf{r}})(\bar{\mathbf{r}}d\mathbf{r} - \mathbf{r}d\bar{\mathbf{r}}) = \frac{\mathbf{r}_0^2}{4}[(\bar{\mathbf{r}}_0 d\mathbf{r})^2 - (\mathbf{r}_0 d\bar{\mathbf{r}})^2]. \tag{23}$$

In accordance with above,

$$\mathbf{r}_0(\mathbf{d}\bar{\mathbf{r}} \wedge \bar{\mathbf{d}}\mathbf{r}) = \mathbf{r}_0(\mathbf{r}_0 d\bar{\mathbf{r}} \wedge \bar{\mathbf{r}}_0 d\mathbf{r}) = 2\mathbf{r}_{\perp}drd\varphi, \tag{24}$$

since $\overline{\mathbf{d}\bar{\mathbf{r}}} = \bar{\mathbf{r}}_0 d\mathbf{r} = \bar{\mathbf{d}}\mathbf{r}$. The vector identity just derived can be obtained explicitly, if we introduce the determinant of the *Jacobi* matrix (*Jacobian*) of the bijective mapping $\mathbf{V}_2 \to \mathbf{V}_2$, defined by the system of vector equations $2\ln r\mathbf{e} = \log(\mathbf{r}\bar{\mathbf{r}})$ and $2\varphi\hat{\mathbf{e}} = \log(\mathbf{r}/\bar{\mathbf{r}})$, as follows

$$\mathbf{J} = \begin{vmatrix} \partial_r\log(\mathbf{r}\bar{\mathbf{r}}) & \partial_{\varphi}\log(\mathbf{r}\bar{\mathbf{r}}) \\ \partial_r\log(\mathbf{r}/\bar{\mathbf{r}}) & \partial_{\varphi}\log(\mathbf{r}/\bar{\mathbf{r}}) \end{vmatrix} = \begin{vmatrix} 2r^{-1}\mathbf{e} & \mathbf{0} \\ \mathbf{0} & 2\hat{\mathbf{e}} \end{vmatrix} = \frac{4}{r}\hat{\mathbf{e}}. \tag{25}$$

In this case, $4\mathbf{r}\mathbf{r}_{0\perp}drd\varphi = \mathbf{r}^2\mathbf{J}drd\varphi = \mathbf{r}^2 d\log(\mathbf{r}\bar{\mathbf{r}})d\log(\mathbf{r}/\bar{\mathbf{r}})$, which leads to (22). The vector $d\mathbf{S} = (\mathbf{d}\bar{\mathbf{r}} \wedge \bar{\mathbf{d}}\mathbf{r})/2 = rdrd\varphi\mathbf{r}_{0\perp}\bar{\mathbf{r}}_0 = dS\hat{\mathbf{e}}$, corresponding to the bivector $rdrd\varphi\hat{\mathbf{1}}$ of the bivector field $\mathbb{R}^2$, is the *Lebesgue* measure of the infinitesimal surface of the field of vectors $\mathbf{V}_2$. In accordance with above, the

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 20 February 2025 doi:10.20944/preprints202502.1614.v1

vector integral operator over the closed smooth *Jordan* curve γ, which is the boundary of an arbitrary region G in the vector field $\mathbf{V}_2$, is defined as follows

$$vt \oint_{\gamma^+} \bar{\mathbf{r}}_0(\mathbf{d} + \mathbf{d}_\perp) = 2vt \oint_{\gamma^+} d\mathbf{r}\eth_{\mathbf{r}} = vt \oint_{\gamma^+} [(d\mathbf{r} \circ \text{ grad}) - (d\mathbf{r} \wedge \text{ grad})] \,, \tag{26}$$

where vt denotes the total value of an improper integral [9]-[14], such that

$$vt \oint_{\gamma^+} d\mathbf{r}\eth_{\mathbf{r}} = \lim_{\varepsilon \to 0^+} \oint_{\gamma_\varepsilon^+} d\mathbf{r}\eth_{\mathbf{r}} = \lim_{\varepsilon \to 0^+} (\oint_{\partial S} d\mathbf{r}\eth_{\mathbf{r}} + \sum_{i=1}^{n} \oint_{\partial G_i} d\mathbf{r}\eth_{\mathbf{r}}) = \tag{27}$$

$$= vp \oint_{\gamma} d\mathbf{r}\eth_{\mathbf{r}} + \sum_{i=1}^{n_\gamma} \lim_{\varepsilon \to 0^+} \int_{c(\boldsymbol{r}_i,\varepsilon)}^{\overset{\frown}{\boldsymbol{r}_{2_i}\boldsymbol{r}_{1_i}}} d\mathbf{r}\eth_{\mathbf{r}} = vt \oint_{\gamma^-} d\mathbf{r}\eth_{\mathbf{r}} + \sum_{i=1}^{n_\gamma} \lim_{\varepsilon \to 0^+} \oint_{c(\boldsymbol{r}_i,\varepsilon)} d\mathbf{r}\eth_{\mathbf{r}},$$

and vp denotes the *Cauchy* principal value of an improper integral, $\boldsymbol{r}_i$ are isolated points on the curve γ, surrounded by circles $c(\boldsymbol{r}_i, \varepsilon)$ centered at the points $\boldsymbol{r}_i$ and with an arbitrarily small radius ε, which at the points $\boldsymbol{r}_{1_i}$ and $\boldsymbol{r}_{2_i}$ intersect the curve γ and do not intersect each other. The set of points $\boldsymbol{r}_i$, of *Lebesgue* measure zero, is the set of singular points, on curve γ, of a field on which the vector integral operator is applied. If also in region G, bounded by the curve γ, there are isolated singular points ϱ_j, which can be surrounded by circles $c(\varrho_j, \varepsilon)$, which do not intersect each other, then it is possible to form a simply connected region S, within which all singularities are, by connecting the circles $c(\varrho_j, \varepsilon)$, successively, the first with the second, the second with the third, etc., using parallel straight line segments l_{j_1} and l_{j_2}, at a mutual distance $\delta_j \ll \varepsilon$, as well as by connecting the circles $c(\boldsymbol{r}_i, \varepsilon)$ on the curve γ with the circles $c(\varrho_j, \varepsilon)$, using the parallel straight line segments l_{i_1} and l_{i_2}, at a mutual distance $\delta_i \ll \varepsilon$. The boundary ∂S of the singular region S (blue region in Figure 1), inside region G, divides region G into n subregions G_i.

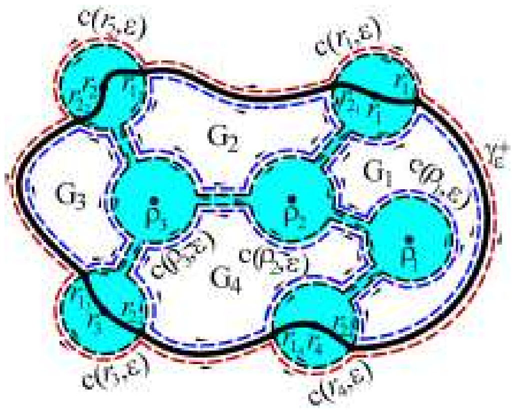

Figure 1.

$$\lim_{\varepsilon \to 0^+} \oint_{\partial S} d\mathbf{r}\eth_{\mathbf{r}} = 2\pi\hat{\mathbf{e}} \sum_{i=1}^{n_\gamma} Res(\eth_{\mathbf{r}}, \boldsymbol{r}_i) + \sum_{i=1}^{n_\gamma} (\int_{l_i^+} d\mathbf{r}\eth_{\mathbf{r}} - \int_{l_i^-} d\mathbf{r}\eth_{\mathbf{r}}) + \tag{28}$$

$$+ 2\pi\hat{\mathbf{e}} \sum_{j=1}^{n_G} Res(\eth_{\mathbf{r}}, \varrho_j) + \sum_{j=1}^{n_G - 1} (\int_{l_j^+} d\mathbf{r}\eth_{\mathbf{r}} - \int_{l_j^-} d\mathbf{r}\eth_{\mathbf{r}}),$$

where

$$\lim_{\varepsilon \to 0^+} \oint_{c(\boldsymbol{r}_i,\varepsilon)} d\mathbf{r}\eth_{\mathbf{r}} = 2\pi\hat{\mathbf{e}} Res(\eth_{\mathbf{r}}, \boldsymbol{r}_i), \ \lim_{\varepsilon \to 0^+} \oint_{c(\varrho_j,\varepsilon)} d\mathbf{r}\eth_{\mathbf{r}} = 2\pi\hat{\mathbf{e}} Res(\eth_{\mathbf{r}}, \varrho_j), \tag{29}$$

$$\int_{l_i^+} d\mathbf{r}\eth_{\mathbf{r}} = \lim_{\varepsilon \to 0^+} \int_{l_{i_1}} d\mathbf{r}\eth_{\mathbf{r}} \text{ and } \int_{l_i^-} d\mathbf{r}\eth_{\mathbf{r}} = \lim_{\varepsilon \to 0^+} \int_{l_{i_2}} d\mathbf{r}\eth_{\mathbf{r}},$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 20 February 2025 doi:10.20944/preprints202502.1614.v1

is the residue operator in region G of the field of vectors $\mathbf{V}_2$. The vector integral operator in region G is as follows

$$2vt\iint\limits_{G^+}(\mathbf{d\bar{r}}\wedge\mathbf{\bar{d}r})\eth^2_{\mathbf{r\bar{r}}} = 2vp\iint\limits_{G}(\mathbf{d\bar{r}}\wedge\mathbf{\bar{d}r})\eth^2_{\mathbf{r\bar{r}}} + \lim_{\varepsilon\to0^+}\int_{\partial S}^{\circlearrowleft} d\mathbf{r}\eth_{\mathbf{r}} = \tag{30}$$

$$= vp\iint\limits_{G} rdrd\varphi(\mathbf{\hat{r}}_0\mathbf{r}_0 \text{ div grad} + \mathbf{\hat{r}}_0\mathbf{r}_0\times \text{ curl grad}) + \lim_{\varepsilon\to0^+}\int_{\partial S}^{\circlearrowleft} d\mathbf{r}\eth_{\mathbf{r}},$$

where $vp\iint\limits_{G}(\mathbf{d\bar{r}}\wedge\mathbf{\bar{d}r})\eth^2_{\mathbf{r\bar{r}}} = \lim_{\varepsilon\to0^+}\sum_{i=1}^{n}\iint\limits_{G_i}(\mathbf{d\bar{r}}\wedge\mathbf{\bar{d}r})\eth^2_{\mathbf{r\bar{r}}}$, $2\mathbf{\hat{r}}drd\varphi = (\mathbf{d\bar{r}}\wedge\mathbf{\bar{d}r})\mathbf{\bar{r}}_0$, $vt\iint\limits_{G^+}(\mathbf{d\bar{r}}\wedge\mathbf{\bar{d}r})\eth^2_{\mathbf{r\bar{r}}} - vt\iint\limits_{G^-}(\mathbf{d\bar{r}}\wedge\mathbf{\bar{d}r})\eth^2_{\mathbf{r\bar{r}}} = 2\pi\mathbf{\hat{e}}\sum_{i=1}^{n_\gamma} Res(\eth_{\mathbf{r}}, r_i)$ and

$$\eth^2_{\mathbf{r\bar{r}}} = \eth_{\mathbf{\bar{r}}}\eth_{\mathbf{r}} = \frac{\mathbf{\bar{r}}_0\mathbf{r}_0}{r^2}(\mathbf{\bar{r}}\eth_{\mathbf{\bar{r}}}\cdot\mathbf{\bar{r}}\eth_{\mathbf{\bar{r}}}) + \frac{\mathbf{\bar{r}}_0\mathbf{r}_0}{r^2}\times(\mathbf{\bar{r}}\eth_{\mathbf{\bar{r}}}\times\mathbf{\bar{r}}\eth_{\mathbf{\bar{r}}}) = \tag{31}$$

$$= \frac{\mathbf{\bar{r}}_0}{4r^2}[\mathbf{r}_0[r\partial_r(r\partial_r) + \partial^2_{\varphi^2}] + \mathbf{r}_\perp(\partial^2_{r\varphi} - \partial^2_{\varphi r})] = \frac{\mathbf{\bar{r}}_0}{4}(\mathbf{r}_0 \text{ div grad} + \mathbf{r}_0\times \text{ curl grad}).$$

Here,

$$\text{div grad} = \frac{4}{r^2}(\mathbf{\bar{r}}\eth_{\mathbf{\bar{r}}}\cdot\mathbf{\bar{r}}\eth_{\mathbf{\bar{r}}}) = \frac{1}{r^2}[r\partial_r(r\partial_r) + \partial^2_{\varphi^2}] \text{ and} \tag{32}$$

$$\text{curl grad} = \frac{4}{r^2}(\mathbf{\bar{r}}\eth_{\mathbf{\bar{r}}}\times\mathbf{\bar{r}}\eth_{\mathbf{\bar{r}}}) = \frac{\mathbf{r}_0\times\mathbf{r}_\perp}{r^2}(\partial^2_{\varphi r} - \partial^2_{r\varphi}). \tag{33}$$

If the set of singular points, either on the contour γ or in the region G, is an empty set, the choice of a representative point ($\boldsymbol{r}$ on the contour γ or $\boldsymbol{\varrho}$ in the region G, respectively) is arbitrary. If the field is uniform [7], then $\int_{l_i^+} d\mathbf{r}\eth_{\mathbf{r}} = \int_{l_i^-} d\mathbf{r}\eth_{\mathbf{r}}$, so that the choice of representative points is not necessary.

Finally, on the basis of the result of the *Kelvin-Stokes* theorem (*Green's* theorem) [6],

$$vt\int_{\gamma^+}^{\circlearrowleft} d\mathbf{r}\eth_{\mathbf{r}} = vt\int_{\gamma^+}^{\circlearrowleft}[(d\mathbf{r}\circ\eth_{\mathbf{\bar{r}}}) - (d\mathbf{r}\wedge\eth_{\mathbf{\bar{r}}})] = \tag{34}$$

$$= \frac{1}{2}vt\iint\limits_{G^+}\mathbf{\hat{r}}drd\varphi(\mathbf{r}_0 \text{ div grad} + \mathbf{r}_0\times \text{ curl grad}) = vt\iint\limits_{G^+}(\mathbf{d\bar{r}}\wedge\mathbf{\bar{d}r})\eth^2_{\mathbf{r\bar{r}}}.$$

If there is a limit $\lim_{\varepsilon\to0^+}\int_{\gamma_\varepsilon^+}^{\circlearrowleft} d\mathbf{r}\eth_{\mathbf{r}}$, and $\int_{\partial S}^{\circlearrowleft} d\mathbf{r}\eth_{\mathbf{r}}$ tends to infinity as ε tends to zero, then the limit $\lim_{\varepsilon\to0^+}\sum_{i=1}^{n}\int_{\partial G_i}^{\circlearrowleft} d\mathbf{r}\eth_{\mathbf{r}} = vp\iint\limits_{G}(\mathbf{d\bar{r}}\wedge\mathbf{\bar{d}r})\eth^2_{\mathbf{r\bar{r}}}$ is also infinite. In this emphasized case, the limit $\lim_{\varepsilon\to0^+}\int_{\gamma_\varepsilon^+}^{\circlearrowleft} d\mathbf{r}\eth_{\mathbf{r}}$ leads to the indeterminate form of the difference of two infinities, which has a finite value. According to (34), since

$$(\mathbf{\bar{d}r}\wedge\mathbf{d\bar{r}})(\eth^2_{\mathbf{r\bar{r}}} - \eth^2_{\mathbf{\bar{r}r}}) = -\mathbf{\hat{r}}_0\mathbf{r}_{0\perp}rdrd\varphi\|\text{ curl grad}\| = \mathbf{\bar{r}}_0\mathbf{r}_0drd\varphi(\partial^2_{r\varphi} - \partial^2_{\varphi r}), \tag{35}$$

it follows that

$$\mathbf{\bar{r}}_0\mathbf{r}_0vt\int_{\gamma^+}^{\circlearrowleft} d = vt\int_{\gamma^+}^{\circlearrowleft}(d\mathbf{r}\eth_{\mathbf{r}} + d\mathbf{\bar{r}}\eth_{\mathbf{\bar{r}}}) = vt\iint\limits_{G^+}(\mathbf{d\bar{r}}\wedge\mathbf{\bar{d}r})(\eth^2_{\mathbf{r\bar{r}}} - \eth^2_{\mathbf{\bar{r}r}}) = \tag{36}$$

$$= \mathbf{\bar{r}}_0\mathbf{r}_0vt\iint\limits_{G^+} drd\varphi(\partial^2_{\varphi r} - \partial^2_{r\varphi}).$$

If $\partial^2_{\varphi r} = \partial^2_{r\varphi}$ and $r\partial_r(r\partial_r) = -\partial^2_{\varphi^2}$, then $\eth^2_{\mathbf{r\bar{r}}} = \eth^2_{\mathbf{\bar{r}r}} = \mathbf{0}$. In addition,

$$\eth^2_{\mathbf{r}^2} = \eth_{\mathbf{r}}\eth_{\mathbf{r}} = \frac{1}{2}\frac{\mathbf{\bar{r}}_0}{r}[\mathbf{\bar{r}}\partial^2_{r^2} - \mathbf{\hat{r}}_0(\partial^2_{\varphi r} - \frac{1}{r}\partial_\varphi)] = \mathbf{\bar{r}}_0\partial_r(\eth_{\mathbf{r}}), \tag{37}$$

since

$$\eth^2_{\mathbf{r}^2} = \eth_{\mathbf{r}}\eth_{\mathbf{r}} = \frac{1}{4}(\mathbf{\bar{r}}_0\partial_r - \mathbf{\hat{r}}_0\frac{1}{r}\partial_\varphi)(\mathbf{\bar{r}}_0\partial_r - \mathbf{\hat{r}}_0\frac{1}{r}\partial_\varphi) = \tag{38}$$

 doi:10.20944/preprints202502.1614.v1

$$= \frac{\bar{\mathbf{r}}_0}{4r^2}[\bar{\mathbf{r}}_0[r\partial_r(r\partial_r) - \partial^2_{\varphi^2} - 2r\partial_r] - \hat{\mathbf{r}}(\partial^2_{\varphi r} + \partial^2_{r\varphi} - \frac{2}{r}\partial_\varphi)].$$

Clearly, $\eth^2_{\bar{\mathbf{r}}^2} = \bar{\eth}^2_{\mathbf{r}^2}$.

2.1. Integrals of Scalar and Vector Fields

The vector differential of a scalar field $F : \mathbf{V}_2 \to \mathbb{R}$ is as follows

$$\mathbf{d}F = \mathbf{r}_0(\eth_{\mathbf{r}}Fd\mathbf{r} + \eth_{\bar{\mathbf{r}}}Fd\bar{\mathbf{r}}) = \mathbf{r}_0(\boldsymbol{f}d\mathbf{r} + \bar{\boldsymbol{f}}d\bar{\mathbf{r}}), \tag{39}$$

where $\boldsymbol{f} = \eth_{\mathbf{r}}F$. The second vector partial derivative of F is the first vector partial derivative of the vector field $\boldsymbol{f}$, so that

$$\eth_{\mathbf{r}}\boldsymbol{f} = \eth^2_{\mathbf{r}^2}F = \frac{\bar{\mathbf{r}}_0}{4r^2}[[r\partial_r(r\partial_r F) - \partial^2_{\varphi^2}F - 2r\partial_r F])\bar{\mathbf{r}}_0 - \tag{40}$$

$$-[(\partial^2_{\varphi r}F + \partial^2_{r\varphi}F) - \frac{2}{r}\partial_\varphi F]\hat{\mathbf{r}}] \text{ and}$$

$$\eth_{\bar{\mathbf{r}}}\boldsymbol{f} = \eth^2_{\mathbf{r}\bar{\mathbf{r}}}F = \frac{\bar{\mathbf{r}}_0}{4r^2}[[r\partial_r(r\partial_r F) + \partial^2_{\varphi^2}F]\mathbf{r}_0 + (\partial^2_{r\varphi}F - \partial^2_{\varphi r}F)\mathbf{r}_\perp]. \tag{41}$$

If $\boldsymbol{f} = \eth_{\mathbf{r}}F$ and $\bar{\boldsymbol{f}} = \eth_{\bar{\mathbf{r}}}F$ are uniform vector fields, then by applying the vector integral operator (36) to the scalar field F, a vector integral identity is obtained

$$\bar{\mathbf{r}}_0\mathbf{r}_0 vt\int_{\gamma^+}^{\circlearrowleft} dF \ = vt\int_{\gamma^+}^{\circlearrowleft} \eth_{\mathbf{r}}Fd\mathbf{r} + \eth_{\bar{\mathbf{r}}}Fd\bar{\mathbf{r}} = vt\int_{\gamma^+}^{\circlearrowleft} \boldsymbol{f}d\mathbf{r} + \bar{\boldsymbol{f}}d\bar{\mathbf{r}} = \tag{42}$$

$$= vt\iint_{G^+}(\eth^2_{\mathbf{r}\bar{\mathbf{r}}}F - \eth^2_{\bar{\mathbf{r}}\mathbf{r}}F)(\mathbf{d}\bar{\mathbf{r}} \wedge \bar{\mathbf{d}}\mathbf{r}) = \bar{\mathbf{r}}_0\mathbf{r}_0 vt\iint_{G^+}(\partial^2_{\varphi r}F - \partial^2_{r\varphi}F)drd\varphi =$$

$$= (2\pi\hat{\mathbf{e}})[\sum_{i=1}^{n_\gamma} Res(\boldsymbol{f}, \boldsymbol{r}_i) + \sum_{j=1}^{n_G} Res(\boldsymbol{f}, \boldsymbol{\varrho}_j) + \sum_{i=1}^{n_\gamma} \overline{Res}(\bar{\boldsymbol{f}}, \boldsymbol{r}_i) + \sum_{j=1}^{n_G} \overline{Res}(\bar{\boldsymbol{f}}, \boldsymbol{\varrho}_j)],$$

where $\lim_{\varepsilon\to 0^+}\int_{\partial S}^{\circlearrowleft}\bar{\boldsymbol{f}}d\bar{\mathbf{r}} = (2\pi\hat{\mathbf{e}})[\sum_{i=1}^{n_\gamma}\overline{Res}(\bar{\boldsymbol{f}}, \boldsymbol{r}_i) + \sum_{j=1}^{n_G}\overline{Res}(\bar{\boldsymbol{f}}, \boldsymbol{\varrho}_j)]$. The integral identity of complex analysis, which is an analogue of the vector integral identity (42), is the integral identity of *Cauchy's* integral theorem [7].

As $\partial^2_{\varphi r}F = \partial^2_{r\varphi}F$ and if, in addition, $r\partial_r(r\partial_r F) = -\partial^2_{\varphi^2}F$, then $\eth_{\mathbf{r}}\bar{\boldsymbol{f}} = \eth_{\bar{\mathbf{r}}}\boldsymbol{f} = \mathbf{0}$, that is,

$$\mathbf{r}\partial_r\boldsymbol{f} = -\mathbf{r}_{0\perp}\partial_\varphi\boldsymbol{f} \text{ and} \tag{43}$$

$$\mathbf{r}\eth_{\mathbf{r}}\boldsymbol{f} = \mathbf{r}\eth^2_{\mathbf{r}^2}F = r\partial_r(\eth_{\mathbf{r}}F) = r\partial_r\boldsymbol{f}. \tag{44}$$

A vector field $\boldsymbol{f}$, satisfying the *Cauchy-Riemann* condition $\mathbf{r}\partial_r\boldsymbol{f} = -\mathbf{r}_{0\perp}\partial_\varphi\boldsymbol{f}$, is said, analogous to complex analytic functions, to be an analytic vector field. Hence, an analytic vector field is a vector derivative of the *Laplace* scalar field F. Clearly, the coordinate components of the analytic vector field $\boldsymbol{f}$ are also *Laplace* scalar fields.

Assume that the analytic vector field $\boldsymbol{f}$, as the vector derivative of the *Laplace* scalar field F, is not defined at the point $\boldsymbol{\varrho} \in int.G$, where G is a region in the field of vectors $\mathbf{V}_2$, bounded by a closed smooth *Jordan* curve γ, as well as at point $\boldsymbol{r}$ on curve γ. The vector integral identity

$$\bar{\mathbf{r}}_0\mathbf{r}_0 vt\int_{\gamma^+}^{\circlearrowleft} dF \ = vt\int_{\gamma^+}^{\circlearrowleft} \eth_{\mathbf{r}}Fd\mathbf{r} = vt\int_{\gamma^+}^{\circlearrowleft} \boldsymbol{f}d\mathbf{r} = \tag{45}$$

$$= (2\pi\hat{\mathbf{e}})[Res(\boldsymbol{f}, \boldsymbol{r}) + Res(\boldsymbol{f}, \boldsymbol{\varrho})] + \int_{l^+}\boldsymbol{f}d\mathbf{r} - \int_{l^-}\boldsymbol{f}d\mathbf{r},$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 20 February 2025 doi:10.20944/preprints202502.1614.v1

is a vector analogue of the integral identity of *Cauchy's* integral theorem, which is slightly generalized, since in this emphasized case

$$vt\iint\limits_{G^+} \eth^2_{\mathbf{r}\bar{\mathbf{r}}} f(\mathbf{d}\bar{\mathbf{r}}\wedge\bar{\mathbf{d}}\mathbf{r}) = \lim_{\varepsilon\to 0^+}\int_{\partial S}^{\circlearrowleft} f d\mathbf{r} = \tag{46}$$

$$= (2\pi\hat{\mathbf{e}})[Res(f,r) + Res(f,\varrho)] + \int_{l^+} f d\mathbf{r} - \int_{l^-} f d\mathbf{r}.$$

If f is a differentiable (regular) vector field, but not an analytic vector field, in an arbitrary region G of the vector field $\mathbf{V}_2$, bounded by a closed smooth *Jordan* curve γ, the integral identity

$$\lim_{\mathbf{S}_\gamma\to\varrho_k}\frac{1}{\mathbf{S}_\gamma}\int_\gamma^{\circlearrowleft} f d\mathbf{r} = 2\eth_{\bar{\mathbf{r}}} f(\varrho_k) = \operatorname{grad} f(\varrho_k) \tag{47}$$

where $2\mathbf{S}_\gamma = \int_\gamma^{\circlearrowleft}\mathbf{r}\wedge d\mathbf{r}$, is a vector analogue, in the field of vectors $\mathbf{V}_2$, of the surface (spatial) derivative, which was introduced, into complex analysis, by *Pompeiu* [8], originally calling it the areolar derivative. Similarly, based on the vector identity (17), the so-called cumulative surface (spatial) derivative of the vector field f can be defined as follows

$$\lim_{\mathbf{S}_\gamma\to\varrho_k}\frac{1}{\mathbf{S}_\gamma}\int_\gamma^{\circlearrowleft} f d\bar{\mathbf{r}} = -2\eth_{\mathbf{r}} f(\varrho_k) = \bar{\mathbf{r}}_0\mathbf{r}_0\times\operatorname{curl} f(\varrho_k) - \bar{\mathbf{r}}_0\mathbf{r}_0\operatorname{div} f(\varrho_k). \tag{48}$$

According to (47), if f is a regular and uniform vector field in the ε-neighborhood $C_\varepsilon^0\backslash\{\mathbf{0}\}$ of its singular point $\varrho = \mathbf{0}$ and $\lim_{\mathbf{r}\to\mathbf{0}}\eth_{\bar{\mathbf{r}}}(r^2 f) = \varrho_0\in\mathbf{V}_2$, then

$$Res(f,\mathbf{0}) = (2\pi\hat{\mathbf{e}})^{-1}\lim_{\varepsilon\to 0^+}\int_{c_\varepsilon^0}^{\circlearrowleft} f d\mathbf{r} = -\lim_{\varepsilon\to 0^+}\frac{\hat{\mathbf{e}}}{2\pi\varepsilon^2}\int_{c_\varepsilon^0}^{\circlearrowleft} r^2 f d\mathbf{r} = \tag{49}$$

$$= \lim_{\varepsilon\to 0^+}\frac{1}{\mathbf{S}_{c_\varepsilon^0}} vt\iint\limits_{C_\varepsilon^0}\eth_{\bar{\mathbf{r}}}(r^2 f)d\mathbf{S} = \lim_{\mathbf{r}\to\mathbf{0}}\eth_{\bar{\mathbf{r}}}(r^2 f) = \varrho_0.$$

If $\eth_{\bar{\mathbf{r}}} f = \mathbf{0}$, then $Res(f,\mathbf{0}) = \lim_{\mathbf{r}\to\mathbf{0}}\mathbf{r}f = \varrho_0$, which is another vector analogy to the well-known result of complex analysis. Let $\boldsymbol{F}$ be an analytic vector field, such that $\lim_{r\to 0^+}(\boldsymbol{F}/\ln r)$ leads to the determinate form only after the application of *L'Hospital's* rule n times. Then, the vector formula for $Res(f,\mathbf{0})$, being analogous to the complex analysis formula, can be obtained via the vector identity $\mathbf{r}f = r\partial_r\boldsymbol{F}$, see (44), where $f = \eth_{\mathbf{r}}\boldsymbol{F}$. Namely, since the same vector identity applies to the analytic vector field $\mathbf{r}^n f = r^n\partial_r\boldsymbol{F}\underbrace{\mathbf{r}_0\ldots\mathbf{r}_0}_{n-1}$, it follows that

$$\eth_{\mathbf{r}}(\mathbf{r}^n f) = (nr^{n-1}\partial_r\boldsymbol{F} + r^n\partial^2_{r^2}\boldsymbol{F})\underbrace{\mathbf{r}_0\ldots\mathbf{r}_0}_{n-2}\ \text{and} \tag{50}$$

$$\eth^2_{\mathbf{r}^2}(\mathbf{r}^n f) = [n(n-1)r^{n-2}\partial_r\boldsymbol{F} + 2nr^{n-1}\partial^2_{r^2}\boldsymbol{F} + r^n\partial^3_{r^3}\boldsymbol{F}]\underbrace{\mathbf{r}_0\ldots\mathbf{r}_0}_{n-3}. \tag{51}$$

Accordingly, applying *L'Hospital's* rule,

$$\frac{1}{(n-1)!}\lim_{\mathbf{r}\to\mathbf{0}}\eth^{n-1}_{\mathbf{r}^{n-1}}(\mathbf{r}^n f) = \frac{n!}{(n-1)!}\lim_{r\to 0^+}\sum_{k=1}^{n}\binom{n-1}{n-k}\frac{r^k\partial^k_{r^k}\boldsymbol{F}}{k!} = \tag{52}$$

$$= [\sum_{k=1}^{n}(-1)^{n-k}\binom{n}{n-k}]\frac{\lim_{r\to 0^+} r^n\partial^n_{r^n}\boldsymbol{F}}{(n-1)!} = \frac{(-1)^{n-1}}{(n-1)!}\lim_{r\to 0^+} r^n\partial^n_{r^n}\boldsymbol{F}.$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 20 February 2025 doi:10.20944/preprints202502.1614.v1

Further, since the vector field $f = \bar{\mathbf{r}}_0 \partial_r F$ is an analytic vector field, it follows that

$$\eth^{n-1}_{\mathbf{r}^{n-1}} f = (\frac{r}{\mathbf{r}})^n \partial^n_{r^n} F \text{ and} \tag{53}$$

$$-\frac{1}{(n-1)!} \lim_{\mathbf{r}\to\mathbf{0}} (-\mathbf{r})^n \eth^{n-1}_{\mathbf{r}^{n-1}} f = \frac{(-1)^{n-1}}{(n-1)!} \lim_{r\to 0^+} r^n \partial^n_{r^n} F = \lim_{\mathbf{r}\to\mathbf{0}} \mathbf{r} f. \tag{54}$$

This means that *L'Hospital's* rule can be explicitly applied to the vector field $\mathbf{r}f$.

If some analytic vector field f is regular in an arbitrary region G bounded by a closed smooth *Jordan* curve γ, then for the vector field

$$f(\mathbf{r} + \varrho_A) - [f(\varrho_A) + \sum_{k=1}^{n-1} \eth^k_{\mathbf{r}^k} f(\varrho_A) \frac{\mathbf{r}^k}{k!}],$$

where $\{\mathbf{0}, \varrho_A\} \subset int.G$, according to (45), (52) and (54), the following is true

$$\lim_{\varepsilon\to 0^+} vt\oint_{c^0_\varepsilon} \{f(\mathbf{r} + \varrho_A) - [f(\varrho_A) + \sum_{k=1}^{n-1} \eth^k_{\mathbf{r}^k} f(\varrho_A) \frac{\mathbf{r}^k}{k!}]\} \frac{d\mathbf{r}}{\mathbf{r}^{n+1}} = \tag{55}$$

$$= 2\pi\hat{\mathbf{e}} \lim_{\mathbf{r}\to\mathbf{0}} \frac{1}{\mathbf{r}^n} \{f(\mathbf{r} + \varrho_A) - [f(\varrho_A) + \sum_{k=1}^{n-1} \eth^k_{\mathbf{r}^k} f(\varrho_A) \frac{\mathbf{r}^k}{k!}]\} = 2\pi\hat{\mathbf{e}} \frac{\eth^n_{\mathbf{r}^n} f(\varrho_A)}{n!}.$$

Hence

$$\frac{n!}{2\pi\hat{\mathbf{e}}} vt\oint_\gamma \frac{f(\mathbf{r} + \varrho_A)}{\mathbf{r}^{n+1}} d\mathbf{r} = \eth^n_{\mathbf{r}^n} f(\varrho_A), \tag{56}$$

since $vt\oint_\gamma \mathbf{r}^{-n} d\mathbf{r} = \mathbf{0}$, whenever $n \geq 2$. This is the vector analogue of the well-known *Cauchy's* integral formula.

If the vector field $\mathbf{F} = F\mathbf{r}_0 + F_\perp \mathbf{r}_{0\perp}$ is such that the scalar fields F and $F_\perp$ have continuous first partial derivatives in region G, bounded by the closed smooth *Jordan* curve γ, almost everywhere (everywhere except on the singular set of points of *Lebesgue* measure zero), then by applying the vector integral operator (36) to the vector field $\mathbf{F}$, one comes to the following vector integral identity

$$vt\oint_{\gamma^+} d\mathbf{F} = vt\oint_{\gamma^+} (\eth_{\mathbf{r}}\mathbf{F} d\mathbf{r} + \eth_{\bar{\mathbf{r}}}\mathbf{F} d\bar{\mathbf{r}}) = vt\iint_{G^+} (\eth^2_{\mathbf{r}\bar{\mathbf{r}}}\mathbf{F} - \eth^2_{\bar{\mathbf{r}}\mathbf{r}}\mathbf{F})(d\bar{\mathbf{r}} \wedge \bar{\mathbf{d}}\mathbf{r}) = \tag{57}$$

$$= (2\pi\hat{\mathbf{e}})[\sum_{i=1}^{n_\gamma} Res(\eth_{\mathbf{r}}\mathbf{F}, r_i) + \sum_{j=1}^{n_G} Res(\eth_{\mathbf{r}}\mathbf{F}, \varrho_j) + \sum_{i=1}^{n_\gamma} \overline{Res}(\eth_{\bar{\mathbf{r}}}\mathbf{F}, r_i) + \sum_{j=1}^{n_G} \overline{Res}(\eth_{\bar{\mathbf{r}}}\mathbf{F}, \varrho_j)],$$

since

$$\partial^2_{\varphi r}\mathbf{F} - \partial^2_{r\varphi}\mathbf{F} = \mathbf{r}_0(\partial^2_{\varphi r}F - \partial^2_{r\varphi}F) + \mathbf{r}_{0\perp}(\partial^2_{\varphi r}F_\perp - \partial^2_{r\varphi}F_\perp) = \mathbf{0} \text{ and} \tag{58}$$

$$\eth^2_{\bar{\mathbf{r}}\mathbf{r}}\mathbf{F} = \eth^2_{\mathbf{r}\bar{\mathbf{r}}}\mathbf{F} = \frac{1}{2}\eth_{\bar{\mathbf{r}}}(\bar{\mathbf{r}}_0\mathbf{r}_0 \operatorname{div}\mathbf{F} + \operatorname{curl}\mathbf{F} \times \bar{\mathbf{r}}_0\mathbf{r}_0) = \tag{59}$$

$$= \frac{r\partial_r(r\partial_r\mathbf{F}) + \partial^2_{\varphi^2}\mathbf{F}}{4r^2} = \frac{\operatorname{div}\operatorname{grad}\mathbf{F}}{4}.$$

Clearly, in the general case $\operatorname{curl}\operatorname{grad}\mathbf{F} = 4(\bar{\mathbf{r}}\eth_{\bar{\mathbf{r}}} \times \bar{\mathbf{r}}\eth_{\bar{\mathbf{r}}})\mathbf{F}/r^2 = \mathbf{0}$ is not the same as $\operatorname{curl}(\bar{\mathbf{r}}_0 \operatorname{grad}\mathbf{F}) = 4(\bar{\mathbf{r}}\eth_{\bar{\mathbf{r}}} \times \bar{\mathbf{r}}\bar{\mathbf{r}}_0\eth_{\bar{\mathbf{r}}}\mathbf{F})/r^2$. Namely,

$$\frac{\mathbf{r}_0}{r^2}(\bar{\mathbf{r}}\eth_{\bar{\mathbf{r}}} \circ \bar{\mathbf{r}}\bar{\mathbf{r}}_0 \operatorname{grad}\mathbf{F}) = \frac{\mathbf{r}_0}{2}[\partial^2_{r^2}F + \frac{1}{r^2}(\partial^2_{\varphi^2}F - \partial_\varphi F_\perp)] \text{ and} \tag{60}$$

$$\frac{\mathbf{r}_0}{r^2}(\bar{\mathbf{r}}\eth_{\bar{\mathbf{r}}} \wedge \bar{\mathbf{r}}\bar{\mathbf{r}}_0 \operatorname{grad}\mathbf{F}) = \frac{\mathbf{r}_{0\perp}}{2}[\partial^2_{r^2}F_\perp + \frac{1}{r^2}(\partial^2_{\varphi^2}F_\perp + \partial_\varphi F)]. \tag{61}$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 20 February 2025 doi:10.20944/preprints202502.1614.v1

So, $\text{div grad}\mathbf{F} = 4(\bar{\mathbf{r}}\eth_{\bar{\mathbf{r}}} \cdot \bar{\mathbf{r}}\eth_{\bar{\mathbf{r}}})\mathbf{F}/r^2$ differs from $\text{div}(\bar{\mathbf{r}}_0\,\text{grad}\mathbf{F}) = 4(\bar{\mathbf{r}}\eth_{\bar{\mathbf{r}}} \cdot \bar{\mathbf{r}}\bar{\mathbf{r}}_0\eth_{\bar{\mathbf{r}}}\mathbf{F})/r^2$. Accordingly,

$$4\eth^2_{\bar{\mathbf{r}}\mathbf{r}}\mathbf{F} = 2\eth_{\mathbf{r}}\,\text{grad}\mathbf{F} = \text{div grad}\mathbf{F} = \frac{2\mathbf{r}_0}{r^2}(\bar{\mathbf{r}}\eth_{\bar{\mathbf{r}}} \circ \bar{\mathbf{r}}\bar{\mathbf{r}}_0\,\text{grad}\mathbf{F} + \bar{\mathbf{r}}\eth_{\bar{\mathbf{r}}} \wedge \bar{\mathbf{r}}\bar{\mathbf{r}}_0\,\text{grad}\mathbf{F}) + \tag{62}$$

$$+\frac{\bar{\mathbf{r}}_0}{r}\,\text{grad}\mathbf{F} = \mathbf{r}_0\,\text{div}(\bar{\mathbf{r}}_0\,\text{grad}\mathbf{F}) + \text{curl}(\bar{\mathbf{r}}_0\,\text{grad}\mathbf{F}) \times \mathbf{r}_0 + \frac{\bar{\mathbf{r}}_0}{r}\,\text{grad}\mathbf{F},$$

since

$$\mathbf{r}_0(\bar{\mathbf{r}}_0\eth_{\bar{\mathbf{r}}} \circ \bar{\mathbf{r}}_0\bar{\mathbf{r}}_0\,\text{grad}\mathbf{F} + \bar{\mathbf{r}}_0\eth_{\bar{\mathbf{r}}} \wedge \bar{\mathbf{r}}_0\bar{\mathbf{r}}_0\,\text{grad}\mathbf{F}) = \mathbf{r}_0[\mathbf{r}_0\eth_{\mathbf{r}}(\bar{\mathbf{r}}_0\bar{\mathbf{r}}_0\,\text{grad}\mathbf{F})] = \tag{63}$$

$$= -\frac{\bar{\mathbf{r}}_0}{r}\,\text{grad}\mathbf{F} + \eth_{\mathbf{r}}\,\text{grad}\mathbf{F} \text{ and}$$

$$2\eth_{\mathbf{r}}(\bar{\mathbf{r}}_0\,\text{grad}\mathbf{F}) = \bar{\mathbf{r}}_0\mathbf{r}_0\,\text{div}(\bar{\mathbf{r}}_0\,\text{grad}\mathbf{F}) + \text{curl}(\bar{\mathbf{r}}_0\,\text{grad}\mathbf{F}) \times \bar{\mathbf{r}}_0\mathbf{r}_0, \tag{64}$$

which can be explicitly obtained if in (17) $\mathbf{F}$ is formally replaced by $\bar{\mathbf{r}}_0\,\text{grad}\mathbf{F}$. Therefore, the two identities 5. and 6., on page 85., in Section 3.16., Chapter 3., in [15], should be replaced by: 5. curl $\text{grad}\mathbf{F} \equiv \mathbf{0}$ and 6. $\text{div grad}\mathbf{F} = \mathbf{0}$ if $\mathbf{F}$ is either an analytic vector field ($\eth_{\bar{\mathbf{r}}}\mathbf{F} = \mathbf{0}$) or a *Laplace* vector field ($\eth_{\mathbf{r}}\mathbf{F} = \mathbf{0}$). In both of these cases, the vector field $\mathbf{F}$ satisfies *Laplace's* equation $r\partial_r(r\partial_r\mathbf{F}) = -\partial^2_{\varphi^2}\mathbf{F}$.

Consequently,

$$vt\int^{\circlearrowleft}_{\gamma^+}\frac{\bar{\mathbf{r}}_0}{2}(\mathbf{dF} + \mathbf{d}_\perp\mathbf{F}) = vt\int^{\circlearrowleft}_{\gamma^+}\eth_{\mathbf{r}}\mathbf{F}d\mathbf{r} = \tag{65}$$

$$= vt\iint\limits_{G^+}\eth^2_{\mathbf{r}\bar{\mathbf{r}}}\mathbf{F}(\mathbf{d}\bar{\mathbf{r}} \wedge \bar{\mathbf{d}}\mathbf{r}) = \frac{\bar{\mathbf{r}}_0\mathbf{r}_{0\perp}}{2}vt\iint\limits_{G^+} r\,\text{div grad}\mathbf{F}dr d\varphi.$$

On the other hand, let $\mathbf{F} = F\mathbf{r}_0 + F_\perp\mathbf{r}_{0\perp}$ be continuous in an arbitrary region G bounded by a closed smooth *Jordan* curve γ, in which the partial derivatives $\partial_r F$, $\partial_\varphi F$, $\partial_r F_\perp$and $\partial_\varphi F_\perp$ exist and satisfy the *Cauchy-Riemann* equations

$$\partial_r F = \frac{1}{r}\partial_\varphi F_\perp \text{ and } \partial_r F_\perp = -\frac{1}{r}\partial_\varphi F. \tag{66}$$

Then, according to the *Looman-Menchoff* theorem [1], both the analytic vector field $\bar{\mathbf{r}}_0\mathbf{F}$ and the *Laplace* vector field $\mathbf{r}_0\bar{\mathbf{F}}$ can be said to be regular (holomorphic) vector fields in G. Therefore, on the basis of (56),

$$(2\pi\hat{\mathbf{e}})^{-1}\int^{\circlearrowleft}_{\gamma}\frac{\bar{\mathbf{r}}_0\mathbf{F}}{\mathbf{r}^{n+1}}d\mathbf{r} = Res(\frac{\bar{\mathbf{r}}_0\mathbf{F}}{\mathbf{r}^{n+1}}, \mathbf{0}) = \frac{\lim_{\mathbf{r}\to\mathbf{0}}\eth^n_{\mathbf{r}^n}(\bar{\mathbf{r}}_0\mathbf{F})}{n!}. \tag{67}$$

In addition,

$$(2\pi\hat{\mathbf{e}})^{-1}[\int^{\circlearrowleft}_{\gamma}\frac{\mathbf{F}}{\mathbf{r}}d\mathbf{r} - vp\iint\limits_{G}\frac{\bar{\mathbf{r}}_0\,\text{grad}\mathbf{F}}{r}d\mathbf{S}] = \mathbf{F}(\mathbf{0}) \text{ and} \tag{68}$$

$$(2\pi\hat{\mathbf{e}})^{-1}[\int^{\circlearrowleft}_{\gamma}\frac{\bar{\mathbf{F}}}{\bar{\mathbf{r}}}d\bar{\mathbf{r}} + vp\iint\limits_{G}\frac{\mathbf{r}_0\,\text{div}\bar{\mathbf{F}} + \text{curl}\bar{\mathbf{F}} \times \mathbf{r}_0}{r}d\mathbf{S}] = -\bar{\mathbf{F}}(\mathbf{0}), \tag{69}$$

where $\bar{\mathbf{r}}\,\text{grad}\mathbf{F} = -\mathbf{F}$ i $\mathbf{r}\,\text{div}\bar{\mathbf{F}} - \mathbf{r} \times \text{curl}\bar{\mathbf{F}} = -\bar{\mathbf{F}}$. These vector integral formulas are analogous to the *Cauchy-Pompeiu* integral formula of complex analysis [17].

On the basis of the previous results one can say that there is a complete analogy between complex analysis in $\mathbb{C}$ and real vector analysis in $\mathbf{V}_2$, thus all the results of complex analysis are applicable to scalar and vector fields in $\mathbf{V}_2$ and vice versa. In doing so, z is formally replaced by $\mathbf{r}$, and the imaginary unit i, more precisely the *i*vector $\boldsymbol{\imath}$, is replaced by the vector $\hat{\mathbf{e}}$ and vice versa ($z \rightleftarrows \mathbf{r}$ and $\boldsymbol{\imath} \rightleftarrows \hat{\mathbf{e}}$). This conclusion can be even more obvious if a formally analogous method of deriving previously obtained vector identities is applied to the field of complex vectors $\mathbf{V}_\mathbb{C}$, which corresponds to the *i*vector field (field of complex numbers) $\mathbb{C}$, in the sense of the correspondence: $\mathbf{1} \risingdotseq \mathbf{e}$ and $\boldsymbol{\imath} \risingdotseq \hat{\mathbf{e}}$, where the unit vector $\mathbf{e}$ and the pseudo-unit vector $\hat{\mathbf{e}}$ ($\hat{\mathbf{e}} \cdot \hat{\mathbf{e}} = -1$) form an orthogonal basis of the field of complex vectors $\mathbf{V}_{\mathbb{C}}$, whose algebraic structure is based on the geometric product of two complex vectors $\mathbf{r}_1 = a\mathbf{e} + b\hat{\mathbf{e}}$ and $\mathbf{r}_2 = c\mathbf{e} + d\hat{\mathbf{e}}$, as follows [9]

$$\bar{\mathbf{r}}_1\mathbf{r}_2 = \frac{1}{2}(\bar{\mathbf{r}}_1\mathbf{r}_2 + \mathbf{r}_1\bar{\mathbf{r}}_2) + \frac{1}{2}(\bar{\mathbf{r}}_1\mathbf{r}_2 - \mathbf{r}_1\bar{\mathbf{r}}_2) := (\mathbf{r}_1 \cdot \bar{\mathbf{r}}_2)\mathbf{e} - (\bar{\mathbf{r}}_1 \times \bar{\mathbf{r}}_2) \times \mathbf{e} = \tag{70}$$

$$= (\bar{\mathbf{r}}_1 \cdot \mathbf{r}_2)\mathbf{e} + (\mathbf{r}_1 \times \mathbf{r}_2) \times \mathbf{e} = [(a\mathbf{e} - b\hat{\mathbf{e}}) \cdot (c\mathbf{e} + d\hat{\mathbf{e}})]\mathbf{e} + [(a\mathbf{e} + b\hat{\mathbf{e}}) \times (c\mathbf{e} + d\hat{\mathbf{e}})] \times \mathbf{e}.$$

References

1. Arsova M., The *Looman-Menchoff* theorem and some subharmonic function analogues, 1955, *Proc. Amer. Math. Soc.*, **6** , 94-105
2. Clifford W., Applications of *Grassmann's* Extensive Algebra, *Amer. J. Math.*, 1878, **1** (**4**): 350-358
3. Grassmann, H., Die lineale Ausdehnungslehre ein neuer Zweig der Mathematik: dargestellt und durch Anwendungen auf die übrigen Zweige der Mathematik, wie auch auf die Statik, Mechanik, die Lehre vom Magnetismus und die Krystallonomie erläutert, Leipzig: O. Wigand, 1844
4. Hestenes D., Multivector Calculus, *J. Math. Anal. Appl.*, 1968, **24**, 313-325
5. Hestenes D., Multivector Functions, *J. Math. Anal. Appl.*, 1968, **24**, 467-473
6. Marsden E. J., Tromba J. A., Vector Calculus, 6th Edition. New York: W. H. Freeman and Company, 2012
7. Mitrinović S. D., Kečkić D. J., The *Cauchy* Method of Residues: Theory and Applications, D. Reidel Publishing Company, 1984
8. Pompeiu D., Sur une classe de fonctions d'une variable complexe. *Rendiconti del Circolo Matematico di Palermo*,1912, **33**(**1**), 108-113
9. Sarić, B., The *Fourier* series of one class of functions with discontinuities, Dissertation, Date of defence: October 20, 2009, at the University of Novi Sad, Faculty of Science, Department of Mathematics and Informatics.
10. Sarić, B., *Cauchy's* residue theorem for a class of real valued functions, *Czech. Math. J.*, 2010, **60**(**4**), 1043-1048
11. Sarić, B., On totalization of the H_1-integral, *Taiw. J. Math.*, 2011, **15**(**4**), 1691-1700
12. Sarić, B., On totalization of the *Henstock-Kurzweil* integral in the multidimensional space, *Czech. Math. J.*, 2011, **61**(**4**), 1017-1022
13. Sarić, B., On an integral as an interval function, *Sci. Bull.*, *Series A*, 2016, **78**(**4**), 53-56
14. Sarić, B., On the H_N-integration of spatial (integral) derivatives of multivector fields with singularities in $\mathbb{R}^N$, *Filomat*, 2017, **31**(**8**), 2433-2439
15. Spiegel, R. M., Lipschutz, S., Schiller, J. J. and Spellman, D., Schaum's outline of theory and problems of complex variables: with an introduction to conformal mapping and its application, 2th Edition, London: McGraw-Hill Book Company, 1974
16. Tung C. C., On *Wirtinger* derivations, the adjoint of the operator $\bar{\partial}$, and applications, *Izv. Math.*, 2018, **82**(**6**), 1239-1264
17. Tutschke W., Interactions between partial differential equations and generalized analytic functions, Cubo A Math. J., 2004, **6**(**1**), 281-292