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Review of Rough, Soft, Fuzzy, and Functorial SuperHyperStructures

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Review of Rough, Soft, Fuzzy, and Functorial SuperHyperStructures

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Abstract

Hyperstructures and their hierarchical extensions—*SuperHyperStructures*—offer a flexible framework for representing multi-layered and intricate systems [1,2]. This paper explores several extended variants of the classical SuperHyperStructure, including *Rough*, *Soft*, *Fuzzy*, and *Functorial SuperHyper-Structures*. Some of these investigations revisit earlier concepts, while others broaden the theoretical scope. The overall aim is not only to provide new insights but also to promote the study and dissemination of SuperHyperStructures and their related families within the broader mathematical community.

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Structure of this paper

The format of this paper is described below.

1. Preliminaries

In this section we collect the notation and basic notions used later. Unless explicitly stated otherwise, all underlying sets are *finite*.

1.1. SuperHyperStructure

A *Classical Structure* refers to a mathematical or real-world construct—such as logic, probability, statistics, algebra, geometry, graph theory, or automata. A *HyperStructure* enlarges this setting by replacing a base set S with its powerset $\mathcal{P}(S)$ and by allowing *hyperoperations* that take subsets to subsets, thereby expressing higher–order interactions [3–5]. Definitions are stated below.

Definition 1.1 (Classical Structure)**.** *A* Classical Structure $\mathcal{C}$ *is a mathematical object drawn from domains such as set theory, logic, probability, statistics, algebra, geometry, graph theory, automata theory, or game theory. Formally,*

$$\mathcal{C} = (H, \{\#^{(m)}\}_{m \in \mathcal{I}}),$$

where:

- H *is a nonempty* carrier set*;*
- *for each* $m \in \mathcal{I} \subseteq \mathbb{Z}_{>0}$*, there is an* m*-ary operation* $\#^{(m)} : H^m \to H$*, subject to the axioms appropriate to the particular structure (e.g., associativity, commutativity, identities).*

We call $\mathcal{C}$ *a structure of* type $\{\#^{(m)} : m \in \mathcal{I}\}$*. Typical instances include:*

- Set*:* $(S, \varnothing)$*, a carrier possibly equipped with distinguished elements or relations [6].*
- Logic*:* $(L, \wedge, \vee, \neg)$ *with binary* $\wedge, \vee$ *and unary* $\neg$*, satisfying the usual logical laws [7].*
- Probability*:* $(\Omega, \mathcal{F}, P)$*, where* $P : \mathcal{F} \to [0, 1]$ *is a measure on a* σ*-algebra* $\mathcal{F} \subseteq \mathcal{P}(\Omega)$ *[8].*
- Statistics*:* $(X, \mathcal{A}, \theta)$*, with* θ *mapping data to parameters [9].*
- Algebra*:*
  - Group $(G, *)$ *with* $* : G \times G \to G$ *and the group axioms [10,11];*
  - Ring $(R, +, \times)$ *with two binary operations satisfying the ring axioms [12,13];*
  - Vector space $(V, +, \cdot)$ *over a field* $\mathbb{F}$ *with scalar multiplication* $\cdot : \mathbb{F} \times V \to V$ *[14,15].*
- Geometry*:* (X, dist) *with a metric* $\mathrm{dist} : X \times X \to \mathbb{R}$ *[16,17].*
- Graph*:* (V, E) *with* $E \subseteq \{\{u, v\} : u, v \in V\}$ *(undirected) or* $E \subseteq V \times V$ *(directed); adjacency and incidence are defined as usual [18,19].*
- Automaton*:* $(Q, \Sigma, \delta, q_0, F)$ *with finite state set* Q*, input alphabet* Σ*, transition function* $\delta : Q \times \Sigma \to Q$*, start state* $q_0 \in Q$*, and accept set* $F \subseteq Q$ *[20,21].*
- Game*:* $(N, \{A_i\}_{i \in N}, \{u_i\}_{i \in N})$*, where* N *is the player set,* A_i *the action set of player* i*, and* $u_i : \prod_{j \in N} A_j \to \mathbb{R}$ *the payoff of player* i *[22,23].*

Definition 1.2 (Powerset)**.** (cf. [24,25]) *For a set* S*, the* powerset $\mathcal{P}(S)$ *is the family of all subsets of* S*, including* $\varnothing$ *and* S*:*

$$\mathcal{P}(S) = \{ A \mid A \subseteq S \}.$$

Definition 1.3 (Hyperoperation)**.** (cf. [26,27]) *A* hyperoperation *on* S *is a binary rule whose output is a subset of* S *rather than a single element; formally,*

$$\circ : S \times S \longrightarrow \mathcal{P}(S).$$

Definition 1.4 (Hyperstructure)**.** (cf. [28–30]) *A* Hyperstructure *upgrades a classical structure by working on the powerset of the carrier. It is given by*

$$\mathcal{H} = (\mathcal{P}(S), \circ),$$

where $\circ$ *is a hyperoperation on subsets of* S*. In this setting, operations combine collections into new collections.*

Example 1.5 (HyperStructure on $\mathbb{Z}_4$)**.** *Let* $H = \mathbb{Z}_4 = \{0, 1, 2, 3\}$ *and define a binary hyperoperation*

$$x \boxplus y := \{ x + y,\ x + y + 1 \} \pmod 4.$$

Concrete computations.

$$2 \boxplus 3 = \{5, 6\} \pmod 4 = \{1, 2\}, \qquad \{1, 3\} \boxplus \{2\} = (1 \boxplus 2) \cup (3 \boxplus 2) = \{3, 0\} \cup \{1, 2\} = \{0, 1, 2, 3\}.$$

Hence $(H, \boxplus)$ *is a concrete* HyperStructure*.*

A *SuperHyperStructure* pushes this idea further by iterating the powerset construction n times. Operations act on nested families of subsets, capturing hierarchical, multi-level interactions [2,31]. Related notions include SuperHyperAlgebra [32,33], SuperHyperGraph [34–36], and other super-level algebraic and combinatorial frameworks.

Definition 1.6 (n-th Powerset)**.** ([31,37]) *For a set H, define inductively*

$$\mathcal{P}^0(H) = H, \qquad \mathcal{P}^{k+1}(H) = \mathcal{P}(\mathcal{P}^k(H)) \quad (k \geq 0).$$

Thus $\mathcal{P}^1(H) = \mathcal{P}(H)$ and $\mathcal{P}^2(H) = \mathcal{P}(\mathcal{P}(H))$, etc. If one excludes the empty set at each stage, write $\mathcal{P}^*(X) = \mathcal{P}(X) \setminus \{\emptyset\}$ and set

$$\mathcal{P}^{*1}(H) = \mathcal{P}^*(H), \qquad \mathcal{P}^{*(k+1)}(H) = \mathcal{P}^*(\mathcal{P}^{*k}(H)).$$

Example 1.7 (n-th Powerset (Definition 1.6))**.** *Let $H = \{a, b\}$. Then*

$$\mathcal{P}^0(H) = H = \{a, b\}, \qquad \mathcal{P}^1(H) = \mathcal{P}(H) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}.$$

The second iterated powerset is

$$\mathcal{P}^2(H) = \mathcal{P}(\mathcal{P}(H)) = \left\{ A \;\middle|\; A \subseteq \{\emptyset, \{a\}, \{b\}, \{a, b\}\} \right\}.$$

If we exclude $\emptyset$ at each stage, we obtain

$$\mathcal{P}^{*1}(H) = \mathcal{P}(H) \setminus \{\emptyset\} = \{\{a\}, \{b\}, \{a, b\}\},$$

and

$$\mathcal{P}^{*2}(H) = \mathcal{P}^*(\mathcal{P}^{*1}(H)) = \left\{ A \;\middle|\; \emptyset \neq A \subseteq \{\{a\}, \{b\}, \{a, b\}\} \right\}.$$

Definition 1.8 (SuperHyperOperations)**.** (cf. [28]) *Let $H \neq \emptyset$ and define $\mathcal{P}^k(H)$ as above. An (m, n)-*SuperHyperOperation *is an m-ary map*

$$\circ^{(m,n)} : H^m \longrightarrow \mathcal{P}^n_*(H),$$

where $\mathcal{P}^n_*(H)$ denotes either the full n-th powerset $\mathcal{P}^n(H)$ or its nonempty variant $\mathcal{P}^n(H) \setminus \{\emptyset\}$. When the empty set is excluded, we speak of a classical-type *SuperHyperOperation; when it is allowed, we refer to a* neutrosophic *SuperHyperOperation.*

Example 1.9 (SuperHyperOperations)**.** *Let $H = \{0, 1, 2\}$. Define a* unary *$(m, n) = (1, 2)$ superhyperoperation*

$$\circ^{(1,2)} : \mathcal{P}^1(H) = \mathcal{P}(H) \longrightarrow \mathcal{P}^2(H) = \mathcal{P}(\mathcal{P}(H))$$

by

$$\circ^{(1,2)}(A) := \{ A,\ H \setminus A \}.$$

Then $\circ^{(1,2)}(A)$ is a set of subsets *of H, hence an element of $\mathcal{P}^2(H)$. For instance, with $A = \{0, 2\}$,*

$$\circ^{(1,2)}(\{0, 2\}) = \{\{0, 2\}, \{1\}\} \in \mathcal{P}^2(H).$$

This provides a concrete $(1, 2)$-SuperHyperOperation.

Definition 1.10 (*n*-Superhyperstructure)**.** (cf. [28,31,38,39]) *An* n-Superhyperstructure *is a hyperstructure built on the n-fold iterated powerset of S:*

$$\mathcal{SH}_n = (\mathcal{P}^n(S), \circ),$$

with $\circ$ *an operation on* $\mathcal{P}^n(S)$*. This yields a hierarchy in which operations act on increasingly nested collections.*

Example 1.11 (*n*-Superhyperstructure)**.** *Let* $S = \{x, y\}$ *and take* $n = 2$*. Consider the carrier* $\mathcal{P}^2(S) = \mathcal{P}(\mathcal{P}(S))$*. Define*

$$\odot : \mathcal{P}^2(S) \times \mathcal{P}^2(S) \longrightarrow \mathcal{P}^2(S), \qquad \odot(X, Y) := \{ A \cup B \mid A \in X, B \in Y \}.$$

Since each $A \cup B$ *is a subset of S, the right-hand side is a subset of* $\mathcal{P}(S)$*, i.e. an element of* $\mathcal{P}^2(S)$*. Hence*

$$\mathcal{SH}_2 = (\mathcal{P}^2(S), \odot)$$

is a valid 2*-Superhyperstructure. Concretely, let*

$$X = \{\{x\}, \{y\}\}, \qquad Y = \{\{y\}, \{x, y\}\}.$$

Then

$$\odot(X, Y) = \{\{x\} \cup \{y\}, \{x\} \cup \{x, y\}, \{y\} \cup \{y\}, \{y\} \cup \{x, y\}\} = \{\{x, y\}, \{x, y\}, \{y\}, \{x, y\}\} = \{\{y\}, \{x, y\}\} \in \mathcal{P}^2(S). \quad (1)$$

Definition 1.12 (SuperHyperStructure of order (m, n))**.** (cf. [2,40]) *Let* $S \neq \emptyset$ *and* $m, n \geq 0$*. A* (m, n)-SuperHyperStructure *of arity s is any map*

$$\odot^{(m,n)} : (\mathcal{P}^m(S))^s \longrightarrow \mathcal{P}^n(S).$$

Classical situations are recovered as special cases: $m = n = 0$ *yields ordinary s-ary operations;* $m = 0, n = 1$ *gives hyperoperations; and* $s = 1$ *corresponds to* superhyperoperations.

Example 1.13 (SuperHyperStructure of order (m, n))**.** *Let* $S = \{0, 1, 2\}$*, set* $(m, n) = (1, 2)$ *and arity* $s = 2$*. Define*

$$\odot^{(1,2)} : (\mathcal{P}^1(S))^2 \longrightarrow \mathcal{P}^2(S), \qquad \odot^{(1,2)}(A, B) := \{A, B, A \cup B\}.$$

By construction, $\odot^{(1,2)}(A, B)$ *is a set of subsets of S, hence an element of* $\mathcal{P}^2(S)$*. For example, with* $A = \{0, 1\}$ *and* $B = \{1, 2\}$*,*

$$\odot^{(1,2)}(A, B) = \{\{0, 1\}, \{1, 2\}, \{0, 1, 2\}\} \in \mathcal{P}^2(S).$$

Thus $(\mathcal{P}^1(S), \odot^{(1,2)})$ *realizes a concrete* $(m, n) = (1, 2)$ *SuperHyperStructure of arity* $s = 2$*.*

1.2. Soft HyperStructure

A *Soft HyperStructure* consists of a parameter set A and a mapping $F : A \to \mathcal{P}(H)$, where each $a \in A$ yields a sub-hyperstructure $F(a) \subseteq H$ (cf.[41–43]).

Definition 1.14 (Soft HyperStructure)**.** *Let H be a hyperalgebra (hyperstructure) with signature* $\Sigma = \{ \star_i : H^{n_i} \to \mathcal{P}^*(H) \mid i \in I\}$*, where* $\mathcal{P}^*(H)$ *denotes the set of all nonempty subsets of H. A subset* $K \subseteq H$ *is a* subhyperstructure *of H if for every* $i \in I$ *and every* $(x_1, \ldots, x_{n_i}) \in K^{n_i}$ *one has*

$$\star_i(x_1, \ldots, x_{n_i}) \subseteq K.$$

Let (F, A) be a non-null soft set over H, i.e. $A \neq \emptyset$, $F : A \to \mathcal{P}^*(H)$ and the support $\mathrm{Supp}(F, A) := \{a \in A \mid F(a) \neq \emptyset\} \neq \emptyset$. *Then (F, A) is called a* Soft HyperStructure over H *iff for every $a \in \mathrm{Supp}(F, A)$, the set $F(a)$ is a subhyperstructure of H.*

Example 1.15 (Soft HyperStructure on $\mathbb{Z}_3$ with left-projection)**.** *Let $H = \mathbb{Z}_3 = \{0, 1, 2\}$ and define a (degenerate but valid) hyperoperation*

$$x \triangleright y := \{x\} \quad (\textit{left projection}).$$

Take the parameter set $A = \{\mathrm{low}, \mathrm{high}\}$ and define the soft set $F : A \to \mathcal{P}(H)$ by

$$F(\mathrm{low}) = \{0, 1\}, \qquad F(\mathrm{high}) = \{1, 2\}.$$

Closure check (each parameter induces a subhyperstructure). *If $x, y \in F(a)$, then $x \triangleright y = \{x\} \subseteq F(a)$ because $x \in F(a)$. Hence $F(\mathrm{low})$ and $F(\mathrm{high})$ are both closed under $\triangleright$, so (F, A) is a concrete* Soft HyperStructure *over $(H, \triangleright)$.*

1.3. Fuzzy HyperStructure

A *Fuzzy HyperStructure* assigns to each hyperoperation outcome a membership function $\mu : H \to [0, 1]$, modeling graded belongingness of elements to hyperoperation results (cf.[44–47]).

Definition 1.16 (Fuzzy hyperoperation)**.** *(cf.[48,49]) Let H be a nonempty set and let $\mathcal{F}(H) := \{ \mu : H \to [0, 1] \}$ be the set of all fuzzy subsets of H. A* fuzzy hyperoperation *on H is a map*

$$\star : H \times H \longrightarrow \mathcal{F}(H), \qquad (x, y) \longmapsto \mu_{x,y}(\cdot),$$

so that for each $(x, y) \in H \times H$, $\mu_{x,y} : H \to [0, 1]$ is a membership function on H. The pair $(H, \star)$ is called a fuzzy hypergroupoid.

Definition 1.17 (Fuzzy HyperStructure)**.** *A* Fuzzy HyperStructure *is a pair $(H, (\star_i)_{i \in I})$ where each $\star_i$ is a fuzzy hyperoperation on H (possibly of specified arity). When $I = \{1\}$ we simply write $(H, \star)$.*

Remark 1.18 (Cut–set representation (and reconstruction))**.** *For $\alpha \in (0, 1]$, the α-cut of $\star$ induces a (crisp) hyperoperation*

$$x \circ_\alpha y := \{ z \in H : \mu_{x,y}(z) \geq \alpha \} \in \mathcal{P}^*(H),$$

hence $(H, \circ_\alpha)$ is a (crisp) hyperstructure for each α. Conversely, a family $\{ \circ_\alpha : \alpha \in (0, 1] \}$ of hyperoperations that is monotone in α (i.e. $\alpha \leq \beta \Rightarrow x \circ_\beta y \subseteq x \circ_\alpha y$ for all x, y) uniquely determines $\star$ by

$$\mu_{x,y}(z) = \sup\{ \alpha \in (0, 1] : z \in x \circ_\alpha y \}.$$

Example 1.19 (Fuzzy HyperStructure on a three-element set)**.** *Let $S = \{x, y, z\}$. Define a fuzzy hyperoperation $\star : S \times S \to [0, 1]^S$ by specifying membership vectors $\star(a, b) = (\mu_{ab}(x), \mu_{ab}(y), \mu_{ab}(z))$. For instance,*

$$\mu_{xy} = (0.7, 0.2, 0.0), \quad \mu_{xx} = (0.1, 0.9, 0.3), \quad \mu_{yz} = (0.0, 0.4, 0.8).$$

α-cut illustration. *At level $\alpha = 0.5$,*

$$(x \star y)_{0.5} = \{x\}, \qquad (x \star x)_{0.5} = \{y\}, \qquad (y \star z)_{0.5} = \{z\}.$$

Thus $(S, \star)$ is a concrete Fuzzy HyperStructure *with explicit graded outputs.*

1.4. Functorial Structure

A *Functorial Structure* is defined as a covariant functor $F : \mathcal{C} \to \mathbf{Set}$, assigning sets to objects and functions to morphisms, ensuring functoriality [50].

Definition 1.20 (Functorial Set)**.** *[50] Let $\mathcal{C}$ be a category and*

$$F: \mathcal{C} \longrightarrow \mathbf{Set}$$

*be a (covariant) endofunctor. For any object $X \in \mathrm{Ob}(\mathcal{C})$, an F-*set over *X is an element*

$$s \in F(X).$$

We denote the collection of all F-sets over X simply by $F(X)$. A morphism $f: X \to Y$ in $\mathcal{C}$ induces a pushforward

$$F(f): F(X) \longrightarrow F(Y), \qquad s \mapsto F(f)(s).$$

Example 1.21 (Concrete Example of a Functorial Set)**.** *Let $\mathcal{C} = \mathbf{FinSet}$, the category of finite sets and functions. Define the functor*

$$F : \mathbf{FinSet} \longrightarrow \mathbf{Set}$$

by

$$F(X) = \mathcal{P}(X), \qquad F(f)(A) = f[A] = \{ f(x) \mid x \in A \}$$

for any finite set X and function $f : X \to Y$.
Example instance. *Take $X = \{1,2,3\}$ and $f : X \to Y = \{a,b\}$ defined by $f(1) = a$, $f(2) = b$, $f(3) = a$. Then*

$$F(X) = \mathcal{P}(\{1,2,3\}) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \dots, \{1,2,3\}\}.$$

For $A = \{1,3\} \in F(X)$, the pushforward is

$$F(f)(A) = f[\{1,3\}] = \{a,a\} = \{a\}.$$

Thus A is an F-set over X, and $F(f)(A)$ is the corresponding F-set over Y.
Hence $(\mathbf{FinSet}, F)$ *provides a concrete example of a* Functorial Set.

Definition 1.22 (Functorial Structure)**.** *[50] Let $\mathcal{C}$ be a category. A* Functorial Structure *on $\mathcal{C}$ is simply a covariant functor*

$$F : \mathcal{C} \longrightarrow \mathbf{Set}.$$

For each object $X \in \mathrm{Ob}(\mathcal{C})$, an element

$$s \in F(X)$$

*is called an F-*structure on *X. Every morphism $f: X \to Y$ in $\mathcal{C}$ induces a* pushforward

$$F(f): F(X) \longrightarrow F(Y), \qquad s \longmapsto F(f)(s),$$

and the usual functoriality conditions $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ and $F(g \circ f) = F(g) \circ F(f)$ hold.

Example 1.23 (Functorial Structure: the list functor on FinSet)**.** *Let $\mathcal{C}$ = FinSet and define the functor $F : \mathcal{C} \to \mathbf{Set}$ by*

$$F(X) := X^* \quad (\textit{finite words over } X), \qquad F(f) : X^* \to Y^*, \;\; F(f)(x_1 \cdots x_k) := f(x_1) \cdots f(x_k).$$

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Functoriality (concrete check). *Take $X = \{a, b\}$, $Y = \{0, 1\}$, $Z = \{u, v\}$, with $f : X \to Y$, $f(a) = 1$, $f(b) = 0$, and $g : Y \to Z$, $g(1) = u$, $g(0) = v$. For $w = [a, b, a] \in X^*$,*

$$F(f)(w) = [1, 0, 1], \qquad F(g)\big(F(f)(w)\big) = [u, v, u].$$

Since $(g \circ f)(a) = u$, $(g \circ f)(b) = v$, we get

$$F(g \circ f)(w) = [u, v, u] = F(g)\big(F(f)(w)\big), \quad F(\mathrm{id}_X)(w) = w.$$

Thus F is a (covariant) functor: a concrete Functorial Structure*.*

2. Main Results

In this section, we examine and discuss newly introduced structures and related developments.

2.1. Rough HyperStructure

A *Rough HyperStructure* is a hyperstructure $(H, \circ)$ endowed with an indiscernibility relation R, so operations are defined on equivalence classes $[x]_R$ approximating uncertainty.

Notation 2.1. *Let $H \neq \emptyset$, $n \geq 2$, and $\mathcal{P}^*(H) := \mathcal{P}(H) \setminus \{\emptyset\}$. An n-ary hyperoperation is $f : H^n \to \mathcal{P}^*(H)$. For $A_1, \ldots, A_n \subseteq H$, its set–extension is*

$$f(A_1, \ldots, A_n) := \bigcup \{\, f(a_1, \ldots, a_n) : a_i \in A_i \,\}.$$

Fix a reflexive relation $R \subseteq H \times H$ and write $R(x) := \{y \in H : xRy\}$. Define rough approximations for $X \subseteq H$ by

$$\ell_R(X) := \{\, x : R(x) \subseteq X \,\}, \qquad u_R(X) := \{\, x : R(x) \cap X \neq \emptyset \,\}.$$

Note $\ell_R(X) \subseteq X \subseteq u_R(X)$, and ℓ_R, u_R are monotone and idempotent.

Notation 2.2 (Rough pairs and lifting)**.** *Let*

$$\mathsf{RP}_R(H) := \{(L, U) \in \mathcal{P}(H)^2 : \ L \subseteq U, \ \ell_R(L) = L, \ u_R(U) = U\}.$$

For $(L_i, U_i) \in \mathsf{RP}_R(H)$ define the lifted hyperoperation

$$\widehat{f}\big((L_1, U_1), \ldots, (L_n, U_n)\big) := \big(\, \ell_R(f(L_1, \ldots, L_n)),\ u_R(f(U_1, \ldots, U_n)) \,\big).$$

Definition 2.3 (Rough HyperStructure)**.** *The structure*

$$\mathbf{RH}(H, f; R) := \big(\mathsf{RP}_R(H), \widehat{f}\big)$$

is called the Rough HyperStructure *determined by (H, f) and the reflexive relation R.*

Example 2.4 (Rough HyperStructure via parity on $\mathbb{Z}_4$)**.** *Let $H = \mathbb{Z}_4$ with the hyperoperation $\boxplus$ above. Equip H with the equivalence (indiscernibility) relation R "same parity," i.e.*

$$[0]_R = \{0, 2\}, \quad [1]_R = \{1, 3\}.$$

For $X \subseteq H$ define the rough approximations

$$\ell_R(X) := \{x \in H : [x]_R \subseteq X\}, \qquad u_R(X) := \{x \in H : [x]_R \cap X \neq \emptyset\}.$$

Concrete approximations. *$\ell_R(\{0, 2\}) = \{0, 2\}$ and $u_R(\{0, 2\}) = \{0, 2\}$, while $\ell_R(\{0\}) = \emptyset$ and $u_R(\{0\}) = \{0, 2\}$.*

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Following the standard lift, define the rough-pair carrier

$$\mathsf{RP}_R(H) := \{(L, U) \subseteq H \times H : \; L \subseteq U, \; \ell_R(L) = L, \; u_R(U) = U\}.$$

Lift $\boxplus$ *by*

$$\widehat{\boxplus}((L_1, U_1), (L_2, U_2)) := \Big(\, \ell_R(L_1 \boxplus L_2), \;\; u_R(U_1 \boxplus U_2) \, \Big),$$

where $A \boxplus B := \bigcup_{a \in A, b \in B}(a \boxplus b)$. ***Concrete computation.*** *With* $(L_1, U_1) = (\{0, 2\}, \{0, 2\})$ *and* $(L_2, U_2) = (\{0, 2\}, \{0, 2\})$,

$$L_1 \boxplus L_2 = \{0, 1, 2, 3\} = H \;\Rightarrow\; \widehat{\boxplus}((L_1, U_1), (L_2, U_2)) = (\ell_R(H), u_R(H)) = (H, H).$$

Thus $(\mathsf{RP}_R(H), \widehat{\boxplus})$ *is a concrete* Rough HyperStructure.

Proposition 2.5 (Closure (well-definedness))**.** $\widehat{f}$ *maps* $\mathsf{RP}_R(H)^n$ *into* $\mathsf{RP}_R(H)$*; hence it is an n-ary hyperoperation on* $\mathsf{RP}_R(H)$.

Proof. Monotonicity of f (set–extension) gives $f(L_1, \ldots, L_n) \subseteq f(U_1, \ldots, U_n)$. Applying monotone ℓ_R, u_R yields $\ell_R(f(L_1, \ldots, L_n)) \subseteq u_R(f(U_1, \ldots, U_n))$. Idempotence of ℓ_R, u_R ensures $\ell_R(\cdot)$– and $u_R(\cdot)$–closure. □

Theorem 2.6 (Generalization of hyperstructures)**.** *Let* (H, f) *be an n-ary hypergroupoid and take R to be equality on H. Define the embedding* $\eta : H \to \mathsf{RP}_=(H)$ *by* $\eta(x) := (\{x\}, \{x\})$*. Then for all* $x_1, \ldots, x_n \in H$,

$$\widehat{f}(\eta(x_1), \ldots, \eta(x_n)) = (f(x_1, \ldots, x_n), \, f(x_1, \ldots, x_n)).$$

Consequently, collapsing exact pairs $(S, S) \mapsto S$ *recovers* (H, f)*; hence* $\mathbf{RH}(H, f; R)$ *strictly generalizes* (H, f).

Proof. For $R = =$, $\ell_=(X) = X = u_=(X)$. Using the set–extension with singletons gives $f(\{x_1\}, \ldots, \{x_n\}) = f(x_1, \ldots, x_n)$, whence the identity above. □

3. Fuzzy SuperHyperStructure

A *Fuzzy SuperHyperStructure* assigns fuzzy membership values in $[0, 1]$ to superhyperoperations on iterated powersets, modeling hierarchical interactions with graded uncertainty semantics.

Notation 3.1 (Iterated powersets and fuzzy powersets)**.** *Let* $S \neq \emptyset$*. Define the iterated powersets*

$$\mathcal{P}^0(S) := S, \qquad \mathcal{P}^{k+1}(S) := \mathcal{P}(\mathcal{P}^k(S)) \quad (k \geq 0).$$

For any set X, its fuzzy powerset *is*

$$\mathsf{Fuzz}(X) := [0, 1]^X = \{ \, \mu : X \to [0, 1] \, \}.$$

For $\mu \in [0, 1]^X$ *and* $\alpha \in (0, 1]$*, the* α-cut *is*

$$X_\alpha(\mu) := \{ \, x \in X : \; \mu(x) \geq \alpha \, \}.$$

Definition 3.2 (Typed fuzzy superhyperoperation)**.** *Fix integers* $m, n \geq 0$ *and an arity* $s \in \mathbb{N}$*. A* fuzzy superhyperoperation of type (m, n) and arity s *on S is a map*

$$\widetilde{\odot}^{(m,n)} : \; (\mathcal{P}^m(S))^s \longrightarrow \mathsf{Fuzz}(\mathcal{P}^n(S)).$$

For inputs $\mathbf{A} = (A_1, \ldots, A_s) \in (\mathcal{P}^m(S))^s$ *we write* $\mu_{\mathbf{A}} := \widetilde{\odot}^{(m,n)}(\mathbf{A}) \in [0,1]^{\mathcal{P}^n(S)}$.

Definition 3.3 (Fuzzy SuperHyperStructure (FSuHS))**.** *Let* $\{(m_j, n_j, s_j)\}_{j \in J}$ *be a finite signature of types. A* Fuzzy SuperHyperStructure *on* S *is a family*

$$\mathbf{FSuH}(S) := \{\, \widetilde{\odot}_j^{(m_j,n_j)} : (\mathcal{P}^{m_j}(S))^{s_j} \to \mathsf{Fuzz}(\mathcal{P}^{n_j}(S)) \;\mid\; j \in J \,\}.$$

Notation 3.4 (α-cut collapse and support)**.** *Given* $\widetilde{\odot}^{(m,n)}$ *as above and* $\alpha \in (0,1]$, *define the* α-cut crispization

$$\odot_\alpha^{(m,n)} : \; (\mathcal{P}^m(S))^s \longrightarrow \mathcal{P}(\mathcal{P}^n(S)), \quad \odot_\alpha^{(m,n)}(\mathbf{A}) := \{\, Y \in \mathcal{P}^n(S) : \; \mu_{\mathbf{A}}(Y) \geq \alpha \,\}.$$

Its support *is* $\mathrm{supp}(\mu_{\mathbf{A}}) := \odot_{0^+}^{(m,n)}(\mathbf{A}) := \{Y : \mu_{\mathbf{A}}(Y) > 0\}$.

Example 3.5 (Fuzzy SuperHyperStructure on $S = \{x, y\}$)**.** *Work at type* $(1,1)$ *(inputs and outputs on* $\mathcal{P}^*(S)$*). Define a fuzzy superhyperoperation* $\widetilde{\odot}^{(1,1)} : (\mathcal{P}^*(S))^2 \to [0,1]^{\mathcal{P}^*(S)}$ *by*

$$\mu_{A,B}(Y) \;:=\; \frac{|\, Y \cap (A \cup B) \,|}{|A \cup B|} \qquad (A, B, Y \in \mathcal{P}^*(S)).$$

α***-cuts.*** *For* $\alpha = \frac{1}{2}$, *if* $A = \{x\}$ *and* $B = \{y\}$ *then* $A \cup B = \{x, y\}$ *and*

$$(\widetilde{\odot}^{(1,1)}(A, B))_{0.5} = \{\{x\}, \{y\}, \{x, y\}\}.$$

Thus $\{\widetilde{\odot}^{(1,1)}\}$ *is a concrete* Fuzzy SuperHyperStructure.

Lemma 3.6 (α-cut reconstruction)**.** *Let* $\widetilde{\odot}^{(m,n)}$ *be a fuzzy superhyperoperation and fix* $\mathbf{A}$. *Then for every* $Y \in \mathcal{P}^n(S)$,

$$\mu_{\mathbf{A}}(Y) \;=\; \sup \{\, \alpha \in (0,1] : \; Y \in \odot_\alpha^{(m,n)}(\mathbf{A}) \,\}.$$

Moreover, $\alpha \leq \beta \;\Rightarrow\; \odot_\beta^{(m,n)}(\mathbf{A}) \subseteq \odot_\alpha^{(m,n)}(\mathbf{A})$ *(nested cuts).*

Proof. By definition, $Y \in \odot_\alpha^{(m,n)}(\mathbf{A})$ iff $\mu_{\mathbf{A}}(Y) \geq \alpha$. Hence $\sup\{\alpha : \; Y \in \odot_\alpha^{(m,n)}(\mathbf{A})\} = \mu_{\mathbf{A}}(Y)$. If $\alpha \leq \beta$ and $\mu_{\mathbf{A}}(Y) \geq \beta$ then $\mu_{\mathbf{A}}(Y) \geq \alpha$, proving the nesting. □

Definition 3.7 (Underlying crisp superoperations at level α)**.** *For a FSuHS* $\mathbf{FSuH}(S)$ *and fixed* $\alpha \in (0,1]$, *the* α-cut superhyperstructure *is*

$$\mathbf{SH}_\alpha(S) := \{\, \odot_{j,\alpha}^{(m_j,n_j)} : \; (\mathcal{P}^{m_j}(S))^{s_j} \to \mathcal{P}(\mathcal{P}^{n_j}(S)) \;\mid\; j \in J \,\},$$

with $\odot_{j,\alpha}^{(m_j,n_j)}$ *defined from* $\widetilde{\odot}_j^{(m_j,n_j)}$ *as above.*

Theorem 3.8 (FSuHS generalizes SuperHyperStructure)**.** *Let* $\mathbf{SH}(S)$ *be a (crisp) superhyperstructure consisting of maps*

$$\odot_j^{(m_j,n_j)} : \; (\mathcal{P}^{m_j}(S))^{s_j} \longrightarrow \mathcal{P}^{n_j}(S).$$

Define an embedding $\mathcal{J}$ *that sends each* $\odot_j^{(m_j,n_j)}$ *to the fuzzy superhyperoperation*

$$\widetilde{\odot}_j^{(m_j,n_j)}(\mathbf{A}) \;:=\; \mathbf{1}_{\odot_j^{(m_j,n_j)}(\mathbf{A})} \;\in [0,1]^{\mathcal{P}^{n_j}(S)},$$

i.e. $\mu_{\mathbf{A}}(Y) = 1$ *if* $Y \in \odot_j^{(m_j,n_j)}(\mathbf{A})$ *and* 0 *otherwise. Then, for every* $\alpha \in (0,1]$ *and every input* $\mathbf{A}$,

$$\odot_{j,\alpha}^{(m_j,n_j)}(\mathbf{A}) \;=\; \odot_j^{(m_j,n_j)}(\mathbf{A}),$$

so the α*-cut of* $\mathcal{J}(\mathbf{SH}(S))$ *recovers* $\mathbf{SH}(S)$ *exactly. In particular,* $\mathcal{J}$ *is injective on superoperations.*

Proof. By construction, $\mu_{\mathbf{A}}(Y) \in \{0,1\}$ and $Y \in \odot_{j,\alpha}^{(m_j,n_j)}(\mathbf{A})$ iff $\mu_{\mathbf{A}}(Y) \geq \alpha$. Since $\alpha \in (0,1]$, this is equivalent to $\mu_{\mathbf{A}}(Y) = 1$, i.e. $Y \in \odot_j^{(m_j,n_j)}(\mathbf{A})$. Thus $\odot_{j,\alpha}^{(m_j,n_j)} = \odot_j^{(m_j,n_j)}$ for all α. If two crisp operations differed on some input, their characteristic functions would differ at that input, proving injectivity. □

Definition 3.9 (Fuzzy HyperStructure (baseline))**.** *A* Fuzzy HyperStructure *on* S *with arity* s *is a map*

$$\star : \; S^s \;\longrightarrow\; [0,1]^S,$$

assigning to each $\mathbf{x} \in S^s$ *a fuzzy subset* $\star(\mathbf{x})$ *of* S.

Theorem 3.10 (FSuHS generalizes Fuzzy HyperStructure)**.** *Let* $(S,\star)$ *be a Fuzzy HyperStructure of arity* s*. Regard it as a special case of FSuHS by choosing* $m = 0$ *and* $n = 0$ *(so* $\mathcal{P}^0(S) = S$*) and defining*

$$\widetilde{\odot}^{(0,0)} : \; S^s \;\longrightarrow\; [0,1]^S, \qquad \widetilde{\odot}^{(0,0)} \;:=\; \star.$$

Then $(S,\star)$ *and* $\{\,\widetilde{\odot}^{(0,0)}\,\}$ *are the same data. Conversely, any FSuHS operation of type* $(0,0)$ *is precisely a fuzzy hyperoperation on* S.

Proof. If $m = n = 0$, the domain is S^s and the codomain is $\mathsf{Fuzz}(\mathcal{P}^0(S)) = [0,1]^S$, which matches exactly the definition of $\star$. Thus the identifications are tautological in both directions. □

Proposition 3.11 (Crisp supports and thresholded superstructures)**.** *For any FSuHS and any* $\alpha \in (0,1]$*, the family* $\mathbf{SH}_\alpha(S)$ *is a (crisp) typed superhyperstructure. Moreover, for each input* $\mathbf{A}$ *and output* Y,

$$\mu_{\mathbf{A}}(Y) \;=\; \sup\{\, \alpha \in (0,1] : \; Y \in \odot_\alpha^{(m,n)}(\mathbf{A}) \,\},$$

so the fuzzy data are uniquely determined by the tower of crisp α*-cuts.*

Proof. Each $\odot_\alpha^{(m,n)}$ maps inputs to subsets of the appropriate codomains by definition, thus forming crisp (super)operations of the same types; composition/typing constraints are inherited verbatim. The reconstruction formula is the α-cut reconstruction lemma applied pointwise in Y. □

3.1. Rough SuperHyperStructure

A *Rough SuperHyperStructure* combines superhyperoperations with rough approximations, employing lower and upper operators on iterated powersets to handle uncertainty in multi-level systems.

Notation 3.12 (Levelwise rough approximation)**.** *For each level* $m \in \mathbb{N}_0$ *fix a reflexive relation* $R^{\langle m\rangle}$ *on* $H^{\langle m\rangle}$ *and write*

$$R^{\langle m\rangle}(x) := \{\, y \in H^{\langle m\rangle} : x\, R^{\langle m\rangle}\, y \,\}.$$

For $X \subseteq H^{\langle m\rangle}$ *define the lower/upper approximations*

$$\ell_m(X) := \{\, x : R^{\langle m\rangle}(x) \subseteq X \,\}, \qquad u_m(X) := \{\, x : R^{\langle m\rangle}(x) \cap X \neq \varnothing \,\}.$$

Then ℓ_m, u_m *are monotone, idempotent, and satisfy* $\ell_m(X) \subseteq X \subseteq u_m(X)$.

Notation 3.13 (Rough pairs per level and lifting of operations). *Let*

$$\mathsf{RP}_m := \{\, (L,U) \in \mathcal{P}(H^{\langle m\rangle})^2 :\ L \subseteq U,\ \ell_m(L) = L,\ u_m(U) = U \,\}.$$

Given F_j of type $(\ell_{j,1},\dots,\ell_{j,n_j}) \to r_j$, define the lifted operation

$$\widehat{F}_j : \prod_{k=1}^{n_j} \mathsf{RP}_{\ell_{j,k}} \longrightarrow \mathsf{RP}_{r_j},$$

$$\widehat{F}_j((L_1,U_1),\dots,(L_{n_j},U_{n_j})) := \Big(\ell_{r_j}(F_j(L_1,\dots,L_{n_j})),\ \ u_{r_j}(F_j(U_1,\dots,U_{n_j})) \Big).$$

Definition 3.14 (Rough SuperHyperStructure). *The structure*

$$\mathbf{RSH}(H,\mathcal{F};\mathcal{R}) := \Big(\prod_{m\geq 0} \mathsf{RP}_m,\ \{\widehat{F}_j\}_{j\in J} \Big)$$

with $\mathcal{R} = \{R^{\langle m\rangle}\}_{m\geq 0}$ is called the Rough SuperHyperStructure *determined by the superoperations $\mathcal{F}$ and the levelwise rough data $\mathcal{R}$.*

Example 3.15 (Rough SuperHyperStructure on $\mathbb{Z}_4$ at level 1). *Let $H = \mathbb{Z}_4 = \{0,1,2,3\}$ and $R^{\langle 0\rangle}$ be "same parity": $[0] = \{0,2\}$, $[1] = \{1,3\}$. Lift to level 1 by $R^{\langle 1\rangle}(A) := \{B \subseteq H :\ \exists x \in A,\ \exists y \in B,\ xR^{\langle 0\rangle}y\}$. Define lower/upper approximations on level 1:*

$$\ell_1(\mathcal{X}) = \{A :\ R^{\langle 1\rangle}(A) \subseteq \mathcal{X}\}, \qquad u_1(\mathcal{X}) = \{A :\ R^{\langle 1\rangle}(A) \cap \mathcal{X} \neq \emptyset\}.$$

Let the (crisp) superoperation at level 1 be $F(A,B) := A \triangle B$ (symmetric difference). Form rough pairs $\mathsf{RP}_1 = \{(L,U) : L \subseteq U,\ \ell_1(L) = L,\ u_1(U) = U\}$ and lift

$$\widehat{F}((L_1,U_1),(L_2,U_2)) := \Big(\ell_1(F(L_1,L_2)),\ u_1(F(U_1,U_2)) \Big).$$

Concrete check. *Take $L_1 = U_1 = \{\{0\},\{2\}\}$ and $L_2 = U_2 = \{\{1\},\{3\}\}$. Then $F(L_1,L_2) = \{\{0,1\},\{0,3\},\{1,2\},\{2,3\}\}$, which is $R^{\langle 1\rangle}$-saturated; hence $\widehat{F}((L_1,U_1),(L_2,U_2)) = (F(L_1,L_2),F(L_1,L_2))$. Thus $(\mathsf{RP}_1,\widehat{F})$ yields a* Rough SuperHyperStructure.

Proposition 3.16 (Closure/well-definedness). *For each $j \in J$, the output of $\widehat{F}_j$ lies in RP_{r_j}; hence every $\widehat{F}_j$ is a well-defined superhyperoperation on rough pairs.*

Proof. Let $(L_k,U_k) \in \mathsf{RP}_{\ell_{j,k}}$. By monotonicity of the set–extension, $F_j(L_1,\dots,L_{n_j}) \subseteq F_j(U_1,\dots,U_{n_j})$. Applying monotone ℓ_{r_j}, u_{r_j} gives

$$\ell_{r_j}(F_j(L_1,\dots,L_{n_j})) \ \subseteq\ u_{r_j}(F_j(U_1,\dots,U_{n_j})).$$

Idempotence yields $\ell_{r_j}(\cdot)$– and $u_{r_j}(\cdot)$–closure, i.e., the pair belongs to RP_{r_j}. □

Theorem 3.17 (Rough SuperHyperStructure generalizes SuperHyperStructure). *Assume $R^{\langle m\rangle}$ is equality on $H^{\langle m\rangle}$ for all m. Define the levelwise embedding*

$$\eta_m :\ H^{\langle m\rangle} \longrightarrow \mathsf{RP}_m, \qquad \eta_m(x) := (\{x\},\{x\}).$$

Then for every $j \in J$ and $x_k \in H^{\langle \ell_{j,k}\rangle}$,

$$\widehat{F}_j(\eta_{\ell_{j,1}}(x_1),\dots,\eta_{\ell_{j,n_j}}(x_{n_j})) = (F_j(x_1,\dots,x_{n_j}),\ F_j(x_1,\dots,x_{n_j})).$$

Collapsing exact pairs $(S,S) \mapsto S$ *recovers* $\mathbf{SH}(H,\mathcal{F})$ *from* $\mathbf{RSH}(H,\mathcal{F};\mathcal{R})$.

Proof. For equality, $\ell_m = u_m = \mathrm{id}$. Using the set–extension on singletons, $F_j(\{x_1\},\ldots,\{x_{n_j}\}) = F_j(x_1,\ldots,x_{n_j})$. Hence the displayed identity holds, and the collapse map gives back the original outputs of F_j. □

Theorem 3.18 (Rough SuperHyperStructure generalizes Rough HyperStructure)**.** *Let* (H,f) *be an* n*-ary hypergroupoid (one-level hyperstructure). Regard it as a superstructure with a single operation* F *of type* $(0,\ldots,0) \to 0$*. Fix a reflexive relation* $R^{\langle 0\rangle} =: R$ *on* H *and ignore higher levels (or set* $R^{\langle m\rangle}$ *arbitrary). Then the level-*0 *component of* $\mathbf{RSH}(H,\{F\};\mathcal{R})$ *is canonically isomorphic to the Rough HyperStructure*

$$\mathbf{RH}(H,f;R) = \big(\mathrm{RP}_R(H),\ \widehat{f}\big),$$

with the identification $\mathrm{RP}_0 \equiv \mathrm{RP}_R(H)$ *and*

$$\widehat{F}((L_1,U_1),\ldots,(L_n,U_n)) = \Big(\ \ell_0\big(f(L_1,\ldots,L_n)\big),\ u_0\big(f(U_1,\ldots,U_n)\big)\ \Big) = \widehat{f}(\cdots).$$

Proof. By construction, $H^{\langle 0\rangle} = H$, $\ell_0 = \ell_R$, $u_0 = u_R$, and the set–extension of F coincides with that of f. Therefore the lifted operation at level 0 matches $\widehat{f}$ exactly, yielding the stated identification. □

3.2. Soft SuperHyperStructure

A *Soft SuperHyperStructure* equips a superhyperstructure with parameterized families of sub-superhyperstructures, where each parameter selects context-dependent subsets closed under superhyperoperations.

Notation 3.19 (Soft sets over the levelled universe)**.** *Let* A *be a nonempty parameter set. A* soft set over the levelled universe *is a map*

$$S : A \longrightarrow \prod_{m\ge 0} \mathcal{P}(H^{\langle m\rangle}), \qquad a \longmapsto \big(S_a^{\langle m\rangle}\big)_{m\ge 0},$$

with support $\mathrm{Supp}(S) := \{a \in A :\ \exists m,\ S_a^{\langle m\rangle} \neq \varnothing\}$*. We assume* $\mathrm{Supp}(S) \neq \varnothing$ *and, for all* a*, that every* $S_a^{\langle m\rangle}$ *is either empty or nonempty as specified below.*

Definition 3.20 (Soft SuperHyperStructure)**.** *Let* $\mathbf{SH}(H,\mathcal{F})$ *be as above. A soft set* S *over the levelled universe is a* Soft SuperHyperStructure over $\mathbf{SH}(H,\mathcal{F})$ *if for every* $a \in \mathrm{Supp}(S)$ *and every operation* $F_j \in \mathcal{F}$ *one has the levelwise closure condition*

$$F_j\Big(S_a^{\langle \ell_{j,1}\rangle},\ldots,S_a^{\langle \ell_{j,n_j}\rangle}\Big) \ \subseteq\ S_a^{\langle r_j\rangle}. \tag{2}$$

Equivalently, for each a*, the family* $\big(S_a^{\langle m\rangle}\big)_{m\ge 0}$ *defines a* sub-superhyperstructure *of* $\mathbf{SH}(H,\mathcal{F})$ *under the restrictions of all* F_j*. We write* $\mathbf{SSH}(H,\mathcal{F};S)$ *for such a soft structure.*

Remark 3.21. *Only levels appearing in the types* $\{\ell_{j,k}, r_j\}$ *are relevant; the definition is independent of arbitrary values* $S_a^{\langle m\rangle}$ *at unused levels.*

Example 3.22 (Soft SuperHyperStructure on $\mathbb{Z}_3$)**.** *Let* $H = \mathbb{Z}_3 = \{0,1,2\}$ *and work at level* 1 *with the superoperation*

$$\odot(A,B) := A \cup B \qquad (A,B \in \mathcal{P}^*(H)).$$

Let $A_{\text{par}} = \{\text{low}, \text{high}\}$ *and define the soft selection*

$$S^{\langle 1 \rangle}_{\text{low}} = \{\{0\},\{1\},\{0,1\}\}, \qquad S^{\langle 1 \rangle}_{\text{high}} = \{\{1\},\{2\},\{1,2\}\}.$$

Closure. *For each parameter* $p \in A_{\text{par}}$ *and* $A, B \in S^{\langle 1 \rangle}_p$, $\odot(A,B) = A \cup B \in S^{\langle 1 \rangle}_p$ *by construction; hence every* $S^{\langle 1 \rangle}_p$ *is a sub-superhyperstructure. Therefore* $(A_{\text{par}}, S^{\langle 1 \rangle})$ *is a* Soft SuperHyperStructure *over* $(\mathcal{P}^*(H), \odot)$.

Proposition 3.23 (Basic closure). *If* S *satisfies* (2), *then for each* $a \in \text{Supp}(S)$ *and all* $j \in J$,

$$\forall\, X_k \subseteq S^{\langle \ell_{j,k} \rangle}_a \quad F_j(X_1, \ldots, X_{n_j}) \subseteq S^{\langle r_j \rangle}_a.$$

Proof. By monotonicity of the set–extension (Notation), $X_k \subseteq S^{\langle \ell_{j,k} \rangle}_a$ implies $F_j(X_1, \ldots, X_{n_j}) \subseteq F_j(S^{\langle \ell_{j,1} \rangle}_a, \ldots, S^{\langle \ell_{j,n_j} \rangle}_a) \subseteq S^{\langle r_j \rangle}_a$ by (2). □

Theorem 3.24 (Soft SuperHyperStructure generalizes Soft HyperStructure). *Suppose all operations act at level* 0, *i.e.* $\ell_{j,k} = r_j = 0$ *for all* j, k. *Let* $f_j := F_j \restriction_{H^{\langle 0 \rangle}} : H^{n_j} \to \mathcal{P}^*(H)$ *and* $(H, \{f_j\}_{j \in J})$ *be the underlying hyperstructure. Then* S *is a Soft SuperHyperStructure over* $\mathbf{SH}(H, \mathcal{F})$ *iff the projected soft set* $S^{(0)} : A \to \mathcal{P}(H)$ *given by* $S^{(0)}(a) := S^{\langle 0 \rangle}_a$ *is a* Soft HyperStructure *over* $(H, \{f_j\})$, *i.e.*

$$\forall a \in \text{Supp}(S),\ \forall j \in J: \quad f_j(S^{(0)}(a), \ldots, S^{(0)}(a)) \subseteq S^{(0)}(a).$$

Proof. ($\Rightarrow$) With $\ell_{j,k} = r_j = 0$, (2) gives $F_j(S^{\langle 0 \rangle}_a, \ldots, S^{\langle 0 \rangle}_a) \subseteq S^{\langle 0 \rangle}_a$. Since F_j on level 0 equals f_j (by definition), this is the soft hyperstructure closure for $S^{(0)}$.
($\Leftarrow$) Conversely, assuming $S^{(0)}$ satisfies the soft hyperstructure closure for all f_j, we have $f_j(S^{\langle 0 \rangle}_a, \ldots, S^{\langle 0 \rangle}_a) \subseteq S^{\langle 0 \rangle}_a$, which is precisely (2) in the present (level-0) typing. Thus S is a Soft SuperHyperStructure. □

Theorem 3.25 (SuperHyperStructures embed into Soft SuperHyperStructures). *Let* $\mathbf{SH}(H, \mathcal{F})$ *be a SuperHyperStructure. Fix a singleton parameter set* $A = \{\bullet\}$ *and define*

$$S(\bullet)^{\langle m \rangle} := H^{\langle m \rangle} \quad (\textit{for all } m).$$

Then S *is a Soft SuperHyperStructure over* $\mathbf{SH}(H, \mathcal{F})$. *Moreover, the* collapse *functor*

$$\mathfrak{C} : \mathbf{SSH}(H, \mathcal{F}; S) \longrightarrow \mathbf{SH}(H, \mathcal{F}), \qquad \mathfrak{C} \text{ “forgets } A \text{ and keeps the underlying operations”},$$

recovers $\mathbf{SH}(H, \mathcal{F})$ *(i.e. is identity-on-operations).*

Proof. For every $j \in J$ and all inputs $X_k \subseteq S(\bullet)^{\langle \ell_{j,k} \rangle} = H^{\langle \ell_{j,k} \rangle}$, the output satisfies $F_j(X_1, \ldots, X_{n_j}) \subseteq H^{\langle r_j \rangle} = S(\bullet)^{\langle r_j \rangle}$, so (2) holds. The collapse functor simply discards the (trivial) soft parameterization and leaves the operations $\{F_j\}$ untouched, yielding the original $\mathbf{SH}(H, \mathcal{F})$. □

3.3. Functorial HyperStructure

A *Functorial HyperStructure* encodes hyperoperations as natural transformations between functors $U^{\times n}$ and $V \circ U$, preserving categorical structure and morphism compatibility.

Notation 3.26 (Underlying and value functors). *Let* $\mathcal{C}$ *be a category. Fix a (covariant) functor*

$$U : \mathcal{C} \longrightarrow \mathbf{Set}$$

that assigns to every object X *its underlying set* $U(X)$. *Let* $V : \mathbf{Set} \to \mathbf{Set}$ *be another (covariant) endofunctor (the* value functor*). For* $n \in \mathbb{N}$, *write* $U^{\times n} : \mathcal{C} \to \mathbf{Set}$ *for the functor* $X \mapsto U(X)^n$.

Definition 3.27 (Functorial hyperoperation and Functorial HyperStructure). *A* Functorial n-ary hyperoperation over (U, V) *is a natural transformation*

$$\Phi: \; U^{\times n} \implies V \circ U,$$

i.e. for each $X \in \mathrm{Ob}(\mathcal{C})$ *a map* $\Phi_X : U(X)^n \to V(U(X))$ *such that for every* $f : X \to Y$ *in* $\mathcal{C}$ *the naturality square*

$$\begin{array}{ccc} U(X)^n & \xrightarrow{\Phi_X} & V(U(X)) \\ {\scriptstyle U(f)^{\times n}}\downarrow & & \downarrow \quad (U(f))V \\ U(Y)^n & \xrightarrow{\Phi_Y} & V(U(Y)) \end{array}$$

commutes. A Functorial HyperStructure (FHS) *on* $(\mathcal{C}; U, V)$ *is a family* $\mathbf{F} = \{\Phi_j : U^{\times n_j} \Rightarrow V \circ U\}_{j \in J}$ *of such natural transformations (finite signature allowed).*

Remark 3.28 (Reading the outputs). *When* $V = \mathcal{P}^*$ *(nonempty powerset),* $\Phi_X(\vec{x}) \in \mathcal{P}^*(U(X))$ *is a genuine* hyper *output; when* $V = \mathrm{Id}_{\mathbf{Set}}$, $\Phi_X(\vec{x}) \in U(X)$ *is a usual (single-valued) operation. Other choices of* V *encode fuzzy, rough, or soft outputs (see Theorems below).*

Example 3.29 (Functorial HyperStructure on FinSet). *Let* $\mathcal{C} = \mathit{FinSet}$, $U = \mathrm{Id}_{\mathcal{C}}$, $V = \mathcal{P}^*$. *Define a natural transformation (functorial hyperoperation)*

$$\Phi: \; U^{\times 2} \Rightarrow V \circ U, \qquad \Phi_X(x, y) = \{x, y\} \in \mathcal{P}^*(X).$$

Naturality. *For* $f : X \to Y$,

$$(VU(f))\Phi_X(x, y) = f[\{x, y\}] = \{f(x), f(y)\} = \Phi_Y(U(f)(x), U(f)(y)).$$

Hence $(\mathcal{C}; U, V, \Phi)$ *is a* Functorial HyperStructure.

Theorem 3.30 (FHS generalizes Functorial Structure). *Given a functor* $F : \mathcal{C} \to \mathbf{Set}$ *(Definition 1.22), there is a canonical FHS whose "structures over X" are exactly elements of* $F(X)$.

Proof. Take $U := \mathrm{Id}_{\mathcal{C}}$ and $V := F$. A choice of $s \in F(X)$ is the same as a component of a *nullary* natural transformation $\sigma : \mathbf{1} \Rightarrow F$ (where $\mathbf{1}$ is the terminal functor), with $\sigma_X(*) = s$. Thus a Functorial HyperStructure with a single nullary operation σ reproduces the notion of an F-structure on X. □

Theorem 3.31 (FHS generalizes hyperstructures). *Let* $(H, \{f_j : H^{n_j} \to \mathcal{P}^*(H)\})$ *be a hyperstructure. Set* $\mathcal{C} := \mathbf{1}$ *(one-object category), choose* $U(*) = H$, *and* $V := \mathcal{P}^*$. *For each* j, *define* Φ_j *by the single component* $\Phi_{j,*} : H^{n_j} \to \mathcal{P}^*(H)$, $\Phi_{j,*} = f_j$. *Then* $\{\Phi_j\}_{j \in J}$ *is an FHS, and this assignment is an isomorphism of data.*

Proof. With $\mathcal{C} = \mathbf{1}$ there are no nontrivial morphisms, so naturality is automatic. The componentwise identification $\Phi_{j,*} = f_j$ gives a bijection between signatures. □

Notation 3.32 (Rough-pair functor). *For a set* Y *equipped with a fixed reflexive relation* $R_Y \subseteq Y \times Y$, *write* $\mathsf{RP}(Y) := \{(L, U) \mid L \subseteq U,\ \ell_{R_Y}(L) = L,\ u_{R_Y}(U) = U\}$. *For a map* $g : Y \to Y'$ *define*

$$g_\sharp : \; \mathsf{RP}(Y) \longrightarrow \mathsf{RP}(Y'), \qquad (L, U) \longmapsto \big(\, \ell_{R_{Y'}}(g[L]),\ u_{R_{Y'}}(g[U]) \,\big).$$

This makes $V_{\mathrm{rough}} : \mathbf{Set} \to \mathbf{Set}$, $Y \mapsto \mathsf{RP}(Y)$, *a functor (naturality follows from functoriality of direct image and idempotence/monotonicity of* ℓ, u*).*

Theorem 3.33 (FHS generalizes Rough HyperStructure). *Let $(H, \{\widehat{f_j}\})$ be a rough hyperstructure on the rough pairs of H (e.g. as constructed from a base hyperstructure via lower/upper approximation). Set $\mathcal{C} := \mathbf{1}$, $U(*) = H$, and $V := V_{\text{rough}}$. Defining $\Phi_{j,*} : H^{n_j} \to \mathsf{RP}(H)$ by $\Phi_{j,*} = \widehat{f_j}$ yields an FHS that is equivalent to the given Rough HyperStructure.*

Proof. Again, with $\mathcal{C} = \mathbf{1}$, naturality is vacuous. The component $\Phi_{j,*}$ is exactly the lifted rough operation $\widehat{f_j}$, so the structures coincide. □

Notation 3.34 (Fuzzy-set functor). *Let $V_{\text{fuz}} : \mathbf{Set} \to \mathbf{Set}$ be the functor $Y \mapsto [0,1]^Y$, and for $g : Y \to Y'$ define the Zadeh pushforward $(g_\sharp \mu)(y') := \sup\{\mu(y) : g(y) = y'\}$.*

Theorem 3.35 (FHS generalizes Fuzzy HyperStructure). *Given a fuzzy hyperstructure $(H, \{\star_j\})$ with $\star_j : H^{n_j} \to [0,1]^H$, choose $\mathcal{C} := \mathbf{1}$, $U(*) = H$, and $V := V_{\text{fuz}}$. Set $\Phi_{j,*} = \star_j$. Then $\{\Phi_j\}$ is an FHS equivalent to the given fuzzy hyperstructure.*

Proof. Identical to Theorem 3.31, replacing $\mathcal{P}^*$ by $[0,1]^{(-)}$. □

Notation 3.36 (Soft-set functor (fixed parameter set $A \neq \emptyset$)). *Define $V_{\text{soft}}(Y) := (\mathcal{P}^*(Y))^A$ and for $g : Y \to Y'$ set $(g_\sharp F)(a) := g[F(a)]$ $(a \in A)$.*

Theorem 3.37 (FHS captures Soft HyperStructure). *Let $(H, \{f_j\})$ be a hyperstructure and (F, A) a soft set with each $F(a) \subseteq H$ a subhyperstructure. Consider $\mathcal{C} := \mathbf{1}$, $U(*) = H$, $V := V_{\text{soft}}$, and the FHS consisting of:*

- *the original hyperoperations encoded as $\Phi_j^{\text{hyp}} : H^{n_j} \to \mathcal{P}^*(H)$ via $V = \mathcal{P}^*$ (as in Theorem 3.31); and*
- *for each $a \in A$, a* nullary natural transformation *$\sigma_a : \mathbf{1} \Rightarrow V_{\text{soft}} \circ U$ with $\sigma_{a,*}(*) = S_a$, where $S_a \in (\mathcal{P}^*(H))^A$ is the "Dirac" soft set selecting $F(a)$ in the a-th coordinate and H elsewhere.*

Then the soft-closure condition "$F(a)$ is a subhyperstructure for all a" is equivalent to the family of FHS-axioms

$$\Phi_{j,*}^{\text{hyp}}(F(a), \ldots, F(a)) \subseteq F(a) \qquad \text{for all } j \in J,\ a \in A.$$

Proof. ($\Rightarrow$) If $F(a)$ is a subhyperstructure, closure under each f_j gives the displayed inclusion, which is exactly the compatibility between Φ_j^{hyp} and the constant soft selections σ_a. ($\Leftarrow$) Conversely, the inclusions state precisely that each $F(a)$ is closed under all f_j, i.e. each $F(a)$ is a subhyperstructure. □

Remark 3.38 (Unifying view). *Definitions 3.27 and Theorems 3.30–3.37 show that by an appropriate choice of the value functor V and of nullary/finite-arity natural transformations,* Functorial HyperStructures *subsume:*

- *ordinary functorial structures ($V = F$, nullary σ);*
- *crisp hyperstructures ($V = \mathcal{P}^*$);*
- *rough hyperstructures ($V = \mathsf{RP}(-)$);*
- *fuzzy hyperstructures ($V = [0,1]^{(-)}$);*
- *soft hyperstructures ($V = (\mathcal{P}^*(-))^A$ plus nullary selectors).*

Functoriality packages the pushforward of structure along morphisms of $\mathcal{C}$ as the naturality of the operations.

3.4. Functorial SuperHyperStructure

A *Functorial SuperHyperStructure* is a family of multi-level hyperoperations encoded as natural transformations between functors, unifying hierarchical and categorical structure.

Notation 3.39 (Level functors and value functors). *Let $\mathcal{C}$ be a category. A* level system *on $\mathcal{C}$ is a family of covariant functors*

$$\mathcal{U} := \{\, U^{\langle m \rangle} : \mathcal{C} \to \mathbf{Set} \;\big|\; m \in \mathbb{N}_0 \,\},$$

and a value system *is a family of endofunctors on* **Set**

$$\mathcal{V} := \{\, V^{\langle m \rangle} : \mathbf{Set} \to \mathbf{Set} \;\mid\; m \in \mathbb{N}_0 \,\}.$$

For a multiindex $\tau = (\ell_1, \ldots, \ell_n | \, r)$ *(input levels* ℓ_k *and output level* r*), write*

$$U^{\tau} \;:=\; \prod_{k=1}^{n} U^{\langle \ell_k \rangle} \quad \text{and} \quad W^{\tau} \;:=\; V^{\langle r \rangle} \circ U^{\langle r \rangle}.$$

Definition 3.40 (Typed functorial superhyperoperation). *Fix a type* $\tau = (\ell_1, \ldots, \ell_n \mid r)$*. A* typed functorial superhyperoperation of type τ *is a natural transformation*

$$\Phi : \; U^{\tau} \implies W^{\tau},$$

i.e. for every $X \in \mathrm{Ob}(\mathcal{C})$ *a function* $\Phi_X : \prod_{k=1}^{n} U^{\langle \ell_k \rangle}(X) \to V^{\langle r \rangle}(U^{\langle r \rangle}(X))$ *such that for every* $f : X \to Y$ *in* $\mathcal{C}$*, the square*

$$\begin{array}{ccc} \prod_k U^{\langle \ell_k \rangle}(X) & \xrightarrow{\Phi_X} & V^{\langle r \rangle}(U^{\langle r \rangle}(X)) \\ {\scriptstyle \prod_k U^{\langle \ell_k \rangle}(f)} \downarrow & & \downarrow \quad {}^{\langle r \rangle}(U^{\langle r \rangle}(f))V \\ \prod_k U^{\langle \ell_k \rangle}(Y) & \xrightarrow{\Phi_Y} & V^{\langle r \rangle}(U^{\langle r \rangle}(Y)) \end{array}$$

commutes.

Definition 3.41 (Functorial SuperHyperStructure (FSS)). *A* Functorial SuperHyperStructure *on* $(\mathcal{C}; \mathcal{U}, \mathcal{V})$ *is a finite signature* $\Sigma = \{\tau_j\}_{j \in J}$ *of types together with a family*

$$\mathbf{FSH} \;:=\; \{\, \Phi_j : U^{\tau_j} \Rightarrow W^{\tau_j} \;\mid\; j \in J \,\}$$

of typed functorial superhyperoperations. For each $X \in \mathrm{Ob}(\mathcal{C})$*, the* fiber at X *consists of the components* $\{\, (\Phi_j)_X \;:\; j \in J \,\}$.

Remark 3.42 (Reading classical cases from $\mathcal{V}$). *Typical choices of value functors* $V^{\langle m \rangle}$ *include:*

- Crisp hyper: $V^{\langle m \rangle} = \mathcal{P}^*$ *(nonempty powerset).*
- Fuzzy: $V^{\langle m \rangle}(Y) = [0,1]^Y$ *with pushforward* $(g_\sharp \mu)(y') = \sup\{\mu(y) : g(y) = y'\}$.
- Rough: $V^{\langle m \rangle}(Y) = \mathrm{RP}_{R_Y^{\langle m \rangle}}(Y)$ *(rough pairs for a fixed reflexive* $R_Y^{\langle m \rangle}$*) with* $g_\sharp(L, U) := (\ell_{R_{Y'}^{\langle m \rangle}}(g[L]),\, u_{R_{Y'}^{\langle m \rangle}}(g[U]))$.
- Soft (fixed $A \neq \emptyset$): $V^{\langle m \rangle}(Y) = (\mathcal{P}^*(Y))^A$ *with* $(g_\sharp F)(a) := g[F(a)]$.

All are functorial (composition and identities are preserved).

Example 3.43 (Functorial SuperHyperStructure on FinSet). *Let* $\mathcal{C} = \mathit{FinSet}$*,* $U^{\langle 0 \rangle} = \mathrm{Id}_{\mathcal{C}}$ *and* $U^{\langle 1 \rangle} = \mathcal{P}^*$ *(nonempty powerset functor). Set* $V^{\langle 1 \rangle} = \mathcal{P}^*$ *and consider the type* $\tau = (1, 1 \mid 1)$*. Define a natural transformation (typed functorial superhyperoperation)*

$$\Phi : \; U^{\langle 1 \rangle} \times U^{\langle 1 \rangle} \Longrightarrow V^{\langle 1 \rangle} \circ U^{\langle 1 \rangle}, \qquad \Phi_X(A, B) := \{\, A \cup B \,\}.$$

Naturality. *For* $f : X \to Y$*,*

$$\big(V^{\langle 1 \rangle} U^{\langle 1 \rangle}(f)\big)\Phi_X(A, B) = \{\, f[A \cup B] \,\} = \{\, f[A] \cup f[B] \,\} = \Phi_Y\big(U^{\langle 1 \rangle}(f)(A), U^{\langle 1 \rangle}(f)(B)\big).$$

Thus $(\mathcal{C}; \{U^{\langle m \rangle}\}, \{V^{\langle m \rangle}\}, \{\Phi\})$ *is a* Functorial SuperHyperStructure.

Theorem 3.44 (Reduction to Functorial Structure)**.** *Let $F : \mathcal{C} \to \mathbf{Set}$ be a covariant functor (Definition 1.22). Choose one level and no operations:*

$$\mathcal{U} = \{U^{\langle 0 \rangle} := \mathrm{Id}_{\mathcal{C}}\}, \qquad \mathcal{V} = \{V^{\langle 0 \rangle} := F\}, \qquad \Sigma = \varnothing.$$

Then the FSS data $(\mathcal{C}; \mathcal{U}, \mathcal{V}, \Sigma)$ carries exactly the same information as the functor F; in particular, the fiber at X is $F(X)$.

Proof. By definition $W^{\langle 0 \rangle} = V^{\langle 0 \rangle} \circ U^{\langle 0 \rangle} = F$. With $\Sigma = \varnothing$ (no operations), the only remaining data is the functor F. □

Theorem 3.45 (Reduction to SuperHyperStructure)**.** *Let $\mathbf{SH}(H, \mathcal{F})$ be a (typed) SuperHyperStructure on a single set H, with levels $H^{\langle 0 \rangle} = H$ and $H^{\langle m+1 \rangle} = \mathcal{P}^*(H^{\langle m \rangle})$, and operations $F_j : \prod_k H^{\langle \ell_{j,k} \rangle} \to \mathcal{P}^*(H^{\langle r_j \rangle})$. Take $\mathcal{C} := \mathbf{1}$ (one-object category), define*

$$U^{\langle m \rangle}(*) = H^{\langle m \rangle}, \qquad V^{\langle m \rangle} := \mathcal{P}^*,$$

and let Φ_j be the unique natural transformations with components $(\Phi_j)_* = F_j$. Then $(\mathcal{C}; \mathcal{U}, \mathcal{V}, \{\Phi_j\})$ is an FSS whose fiber at $*$ is exactly $\mathbf{SH}(H, \mathcal{F})$.

Proof. Naturality is automatic in $\mathbf{1}$. The components reproduce the given F_j. □

Theorem 3.46 (Reduction to Rough SuperHyperStructure)**.** *Let $\mathbf{RSH}(H, \mathcal{F}; \mathcal{R})$ be a Rough SuperHyperStructure with levelwise reflexive relations $\mathcal{R} = \{R^{\langle m \rangle}\}$ and lifted operations $\widehat{F}_j$ on rough pairs. For $\mathcal{C} := \mathbf{1}$ set*

$$U^{\langle m \rangle}(*) = H^{\langle m \rangle}, \qquad V^{\langle m \rangle}(Y) = \mathsf{RP}_{R_Y^{\langle m \rangle}}(Y),$$

and define Φ_j by $(\Phi_j)_* = \widehat{F}_j$. Then the resulting FSS is (componentwise) identical to $\mathbf{RSH}(H, \mathcal{F}; \mathcal{R})$.

Proof. Same as Theorem 3.45, using the rough-pair value functors. □

Theorem 3.47 (Reduction to Fuzzy SuperHyperStructure)**.** *Let $\mathbf{FuSH}(H, \mathcal{F})$ be a fuzzy superhyperstructure with operations $F_j^{\mathrm{fuz}} : \prod_k H^{\langle \ell_{j,k} \rangle} \to [0,1]^{H^{\langle r_j \rangle}}$. With $\mathcal{C} := \mathbf{1}$, take $U^{\langle m \rangle}(*) = H^{\langle m \rangle}$ and $V^{\langle m \rangle}(Y) = [0,1]^Y$, and set $(\Phi_j)_* = F_j^{\mathrm{fuz}}$. Then we obtain an FSS whose fiber equals $\mathbf{FuSH}(H, \mathcal{F})$.*

Proof. Identical to Theorem 3.45, now with fuzzy value functors. □

Theorem 3.48 (Reduction to Functorial HyperStructure)**.** *Let $\mathbf{FH}(\mathcal{C}; U, V, \{\Phi_j\})$ be a Functorial HyperStructure (single level) with $\Phi_j : U^{\times n_j} \Rightarrow V \circ U$. Choose the one-level systems $\mathcal{U} = \{U^{\langle 0 \rangle} := U\}$ and $\mathcal{V} = \{V^{\langle 0 \rangle} := V\}$, and keep the same family $\{\Phi_j\}$. Then $(\mathcal{C}; \mathcal{U}, \mathcal{V}, \{\Phi_j\})$ is an FSS whose data coincide with* **FH**.

Proof. This is Definition 3.40 specialized to a single level. □

Theorem 3.49 (Reduction to Soft SuperHyperStructure)**.** *Let $\mathbf{SSH}(H, \mathcal{F}; S)$ be a Soft SuperHyperStructure with parameter set $A \neq \varnothing$ and per-parameter sub-superhyperstructures $S_a^{\langle m \rangle} \subseteq H^{\langle m \rangle}$ satisfying the closure condition*

$$F_j\big(S_a^{\langle \ell_{j,1} \rangle}, \ldots, S_a^{\langle \ell_{j,n_j} \rangle}\big) \subseteq S_a^{\langle r_j \rangle} \qquad (\forall j,\ \forall a \in A).$$

Fix $\mathcal{C} := \mathbf{1}$, set $U^{\langle m \rangle}(*) = H^{\langle m \rangle}$ and $V^{\langle m \rangle}(Y) = (\mathcal{P}^*(Y))^A$, and let $(\Phi_j)_*$ be the coordinatewise images of F_j under the inclusion into the A-indexed product. For each $a \in A$ introduce a nullary *natural transformation*

$\sigma_a : \mathbf{1} \Rightarrow V^{\langle m \rangle} \circ U^{\langle m \rangle}$ with component $(\sigma_a)_*(*) = (S_a^{\langle m \rangle})_{m \geq 0}$. Then $(\mathcal{C}; \mathcal{U}, \mathcal{V}, \{\Phi_j\} \cup \{\sigma_a\}_{a \in A})$ is an FSS, and the soft-closure condition above is equivalent to the FSS equations

$$(\Phi_j)_*\big(\underbrace{\sigma_a, \ldots, \sigma_a}_{n_j}\big) \preceq \sigma_a \quad \text{(coordinatewise in } A \text{ and levelwise in } m\text{)}.$$

Consequently, $\mathbf{SSH}(H, \mathcal{F}; S)$ *is (componentwise) recovered from this FSS.*

Proof. In $\mathbf{1}$, naturality is trivial. The inequality displayed is exactly the statement that applying F_j to the selected subsets $S_a^{\langle \ell_{j,k} \rangle}$ lands inside $S_a^{\langle r_j \rangle}$ for each a, which is the definition of soft closure. Conversely, those inclusions reproduce the soft structure. □

4. Conclusion

This paper has examined several extended variants of the classical SuperHyperStructure, including *Rough*, *Soft*, *Fuzzy*, and *Functorial SuperHyperStructures*. It is our hope that future work will investigate practical applications of these concepts in real-world contexts, as well as their deeper mathematical properties, accompanied by rigorous quantitative analyses. We also envision further studies on possible extensions employing the framework of *Plithogenic Sets* [51–53], thereby enriching the theoretical landscape and expanding potential applications.

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Disclaimer: The ideas presented here are theoretical and have not yet been validated through empirical testing. While we have strived for accuracy and proper citation, inadvertent errors may remain. Readers should verify any referenced material independently. The opinions expressed are those of the authors and do not necessarily reflect the views of their institutions.

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