Article

Not peer-reviewed version

Beyond Additivity and Extensivity of Entropy for Black Hole and Cosmological Horizons

Mariusz P. Dąbrowski *

Posted Date: 8 August 2024

doi: 10.20944/preprints202408.0564.v1

Keywords: thermodynamics; entropy; additivity; extensivity; long-range interactions; horizons

Article

Beyond Additivity and Extensivity of Entropy for Black Hole and Cosmological Horizons

Mariusz P. Dąbrowski [1,2,3]

[1] Institute of Physics, University of Szczecin, Wielkopolska 15, 70-451 Szczecin, Poland; Mariusz.Dabrowski@usz.edu.pl
[2] National Centre for Nuclear Research, Andrzeja Sołtana 7, 05-400 Otwock, Poland;
[3] Copernicus Center for Interdisciplinary Studies, Szczepańska 1/5, 31-011 Kraków, Poland

Abstract: The extensions of Boltzmann-Gibbs classical entropic form are presented. The focus is on the thermodynamical entropies which do not obey classical definition of additivity and extensity with main motivation coming from the gravitational systems admitting long-range interactions both in astrophysics (black holes) and in cosmology. Horizon entropies are the main concern. In the paper, the plethora of non-Boltzmann-Gibbs entropies are studied and the comparative analysis of their properties with respect to additivity and extensivity is presented.

Keywords: thermodynamics; entropy; additivity; extensivity; long-range interactions; horizons

---

1. Introduction

It is widely known that the Boltzmann-Gibbs thermodynamics (from now on BG) and statistical mechanics are additive and extensive [1]. The core physical quantity which relates to these theories is the entropy which is assumed to be extensive since it relates to the negligence of the long-range forces between thermodynamic sub-systems. This assumption is justified only when the size of the system exceeds the range of the interaction between its components. As a result, the total entropy of a composite system is equal to the sum of the entropies of the individual subsystems (additivity) and the entropy grows with the size of the system or its configuration space (extensivity).

However, contemporary physics exhibits a number of systems for which the long-range forces are important. The very examples of such systems are gravitational systems since gravity is long-range interactive, and besides, it is strongly non-linear when its extreme regimes are taken into account. Strong gravity characterises all the compact astrophysical objects in the Universe like white dwarfs, neutron and boson stars, quark stars etc. with the most extreme and most intriguing - the black holes. The latter are surrounded by the horizons to which we can associate entropy and formulate the appropriate thermodynamics according to early considerations by Bekenstein and Hawking [2,3]. Black holes can be associated with Bekenstein entropy which *scales with the area* and not with the volume (size), and is consequently a nonextensive quantity [4–9]. In addition to that, because of a long-range interaction nature of gravity, Bekenstein entropy is also nonadditive.

In fact, the number of nonadditive and/or nonextensive thermodynamics have been proposed in the literature [10–16] – some of them applied to gravitational systems both in astrophysics and in cosmology [17,18].

In this paper, we explore the topic of additivity and extensivity of entropies which go beyond the standard BG thermodynamics being strongly motivated by gravitational interaction. Our focus will be put on non-standard, but better fitting to gravitational systems entropies, such as: Bekenstein entropy [2,3], Tsallis q-entropy [10,19], Tsallis-Cirto δ-entropy [4], Barrow Δ-entropy [16], Tsallis q,δ entropy, Rényi entropy [11], Landsberg U-entropy [20], Sharma-Mittal entropy [12,13] , and Kaniadakis entropy [14,15] .

The following is the outline of the paper. In Sec. 2, we define additivity and extensivity in the context of the BG thermodynamics. In Sec. 3 we go beyond the definitions of additivity and extensivity in thermodynamics. In Sec. 4 we introduce the plethora of nonadditive and/or nonextensive entropies together with accompanying nonadditive and/or nonextensive thermodynamical quantities and make the classification of the entropies under study with respect to additivity and extensivity properties as well as with the application of the appropriate generalized additivity rules. Finally, in Sec. 5, we summarize the paper.

2. Boltzmann-Gibbs Thermodynamics and Statistical Mechanics

Boltzmann-Gibbs thermodynamics and statistical mechanics are based on two key hypotheses which are that the entropy is extensive and that the internal energy and entropy follow the additive composition rule. All thermodynamic relations in BG statistical mechanics are defined in light of these conditions, which in fact rely on ignoring long-range forces between thermodynamic subsystems.

The Boltzmann-Gibbs (BG) entropy is defined as [1]

$$S_{BG} = -k_B \sum_{i=1}^{n} p_i \ln p_i = k_B \sum_{i=1}^{n} p_i \ln \frac{1}{p_i}. \quad (1)$$

where p_i is the probability distribution defined on a configuration space Ω with the number of degrees of freedom (states) n, k_B is the Boltzmann constant, and the condition that the total probability must be equal to one $\sum p_i = 1$ is fulfilled. For the case of all probabilities equal, i.e. for $p_i = \text{const.} = p$, we get

$$\sum_{i=1}^{n} p_i = 1 = np \quad \Rightarrow p = 1/n. \quad (2)$$

After applying (2) to (1), one obtains that

$$S_{BG} = k_B \ln n, \quad (3)$$

which means that the entropy is proportional to the number of states n in the configuration space Ω.

In view of the key properties of BG thermodynamics and in the context of our investigations beyond these properties we will define additivity and extensivity in quite a general way following some literature [20–22] as below.

2.1. *Additivity*

Additivity means that for a given physical or thermodynamical quantity f, the following composition rule is fulfilled:

$$f(A+B) = f(p_{A\cup B}) = f(p_A p_B) = f(p_A) + f(p_B) = f(A) + f(B), \quad (4)$$

where A, B are independent subsystems, equipped with the sets of configuration space degrees of freedom Ω_A and Ω_B, and corresponding probabilities p_A and p_B. The composite system $A \cup B$ allows the probability $p_{A\cup B}$ and it is equipped with the set of configuration space degrees of freedom $\Omega_{A\cup B}$. If the subsystems A and B are assumed to be independent, then it happens that the probabilities are related by $p_{A\cup B} = p_A p_B$ which allows the transition leading to the additivity rule (4) [23].

If a particular case of the entropy is taken into account, then (4) reads

$$S(A+B) = S(A) + S(B) \quad (5)$$

2.2. *Extensivity*

Let us assume that there is a set of physical quantities $(X_0, X_1, X_2, \ldots, X_k)$ such that $X_0 = f(X_1, X_2, \ldots, X_k)$. Extensivity of a selected physical quantity means that the function f which describes this quantity is *homogeneous degree one* [1,20,21] i.e. that

$$f(aX_1, aX_2, ..., aX_k) = af(X_1, X_2, ..., X_k) \quad (6)$$

for every positive real number $a > 0$, for all $X_1, X_2, ...X_k$. Taking $k = 3$, so that we have only four quantities X_0, X_1, X_2, X_3, and assuming that they are the entropy S, the energy E, the volume V, and the mole number N accordingly, we can obtain the standard Boltzmann-Gibbs thermodynamical extensivity relation for entropy [20]

$$S(aE, aV, aN) = aS(E, V, N). \tag{7}$$

In fact, the property (6) is called 'homogeneity' and is considered the most general definition of extensivity (cf. [20]).

In standard textbooks of thermodynamics one commonly uses less general definition of an extensive quantity which says that if a system's total number of states in the configuration space Ω is proportional to its number of degrees of freedom, then this quantity (such as the entropy, for example) is extensive. For BG entropy, as we have shown in (3), one has that $S_{BG}(n) = k_B \ln(n) \propto n$, where n is the total number of states in the system.

The advantage of definition (6) is that one does not refer to any kind of geometrical or bulk properties of a system such as the 'size', though the geometrical size of a system seems intuitively to be related to the number of states or degrees of freedom.

2.3. Concavity

Concavity is the feature of the functions which read as [4,20]

$$f(ax + (1-a)y) \geq af(x) + (1-a)f(y) \quad (a > 0). \tag{8}$$

In the context of thermodynamics concavity of entropy guarantees that the system in thermodynamic equilibrium is *stable*.

3. Beyond Boltzmann-Gibbs Thermodynamics

3.1. Beyond Additivity

Additivity is violated if the rule (5) does not hold. In such a case, one can have two options [20]. The first one is when

$$S(A + B) \geq S(A) + S(B), \tag{9}$$

which is called *superadditivity*, and it leads to the tendency of the system to clump its pieces/subsystems. The second one is when

$$S(A + B) < S(A) + S(B), \tag{10}$$

which is *subadditivity*, and it tends to fragment the system into pieces rather than clump. A cosmological similarity of such a system is phantom [24,25] since it splits spontaneously into pieces under (anti)gravity beginning with the largest size objects and terminating at the smallest [26].

In the literature, there are a number of rules for nonadditivity which we introduce later. One of them, which generalizes the additive composition rule (5) into a nonadditive case is the Abé rule [27–29]. If applied to entropy, it reads as follows

$$S(A + B) = S(A) + S(B) + \frac{\Upsilon}{k_B} S(A) S(B), \tag{11}$$

where Υ takes numerical values according to a statistical definition of a specific entropy type which we will present later. For BG entropy, one just has $\Upsilon = 0$. With the assumption that all the entropies in (11) are positive, one deals with superadditivity for $\Upsilon \geq 0$ and with subadditivity for $\Upsilon < 0$, In fact, the physical interpretation of Υ is the result of the long-range interactions between subsystems which leads to nonadditivity.

3.2. Beyond Extensivity

In BG thermodynamics additivity and extensivity are closely related - additivity implies extensivity and extensivity implies additivity [1]. This is not the case in general and so extensivity and additivity may not be related, i.e. extensivity may not imply additivity and vice verse. An example of such a kind of a quantity which is based on the definition (6) is given by the function $f(X_1, X_2) = x_1^2/\sqrt{X_1^2 + X_2^2}$ which fulfils extensivity, but not additivity [20].

Generally, the entropy S is nonextensive if

$$S(aX) \neq aS(X), \tag{12}$$

where X is a thermodynamical quantity and $a > 0$, i.e. when the relations (6) and (7) are violated.

4. The Plethora of Nonextensive Entropies

4.1. Bekenstein Entropy

Bekenstein entropy is not motivated by anything like statistical mechanics, but it is well established in gravity [2]. For a Schwarzschild black hole it reads

$$S_{Bek} = 4\pi k_B \left(\frac{M}{m_p}\right)^2 = \frac{4\pi k_B G M^2}{\hbar c} \tag{13}$$

and it is usually presented with its accompanying Hawking temperature which reads

$$T_H = \frac{\hbar c^3}{8\pi G k_B M}, \tag{14}$$

where M is the mass of a black hole, c is the speed of light, G is the gravitational constant, $\hbar$ is the reduced Planck constant, and m_p is the Planck mass. In fact, the temperature (14) can be calculated from the entropy (13) by applying the Clausius formula

$$\frac{k_B}{T} = \frac{\partial S}{\partial E} \tag{15}$$

and using the Einstein mass-energy equivalence $E = Mc^2$.

It is not always understood in the literature that because of the area rather than volume scaling, Bekenstein entropy is nonextensive and obeys the following nonadditive composition rule (see e.g. [30])

$$S_{A+B} = S_A + S_B + 2\sqrt{S_A}\sqrt{S_B}, \tag{16}$$

which we will call *square root rule* from now on. This rule comes directly as a consequence of the formula (13) according to which the entropy $S_{Bek} \propto M^2$, so that $S_A \propto M_A^2$ and $S_B \propto M_B^2$. If black holes merge in an adiabatic way, then their mass after merging is the sum $M_{A+B} = M_A + M_B$, but the entropy $S_{A+B} \propto M_{A+B}^2$, giving an extra term $1/2 M_A M_B$ which is an extra nonadditive term in (16).

Curiously, after making a redifinition of the Bekenstein entropies as $\tilde{S}_{A+B} \equiv \sqrt{S_{A+B}} = M_{A+B} = M_A + M_B$, $\sqrt{S_A} = M_A \equiv \tilde{S}_A$, and $\sqrt{S_B} = M_B \equiv \tilde{S}_B$ one can rewrite the rule (16) in an additive way

$$\tilde{S}_{A+B} = \tilde{S}_A + \tilde{S}_B, \tag{17}$$

but this is not of any physical meaning.

In conclusion, Bekenstein entropy addition formula (16) does not fulfil Abé rule (11) though it looks quite similar. We will comment on this point later in relation to Tsallis entropy.

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 8 August 2024 doi:10.20944/preprints202408.0564.v1

4.2. Tsallis q, Tsallis δ and Tsallis q, δ Entropies

4.2.1. Tsallis q-Entropy

Tsallis [10,31] generalized BG entropy for nonextensive systems in order to encompass and address the issue of long-range interaction by introducing a new nonextensivity parameter q ($q \in R$) into the BG entropy definition (1) with the standard BG condition that the sum of all the probabilities is equal to one $\sum p_i = 1$, as follows

$$\mathcal{S}_q = k_B \sum_{i=1}^{n} p_i \ln_q \frac{1}{p_i} = -k_B \sum_{i=1}^{n} (p_i)^q \ln_q p_i = -k_B \sum_{i=1}^{n} \ln_{2-q} p_i, \tag{18}$$

where a newly defined q-logarithmic function $\ln_q p$ was introduced

$$\ln_q p \equiv \frac{p^{1-q} - 1}{1-q} \tag{19}$$

with the limit $q \to 1$ being the standard logarithm $\ln_1 p = \ln p$. It is important to keep in mind that the q-logarithm does not fulfil the standard logarithm addition rule $\ln ab = \ln a + \ln b$, where a, b some numbers. Instead, it fulfils a nonadditive rule given by

$$\ln_q ab = \ln_q a + \ln_q b + (1-q) \ln_q a \ln_q b, \tag{20}$$

which is in fact the origin of the Abé rule (11). Interestingly, be the introduction of specific $q-$product defined as [32]

$$(x \otimes y)_q \equiv \left[x^{1-q} + y^{1-q} - 1\right]^{\frac{1}{1-q}}, \ (x \geq 0, y \geq 0), \tag{21}$$

one can make the rule additive i.e.

$$\ln_q \left[(x \otimes y)_q\right] = ln_q x + \ln_q y. \tag{22}$$

It is also possible to define the q-exponential function

$$e_q^p \equiv [1 + (1-q)p]^{\frac{1}{1-q}}, \tag{23}$$

which in the limit $q \to 1$ gives $e_1^p = e^p$, and also does not fulfil the standard exponential rule $e^{a+b} = e^a e^b$.

The above formula (18) is presented in three equivalent forms. However, using the definition of q-logarithm (19), all of them can be brought into the following shape (cf. Appendix A)

$$\mathcal{S}_q = k_B \frac{1 - \sum_{i=1}^{n} (p_i)^q}{q-1}. \tag{24}$$

It is important to mention that in order to fulfil the requirements of concavity for S_q according to (8), the nonextensivity parameter $q > 0$ [4].

In the limit $q \to 1$, Tsallis entropy S_q given by (18) or (24) reduces to BG entropy (1). After some check, it is possible to find that Tsallis q-entropy (18) or (24) satisfies the nonadditive composition Abé rule (11), if one defines a nonextensivity parameter as $\Upsilon = 1 - q$ (cf. Appendix B). For equal probability states (2), the formula (24) gives the Tsallis q-entropy as

$$\mathcal{S}_q = k_B \ln_q n = k_B \frac{n^{q-1} - 1}{1-q}, \tag{25}$$

which shows how it generalizes BG entropy (3) via the new parameter q.

4.2.2. Tsallis-Cirto δ-Entropy

Tsallis-Cirto δ-entropy [4,9] sometimes also known in the literature as just Tsallis entropy is yet another generalization of BG entropy (1) by introducing a different nonextensivity δ parameter as follows

$$\mathcal{S}_\delta = k_B \sum_{i=1}^{n} p_i (\ln p_i)^\delta \quad (\delta > 0, \delta \in R), \tag{26}$$

and this difference is easily seen when one compares with Tsallis q-entropy (18) and with BG entropy (1). Under the assumption that all the states are equally probable as in (2), one gets from (26) that

$$\mathcal{S}_\delta = k_B (\ln n)^\delta \equiv k_B \ln^\delta n. \tag{27}$$

Making further assumption that we deal with two independent systems A and B fulfilling the condition $n^{A+B} = n^A \cdot n^B$, one realizes that the composition rule for the Tsallis-Cirto entropy (27) reads

$$\left(\frac{S_{\delta,A+B}}{k_B}\right)^{1/\delta} = \left(\frac{S_{\delta,A}}{k_B}\right)^{1/\delta} + \left(\frac{S_{\delta,B}}{k_B}\right)^{1/\delta}, \tag{28}$$

which is another example of nonadditivity rule, *different* from Abé rule (11). We will call it *δ-addition rule* from now on. In fact, Tsallis and Cirto suggest that [4,9]

$$S_\delta = k_B \left(\frac{S_{Bek}}{k_B}\right)^\delta, \tag{29}$$

where S_{Bek} is the Bekenstein entropy (13). According to a new composition rule (28), one realizes that the Bekenstein entropy as given by $S_{Bek} \propto (S_\delta)^{1/\delta}$ can be additive, while the Tsallis-Cirto entropy S_δ itself is nonadditive. Besides, bearing in mind the definition of Bekenstein entropy for a Schwarzschild black hole (13), one can easily notice that for $\delta = 3/2$ the Tsallis-Cirto entropy (29) is proportional to the volume $S_\delta \propto M^3$ and so it is an extensive quantity in view of the standard definition of extensivity.

For the Tsallis-Cirto δ-entropy, it is easy to calculate the corresponding temperature by using the Clausius relation (15) as follows [23,33]

$$T_\delta = \frac{T_H}{\delta}\left(\frac{S_{Bek}}{k_B}\right)^{1-\delta}, \tag{30}$$

which scales with $1/M^2$ for $\delta = 3/2$, i.e., $T_\delta \propto 1/M^2$ for a Schwarzschild black hole.

4.2.3. Tsallis q, δ-Entropy

Tsallis q, δ-entropy generalizes both Tsallis q-entropy (18) and Tsallis-Cirto δ-entropy (26) as follows [4,9]

$$\mathcal{S}_{q,\delta} = k_B \sum_{i=1}^{n} p_i (\ln_q p_i)^\delta \quad (\delta > 0, q \in R, \delta \in R). \tag{31}$$

Now both q and δ play the role of two independent nonextensivity parameters. By assuming that all the states are equally probable as in (2), one gets from (31)

$$\mathcal{S}_{q,\delta} = k_B (\ln_q n)^\delta \equiv k_B \ln_q^\delta n. \tag{32}$$

The Tsallis q, δ-entropy fulfils neither Abé addition rule nor δ-addition rule though it does the former in the limit $\delta \to 0$ and the latter in the limit $q \to 0$.

Table 1 gives the summary of three different Tsallis invented entropies.

Table 1. Tsallis entropies.

Entropy Type	Extensivity	Additivity	Abé addition rule	$\delta-$addition rule
Boltzmann-Gibbs S_{BG}	yes	yes	yes, $\Upsilon = 0$	yes, $\delta = 1$
Tsallis $S_{q,1} = S_q$	no	no	yes, $\Upsilon = 1 - q$	no
Tsallis-Cirto $S_{1,\delta} = S_\delta$	no	no	no	yes
General Tsallis $S_{q,\delta}$	no	no	no	no

4.3. Barrow Fractal Horizon Δ-Entropy and Its Relation to Bekenstein and Tsallis-Cirto δ-Entropy

Barrow entropy [16] has no statistical roots at all. It is closely tied to black hole horizon geometry influenced by quantum fluctuations which make initially smooth black hole horizon a fractal composed of spheres forming the so-called sphereflake. This structure is characterised by the fractal dimension d_f which in the extreme cases is the surface or the volume i.e. $2 \leq d_f \leq 3$, and results in an effective horizon area of r^{d_f}, where r is the black hole horizon radius. After quantum-motivated modification of the area, the entropy reads

$$S_{Bar} = k_B \left(\frac{A}{A_p} \right)^{1+\frac{\Delta}{2}} = k_B \left(\frac{S_{Bek}}{k_B} \right)^{1+\frac{\Delta}{2}}, \tag{33}$$

where S_{Bek} is Bekenstein entropy, A - the horizon area, A_p - the Planck area, $A_p \propto l_p^2$ with l_p - Planck length, and Δ is the parameter related to the fractal dimension by the relation $\Delta = d_f - 2$. In fact, $0 \leq \Delta \leq 1$ with $\Delta \to 1$ limit yielding maximally fractal structure, where the horizon area behaves effectively like a $3-$dimensional volume, and with $\Delta \to 0$ limit yielding the Bekenstein area law, where no fractalization occurs. Although Barrow entropy has geometrical roots, and is not motivated by thermodynamics, it has the same form as Tsallis-Cirto δ entropy (29) [34] being also related to Bekenstein entropy S_{Bek} as in (13), provided that

$$\delta = 1 + \frac{\Delta}{2}. \tag{34}$$

However, the ranges of parameters δ and Δ are different - δ has only the bound $\delta > 0$ while $0 \leq \Delta \leq 1$ is equivalent to $1 \leq \delta \leq 3/2$. Thus, qualitatively, both entropic forms yield the same temperatures as a function of a black hole mass. Both Tsallis-Cirto entropy limit $\delta \to 3/2$ and Barrow limit $\Delta \to 1$ yield an extensive, but still nonadditive entropy for black holes.

4.4. Landsberg U-Entropy

Landsberg U-entropy is defined in relation to Tsallis $q-$entropy (24) as [20]

$$S_U = \frac{k_B}{1-q} \left(1 - \frac{1}{\sum_{i=1}^n (p_i)^q} \right) = k_B \frac{1 - \sum_{i=1}^n (p_i)^q}{q-1} \frac{1}{(p_i)^q} = \frac{S_q}{\sum_{i=1}^n (p_i)^q}, \tag{35}$$

and it fulfils the Abé rule (11) for $\Upsilon = q - 1$ (cf. Appendix B. By assuming that all the states are equally probable as in (2), it simplifies (35) to the form

$$S_U = n^{q-1} S_q, \tag{36}$$

so it simply relates to Tsallis q-entropy.

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 8 August 2024 doi:10.20944/preprints202408.0564.v1

4.5. *Rényi Entropy*

Rényi entropy [11], which is in fact a measure of entanglement in quantum information theory, is additive and preserves event independence. It is another important generalization of BG entropy, which is defined by

$$S_R = k_B \frac{\ln \sum_{i=1}^{n} (p_i)^q}{1-q}. \tag{37}$$

By assuming that all the states are equally probable as in (2), it reads from (37) that

$$S_R = k_B \ln n, \tag{38}$$

which is the same as BG entropy (1).

In fact, Rényi entropy (37) can be written in terms of Tsallis q-entropy by using the formal logarithm approach [35], on the base of which

$$S_R = \frac{k_B}{1-q} \ln[1 + \frac{1-q}{k_B} \mathcal{S}_q]. \tag{39}$$

Quite a unique feature of Rényi entropy is that *it is additive* which comes from some more general Abé composition rule given as [30]

$$H(S_{A+B}) = H(S_A) + H(S_B) + \frac{\Upsilon}{k_B} H(S_A) H(S_B), \tag{40}$$

together with redefinition using the logarithm in the form

$$L(S) = \frac{k_B}{\Upsilon} \ln \left(1 + \frac{\Upsilon}{k_B} H(S)\right) \tag{41}$$

which applied to (39) gives an additive formula

$$L(S_{A+B}) = L(S_A) + L(S_B), \tag{42}$$

where $L(S)$ corresponds to Rényi entropy and $H(S)$ to Tsallis q-entropy. In principle, one can write that Rényi entropy fulfils less general Abé rule (11) with $\Upsilon = 0$.

In fact, S_R is the equilibrium entropy which corresponds to an equilibrium temperature T_R defined from the equilibrium condition by maximizing the Tsallis entropy (18) according to the Clausius formula (15), and is given by [23]

$$T_R = (1 + \frac{1-q}{k_B} S_q) \frac{1}{k_B}. \tag{43}$$

Besides, Rényi entropy can be defined on the horizon of a black hole [5–8,36] by assuming that the Bekenstein entropy (13) is the Tsallis entropy S_q, and replacing energy E with the mass of a black hole M in equations (39) and (43).

4.6. *Sharma-Mittal Entropy*

Sharma-Mittal (SM) entropy [12,37] generalizes both Rényi and Tsallis $q-$entropies, and is defined as

$$S_{SM} = \frac{k_B}{R} \left[\left(\sum_{i=1}^{n} (p_i)^q \right)^{\frac{R}{1-q}} - 1 \right] \tag{44}$$

where R is another dimensionless parameter apart from q. For equally probable states in (2), one gets from (44) that [38]

$$S_{SM} = \frac{k_B}{R}\left\{\left[1 + \frac{1-q}{k_B}S_T\right]^{\frac{R}{1-q}} - 1\right\}, \tag{45}$$

where $R \to 1-q$ limit yields the Tsallis entropy, and $R \to 0$ limit yields Rényi entropy. The SM entropy obeys the same general nonadditive composition rule of Abé (11) for $\Upsilon = 1$ (cf. Appendix C).

4.7. Kaniadakis Entropy

Kaniadakis entropy [14,15,18] results from taking into account Lorentz transformations of special relativity. It is a single K-parameter $(-1 < K < 1)$ deformation of BG entropy (1 with K parameter being connected to the dimensionless rest energy of the various parts of a multibody relativistic system. The basic definition of Kaniadakis entropy which directly generalizes BG entropy reads

$$S_K = -k_B\sum_{i=1}^{n} p_i \ln_K p_i, \tag{46}$$

where p_i is the probability of the system to be in the i-th state and n is the total number of states. The formula (46) introduces the K-logarithm

$$\ln_K x \equiv \frac{x^K - x^{-K}}{2K} = \frac{1}{K}\sinh\left(K\ln x\right) \tag{47}$$

with some simple properties like $\ln_K x^{-1} = -\ln_K x$ and $\ln_{-K} x = \ln_K x$ and it gives the standard logarithm $\ln x$ in the limit $K \to 0$. An equivalent definition of Kaniadakis entropy which can be obtained after the application of $K-$logarithm (47) reads

$$S_K = -k_B\sum_{i=1}^{n}\frac{(p_i)^{1+K} - (p_i)^{1-K}}{2K}. \tag{48}$$

The $K-$deformed logarithm is associated with the $K-$exponential which reads

$$\exp_K x = \exp\left[\frac{1}{K}\text{arcsinh}(Kx)\right] = \left(\sqrt{1+K^2x^2} + Kx\right)^{1/K} \tag{49}$$

and fulfils some basic relations like $\exp_K(x)\exp_K(-x) = 1$ and $\exp_K(x)\exp_{-K}(x)$ giving the standard exponential function $\exp x$ in the limit $K \to 0$. In fact, $K-$logarithm and $K-$exponential are the inverse functions which means that they fulfil the relation

$$\ln_K\left(\exp_K x\right) = \exp_K(\ln_K x) = x. \tag{50}$$

The $K-$logarithm fulfils a generalized addition rule which reads

$$\ln_K(xy) = \ln_K x\sqrt{1+K^2(\ln_K y)^2} + \ln_K y\sqrt{1+K^2(\ln_K x)^2}, \tag{51}$$

which admits a standard logarithmic addition rule $\ln(xy) = \ln x + \ln y$ in the limit $K \to 0$. The rule comes from the definition of $K-$sum

$$(x \oplus y)_K = x\sqrt{1+K^2y^2} + y\sqrt{1+K^2x^2}, \tag{52}$$

 doi:10.20944/preprints202408.0564.v1

where one replaced $x \to \ln x$ and $y \to \ln y$ and giving standard additivity rule in the limit $K \to 0$. Using the definition of Kaniadakis entropy (46) and $K-$logaritm addition rule we can write down the Kaniadakis entropy additivity rule as follows

$$S_K(A+B) = S_K(A)\sqrt{1+\frac{K^2}{k_B^2}S_K(B)} + S_K(B)\sqrt{1+\frac{K^2}{k_B^2}S_K(A)} \quad (53)$$

which we call $K-$*addition rule*.

It is interesting to note that by the application of the $K-$sum defined as [14]

$$(x \otimes y)_K = \frac{1}{K}\sinh\left[\frac{1}{K}\text{arcsinh}(Kx)\text{arcsinh}(Ky)\right]. \quad (54)$$

By using this definition (54) one has for the $K-$logarithm

$$\ln_K[(x \otimes y)_K] = \ln_K x + \ln_K y, \quad (55)$$

so that applying it to (46), the Kaniadakis entropy (in full analogy to the $q-$product of Tsallis given by (21) can take a completely *additive form* as below (cf. the definition of additivity (4 for statistically independent systems)

$$S_K(A+B)_K = S_K(p_A p_B) = S_K(p_A) + S_K(p_B) = S_K(A) + S_K(B). \quad (56)$$

Using the $K-$product Kaniadakis entropy can also be *extensive*

$$S_K(x^{\otimes r}) = rS_K(x), \quad (57)$$

where $r =$ const., which comes as a result of the identity

$$\ln_K(x^{\otimes r}) = r\ln_{rK}(x). \quad (58)$$

Finally, in analogy to the previous considerations, and under the assumption that all the states are equally probable as in (2), one gets from (46) that

$$\ln_K p_i = -\frac{1}{K}\frac{e^{K\ln n} - e^{-K\ln n}}{2}, \quad (59)$$

which can further be transformed into

$$S_K = \frac{k_B}{K}\sinh\left(\frac{K}{k_B}S\right), \quad (60)$$

where $S = k_B \ln n$ is the BG entropy (3).

4.8. Classification of Entropies

Bearing in mind all the considerations of the whole Section 4, we present the summary of the additivity and extensivity properties of entropies in the Table 2.

Table 2. The additivity and extensivity properties of entropies.

Entropy Type	Extensivity	Additivity	Abé addition rule	$\delta-$addition rule	$K-$addition rule
Boltzmann-Gibbs S_{BG}	yes	yes	yes, $Y = 0$	yes, $\delta = 1$	yes, $K = 0$
Bekenstein S_{Bek}	no	no*	no	no	no
Tsallis q-entropy S_q	no	no	yes, $Y = 1 - q$	no	no
Tsallis δ-entropy S_δ ($\delta \neq \frac{3}{2}$)	no	no	no	yes	no
Tsallis δ-entropy S_δ ($\delta = \frac{3}{2}$)	yes	no	no	yes, $\delta = \frac{3}{2}$	no
Barrow $S_{Bar} = S_{Bek}$ ($\Delta = 0$)	no	no*	no	no	no
Barrow S_{Bar} ($0 < \Delta < 1$)	no	no	no	yes	no
Barrow S_{Bar} ($\Delta = 1$)	yes	no	no	yes, $\delta = \frac{3}{2}$	no
Rényi S_R	no	yes	yes, $Y = 0$	no	no
Landsberg U-entropy S_U	no	no	yes, $Y = q - 1$	no	no
Kaniadakis S_K	no	no	no	no	yes
Sharma-Mittal $S_{SM}(q, R)$	no	no	yes, $Y = R$	no	no
Tsallis q, δ-entropy $S_{q,\delta}$	no	no	no	no	no

* obeys square root rule (16)

5. Summary and Discussion

Beginning with the underlying properties of Boltzmann-Gibbs classical entropy, we have investigated the problem of nonextensity and nonadditivity in thermodynamics aiming towards gravitational systems which admit long-range interactions. The focus was on the different extensions of Boltzmann-Gibbs entropic form which allow various deformations of it via some new parameters modifying the space of microstates Ω. These new parameters are given some interpretations according to a deformation and can be enumerated as: Tsallis nonextensivity $q-$parameter, Tsallis-Cirto nonextensivity $\delta-$parameter which is equivalent to Barrow fractality $\Delta-$parameter, Sharma-Mittal $R-$parameter, and Kaniadakis relativistic $K-$deformation parameter.

The entropies under study may fulfil some analytic additivity rules which are the Abé rule and $\delta-$addition rule, both of which are nonaddtitive. Bekenstein-Hawking entropy obeys some other nonadditive rule called square root rule which is somewhat similar, but not the type of Abé rule. Kaniadakis entropy is additive within a special $K-$deformed algebra which reaches the standard Boltzmann-Gibbs additivity rule in the limit $K \to 0$.

We have presented the comparable tables with the additivity, extensivity, nonadditivity, and nonextensivity properties of the entropies under study. In view of recent interest of both relativists and cosmologists in the application of the plethora of alternative to Boltzmann-Gibbs entropies, this paper may serve a useful guide to these applications.

Funding: This research received no external funding.

Acknowledgments: The discussions with Ilim Çimdiker and Hussain Gohar are acknowledged.

Conflicts of Interest: The author declares no conflict of interest.

Appendix A. Equivalent forms of Tsallis q-Entropy

There are three equivalent forms of Tsallis $q-$entropy which are given by (18) which we will call according to their appearance in (18): $S_{qI}, S_{qII}, S_{qIII}$. We will show this by reducing all of them to the form (24) by the application the condition of probability summation $\sum p_i = 1$ and the formula (19). For S_{qI} one has

$$\mathcal{S}_{qI} \equiv k_B \sum_{i=1}^{n} p_i \ln_q \frac{1}{p_i} = k_B \sum_{i=1}^{n} p_i \frac{\left(\frac{1}{p_i}\right)^{1-q} - 1}{1-q} = k_B \sum_{i=1}^{n} \frac{(p_i)^q - 1}{1-q} = k_B \frac{1 - \sum_{i=1}^{n} (p_i)^q}{q-1}, \tag{A1}$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 8 August 2024 doi:10.20944/preprints202408.0564.v1

which is equivalent to (24). For S_{qII} one has

$$\mathcal{S}_{qII} = -k_B \sum_{i=1}^{n} (p_i)^q \ln_q p_i = -k_B \sum_{i=1}^{n} (p_i)^q \frac{(p_i)^{1-q} - 1}{1-q} = -k_B \frac{\sum_{i=1}^{n} [(p_i) - (p_i)^q]}{1-q} = k_B \frac{1 - \sum_{i=1}^{n} (p_i)^q}{q-1}, \quad \text{(A2)}$$

which is equivalent to (24). For S_{qIII} one has

$$\mathcal{S}_{qIII} = -k_B \sum_{i=1}^{n} \ln_{2-q} p_i = -k_B \sum_{i=1}^{n} (p_i) \frac{(p_i)^{q-1} - 1}{q-1} = -k_B \sum_{i=1}^{n} \frac{(p_i)^q - (p_i)}{q-1} = k_B \frac{1 - \sum_{i=1}^{n} (p_i)^q}{q-1}, \quad \text{(A3)}$$

where we have applied a redefined $q-$logarithm (19) with $q' = 2 - q$. This finishes the prove.

Appendix B. Validity of Abé rule for Tsallis *q*-Entropy and Landsberg *U*-Entropy

We first assume two probabilistically independent systems A and B fulfilling

$$\sum_{i=1}^{n} \sum_{j=1}^{m} p_{ij}^{A+B} = \sum_{i=1}^{n} \sum_{j=1}^{m} p_i^A p_j^B \quad \text{(A4)}$$

for every i, j which gives

$$\sum_{i=1}^{n} \sum_{j=1}^{m} \left(p_{ij}^{A+B}\right)^q = \sum_{i=1}^{n} \sum_{j=1}^{m} \left(p_i^A p_j^B\right)^q = \sum_{i=1}^{n} \sum_{j=1}^{m} \left(p_i^A\right)^q \left(p_j^B\right)^q. \quad \text{(A5)}$$

Following the definition of Tsallis $q-$entropy (24) one can write using (A5):

$$S_q(A+B) = k_B \frac{1 - \sum_{i=1}^{n} \sum_{j=1}^{m} \left(p_{ij}^{A+B}\right)^q}{q-1} = k_B \frac{1 - \sum_{i=1}^{n} \sum_{j=1}^{m} (p_i^A)^q \left(p_j^B\right)^q}{q-1} \quad \text{(A6)}$$

as well as

$$S_q(A) = k_B \frac{1 - \sum_{i=1}^{n} (p_i^A)^q}{q-1} \quad \text{(A7)}$$

and analogously for $S_q(B)$.

Following Abé rule (11) we can then write the right-hand side of it as

$$\begin{aligned} RHS &= S_q(A) + S_q(B) + \frac{\Upsilon}{k_B} S_q(A) S_q(B) = k_B \frac{1 - \sum_{i=1}^{n} (p_i^A)^q}{q-1} + k_B \frac{1 - \sum_{j=1}^{m} \left(p_j^A\right)^q}{q-1} \\ &+ \Upsilon k_B \left(\frac{1 - \sum_{i=1}^{n} (p_i^A)^q}{q-1}\right) \left(\frac{1 - \sum_{j=1}^{m} (p_i^A)^q}{q-1}\right) = \frac{k_B}{q-1} \left[1 - \sum_{i=1}^{n} (p_i^A)^q + 1 - \sum_{i=1}^{n} (p_j^B)^q \right. \\ &\left. + \frac{\Upsilon}{q-1} - \frac{\Upsilon}{q-1} \sum_{i=1}^{n} (p_i^A)^q - \frac{\Upsilon}{q-1} \sum_{i=1}^{n} (p_j^B)^q + \frac{\Upsilon}{q-1} \sum_{i=1}^{n} (p_i^A)^q \sum_{j=1}^{m} (p_j^B)^q \right], \end{aligned} \quad \text{(A8)}$$

which after selecting $\Upsilon = 1 - q$ cancels six out of eight terms giving on the base of (A5) that

$$RHS = k_B \frac{1 - \sum_{i=1}^{n} \sum_{j=1}^{m} (p_i^A)^q \left(p_j^B\right)^q}{q-1} = k_B \frac{1 - \sum_{i=1}^{n} \sum_{j=1}^{m} \left(p_{ij}^{A+B}\right)^q}{q-1} = S_q(A+B). \quad \text{(A9)}$$

The proof of applicability of the Abé rule for Landsberg $U-$entropy (35) proceeds analogously provided that $\Upsilon = q - 1$ instead.

Appendix C. Validity of Abé Rule for Sharma-Mittal Entropy

Let us write the Abé rule for the Sharma-Mittal entropy as follows

$$S_{SM}(A+B) = S_{SM}(A) + S_{SM}(B) + \frac{\Upsilon}{k_B} S_{SM}(A) S_{SM}(B) \tag{A10}$$

with

$$S_{SM}(A+B) = \frac{k_B}{R}\left[\left(\sum_{i=1}^{n}\sum_{j=1}^{m}(p_{ij}^{A+B})^q\right)^{\frac{R}{1-q}} - 1\right] \tag{A11}$$

and

$$S_{SM}(A) = \frac{k_B}{R}\left[\left(\sum_{i=1}^{n}(p_i^A)^q\right)^{\frac{R}{1-q}} - 1\right]. \tag{A12}$$

Calculation of the RHS of (A10) gives

$$RHS = \frac{k_B}{R}\left[\left(\sum_{i=1}^{n}(p_i^A)^q\right)^{\frac{R}{1-q}} - 1\right] + \frac{k_B}{R}\left[\left(\sum_{j=1}^{m}(p_j^B)^q\right)^{\frac{R}{1-q}} - 1\right] + \frac{\Upsilon}{k_B}\frac{k_B}{R}\left[\left(\sum_{i=1}^{n}(p_i^A)^q\right)^{\frac{R}{1-q}} - 1\right] \times$$
$$\frac{k_B}{R}\left[\left(\sum_{j=1}^{m}(p_j^B)^q\right)^{\frac{R}{1-q}} - 1\right] = \frac{k_B}{R}\left[\left(\sum_{i=1}^{n}(p_i^A)^q\right)^{\frac{R}{1-q}} - 1 + \left(\sum_{j=1}^{m}(p_j^B)^q\right)^{\frac{R}{1-q}} - 1\right.$$
$$\left. + \frac{\Upsilon}{R}\left(\sum_{i=1}^{n}(p_i^A)^q\right)^{\frac{R}{1-q}}\left(\sum_{j=1}^{m}(p_j^B)^q\right)^{\frac{R}{1-q}} - \frac{\Upsilon}{R}\left(\sum_{i=1}^{n}(p_i^A)^q\right)^{\frac{R}{1-q}} - \frac{\Upsilon}{R}\left(\sum_{j=1}^{m}(p_j^B)^q\right)^{\frac{R}{1-q}} + \frac{\Upsilon}{R}\right]$$
$$= \frac{k_B}{R}\left[\left(\sum_{i=1}^{n}\sum_{j=1}^{m}(p_i^A)^q(p_j^B)^q\right)^{\frac{R}{1-q}} - 1\right] = \frac{k_B}{R}\left[\left(\sum_{i=1}^{n}\sum_{j=1}^{m}(p_{ij}^{A+B})^q\right)^{\frac{R}{1-q}} - 1\right] = LHS, \tag{A13}$$

where we have taken $\Upsilon = R$ and applied (A5).

References

1. Swendsen, R.H. *An Introduction to Statistical Mechanics and Thermodynamics*; Oxford University Press, Oxford - New York, 2012.
2. Bekenstein, J.D. Black holes and entropy. *Phys. Rev. D* **1973**, *7*, 2333–2346. doi:10.1103/PhysRevD.7.2333.
3. Hawking, S.W. Particle Creation by Black Holes. *Commun. Math. Phys.* **1975**, *43*, 199–220. [Erratum: Commun.Math.Phys. 46, 206 (1976)], doi:10.1007/BF02345020.
4. Tsallis, C.; Cirto, L.J.L. Black hole thermodynamical entropy. *Eur. Phys. J. C* **2013**, *73*, 2487, [arXiv:cond-mat.stat-mech/1202.2154]. doi:10.1140/epjc/s10052-013-2487-6.
5. Biró, T.S.; Czinner, V.G. A q-parameter bound for particle spectra based on black hole thermodynamics with Rényi entropy. *Phys. Lett. B* **2013**, *726*, 861–865, [arXiv:gr-qc/1309.4261]. doi:10.1016/j.physletb.2013.09.032.
6. Czinner, V.G.; Iguchi, H. A Zeroth Law Compatible Model to Kerr Black Hole Thermodynamics. *Universe* **2017**, *3*, 14. doi:10.3390/universe3010014.
7. Czinner, V.G.; Iguchi, H. Rényi Entropy and the Thermodynamic Stability of Black Holes. *Phys. Lett. B* **2016**, *752*, 306–310, [arXiv:gr-qc/1511.06963]. doi:10.1016/j.physletb.2015.11.061.
8. Czinner, V.G.; Iguchi, H. Thermodynamics, stability and Hawking–Page transition of Kerr black holes from Rényi statistics. *Eur. Phys. J. C* **2017**, *77*, 892, [arXiv:gr-qc/1702.05341]. doi:10.1140/epjc/s10052-017-5453-x.
9. Tsallis, C. Black Hole Entropy: A Closer Look. *Entropy* **2019**, *22*, 17. doi:10.3390/e22010017.
10. Tsallis, C. Possible Generalization of Boltzmann-Gibbs Statistics. *J. Statist. Phys.* **1988**, *52*, 479–487. doi:10.1007/BF01016429.
11. Rényi, A. On the dimension and entropy of probability distributions. *Acta Mathematica Academiae Scientiarum Hungaricae* **1959**, *10*, 193–215. doi:10.1007/BF02063299.

12. Sharma, B.D.; Mittal, D.P. New non-additive measures of relative information. *J.Comb.Inf.Syst.Sci.* **1977**, *2*, 122–133.
13. Sharma, B.D.; Mittal, D.P. New non-additive measures of entropy for discrete probability distributions. *J. Math. Sci*, *10*, 28–40.
14. Kaniadakis, G. Statistical mechanics in the context of special relativity. *Phys. Rev. E* **2002**, *66*, 056125, [cond-mat/0210467]. doi:10.1103/PhysRevE.66.056125.
15. Kaniadakis, G. Statistical mechanics in the context of special relativity. II. *Phys. Rev. E* **2005**, *72*, 036108, [cond-mat/0507311]. doi:10.1103/PhysRevE.72.036108.
16. Barrow, J.D. The area of a rough black hole. *Physics Letters B* **2020**, *808*, 135643. doi:10.1016/j.physletb.2020.135643.
17. Nojiri, S.; Odintsov, S.D.; Faraoni, V. Area-law versus Rényi and Tsallis black hole entropies. *Phys. Rev. D* **2021**, *104*, 084030, [arXiv:gr-qc/2109.05315]. doi:10.1103/PhysRevD.104.084030.
18. Drepanou, N.; Lymperis, A.; Saridakis, E.N.; Yesmakhanova, K. Kaniadakis holographic dark energy and cosmology. *Eur. Phys. J. C* **2022**, *82*, 449, [arXiv:gr-qc/2109.09181]. doi:10.1140/epjc/s10052-022-10415-9.
19. Tsallis, C.; Mendes, R.; Plastino, A. The role of constraints within generalized nonextensive statistics. *Physica A: Statistical Mechanics and its Applications* **1998**, *261*, 534–554. doi:10.1016/S0378-4371(98)00437-3.
20. Landsberg, P. Entropies galore! *Brazilian Journal of Physics* **1999**, *29*, 46–49.
21. Swendsen, R.H. How physicists disagree on the meaning of entropy. *American Journal of Physics* **2011**, *79*, 342–348, [https://doi.org/10.1119/1.3536633]. doi:10.1119/1.3536633.
22. Mannaerts, S.H. Extensive quantities in thermodynamics. *European Journal of Physics* **2014**, *35*, 035017. doi:10.1088/0143-0807/35/3/035017.
23. Çimdiker, I.; Dąbrowski, M.P.; Gohar, H. Equilibrium Temperature for Black Holes with Nonextensive Entropy. *European Physical Journal C: Particles and Fields* **2023**, *83*. doi:10.1140/epjc/s10052-023-11317-0.
24. Caldwell, R. A phantom menace? Cosmological consequences of a dark energy component with super-negative equation of state. *Physics Letters B* **2002**, *545*, 23–29. doi:10.1016/S0370-2693(02)02589-3.
25. Caldwell, R.R.; Kamionkowski, M.; Weinberg, N.N. Phantom Energy: Dark Energy with *w*< −1 Causes a Cosmic Doomsday. *Phys. Rev. Lett.* **2003**, *91*, 071301. doi:10.1103/PhysRevLett.91.071301.
26. Dąbrowski, M.P. Puzzles of dark energy in the Universe - phantom. *European Journal of Physics* **2015**, *36*, 065017. doi:10.1088/0143-0807/36/6/065017.
27. Abé, S.; Martınez, S.; Pennini, F.; Plastino, A. Nonextensive thermodynamic relations. *Physics Letters A* **2001**, *281*, 126–130. doi:10.1016/s0375-9601(01)00127-x.
28. Abé, S.; Rajagopal, A.K. Macroscopic thermodynamics of equilibrium characterized by power law canonical distributions. *Europhysics Letters (EPL)* **2001**, *55*, 6–11. doi:10.1209/epl/i2001-00373-4.
29. Abé, S. General pseudoadditivity of composable entropy prescribed by the existence of equilibrium. *Physical Review E* **2001**, *63*. doi:10.1103/physreve.63.061105.
30. Alonso-Serrano, A.; Dabrowski, M.P.; Gohar, H. Nonextensive Black Hole Entropy and Quantum Gravity Effects at the Last Stages of Evaporation. *Phys. Rev. D* **2021**, *103*, 026021, [arXiv:gr-qc/2009.02129]. doi:10.1103/PhysRevD.103.026021.
31. Tsallis, C. *Introduction to Nonextensive Statistical Mechanics: Approaching a Complex World*; Springer New York, NY, 2009. doi:10.1007/978-0-387-85359-8.
32. Tsallis, C. Nonadditive entropy and nonextensive statistical mechanics -an overview after 20 years. *Brazilian Journal of Physics* **2009**, *39*, 337–356. doi:10.1590/S0103-97332009000400002.
33. Çimdiker, I.; Dąbrowski, M.P.; Gohar, H. Generalized uncertainty principle impact on nonextensive black hole thermodynamics. *Classical and Quantum Gravity* **2023**, *40*, 145001. doi:10.1088/1361-6382/acdb40.
34. Abreu, E.M.; Ananias Neto, J. Barrow black hole corrected-entropy model and Tsallis nonextensivity. *Physics Letters B* **2020**, *810*, 135805. doi:10.1016/j.physletb.2020.135805.
35. Biró, T.S.; Ván, P. Zeroth law compatibility of nonadditive thermodynamics. *Phys. Rev. E* **2011**, *83*, 061147. doi:10.1103/PhysRevE.83.061147.
36. Czinner, V.G. Black hole entropy and the zeroth law of thermodynamics. *Int. J. Mod. Phys. D* **2015**, *24*, 1542015. doi:10.1142/S0218271815420158.
37. Masi, M. A step beyond Tsallis and Rényi entropies. *Physics Letters A* **2005**, *338*, 217–224. doi:10.1016/j.physleta.2005.01.094.

38. Sayahian Jahromi, A.; Moosavi, S.A.; Moradpour, H.; Morais Graça, J.P.; Lobo, I.P.; Salako, I.G.; Jawad, A. Generalized entropy formalism and a new holographic dark energy model. *Phys. Lett. B* **2018**, *780*, 21–24, [arXiv:gr-qc/1802.07722]. doi:10.1016/j.physletb.2018.02.052.