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Exceptional Differential Polynomial Systems Formed by Simple Pseudo-Wronskians of Jacobi Polynomials and Their Infinite and Finite X-Orthogonal Reductions

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Abstract: The paper advances the new technique for constructing the exceptional differential polynomial systems (X-DPSs) and their infinite and finite orthogonal subsets. First, using Wronskians of Jacobi polynomials (JPWs) with a common pair of the `indexes, we generate the Darboux-Crum nets of the rational canonical Sturm-Liouville equations (RCSLEs). It is shown that each RCSLE in question has four infinite sequences of quasi-rational solutions (q-RSs) such that the polynomial components from each sequence form a X-Jacobi DPS composed of simple pseudo-Wronskian polynomials (*p*-WPs). For each p-th order rational Darboux Crum transform of the Jacobi-reference (JRef) CSLE used as the starting point, we formulate two rational Sturm-Liouville problems (RSLPs) by imposing the Dirichlet boundary conditions on the solutions of the so-called 'prime' SLE at the ends of the intervals (-1,+1) or (+1,∞). Finally, we demonstrate that the polynomial components of the q-RSs representing the eigenfunctions of these two problems have the form of simple *p*-WPs composed of p Romanovski-Jacobi (R-Jacobi) polynomials with the same pair of the indexes and a single classical Jacobi polynomial or accordingly p classical Jacobi polynomials with the same pair of positive indexes and a single R-Jacobi polynomial. The common fundamentally important feature of all the simple *p*-WPs involved is that they do not vanish at the finite singular endpoints –the main reason of why they were selected for the current analysis in the first place. The discussion is accompanied by a sketch of the one-dimensional quantum-mechanical problems exactly solvable by the aforementioned infinite and finite EOP sequences.

Keywords: rational Sturm-Liouville equation; pseudo-Wronskian polynomial; Darboux-Crum transformation; exceptional differential polynomial system; exceptional orthogonal polynomial system; exceptional orthogonal polynomials; Romanovski-Jacobi polynomials; Dirichlet problem

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1. Introduction

Nearly two decades ago Gȯmez-Ullate, Kamran, and Milson [1,2] discovered new *infinite* sequences of orthogonal polynomial solutions of a second-order differential eigenequation with rational coefficients. Since each sequence either does not start from a constant or lacks the first-degree polynomial, the discovered polynomials are not covered by Bochner's classical proof [3].

However, as stressed by Kwon and Littlewood [4], Bochner himself "did not mention the orthogonality of the polynomial systems that he found. The problem of classifying all classical orthogonal polynomials was handled by many authors thereafter" based on his analysis of possible polynomial solutions of *complex* second-order differential eigenequations. This observation brought the author [5] to the concept of *exceptional* eigenpolynomials which satisfy a differential eigenequation of Bochner type but violate his theorem because each sequence either does not start from a constant

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or lacks the first-degree polynomial. Thereby we refer to these sequences as complex exceptional differential polynomial systems (X-DPSs) with the term 'DPS' used in exactly the same sense it is done by Everitt et al [6,7] for the conventional sequences of eigenpolynomials obeying the Bochner theorem.

It has been proven by Kwon and Littlejohn [4] that all the real field reductions of the complex DPSs constitute quasi-definite orthogonal polynomial sequences [8] and for this reason the cited authors refer to the latter as 'OPSs'. However this is not true for the X-DPS and we thus preserve the term 'X-OPS' solely for the sequences formed by positively definite orthogonal polynomials.

Compared with the rigorous mathematical analysis of the X-OPSs in [9,10], the concept of the X-DPSs put forward by us in [7] represents the parallel direction dealing with the solvable rational CSLEs (RCSLEs) and related X-Bochner ordinary differential equations (ODEs), instead of the (generally irregular) exceptional Bochner (X-Bochner) operators in [10].

The interrelation between the two approaches is closely related to the dual use of the term 'Darboux transformation' (DT), following the discovery by Andrianov et al [11,12] that the renowned transformation of the Schrödinger equation initially suggested by Darboux [13] for the generic second-order canonical differential eigenequation (long before the birth of the quantum mechanics) is equivalent to its intertwining factorization. We refer the reader to a comprehensive overview of this issue in [14].

More recently Gòmez-Ullate et al. [15] initiated the new direction in the theory of the rational Sturm-Liouville equations (RSLEs) by applying the intertwining factorization to the second-order differential eigenoperator. This operation was termed 'Darboux transformation', based on the dualism existent in the particular case of the Schrödinger operator. This innovation followed by its extension to the X_m-Jacobi and X_m-Laguerre OPSs [9,16] laid the foundation for their rigorous theory more recently advanced to the more sophisticated level in [10,17].

The author (being accustomed [18] to the strict use of the mentioned term) took the different turn in the extension of the DTs to the SLEs, based on the nearly forgotten paper by Rudjak and Zakharov [19] in the late eighties. In Appendix A we outline the most essential features of the 'Rudjak-Zakhariev transformations' (RZTs) applied to the generic canonical SLE (CSLE). The RZTs turn into the conventional DTs if both leading coefficient function and weigh are identically equal to 1.

We refer to a RZT of the rational CSLE (RCSLE) as 'rational' (RRZT) , if it uses a quasi-rational transformation function (q-RTF). In the recent papers [20] and [21] the author has demonstrated the power of the suggested approach by constructing X_1- and respectively X_m-Jacobi DPSs and then extracting from them infinite and finite exceptional orthogonal polynomial (EOP) sequences.

We take advantage of Schulze-Halberg's formalism for the so-called 'foreign auxiliary equations' [22] to generalize the notion of the Darboux-Crum [13,23] transformations (DCTs) to the CSLEs. It is proven in Appendix B that sequential RZTs give rise to a DCT defined in such a way. We refer to a DCT of the RCSLE as 'rational' (RDCT) if it uses quasi-rational seed functions.

In this paper we focus solely on the RDCTs using quasi-rational seed functions with polynomial components formed by Jacobi polynomials with a common pair of the indexes [24]. It was shown that the rational Darboux-Crum transform ($\mathrm{RDC}\mathfrak{T}$) of the Jacobi-reference (JRef) CSLE, which is defined via (1)–(3) in Section 2, has four infinite sequences of quasi-rational solutions (q-RSs). While one of these sequences is formed by Jacobi polynomial Wronskians (JPWs), the polynomials components of three others are represented by the so-called [21] 'simple' pseudo-Wronskian polynomials ($\mathfrak{p}$-WPs }. Namely, we refer to a pseudo-Wronskian of Jacobi polynomials [25] as 'simple' if only a single polynomial in the given set of Jacobi polynomials has at least one Jacobi index with a different sign (compared with the sign of the common index of the seed Jacobi polynomials). As proven in this paper, the simple $\mathfrak{p}$-WPs remain finite at the singular points ±1 and as a results obey the X-Bochner differential equations with polynomial coefficients, forming a X-Jacobi DPS.

From our perspective, this is the significant achievement, compared to the paper by Bonneux [26], who studied a more general manifold of the pseudo-Wronskians of Jacobi polynomials (referred

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to as 'generalized Jacobi polynomials'), while completely skipping the discussion of exceptional Jacobi operators [9,10] -- the X-Bochner differential equations in the framework of this paper. Let us remind the reader that the Bochner-type differential equation can be trivially converted to the eigenequation with rational coefficients. As a result, the polynomials in the given X-DPS represent the set of the eigenpolynomials for the corresponding exceptional Jacobi operator. Our understanding is that it is generally a more challenging problem to construct the latter operator (assuming that all of its singular points are regular) if the p-WPs have a more complicated structure.

To pinpoint infinite and finite EOP sequences in the given X-DPS, we [27] put forward the concept of the 'prime' SLEs (*p*-SLEs) chosen in such a way that the two characteristic exponents (ChExps) for the poles at the endpoints differ only by sign. As a result, the energy spectrum of the given Sturm-Liouville problem can be obtained by solving the given *p*-SLE under the Dirichlet boundary conditions (DBCs). This in turn allows one to take advantage of the rigorous theorems proven in [28] for eigenfunctions of the generic SLE solved under the DBCs. As it has been already illustrated in [24] and illuminated more thoroughly here, the new approach allows one to treat in parallel both infinite and finite EOP sequences, as different orthogonal subsets of the same X-DPS.

Before continuing our discussion, let us first point to the dubious use of the term 'EOP' in the literature, similar to the slang use of the term 'orthogonal Jacobi polynomials', instead of 'classical Jacobi polynomials', which disregards the existence of the finite orthogonal subsets formed by the Romanovski-Jacobi (R-Jacobi) polynomials [29–31]. Similarly, Gȯmez-Ullate, Milson et al. [9,10,32] use the term 'EOPs' as the synonym for 'X-OPS', with a few puzzling exceptions. For example, when referring to the studies on the EOPs in [32], the cited authors mentioned the papers [33,34], which deal solely with the problems solved by the finite EOP sequences. This is also true for Refs. [8,16,17] in [10]. The reference to the paper of Ho et al. [35] (and similarly to the study by Yadav et al. [36] on rationally extended Eckart potential) is misleading because the absolute majority of all the rational potentials examined in this paper are quantized by polynomials with degree-dependent indexes and therefore irrelevant to the subject. The only exception is the RDC𝔖s of the hyperbolic Pöschl-Teller (h-PT) potential, which are indeed quantized by finite EOP sequences.

Similarly, Refs. [21,25,27,30,48,49] in their work with Grandati [37] are pointed to the rational extensions of the translationally shape-invariant potentials (TSIPs) of group B in Odake and Sasaki's [38] TSIP classification scheme. The common feature of the potentials of this group is that the corresponding quasi-rational eigenfunctions are composed of Jacobi or Laguerre polynomials with degree-dependent indexes and as a results has nothing to do with the EOPs, contrary to the statement in [37]. We shall come back to this issue, while summarizing the paper results in Section 8.

Let us also stress that our paper solely focuses on the infinite and finite EOP sequences which represent the so-called 'standard examples' in Durán's terms [39], with the absolute value of each Jacobi index being exactly the same for all the Jacobi polynomials forming the pseudo-Wronskian in question.

2. Four Distinguished Infinite Sequences of q-RSs

Let us start our analysis with the Jacobi-reference (JRef) CSLE

$$\left\{\frac{d^2}{d\eta^2}+I^0[\eta;\vec{\lambda}_o]+sgn(1-\eta^2)\,\varepsilon\,\rho[\eta]\right\}\Phi[\eta;\vec{\lambda}_o;\varepsilon]=0 \tag{1}$$

with the single pole density function

$$\rho[\eta]:=\frac{1}{|\eta^2-1|} \tag{2}$$

and the reference polynomial fraction (RefPF) parameterized as follows:

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$$I^{o}[\eta;\vec{\lambda}_{o}] \equiv \sum_{\aleph=\pm} \frac{1-\lambda_{o;\aleph}^{2}}{4(1-\aleph\eta)^{2}} + \frac{1-\lambda_{o;+}^{2}-\lambda_{o;-}^{2}}{4(1-\eta^{2})} \qquad (3)$$

$$= \frac{1}{2(1-\eta^{2})} \sum_{\aleph=\pm} \frac{1-\lambda_{o;\aleph}^{2}}{1-\aleph\eta} - \frac{1}{4(1-\eta^{2})}, \qquad (4)$$

where $\lambda_{o;\pm}$ are the ExpDiffs for the poles at ±1 and the energy reference point is chosen by the requirement that the ExpDiff for the singular point at infinity vanishes at zero energy, i. e.,

$$\lim_{|\eta|\to\infty} \left(\eta^{2} I^{o}[\eta;\vec{\lambda}_{o}]\right) = 1/4. \qquad (5)$$

The energy sign is chosen in such a way:

$$sgn(\varepsilon) = sgn(1-\eta^{2}) \qquad (6)$$

that the sought-for eigenvalues are positive (negative) when the Sturm-Liouville problem in question is formulated on the finite interval $-1<\eta<1$ (or respectively on the positive infinite interval $1<\eta<\infty$). An analysis of solutions of the CSLE (1) on the negative infinite interval $-\infty<\eta<-1$ can be skipped without loss of generality due to the symmetry of the RefPF (2) under reflection of its argument, accompanied by the interchange of the exponents differences (ExpDiffs) $\lambda_{o;\pm}$ for the CSLE poles at ±1.

Let us now consider the gauge transformation

$$F[\eta;\vec{\lambda};\varepsilon] := \Phi[\eta;\vec{\lambda}_{o};\varepsilon] / \phi_{0}[\eta;\vec{\lambda}] \times, \qquad (7)$$

where

$$\phi_{0}[\eta;\vec{\lambda}] := (1+\eta)^{1/2(\lambda_{-}+1)} |1-\eta|^{1/2(\lambda_{+}+1)} \quad (-1<\eta\neq 1). \qquad (8)$$

Keeping in mind that

$$ld\,\phi_{0}[\eta;\vec{\lambda}] := \dot{\phi}_{0}[\eta;\vec{\lambda}] / \phi_{0}[\eta;\vec{\lambda}] = \frac{\lambda_{-}+1}{2(\eta+1)} + \frac{\lambda_{+}+1}{2(\eta-1)}, \qquad (9)$$

coupled with (3), one finds

$$\ddot{\phi}_{0}[\eta;\vec{\lambda}] / \phi_{0}[\eta;\vec{\lambda}] = ld^{2}\phi_{0}[\eta;\vec{\lambda}] + l\dot{d}\,\phi_{0}[\eta;\vec{\lambda}] \qquad (10)$$

$$= -I^{o}[\eta;\vec{\lambda}_{o}] - \frac{(\lambda_{-}+\lambda_{+}+1)^{2}}{4(1-\eta^{2})}, \qquad (11)$$

with dot standing for the derivative with respect to η. i.e., the quasi-rational function (8) is the solution of the JRef CSLE at $|\varepsilon|$ equal to

$$\varepsilon_{0}(\vec{\lambda}) = 1/4(\lambda_{-}+\lambda_{+}+1)^{2}. \qquad (12)$$

It then directly follows from the identity

$$\ddot{\Phi}[\eta;\vec{\lambda};\varepsilon] / \phi_{0}[\eta;\vec{\lambda}] \equiv -\left\{ I^{o}[\eta;\vec{\lambda}_{o}] + \frac{\varepsilon_{0}(\vec{\lambda})}{1-\eta^{2}} \right\} F[\eta;\vec{\lambda};\varepsilon] + \ddot{F}[\eta;\vec{\lambda};\varepsilon]$$

$$+2ld\phi_{0}[\eta;\vec{\lambda}] \times \dot{F}[\eta;\vec{\lambda};\varepsilon] \qquad (13)$$

that the function (7) satisfies the Bochner-type ordinary differential equation (ODE) with the polynomial coefficients:

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$$\left[(\eta^2-1)\frac{d^2}{d\eta^2}+2P_1^{(\lambda_+,\lambda_-)}(\eta)\frac{d}{d\eta}+\tfrac{1}{4}(\lambda_-+\lambda_++1)^2-|\varepsilon|\right]\times F[\eta;\vec{\lambda};\varepsilon]=0 \qquad (14)$$

with the polynomial coefficients. It is essential that the resultant ODE is well-defined for any real values of the variable η, including the border points $|\lambda_-|=1$ or $|\lambda_+|=1$ between the LP and LC regions (which require a special attention and were sidelined for this reason in our current discussion). The ODE (14) turns into the conventional Jacobi equation

$$(\eta^2-1)\ddot{P}_m^{(\lambda_+,\lambda_-)}(\eta)+2P_1^{(\lambda_+,\lambda_-)}(\eta)\dot{P}_m^{(\lambda_+,\lambda_-)}(\eta)+ \qquad (15)$$

$$m(\lambda_++\lambda_-+m+1)P_m^{(\lambda_+,\lambda_-)}(\eta)=0$$

at the energies

$$|\varepsilon|=\varepsilon_m(\vec{\lambda})=\tfrac{1}{4}(\lambda_++\lambda_-+2m+1)^2. \qquad (16)$$

In following [5,20,21], we say that the polynomials in question form the Jacobi DPS.

Note that, in addition with the renowned polynomial solutions, the ODE (14) has 3 other infinite sequences of the q-RSs listed in Table 1 in [26] (or Table 2 in [17]). It is worth pointing out to the difference in our terminology, compared with that in {17,26]. Namely, we restrict the term 'eigenfunction' only to a solution of a Sturm-Liouville problem (SLP), i.e., in our terms only the classical Jacobi polynomials constitute the eigenfunctions of the Sturm-Liouville differential expression ( ), assuming that the corresponding polynomial SLP (PSLP] is formulated on the interval (-1,+1), using the boundary conditions (10) in [9].

By choosing

$$\lambda_-,\ \lambda_+,\ \lambda_-+\lambda_++m\neq -k \quad \text{for any positive integer } k\leq m \qquad (17)$$

(see [42, Chapter 6.72]), we assure that the Jacobi polynomial in question has exactly m simple zeros $\eta_l(\vec{\lambda};m)$, i.e., using its monic form,

$$\hat{P}_m^{(\lambda_+,\lambda_-)}(\eta)=\Pi_m[\eta;\bar{\eta}(\vec{\lambda};m)], \qquad (18)$$

where by definition

$$\Pi_m[\eta;\bar{\eta}]:=\prod_{l=1}^{m}[\eta-\eta_l]. \qquad (19)$$

It is crucial that the Jacobi indexes do not depend on the polynomial degree, in contrast with the general case [40,41]. This remarkable feature of the CSLE under consideration is the direct consequence of the fact that the density function (2) has only simple poles in the finite plane [25] and as a result the ExpDiffs for the CSLE poles at ±1 become energy-independent [1].

We conclude that the JRef CSLE with the density function (2) has four infinite sequences of the q-RSs

$$\phi_m[\eta;\vec{\lambda}]=|1+\eta|^{\frac{1}{2}(\lambda_-+1)}|1-\eta|^{\frac{1}{2}(\lambda_++1)}P_m^{(\lambda_+,\lambda_-)}(\eta) \qquad (20)$$

$$(|\lambda_\pm|=\lambda_{0;\pm})$$

at the energies (16), with the vector parameter $\vec{\lambda}$ restricted to the one of the four quadrants for each sequence.

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Each infinite sequence starts from the q-RS (8) with $\vec{\lambda}$ restricted to the corresponding quadrant. Substituting (9) into the identity

$$\ddot{\phi}_0[\eta;\vec{\lambda}]/\phi_0[\eta;\vec{\lambda}] \equiv ld^2\phi_0[\eta;\vec{\lambda}] + l\dot{d}\,\phi_0[\eta;\vec{\lambda}], \tag{21}$$

where the symbolic expression *ld* f[η] denotes the logarithmic derivative of the function f[η], we find that the function (8) is the solution of the Riccati equation

$$l\dot{d}\,\phi_0[\eta;\vec{\lambda}] - ld^2\phi_0[\eta;\vec{\lambda}] + I^0[\eta;\vec{\lambda}] + \frac{\varepsilon_0(\vec{\lambda})}{1-\eta^2} = 0 \tag{22}$$

3. 'Prime' Forms of J-Ref CSLE on Finite and Infinite Intervals of Orthogonalization

The gauge transformation

$$\Psi_{\wp}[\eta;\vec{\lambda};\varepsilon] = \wp^{-1/2}[\eta;\vec{\lambda}]\Phi[\eta;\vec{\lambda};\varepsilon], \tag{23}$$

with an arbitrarily chosen positive function $\wp[\eta]$, converts the JRef CSLE (1) into the SLE of the generic form:

$$\left\{\frac{d}{d\eta}\wp[\eta;\vec{\lambda}]\frac{d}{d\eta} - q_{\wp}[\eta;\vec{\lambda}] + sgn(1-\eta^2)\varepsilon\, w_{\wp}[\eta;\vec{\lambda}]\right\}\Psi[\eta;\vec{\lambda};\varepsilon] = 0, \tag{24}$$

with the weight

$$w_{\wp}[\eta;\vec{\lambda}] := \rho[\eta]/\wp[\eta;\vec{\lambda}]. \tag{25}$$

The PF representing the zero-energy free term is given by the following generic formula [42]:

$$q_{\wp}[\eta;\vec{\lambda}] = \wp[\eta;\vec{\lambda}]I^0[\eta;\vec{\lambda}] + \mathcal{S}\{\wp[\eta;\vec{\lambda}]\} \tag{26}$$

with

$$\mathcal{S}\{f[\eta]\} := \tfrac{1}{4}\dot{f}^2[\eta]/f[\eta] - \tfrac{1}{2}\ddot{f}[\eta] \tag{27}$$

and the sign of the sought-for spectral parameter ε is dictated by the constraint (6).

Let us choose the leading coefficient function in such a way:

$$\wp[\eta] = \not{\wp}[\eta] := \begin{cases} 1-\eta^2 & \text{for } -1<\eta<+1, \\ \eta-1 & \text{for } \eta>1 \end{cases} \tag{28}$$

that the SLP of our interest can be formulated as the Dirichlet problem:

$$\lim_{\eta\to\eta_\mp} \psi_j[\eta;\vec{\lambda}] = 0 \tag{29}$$

at the ends of the given interval of orthogonalization $\eta_\mp = \mp 1$ or $\eta_- = 1,\ \eta_+ = \infty$. It has been proven in [28] that the eigenfunctions of this Dirichlet problem must be square-integrable:

$$\int_{\eta_-}^{\eta_+} d\eta\, \psi_j^2[\eta;\vec{\lambda}]w[\eta] < \infty \tag{30}$$

and mutually orthogonal:

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$$\int_{\eta_-}^{\eta_+} d\eta\, \psi_j[\eta;\vec{\lambda}_o]\psi_{j'}[\eta;\vec{\lambda}_o]w[\eta] = 0 \quad (j \neq j'), \tag{31}$$

with the weight

$$w[\eta] := w_{\not p}[\eta] = \begin{cases} 1 & \text{for } -1 < \eta < +1, \\ (1+\eta)^{-1} < 1/2 & \text{for } \eta > 1. \end{cases} \tag{32}$$

Due to the very special choice of the leading coefficient function (28) for the 'prime' SLE (24), the two ChExps for each singular endpoint differ by sign, while having exactly the same absolute value, which assures [42] that each DBC unambiguously selects PFS near the given end. In other words, the DBCs (28) unequivocally determine the PFSs near the both singular ends of the given interval of orthogonalization.

Substituting (28) into (27) gives [43]:

$$\Im\{\not p[\eta]\} = \begin{cases} (1-\eta^2)^{-1} & \text{for } -1<\eta<+1, \\ 1/4(\eta-1)^{-1} & \text{for } \eta>1, \end{cases} \tag{33}$$

which shows the free-energy term of the prime SLE with the leading coefficient function (28) has simple poles in the finite plane.

As discussed in Section 7, the concept of the prime SLEs allows one to select the sequences of the nodeless PFSs, which assure that the corresponding X-Bochner operators are regular inside the given interval of orthogonalization. This is one of the most important achievements of this paper.

3.1. Dirichlet Problem on Interval (-1,+1)

The crucial advantage of representing the conventional Jacobi equation in the prime SLE form is that the q-RS

$$\psi_j[[\eta;\vec{\lambda}] = (1+\eta)^{1/2\lambda_-}(1-\eta)^{1/2\lambda_+} P_j^{(\lambda_+,\lambda_-)}(\eta) \tag{34}$$

represents the PFS near the poles at ∓ 1 iff the corresponding Jacobi index $\lambda_\mp$ is positive. In particular, the q-RSs

$$\psi_j[[\eta;\vec{\lambda}_o] = (1+\eta)^{1/2\lambda_{o;-}}(1-\eta)^{1/2\lambda_{o;+}} P_j^{(\lambda_{o;+},\lambda_{o;-})}(\eta) \quad (-1<\eta<1) \tag{35}$$

formed by the classical Jacobi polynomials with positive indexes necessarily satisfy the DBCs at ∓ 1 and as a result constitute the eigenfunctions of the given Dirichlet problem. The orthogonality relations (30) thus turn into the conventional orthogonality relations for the classical Jacobi polynomials

$$\int_{-1}^{+1} d\eta\, P_j^{(\lambda_{o;+},\lambda_{o;-})}(\eta) P_{j'}^{(\lambda_{o;+},\lambda_{o;-})}(\eta) \mathcal{W}_m[\eta;\vec{\lambda}_o] = 0 \quad (j \neq j') \tag{36}$$

with the weight function

$$\mathcal{W}_m[\eta;\vec{\lambda}] := (1+\eta)^{\lambda_-}(1-\eta)^{\lambda_+} \quad \text{for } -1<\eta<1. \tag{37}$$

Since the j^{th}+1-solution has exactly j zeros between -1 and +1 and the positive eigenvalues converges to 0 as the polynomial degree tends to infinity, the Dirichlet problem in question may not have any other eigenfunctions.

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One still needs to prove that the q-RSs (34) form the complete set of the eigenfunctions of the given Dirichlet problem. This can be performed, for example, by converting the JRef CSLE (1) to the hypergeometric equation on the interval (0,+1) and then follow the arguments presented by the author [40] for the exactly solvable JRef CSLE with the properly chosen density function.

On other hand, choosing

$$\wp[\eta;\vec{\lambda}] := (1-\eta^2)\mathrm{W}[\eta;\vec{\lambda}] = (1+\eta)^{\lambda_-+1}(1-\eta)^{\lambda_++1}, \qquad (38)$$

we come to the Sturm-Liouville form [44] of the Jacobi equation:

$$\left\{\frac{d}{d\eta}\wp[\eta;\vec{\lambda}]\frac{d}{d\eta} - q_{\wp}[\eta;\vec{\lambda}] + \varepsilon_j(\vec{\lambda}_o)\mathrm{W}[\eta;\vec{\lambda}]\right\} P_j^{(\lambda_+,\lambda_-)}(\eta) = 0. \qquad (39)$$

It is crucial that the leading coefficient function (38) for $\lambda_\mp > -1$ vanishes at the ends of the interval [-1,+1], which assures that the 'generalized' [45] Wronskian ($\wp$-W) of two classical Jacobi polynomials

$$\mathcal{W}_{\wp}\{P_j^{(\lambda_+,\lambda_{o;-})}(\eta), P_{j'}^{(\lambda_+,\lambda_-)}(\eta)\} := \wp[\eta;\vec{\lambda}]\mathrm{W}\{P_j^{(\lambda_+,\lambda_{o;-})}(\eta), P_{j'}^{(\lambda_+,\lambda_-)}(\eta)\} \qquad (40)$$

$$\text{for } 0 \le j' < j \le j_{max}$$

vanishes at ±1.

Since our approach allows one to formulate the spectral problem only for positive values of the Jacobi indexes, this limitation restricts one's ability to construct the X-Jacobi OPSs formed by the RDC$\mathfrak{I}$s of the classical Jacobi polynomials with negative indexes, as it has already become clear [20,21] in the particular case of X_m-Jacobi OPSs. However, the certain advantage of our approach is that it allows one to treat in parallel the RSLPs for both intervals (-1,+1) and (1,∞) as seen from the simplest example discussed in next subsection.

Our next step is first to consider all the q-RSs of the given prime SLE, which vanish at one of the endpoints of the infinite interval (-1, +1) and then select the subsets of the collected PFSs below the lowest eigenvalue.

In following our olden study [18] on the Darboux transforms (D$\mathfrak{I}$s) of radial potentials, we use the letters **a** and **b** to specify the PFS near the singular endpoints ∓ 1 (cases I and II in Quesne's [46] commonly used classification scheme of q-RSs according to their behavior near the endpoints). We use the letters **c** and **d** [18] to identify the $n_{\mathfrak{c}}$ eigenfunctions and respectively all the q-RSs (34) not vanishing at both ends (case III in Quesne's classification scheme). For the given SLP there is the one-to-one correlation between the labels **t**=**a**, **b**, **c**, **d** and the sign $\sigma_\mp$ of the Jacobi indexes $\lambda_\mp$, as specified in Table 1.

Re-writing the dispersion formula (16) for $\vec{\sigma} = \pm\mp$ as

$$\varepsilon_{\pm\mp,m}(\vec{\lambda}_o) = (\mp\lambda_{o;-} \pm \lambda_{o;+} - 2m - 1)^2 \qquad (41)$$

Table 1. Correlation between labels **t** and signs of Jacobi indexes.

t	**a**	**b**	**c**	**d**
σ_- σ_+	+ -	- +	+ +	+ -

We find the PFSs of types **a** or **b** lie below the lowest eigenvalue

$$\varepsilon_{\mathbf{c},0}(\vec{\lambda}_o) = (\lambda_{o;-} + \lambda_{o;+} + 1)^2 \qquad (42)$$

iff

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$$m < -\lambda_+ = \lambda_{o;+} \quad (\lambda_\mp = \pm\lambda_{o;\mp}) \tag{43}$$

or

$$m < -\lambda_- = \lambda_{o;-} \quad (\lambda_\mp = \mp\lambda_{o;\mp}) \tag{44}$$

accordingly.

It has been proven by us in [27] that the Jacobi polynomials do not have zeros between -1 and +1 iff

$$m \le \tfrac{1}{2}\left(\lambda_{o;-} - \lambda_- + \lambda_{o;+} - \lambda_+\right) \tag{45}$$

And

$$(-)^m \langle\lambda_- + 1\rangle_m \langle\lambda_+ + 1\rangle_m =$$

$$\langle-\lambda_- - m\rangle_m \langle\lambda_+ + 1\rangle_m = \langle\lambda_- + 1\rangle_m \langle-\lambda_+ - m\rangle_m > 0, \tag{46}$$

where $\langle\nu\rangle_m$ is the rising factorial [48]. We come to (43) and (44) by choosing

$$\lambda_+ = -\lambda_-.$$

3.2. Dirichlet Problem on Interval (1,∞)

Examination of the q-RSs

$$\psi_j[[\eta; -\lambda_{o;-}, \lambda_{o;+}] = (1+\eta)^{\frac{1}{2}-\frac{1}{2}\lambda_{o;-}} (\eta-1)^{\frac{1}{2}\lambda_{o;+}} P_j^{(\lambda_{o;+}, -\lambda_{o;-})}(\eta) \tag{47}$$

reveals that they satisfy the DBCs at the both ends of the interval (1,∞) for

$$0 \le j < \tfrac{1}{2}(\lambda_{o;-}, -\lambda_{o;+} - 1) \tag{48}$$

and therefore represent the eigenfunctions of the RSLP in question, which brings us to the orthogonality relations

$$\int_1^\infty d\eta\, P_j^{(\lambda_{o;+}, -\lambda_{o;-})}(\eta) P_{j'}^{(\lambda_{o;+}, -\lambda_{o;-})}(\eta) \mathcal{W}_m[\eta; \vec{\lambda}_o] = 0 \quad (j \ne j') \tag{49}$$

with the weight function

$$\mathcal{W}_m[\eta; \vec{\lambda}_o] := (1+\eta)^{-\lambda_{o;-}} (\eta-1)^{\lambda_{o;+}} \quad \text{for} \quad \eta > 1. \tag{50}$$

One can easily verify that (49) is nothing but another form of the conventional orthogonality relations for the R-Jacobi polynomials

$$\int_0^\infty d\underset{\rightarrow}{z}\; J_j^{(\alpha,\beta)}(\underset{\rightarrow}{z}) J_{j'}^{(\alpha,\beta)}(\underset{\rightarrow}{z})\, \underset{\rightarrow}{\omega}_{\alpha,\beta}[\underset{\rightarrow}{z}] = 0 \quad (j \ne j') \tag{51}$$

with the weight function

$$\underset{\rightarrow}{\omega}_{\alpha,\beta}[\underset{\rightarrow}{z}] := \underset{\rightarrow}{z}^{\alpha} (\underset{\rightarrow}{z}+1)^{-|\beta|} \quad \text{for} \quad \eta \in [1, \infty) \tag{52}$$

under constraint $\alpha > 0$, $\beta < 0$, where we adopted Askey's [49] definition of the R-Jacobi polynomials which, as proven by Chen and Srivastava [50], is equivalent to the elementary formula

$$J_n^{(\alpha,\beta)}(\underset{\rightarrow}{z}) := P_n^{(\alpha,\beta)}(2\underset{\rightarrow}{z}+1) \text{ for } \alpha > -1,\ \beta + 2n < 0, \tag{53}$$

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with

$$\underset{\sim}{z} := \tfrac{1}{2}(\eta - 1). \qquad (54)$$

Note that we [5,20,21,24] (see also [51]) changed the symbol R for J to avoid the confusion with R-Routh (Romanovski/pseudo-Jacobi [30,31] polynomials denoted in the recent publications [51–55] by the same letter 'R'.

Our next step is to determine all the q-RSs vanishing at one of the endpoints of the infinite interval [+1, +∞) and then select the subsets of the collected PFSs below the lowest eigenvalue. To explicitly reveal the behavior of the Jacobi-seed (JS) q-RSs (54) near the singular endpoints in question, we [5,20,21,47] label them as indicated in Table 2 below, with σ_∞ specifying either the decay (+) or growth (-) of the given JS at infinity.

We underline the symbol **†** by tilde to indicate that the classification of the JS solutions is done on the infinite interval (1, ∞). We then mark the given symbol by prime if the polynomial components of the given sequence of the q-RSs do not include a constant. (Note that the 'secondary' sequences of such a type do not exist for the potentials with infinitely many discrete energy levels which were the focal point of Quesne's analysis [46].)

By definition

$$\varepsilon_{\underset{\sim}{\dagger}\vec{\sigma},m},m(\vec{\lambda}_o) \equiv -\varepsilon_{\vec{\sigma},m}(\vec{\lambda}_o), \qquad (55)$$

Note that the PFSs of the series $\underset{\sim}{\mathbf{b}}'$ may exist only if the SLE does not have the discrete energy spectrum. We thus need to consider the three sequences of the quasi-rational PFSs: two *primary* (starting from m=0) sequences $\underset{\sim}{\mathbf{a}}$ and $\underset{\sim}{\mathbf{b}}$ as well as the infinite secondary sequence $\underset{\sim}{\mathbf{a}}'$ starting from $m = n_{\underset{\sim}{c}}$.

Table 2. Classification of JS solutions on the infinite interval (1, ∞) based on their asymptotic behavior near the endpoints.

$\underset{\sim}{\dagger}_{\vec{\sigma},m}$	$\sigma_- \ \sigma_+ \ \sigma_\infty$	m
$\underset{\sim}{\mathbf{a}}$	+ + −	$0 \le m < \infty$
$\underset{\sim}{\mathbf{a}}'$	− + −	$m \ge n_{\underset{\sim}{c}} = j_{max} + 1$
$\underset{\sim}{\mathbf{b}}$	− − +	$0 \le m < \tfrac{1}{2}(\lambda_{o;-} + \lambda_{o;+} - 1)$
$\underset{\sim}{\mathbf{b}}'$	− + +	$0 \le m < \tfrac{1}{2}(\lambda_{o;+} - \lambda_{o;-} - 1)$
$\underset{\sim}{\mathbf{c}}$	− + +	$0 \le m \le j_{max}$
$\underset{\sim}{\mathbf{d}}$	+ − −	$\tfrac{1}{2}(\lambda_{o;+} - \lambda_{o;-} - 1) \le m < \infty$
$\underset{\sim}{\mathbf{d}}'$	− − −	$m > \tfrac{1}{2}(\lambda_{o;-} + \lambda_{o;+} - 1)$

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The primary sequence $\underset{\sim}{a}$ is formed by classical Jacobi polynomials and consequently may not have zeros between 1 and ∞. As expected, all the PFSs of this type lie at the energies

$$\varepsilon_{\underset{\sim}{a},m}(\vec{\lambda}_o) \equiv -\varepsilon_{++,m}(\vec{\lambda}_o) \qquad (56)$$

below the lowest eigenvalue

$$\varepsilon_{\underset{\sim}{c},0}(\vec{\lambda}_o) \equiv -\varepsilon_{+-,0}(\vec{\lambda}_o). \qquad (57)$$

The PFSs from the primary sequence $\underset{\sim}{b}$ at the energies

$$\varepsilon_{\underset{\sim}{b},m}(\vec{\lambda}_o) \equiv -\varepsilon_{--,m}(\vec{\lambda}_o) \qquad (58)$$

for

$$0 \le m < \tfrac{1}{2}(\lambda_{o;-} + \lambda_{o;+} - 1) \qquad (59)$$

do not have real zeros larger than 1 iff

$$\varepsilon_{\underset{\sim}{b},m}(\vec{\lambda}_o) - \varepsilon_{\underset{\sim}{c},0}(\vec{\lambda}_o) = -4(\lambda_{o;-} - m - 1)(\lambda_{o;+} - m) < 0, \qquad (60)$$

i.e., iff

$$0 \le m < \lambda_{o;+} < \lambda_{o;-} - 1. \qquad (61)$$

Similarly the PFSs from the secondary sequence $\underset{\sim}{a}'$ at the energies

$$\varepsilon_{\underset{\sim}{a}',m}(\vec{\lambda}_o) = -\varepsilon_{+-,m}(\vec{\lambda}_o) \quad \text{for} \quad m \ge n_{\underset{\sim}{c}} \qquad 62)$$

do not have real zeros larger than 1 iff

$$\varepsilon_{\underset{\sim}{a}',m}(\vec{\lambda}_o) - \varepsilon_{\underset{\sim}{c},0}(\vec{\lambda}_o) = -4m(\lambda_{o;+} - \lambda_{o;-} + m + 1) < 0 \qquad (63)$$

or, in other words, iff

$$m > \lambda_{o;-} - \lambda_{o;+} - 1. \quad (64)$$

4. RDCT of JRef SLE Using Seed Jacobi Polynomials with Common Pair of Indexes

We call the DCT rational if it uses quasi-rational seed functions. In this Section we focus solely on the RDCTs using the seed functions (20) with the common Jacobi indexes $\vec{\lambda}$. Let us consider the RDCT using an arbitrary set of p seed functions,

$$\bar{M}_p := m_1, \dots, m_p .$$

Denoting the Jacobi polynomial Wronskian (JPW) as

$$W^{(\lambda_+,\lambda_-)}_{\mathfrak{N}(\bar{M}_p)}[\eta \,|\, \bar{M}_p] := W\{P^{(\lambda_+,\lambda_-)}_{m_{k=1,\dots,p}}(\eta)\} \qquad (65)$$

and substituting the Wronskian

$$W\{\phi_{m_{k=1,\dots,p}}[\eta;\vec{\lambda}]\} = \phi_0^p[\eta;\vec{\lambda}]W^{(\lambda_+,\lambda_-)}_{\mathfrak{N}(\bar{M}_p)}[\eta \,|\, \bar{M}_p] \qquad (66)$$

into (A18), we come to the RCSLE

$$\left\{ \frac{d^2}{d\eta^2} + I^o[\eta;\vec{\lambda}|\bar{M}_p] + sgn(1-\eta^2)\varepsilon\,\rho[\eta] \right\} \Phi[\eta;\vec{\lambda};\varepsilon|\bar{M}_p] = 0 \qquad (67)$$

with the RefPF [22]

$$I^o[\eta;\vec{\lambda}|\bar{M}_p] = I^o[\eta;\vec{\lambda}_o] + 2p\sqrt{\rho[\eta]}\frac{d}{d\eta}\frac{ld\,\phi_0[\eta;\vec{\lambda}]}{\sqrt{\rho[\eta]}} - p(p-2)\,\mathfrak{J}\{\rho[\eta]\}$$

$$+2\sqrt{\rho[\eta]}\frac{d}{d\eta}\frac{ld\,W^{(\lambda_+,\lambda_-)}_{\mathfrak{N}(\bar{M}_p)}[\eta\,|\,\bar{M}_p]}{\sqrt{\rho[\eta]}}. \qquad (68)$$

Let us now show that the first three summands can be then re-arranged as

$$I^o[\eta;\vec{\lambda}+p\vec{1}] = I^o[\eta;\vec{\lambda}_o] + 2p\sqrt{\rho[\eta]}\frac{d}{d\eta}\frac{ld\,\phi_0[\eta;\vec{\lambda}]}{\sqrt{\rho[\eta]}} \qquad (69)$$

$$+2\sqrt{\rho[\eta]}\frac{d}{d\eta}\frac{ld\,W^{(\lambda_+,\lambda_-)}_{\mathfrak{N}(\bar{M}_p)}[\eta\,|\,\bar{M}_p]}{\sqrt{\rho[\eta]}} - p(p-2)\mathfrak{J}\{|1-\eta^2|\}$$

and then prove that

$$I^o[\eta;\vec{\lambda}|\bar{M}_p] = I^o[\eta;\vec{\lambda}+p\vec{1}] + 2\,\dot{ld}\,W^{(\lambda_+,\lambda_-)}_{\mathfrak{N}(\bar{M}_p)}[\eta\,|\,\bar{M}_p]$$

$$-ld\,\rho[\eta]\,ld\,W^{(\lambda_+,\lambda_-)}_{\mathfrak{N}(\bar{M}_p)}[\eta\,|\,\bar{M}_p] \qquad (70)$$

which represents one of the most important results of this section. To prove (70), we first re-write the second summand in (69) as

$$2p\sqrt{\rho[\eta]}\frac{d}{d\eta}\frac{ld\,\phi_0[\eta;\vec{\lambda}]}{\sqrt{\rho[\eta]}} = 2p\,\dot{ld}\,\phi_0[\eta;\vec{\lambda}] - p\,ld\,\rho[\eta]\,ld\,\phi_0[\eta;\vec{\lambda}]. \qquad (71)$$

Taking into account (9), coupled with

$$ld\,\rho[\eta] = -\frac{1}{\eta+1} - \frac{1}{\eta-1}, \qquad (72)$$

gives

$$2p\sqrt{\rho[\eta]}\frac{d}{d\eta}\frac{ld\,\phi_0[\eta;\vec{\lambda}]}{\sqrt{\rho[\eta]}} = 2p\,\dot{ld}\,\phi_0[\eta;\vec{\lambda}] - p\,ld\,\rho[\eta]\,ld\,\phi_0[\eta;\vec{\lambda}] \qquad (73)$$

$$= -\frac{p(\lambda_+ +1)}{2(\eta-1)^2} - \frac{p(\lambda_- +1)}{2(\eta+1)^2} + \frac{p(\lambda_- + \lambda_+ +2)}{2(\eta^2-1)}. \qquad (74)$$

Combining (74) with the definition (3) of the RefPF of the JRef CSLE (1), and also taking into account that [ ]

$$\mathfrak{J}\{|1-\eta^2|\} = \frac{1}{4(\eta-1)^2} + \frac{1}{4(\eta+1)^2} - \frac{1}{2(\eta^2-1)}, \qquad (75)$$

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one can directly confirm that the three distinguished singularities appearing in the right-hand side of (74) can be grouped as follows

$$2p\sqrt{\rho[\eta]}\,\frac{d}{d\eta}\,\frac{ld\,\phi_0[\eta;\vec{\lambda}]}{\sqrt{\rho[\eta]}} = I^o[\eta;\vec{\lambda}+p\vec{1}] - I^o[\eta;\vec{\lambda}_o] + \qquad (76)$$

$$+p(p-2)\mathcal{J}\{|1-\eta^2|\}.$$

Before proceeding with the further analysis of the RefPF (66), let us first illuminate some remarkable features of the JPW (65).

Theorem 1: *The JPW (65) is finite at the singular point* ∓ 1 if $\lambda_{\mp} > 0$.

Proof. Let us examine more thoroughly the TF (A19) for the rational RZT (RRZT) applied to the RCSLE (67). Making use of (66), one can easily verify that it has the following quasi-rational form

$$\phi_{m_{p+1}}[\eta;\vec{\lambda}|\bar{M}_p] = \frac{\phi_0[\eta;\vec{\lambda}]W^{(\lambda_+,\lambda_-)}_{\mathfrak{N}(\bar{M}_{p+1})}[\eta\,|\,\bar{M}_{p+1}]}{\rho^{1/2p}[\eta]W^{(\lambda_+,\lambda_-)}_{\mathfrak{N}(\bar{M}_p)}[\eta\,|\,\bar{M}_p]}, \qquad (77)$$

i.e., taking into account (2) and (8),

$$\phi_{m_{p+1}}[\eta;\vec{\lambda}|\bar{M}_p] = \frac{\phi_0[\eta;\vec{\lambda}+p\vec{1}]W^{(\lambda_+,\lambda_-)}_{\mathfrak{N}(\bar{M}_{p+1})}[\eta\,|\,\bar{M}_{p+1}]}{W^{(\lambda_+,\lambda_-)}_{\mathfrak{N}(\bar{M}_p)}[\eta\,|\,\bar{M}_p]}. \qquad (78)$$

If we assume that the JPW in the denominator of the PF on the right remains finite at ∓ 1, then, according to (70), the power exponent of $\eta\pm 1$ coincides with one of the two characteristic exponents for the pole of the RCSLE (67) at ∓ 1. The TF (78) thus represents the PFS near the pole in question iff $\lambda_{\mp} > 0$. This implies that that the numerator of the PF may not have the zero at ∓ 1. This completes the proof of Theorem 1, since it necessarily holds for p=1 due to the constraint (17) imposed on the seed Jacobi polynomial. □

Corollary 1: *The Wronskian of classical Jacobi polynomials with positive indexes may not have zeros at* ∓ 1.

As illuminated in subsection 7.2, this corollary plays the crucial role in the theory of the RDC$\mathfrak{I}$s of the R-Jacobi polynomials using the quasi-rational seed functions formed by the classical Jacobi polynomials with positive indexes.

Preposition 1: *The JPW does not generally have zeros at* ∓ 1, *regardless of the sign of* $\lambda_{\mp}$.

Preposition 2: *The JPWs in the numerator and denominator of the fraction (77) do not have common zeros.*

Theorem 2: *The JPW in the numerator of the fraction (77) has only simple zeros, assuming that both Prepositions 1 and 2 hold.*

Proof. Based on our prepositions, any zero of the JPW in the numerator of the fraction (77) is a regular point of the RCSLE (67) and therefore the polynomial in question may not have zeros of order higher than 1. (Otherwise the solution (77) of the RCSLE (67) and its first derivative would vanish at the same point which is possible only for the trivial solution identically equal to zero). □

Let

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$$\bar{\eta}(\vec{\lambda}\,|\,\bar{M}_p) := \eta_{l=1,\ldots,\mathfrak{N}(\bar{M}_p)}(\vec{\lambda}\,|\,\bar{M}_p) \qquad (79)$$

be the $\mathfrak{N}(\bar{M}_p)$ zeros of the JPW (65), i.e.,

$$\hat{W}^{(\lambda_+,\lambda_-)}_{\mathfrak{N}(\bar{M}_p)}[\eta\,|\,\bar{M}_p] = \Pi_{\mathfrak{N}(\bar{M}_p)}[\eta;\bar{\eta}^{(\mathfrak{N})}(\vec{\lambda}|\bar{M}_p)] :=$$

$$\prod_{\ell=1}^{\mathfrak{N}(\bar{M}_p)} [\eta - \eta_\ell(\vec{\lambda}|\bar{M}_p)] \qquad (80)$$

Re-writing (72) as

$$ld\,\rho[\eta] = -\frac{2\eta}{\eta^2 - 1} \qquad (81)$$

and taking into account that

$$Q[\eta;\bar{\eta}] := \dot{ld}\,\Pi_m[\eta;\bar{\eta}] = -\sum_{l=1}^{m} \frac{1}{[\eta - \eta_l]^2}, \qquad (82)$$

we can decompose the RefPF (70) as follows

$$I^o[\eta;\vec{\lambda}|\bar{M}_p] = I^o[\eta;\vec{\lambda} + p\vec{1}] - \sum_{\ell=1}^{\mathfrak{N}(\bar{M}_p)} \frac{2}{[\eta - \eta_l(\vec{\lambda}|\bar{M}_p)]^2}$$
$$+\frac{2\eta}{\eta^2 - 1} \sum_{\ell=1}^{\mathfrak{N}(\bar{M}_p)} \frac{1}{\eta - \eta_l(\vec{\lambda}|\bar{M}_p)}, \qquad (83)$$

in agreement with (87) in [21] for p=1.

The indicial equation for the extraneous poles of the RCSLE (67) has exactly the same form

$$\rho(\rho\text{-}1)\ \text{-}2 = 0. \qquad 84)$$

The equation has two roots -1 and 2, which implies that the JPW in the numerator of the fraction (78) may have a zero of the third order [56]. However, as it becomes obvious from the analysis presented in [56], this is a relatively exotic case which will be simply disregarded here.

5. Four Infinite Sequences of q-RSs with Polynomial Components Represented by Simple p-WPs

The RCSLE (67) has 4 infinite sequences of the q-RSs:

$$\phi_{\vec{\sigma},j}[\eta;\vec{\lambda}|\bar{M}_p] = \frac{W\{\phi_{m_{k=1,\ldots,p}}[\eta;\vec{\lambda}], \phi_j[\eta;\vec{\lambda}']\}}{\rho^{-1/2p}[\eta]W\{\phi_{m_{k=1,\ldots,p}}[\eta;\vec{\lambda}]\}} \qquad (85)$$

with

$$\lambda'_{\mp} = \sigma_{\mp}\vec{\lambda}_{\mp} \qquad (86)$$

and

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$$|\lambda'_{\mp}| = \lambda_{o;\mp}. \qquad (87)$$

The q-RSs (78) represent the simplest case ($\phi_{\mp} = +$) when the polynomial components turn into the JPWs.

Theorem 3: *The polynomial components of the* q-RSs (74) *can be represented in the form of simple* p*-WPs defined via (92) below.*

Proof. In following [26}, let us first introduce the eigenfunctions of the Jacobi operator:

$$f[\eta;\vec{\lambda}'|\vec{\phi}, \mathrm{j}] := f[\eta;\vec{\lambda}'|\vec{\phi}]\mathrm{P}_{\mathrm{j}}^{(\lambda'_+,\lambda'_-)}(\eta), \qquad (88)$$

where

$$f[\eta;\vec{\lambda}'|\vec{\phi}] := \phi_0[\eta;\vec{\lambda}'] / \phi_0[\eta;\vec{\phi}\times\vec{\lambda}'], \qquad (89)$$

$$= \prod_{\aleph=\pm} |1-\aleph\eta|^{\frac{1}{2}(1-\phi_{\aleph}1)\lambda'_{\aleph}}, \qquad (90)$$

(see Table 1 in [26] for details). We can then re-write the Wronskian in the numerator of the PF (85) as

$$\mathrm{W}\{\phi_{\mathrm{m}_{k=1,\dots,p}}[\eta;\vec{\lambda}], \phi_{\mathrm{j}}[\eta;\vec{\lambda}']\} \qquad (91)$$

$$= \phi_0^{\mathrm{p}+1}[\eta;\vec{\lambda}]\times \mathrm{W}\{\mathrm{P}_{\mathrm{m}_{k=1,\dots,p}}^{(\lambda_+,\lambda_-)}(\eta), f[\eta;\vec{\lambda}|\vec{\phi}, \mathrm{j}]\}.$$

Making use of Jacobi polynomial relations (92) in [25], we can represent the derivatives of functions (4.1) in in the explicitly quasi-rational form:

$$\frac{d^l}{d\eta^l} f[\eta;\vec{\lambda}|\vec{\phi}, \mathrm{j}] = f[\eta;\vec{\lambda}|\vec{\phi}, 0]\Theta_{\vec{\phi}}^{-\mathrm{p}}[\eta] P^{(l)}_{\mathrm{n}_{\mathrm{j};\vec{\phi}}(\mathrm{p}-l)}[\eta;\vec{\lambda}'|\vec{\phi}] \qquad (92)$$

with the polynomial components:

$$P^{(l)}_{\mathrm{n}_{\mathrm{j};|\vec{\phi}|}(k)}[\eta;\vec{\lambda}'|\vec{\phi}] := \mathrm{d}^{(l)}_{\vec{\phi},\mathrm{j}}(\vec{\lambda}')\Theta^{k}_{1-\frac{1}{2}|\vec{\phi}|}[\eta]\, \mathrm{P}^{(\lambda'_+ + \phi_+ l, \lambda'_- + \phi_- l)}_{\mathrm{j}-|\vec{\phi}|\frac{l}{2}}(\eta), \qquad (93)$$

where

$$\Theta_{1-\frac{1}{2}|\vec{\phi}|}[\eta] := \prod_{\aleph=\pm} (1-\aleph\eta)^{\frac{1}{2}(1-\phi_{\aleph}1)}, \qquad (94)$$

$$|\vec{\phi}| := \phi_+ 1 + \phi_- 1, \qquad (95)$$

and

$$\mathrm{n}_{\mathrm{j};|\vec{\phi}|}(k) = \mathrm{j} - 1 + (k+1)(1 - \tfrac{1}{2}|\vec{\phi}|). \qquad (96)$$

The proportionality factors in (93) are determined by the elementary formulas [25]:

$$d^{(l)}_{\vec{\sigma},j}(\vec{\lambda}') = \begin{cases} 2^{-l}\langle \lambda'_+ + \lambda'_- + j+1\rangle_l & \text{if } \vec{\sigma} = ++, \\ (-2)^l \langle j+1\rangle_l & \text{if } \vec{\sigma} = --, \\ (-1)^l \langle j+\lambda'_+\rangle_l & \text{if } \vec{\sigma} = -+. \\ \langle j+\lambda'_-\rangle_l & \text{if } \vec{\sigma} = +-, \end{cases} \quad (97)$$

The listed formulas can be directly verified by expressing the hypergeometric functions in terms of Jacobi polynomials in 2.1(20), 2.1(27), 2.1(24), and 2.1(22) in [57], with a = -m. While all four Jacobi polynomial relations (92), (93), and (97) were obtained in [25] based on the translational shape-invariance of the trigonometric Pöschl-Teller (t-PT) potential, we prefer to refer the reader to the more general relations 2.1(7), 2.1(9), 2.1(8), and 2.1(22) for hypergeometric functions in [57] as the starting point for validating (92). The cited relations are valis are within a broader range of the parameters, beyond the limits of the Liouville transformations implicitly used in [18].

Substituting the derivatives (92) into the Wronskian in the right-hand side of (86), we can represent the quasi-rational form

$$W\{P^{(\lambda_+,\lambda_-)}_{m_{k=1,\dots,p}}(\eta), f[\eta;\vec{\lambda}'|\vec{\sigma},j]\} = f[\eta;\vec{\lambda}'|\vec{\sigma}]\Theta^{-p}_{1-\frac{1}{2}|\vec{\sigma}|}[\eta]\mathcal{P}_{\mathfrak{N}(\vec{\sigma}:\bar{M}_p;j)}[\eta;\vec{\lambda}'|\vec{\sigma}:\bar{M}_p;j], \quad (98)$$

with the polynomial component represented by the polynomial determinant:

$$\mathcal{P}_{\mathfrak{N}(\vec{\sigma}:\bar{M}_p;j)}[\eta;\vec{\lambda}'|\vec{\sigma}:\bar{M}_p;j] := \quad (99)$$

$$\begin{vmatrix} P^{(\lambda_+,\lambda_-)}_{m_1}(\eta) & P^{(\lambda_+,\lambda_-)}_{m_2}(\eta) & \dots & \Theta^p_{1-\frac{1}{2}|\vec{\sigma}|}[\eta]P^{(\lambda'_+,\lambda'_-)}_j(\eta) \\ \frac{d}{d\eta}P^{(\lambda_+,\lambda_-)}_{m_1}(\eta) & \frac{d}{d\eta}P^{(\lambda_+,\lambda_-)}_{m_2}(\eta) & \dots & P^{(1)}_{n_{j;|\vec{\sigma}|}(p-1)}[\eta;\vec{\lambda}'|\vec{\sigma}] \\ & & \dots & \\ \frac{d^p}{d^p\eta}P^{(\lambda_+,\lambda_-)}_{m_1}(\eta) & \frac{d^p}{d^p\eta}P^{(\lambda_+,\lambda_-)}_{m_2}(\eta) & \dots & P^{(p)}_{n_{j;|\vec{\sigma}|}(0)}[\eta;\vec{\lambda}'|\vec{\sigma}] \end{vmatrix}.$$

Keeping in mind that

$$n_{j;|\vec{\sigma}|}(0) = j - \tfrac{1}{2}|\vec{\sigma}|, \quad (100)$$

one verify that the polynomial determinant (99) has the degree not larger than

$$\mathfrak{N}(\vec{\sigma}:\bar{M}_p;j) = \mathfrak{N}(\vec{\sigma}:\bar{M}_p) + j - \tfrac{1}{2}|\vec{\sigma}|, \quad (101)$$

where [58]

$$\mathfrak{N}(\bar{M}_p) = \sum_{k=1}^{p} m_k - \tfrac{1}{2}p(p-1). \quad (102)$$

One can easily verify (101) using the cofactor expansion of the determinant (99) in terms of the ($p-l$, p+1) minors (l =0,…,p). The first term in the sum brings us directly to (101), while the degree of the($p+1-l$)-th column element and degree of the corresponding cofactor polynomial minor then increases and respectively decreases by 1 as l grows. confirming that all the polynomial summands have the common degree. As stressed by Bonneux [26], the determinant degree can be smaller for some particular values of the Jacobi indexes. Here we assume that the degree of the polynomial

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determinant (99) is equal exactly to (101) and we refer the refer the reader to [26] for the discussion of the necessary constraints on the Jacobi indexes.

The numerator of the fraction (85) thus takes the form:

$$W\{\phi_{m_{k=1,\ldots,p}}[\eta;\vec{\lambda}],\phi_j[\eta;\vec{\lambda}']\} = \phi_0^p[\eta;\vec{\lambda}]\Theta_{\vec{\sigma}}^{-p}[\eta]\phi_0[\eta;\vec{\lambda}']$$

$$\times \mathcal{P}_{\mathfrak{N}(\vec{\sigma}:\bar{M}_p)+j}[\eta;\vec{\lambda}'\,|\,\vec{\sigma}:\bar{M}_p;j] \quad (103)$$

Making use of the identity

$$\Theta_{\vec{\sigma}}^{-p}[\eta]\phi_0[\eta;\vec{\lambda}'] = \phi_0[\eta;\vec{\lambda}' + p(\vec{\sigma}\times\vec{1} - \vec{1})], \quad (104)$$

we can then re-write (103) as

$$W\{\phi_{m_{k=1,\ldots,p}}[\eta;\vec{\lambda}],\phi_j[\eta;\vec{\lambda}']\} = \phi_0^p[\eta;\vec{\lambda}]\phi_0[\eta;\vec{\lambda}' + p(\vec{\sigma}\times\vec{1} - \vec{1})]$$

$$\times \mathcal{P}_{\mathfrak{N}(\vec{\sigma}:\bar{M}_p)+j}[\eta;\vec{\lambda}'\,|\,\vec{\sigma}:\bar{M}_p;j], \quad (105)$$

so the fraction takes the sought-for form:

$$\phi_{\vec{\sigma},j}[\eta;\vec{\lambda}|\bar{M}_p] = \phi_0[\eta;\vec{\sigma}\times(\vec{\lambda}+p\,\vec{1})]\frac{\mathcal{P}_{\mathfrak{N}(\vec{\sigma}:\bar{M}_p)+j}[\eta;\vec{\lambda}'\,|\,\vec{\sigma}:\bar{M}_p;j]}{W_{\mathfrak{N}(\bar{M}_p)}^{(\lambda_+,\lambda_-)}[\eta\,|\,\bar{M}_p]}, \quad (106)$$

which completes the proof. □ Setting $p = 1$, $\bar{M}_1 = m$ brings us to (125) in [21], as expected.

In particular, if we choose $\vec{\sigma} = -+$, $\lambda_+ = \alpha$, $\lambda_- = \beta$, i.e.,

$$f[\eta;-\beta,\alpha\,|\,-+] = (1+\eta)^{-\beta} \quad (107)$$

and

$$\Theta_{-+}[\eta] = 1+\eta, \quad (108)$$

we come to the generalized Jacobi polynomials (2.7) in [26], with $r_1 = p$ and $r_2 = 1$.

It is however crucial for our discussion that the power exponents of $\eta \pm 1$ for the weight function of the PF in the right-hand side of (106) coincides with one of the characteristic exponents for the pole of the CSLE (63) at ∓ 1, This is the unique feature of the simple p-WPs which by definition have only one column formed by polynomials (88). As illuminated in Section 6, it assures that the p-WPs in question form a X-DPS.

To illustrate the above assertion, let us represent the Wronskian (103) in the alternative p-W form:

$$W\{\phi_{m_{k=1,2}}[\eta;\vec{\lambda}],\phi_j[\eta;\vec{\lambda}']\} = \Theta_{\vec{\sigma}}^{-4}[\eta]\phi_0[\eta;\vec{\lambda}']\phi_0^2[\eta;\vec{\lambda}] \quad (109)$$

$$\times \mathcal{P}_{\mathfrak{N}(\vec{\sigma}:m_1,m_2;j)}[\eta;\vec{\lambda}\,|\,\vec{\sigma},j;m_1,m_2],$$

where

$$\mathcal{P}_{\mathfrak{N}(\vec{\sigma}:m_1,m_2;j)}[\eta;\vec{\lambda}\,|\,\vec{\sigma},j;m_1,m_2] := \quad (110)$$

$$\begin{vmatrix} \Theta^2_{\vec{\sigma}}[\eta]P^{(\lambda_+,\lambda_-)}_{m_1}(\eta) & \Theta^2_{\vec{\sigma}}[\eta]P^{(\lambda_+,\lambda_-)}_{m_2}(\eta) & P^{(\lambda'_+,\lambda'_-)}_{j}(\eta) \\ P^{(1)}_{m_1-\sigma_+\iota/2-\sigma_-\iota/2}[\eta;\vec{\lambda}\,|\,\vec{\sigma}] & P^{(1)}_{m_2-\sigma_+\iota/2-\sigma_-\iota/2}[\eta;\vec{\lambda}\,|\,\vec{\sigma}] & \dot{P}^{(\lambda'_+,\lambda'_-)}_{j}(\eta) \\ P^{(2)}_{m_1-\sigma_+\iota/2-\sigma_-\iota/2}[\eta;\vec{\lambda}\,|\,\vec{\sigma}] & P^{(2)}_{m_2-\sigma_+\iota/2-\sigma_-\iota/2}[\eta;\vec{\lambda}\,|\,\vec{\sigma}] & \ddot{P}^{(\lambda'_+,\lambda'_-)}_{j}(\eta) \end{vmatrix}$$

(the generalized Jacobi polynomial (2.7) in {26], with $\lambda_+ = \alpha,\ \lambda_- = -\beta,\ r_1 = 1$, and $r_2 = 2$). Comparing (109) with (103) gives

$$\mathcal{P}_{\mathfrak{N}(\vec{\sigma}:m_1,m_2;\,j)}[\eta;\vec{\lambda}\,|\,\vec{\sigma},\,j;m_1,m_2] = \Theta^2_{\vec{\sigma}}[\eta]\mathcal{P}_{\mathfrak{N}(\vec{\sigma}:\bar{M}_p;\,j)}[\eta;\vec{\lambda}'\,|\,\vec{\sigma}:\bar{M}_p;j], \quad (111)$$

i.e., the p-WPs (108) vanish at least at one of the points ∓ 1, other than the trivial case $\vec{\sigma} = ++$, when the polynomials in question turn into the polynomial Wronskians. This is the main reason why we restrict our analysis solely to the simple p-WPs, which do remain finite at both points, at least if the ExpDiffs for the poles of the RCSLE (67) are not positive integers.

6. X-Jacobi DPSs Composed of Simple p-WPs

Our next step is to prove that both JPW (65) and all three p-WPs in the numerator of the fraction in the right-hand side of (106) for $\vec{\sigma} \neq ++$ satisfy the Bochner-type ODEs.

Theorem 4: *The polynomials (99) satisfy the Bochner-type ODEs and therefore form four distinguished X-Jacobi DPSs.*

Proof: Let us consider the four alternative gauge transformations

$$\Phi[\eta;\vec{\lambda}_o;\varepsilon|\bar{M}_p] = \phi_0[\eta;\vec{\lambda}_o|\bar{M}_p;\vec{\sigma}]\times F[\eta;\vec{\lambda}_o;\varepsilon|\bar{M}_p;\vec{\sigma}] \qquad (112)$$

with the gauge function

$$\phi_0[\eta;\vec{\lambda}|\bar{M}_p;\vec{\sigma}] = \frac{\phi_0[\eta;\vec{\sigma}\times(\vec{\lambda}+p\vec{1})]}{W^{(\lambda_+,\lambda_-)}_{\mathfrak{N}(\bar{M}_p)}[\eta\,|\,\bar{M}_p]} \qquad (113)$$

satisfying the RCSLE

$$\ddot{\phi}_0[\eta;\vec{\lambda}|\bar{M}_p;\vec{\sigma}] + \left\{ I^0_0[\eta;\vec{\lambda}|\bar{M}_p] - \frac{\varepsilon_0(\vec{\lambda}'+p\vec{\sigma}\times\vec{1})}{\eta^2-1} \right\}\phi_0[\eta;\vec{\lambda}|\bar{M}_p;\vec{\sigma}] = 0 \quad (114)$$

with the RefPF

$$I^0_0[\eta;\vec{\lambda}|\bar{M}_p] := I^0[\eta;\vec{\lambda}+p\vec{1}] + 2\hat{Q}[\eta;\bar{\eta}^{(\mathfrak{N})}(\vec{\lambda}|\bar{M}_p)] \qquad (115)$$

$$+2ld\,\phi_0[\eta;\vec{\sigma}\times(\vec{\lambda}+p\vec{1})]ld\,W^{(\lambda_+,\lambda_-)}_{\mathfrak{N}(\bar{M}_p)}[\eta\,|\,\bar{M}_p],$$

where [47]

$$\hat{Q}[\eta;\bar{\eta}] := -\tfrac{1}{2}\Pi_m[\eta;\bar{\eta}]\frac{d^2}{d\eta^2}\Pi^{-1}_m[\eta;\bar{\eta}] \qquad (116)$$

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$$= \frac{1}{2}\frac{\ddot{\Pi}_m[\eta;\bar{\eta}]}{\Pi_m[\eta;\bar{\eta}]} - \frac{\dot{\Pi}^2_m[\eta;\bar{\eta}]}{\Pi^2_m[\eta;\bar{\eta}]}. \qquad (117)$$

The PF (116) is related to the Quesne PF [59–61]

$$Q[\eta;\bar{\eta}] := \frac{\ddot{\Pi}_m[\eta;\bar{\eta}]}{\Pi_m[\eta;\bar{\eta})]} - \frac{\dot{\Pi}^2_m[\eta;\bar{\eta}]}{\Pi^2_m[\eta;\bar{\eta}]} \qquad (118)$$

in the elementary fashion:

$$\hat{Q}[\eta;\bar{\eta}] = Q[\eta;\bar{\eta}] + \frac{\ddot{\Pi}_m[\eta;\bar{\eta}]}{2\Pi_m[\eta;\bar{\eta}]} \qquad (119)$$

$$= -\sum_{l=1}^{m}\frac{1}{[\eta-\eta_l]^2} + \frac{\ddot{\Pi}_m[\eta;\bar{\eta}]}{2\Pi_m[\eta;\bar{\eta}]}. \qquad (120)$$

In our earlier works [47,62] we adopted the Quesne PF in the form (118) (see, i.g., (39) in [61], with $g_\mu^{(\alpha)}$ standing for $\Pi_m[\eta;\bar{\eta}]$ here), overlooking its alternative form (82) without any mixed simple poles at $\eta_{l=1,\ldots,m}$.

Substituting (112) into the RCSLE (67) and taking advantage of (114), coupled with (115) and (120), we come to the second-order ODE:

$$\{\mathbf{D}_\eta(\vec{\lambda}|\bar{M}_p;\vec{\phi}) + C_{\mathfrak{N}(\bar{M}_p)}[\eta;\vec{\lambda};\varepsilon\,|\,\bar{M}_p;\vec{\phi}]\}F[\eta;\vec{\lambda}_o;\varepsilon|\bar{M}_p;\vec{\phi}] = 0, \qquad (121)$$

where $\mathbf{D}_\eta(\vec{\lambda}|\bar{M}_p;\vec{\phi})$ is an abbreviated notation for the second-order differential operator in η:

$$\mathbf{D}_\eta(\vec{\lambda}|\bar{M}_p;\vec{\phi}) = (\eta^2-1)\Pi_{\mathfrak{N}(\bar{M}_p)}[\eta;\bar{\eta}^{(\mathfrak{N})}(\vec{\lambda}\,|\,\bar{M}_p)]\frac{d^2}{d\eta^2} \qquad (122)$$

$$+2B_{\mathfrak{N}(\bar{M}_p)+1}[\eta;\vec{\lambda}\,|\,\bar{M}_p;\vec{\phi}]\frac{d}{d\eta}$$

with the polynomial coefficient function of the first derivative

$$B_{\mathfrak{N}(\bar{M}_p)+1}[\eta;\vec{\lambda}\,|\,\bar{M}_p;\vec{\phi}] := (\eta^2-1)\,\Pi_{\mathfrak{N}(\bar{M}_p)}[\eta;\bar{\eta}^{(\mathfrak{N})}(\vec{\lambda}\,|\,\bar{M}_p)]$$

$$\times\left(\sum_{\aleph=\pm}\frac{\phi_\aleph(\lambda_\aleph+1)+1}{2(\eta-\aleph 1)} - \sum_{l=1}^{\mathfrak{N}(\bar{M}_p)}\frac{1}{\eta-\eta_l(\vec{\lambda};\bar{M}_p)}\right). \qquad (123)$$

The ε-dependent polynomial of degree m representing the free term of the ODE (121) is linear in the energy:

$$C_{\mathfrak{N}(\bar{M}_p)}[\eta;\vec{\lambda};\varepsilon|\bar{M}_p;\vec{\phi}] = C_{\mathfrak{N}(\bar{M}_p)}[\eta;\vec{\lambda}|\bar{M}_p;\vec{\phi}] - \varepsilon\Pi_{\mathfrak{N}(\bar{M}_p)}[\eta;\bar{\eta}^{(\mathfrak{N})}(\vec{\lambda}\,|\,\bar{M}_p)], \qquad (124)$$

with the energy-independent part represented by the following polynomial of degree $\mathfrak{N}(\bar{M}_p)$:

$$C_{\mathfrak{N}(\bar{M}_p)}[\eta;\vec{\lambda}|\bar{M}_p;\vec{\phi}] + \varepsilon_0(\vec{\lambda}' + p\vec{\phi}\times\vec{1})\bar{\eta}^{(\mathfrak{N})}(\vec{\lambda}|\bar{M}_p)$$

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$$=(\eta^2-1)W^{(\lambda_+,\lambda_-)}_{\mathfrak{N}(\bar{M}_p)}[\eta\,|\,\bar{M}_p]\left\{I^o[\eta;\vec{\lambda}|\bar{M}_p]-I^o_0[\eta;\vec{\lambda}|\bar{M}_p]\right\} \quad (125)$$

$$=2\left[\eta-P_1^{(\lambda'_++\phi_+1,\lambda'_-+\phi_-1)}(\eta)\right]\dot{W}^{(\lambda_+,\lambda_-)}_{\mathfrak{N}(\bar{M}_p)}[\eta\,|\,\bar{M}_p]$$

$$-(\eta^2-1)\,\ddot{W}^{(\lambda_+,\lambda_-)}_{\mathfrak{N}(\bar{M}_p)}[\eta\,|\,\bar{M}_p]. \quad (126)$$

Making use of the Jacobi equation (15), one can verify that (121) turns into (151) in [121] for p=1, $\bar{M}_1 = m$.

7. 'Prime' Forms of RDCJs of J-Ref CSLE on Finite and Infinite Intervals of Orthogonalization

Starting from this point, we discuss only the admissible sets $\bar{\mathbf{M}}_p = \mathbf{m}_1,\dots,\mathbf{m}_p$ of JS solutions assuring that the corresponding JPWs do not have nodes within the given interval of orthogonalization for the specified ranges of the parameters $\boldsymbol{\lambda}_-,\boldsymbol{\lambda}_+$.

Using the gauge transformations

$$\Psi[\eta;\vec{\lambda};\varepsilon\,|\,\bar{\mathbf{M}}_p]=(1-\eta^2)^{-1/2}\,\Phi[\eta;\vec{\lambda};\varepsilon\,|\,\bar{\mathbf{M}}_p] \quad (127)$$

and

$$\Psi[\eta;\vec{\lambda};\underset{\sim}{\varepsilon}\,|\,\bar{\mathbf{M}}_p]=(\eta-1)^{-1/2}\,\Phi[\eta;\vec{\lambda};\underset{\sim}{\varepsilon}\,|\,\bar{\mathbf{M}}_p], \quad (128)$$

we then convert the RCSLE (67) to its prime forms on the intervals $(-1,+1)$ and $(+1,\infty)$:

$$\left\{\frac{d}{d\eta}(1-\eta^2)\frac{d}{d\eta}-q[\eta;\vec{\lambda}\,|\,\bar{\mathbf{M}}_p]+\varepsilon\right\}\Psi[\eta;\vec{\lambda};\varepsilon\,|\,\bar{\mathbf{M}}_p]=0 \quad (129)$$

for $\eta\in(-1,+1)$

and

$$\left\{\frac{d}{d\eta}(\eta-1)\frac{d}{d\eta}-q[\eta;\vec{\lambda}\,|\,\bar{\mathbf{M}}_p]+\frac{\underset{\sim}{\varepsilon}}{\eta+1}\right\}\Psi[\eta;\vec{\lambda};\underset{\sim}{\varepsilon}\,|\,\bar{\mathbf{M}}_p]=0 \quad (130)$$

for $\eta\in(1,\infty)$,

with the leading coefficient function and weight function defined via to (28) and (32) respectively.

In this paper we only discuss the seed solutions represented by the PFSs near the same endpoint under condition that they lie below the lowest eigenvalue. Since the RDCTs using the seed functions of types + - and - + are specified by same series of the Maya diagrams [25], any RCSLE using an arbitrary combination of these seed functions can be alternatively obtained by considering only infinitely many combinations $\bar{M}_p := \{m_1, m_2,\dots,m_p\}$ of the PFSs of the same type + - or - + [25,63]. In particular, the Jacobi polynomial of order m with the indexes $\vec{\boldsymbol{\lambda}}$ can be represented as the Wronskian of Jacobi polynomials of the sequential degrees $\tilde{m}$=1,...,m with the indexes $-\vec{\boldsymbol{\lambda}}$.

Here we consider only the Wronskian net of the Jacobi polynomials with the indexes

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$$\lambda_{\mp} = \mp\lambda_{o;\mp} \text{ for } |\eta|<1 \text{ and } \vec{\lambda} = \vec{\lambda}_o \text{ for } \eta > 1, \qquad (131)$$

while

$$\lambda'_{\mp} = \mp\lambda_{\mp} \ (\vec{\sigma} = -+) \text{ in both cases.} \qquad (132)$$

We will refer to the X-Jacobi DPS constructed using p seed Jacobi polynomials of the degrees $m_1, m_2, ..., m_p$ as being of series J1(p). The selection (131)-(132) for the p-WEOP sequences under consideration is consistent with (2.9) in [26], with $\alpha = \lambda_{o;+}$, $\beta = \lambda_{o;-}$ However, it is worth noting that the net of X-Jacobi OPSs of our choice starts from the Xm-Jacobi OPSs of series J1, but not with the traditional Xm-Jacobi OPSs [9,16], referred to in our works as being of series J2.

Note that Gómez-Ullate et al. [9,16] took advantage of the Klein formulas [64] to select all the Jacobi polynomials without zeros between -1 and +1 under constraint

$$\lambda_+ = -\alpha - 1 < 0, \ \lambda_- = \beta - 1 > -2, \qquad (133)$$

whereas our approach allows us to identify only the bulk part of those polynomials with one of the first Jacobi indexes restricted solely to positive values. However, to our knowledge there is no theory extending Klein's renowned results to the JPWs. This is why we consider Preposition 2 as one of the most significant achievements of this paper.

Coming back to our discussion, we conclude that the chain of the sequential RRZTs of the prime SLE (129) specified by the Jacobi indexes (131) must be truncated when the ExpDiff for the pole at -1 reaches its minimum value

$$0 < {}^*\lambda_{o;-}^{(p_{max})} = \lambda_{o;-} - p_{max} < 1 \qquad (134)$$

with

$$p_{max} = \lfloor \lambda_{o;-} \rfloor. \quad (135)$$

Below we always assume that p in (129) does not exceed (135), without explicitly mentioning this restriction.

Examination of the q-RS

$$\psi_{-+,j}[\eta;\vec{\lambda}|\bar{\boldsymbol{M}}_p] = \psi_0[\eta;-\lambda_- - p, \lambda_+ + p] \frac{\mathcal{P}_{\mathfrak{N}(\bar{\boldsymbol{M}}_p)+j}[\eta;\vec{\lambda}' | -+ \vdots \bar{\boldsymbol{M}}_p; j]}{W^{(\lambda_+,\lambda_-)}_{\mathfrak{N}(\bar{\boldsymbol{M}}_p)}[\eta | \bar{\boldsymbol{M}}_p]} \qquad (136)$$

$$\text{for } 0 \le j \le j_{max},$$

where

$$\psi_0[\eta;\lambda_-,\lambda_+] = \begin{cases} (1+\eta)^{\frac{1}{2}\lambda_-} (1-\eta)^{\frac{1}{2}\lambda_+} & \text{for } |\eta|<1, \\ (1+\eta)^{\frac{1}{2}(\lambda_-+1)} (\eta-1)^{\frac{1}{2}\lambda_+} & \text{for } \eta>1, \end{cases} \qquad (137)$$

reveals that it obeys the DBCs at the endpoints ±1 and therefore represents the eigenfunction of the prime SLE (129). Consequently it is the PFS of this SLE near the singular endpoint +1 at the energy $\varepsilon_j(\vec{\lambda}')$. The latter assertion also valid for the branch of this q-RS on the interval (1,∞) with the energy $\underset{\sim}{\varepsilon}_j(\vec{\lambda}') = -\varepsilon_j(\vec{\lambda}')$.

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Furthermore, since the functions $\phi_0[\eta;-\lambda_{0,-},\lambda_{0;+}]$ and $\phi_0[\eta;-\lambda_{0,-}-p,\lambda_{0;+}+p]$ have exactly the same asymptotics at infinity and the eigenfunction (47) of the prime SLE (24) on the interval (+1,∞) vanishes at the upper end by definition, we conclude that the q-RS (136) obeys the DBC at infinity and therefore represents the eigenfunction of the corresponding prime SLE (130) at the energy $\underset{\sim}{\varepsilon}_j(\vec{\lambda}')$.

Theorem 5. *A PFS near one of the endpoints ±1 may not have zeros inside the given interval of orthogonalization if it lies below the lowest eigenvalue of the given Sturm-Liouville problem.*

Proof: For the Sturm-Liouville problem on the interval of orthogonalization (-1,+1) the formulated assertion directly follows from the Sturm comparison theorem (see, i.g., Theorem 3.1 in Section XI of Hartman's monograph [65]), keeping in mind that the logarithmic derivatives for the all PFSs (including the eigenfunction in question) have the same asymptotics near the pole in question:

$$\lim_{\eta\to-1}\left[(1-\eta^2)ld\Psi[\eta;\boldsymbol{\lambda}_{0;-},\boldsymbol{\lambda}_+;\varepsilon\,|\,-+\,\vdots\,\bar{\boldsymbol{M}}_p]\right]=-{}^*\lambda^{(p)}_{0;-}, \tag{138}$$

and as a result the condition (3.4) in [65] turns into the identity. To apply the Sturm Theorem to the PFSs near the upper end +1, one simply needs to replace η for the reflected argument -η.

It is a more challenging problem to satisfy Sturm's constraint for the logarithmic derivatives in the limit η→∞ and we refer the reader to the proof of this assertion given in Appendix B in [21] for the PFSs of the prime SLE (24) solved under the DBCs at the ends of the interval (+1,∞). The arguments presented in support of this proof can be equally applied to the prime SLE (130) without any modification. □

Theorem 6. *If the sequence of the eigenfunctions (136) starts from a nodeless eigenfunction, then the set of the seed functions* $\bar{\boldsymbol{M}}_{p+1}$ *is admissible (assuming that the latter is true for* $\bar{\boldsymbol{M}}_p$ *).*

Proof: Let us consider the q-RS

$$\psi_{-+,m_{p+1}}[\eta;\vec{\lambda}|\bar{\boldsymbol{M}}_p]=\psi_0[\eta;-\lambda_{0;-}+p,\lambda_{0;+}+p]\frac{W^{(\lambda_+,\lambda_-)}_{\mathfrak{N}(\bar{\boldsymbol{M}}_{p+1})}[\eta\,|\,\bar{\boldsymbol{M}}_{p+1}]}{W^{(\lambda_+,\lambda_-)}_{\mathfrak{N}(\bar{\boldsymbol{M}}_p)}[\eta\,|\,\bar{\boldsymbol{M}}_p]} \tag{139}$$

used as the TF for the RRZT converting the prime SLEs (129) and (130) into the next SLE in the given chain of the RDC$\mathfrak{I}$s of the prime SLE (24) on the intervals (-1,+1) and (+1,∞) accordingly. Repeating the arguments presented by us for the q-RS (136), we assert that it is the PFS near the pole at +1. Since the eigenfunction (136) is nodeless for j = 0 and the energy of the (p+1)-th seed solution is smaller than the eigenvalue in question, the PFS may not have zeros inside the given interval of orthogonalization and therefore this must be also true for the numerator of the PF in the right-hand side (139). □

The proven theorem represents the very important milestone in our analysis of the admissible JPWs. The concluding step is to confirm that the p-WEOP starting the given sequence does not have zeros inside the interval of orthogonalization. It is proven in Appendix D that the RDC$\mathfrak{I}$s in question constitute the isospectral net of the solvable RSLPs, in particular assuring that the necessary prerequisite for Theorem 5 to be automatically hold.

It has been proven in [28] that the eigenfunctions of the generic SLE solved under the DBCs must be mutually orthogonal with the equation weight function on the interval in question. Therefore

$$\int_{\eta_-}^{\eta_+} d\eta\,\psi_{\mathbf{c},j}[\eta;\vec{\lambda}'\,|\,-+\,\vdots\,\bar{\boldsymbol{M}}_p]\psi_{\mathbf{c},j'}[\eta;\vec{\lambda}'\,|\,-+\,\vdots\,\bar{\boldsymbol{M}}_p]w[\eta;\vec{\boldsymbol{\lambda}}_o]=0 \tag{140}$$

$$\text{for } 0\le j'<j\le j_{max},\ p\le p_{max}.$$

Consequently, the polynomial components of the quasi-rational eigenfunctions (136) must be mutually orthogonal with the weight function

$$W[\eta;\vec{\lambda}'|-+\vdots\bar{\mathbf{M}}_p]:=\frac{\psi_0^2[\eta;-\lambda_- - p,\lambda_+ + p]}{W^{(\lambda_+,\lambda_-)}_{\mathfrak{N}(\bar{M}_p)}[\eta|-+\vdots\bar{\mathbf{M}}_p]}w[\eta;\vec{\lambda}_o]\ (p\le p_{max})\quad(141)$$

for $|\eta|<1$ or $\eta>1$;

namely,

$$\int_{\eta_-}^{\eta_+} d\eta\, \mathcal{P}_{\mathfrak{N}(\vec{\sigma}\vdots\bar{\mathbf{M}}_p;j)}[\eta;\vec{\lambda}'|-+\vdots\bar{\mathbf{M}}_p;j]\mathcal{P}_{\mathfrak{N}(\vec{\sigma}\vdots\bar{\mathbf{M}}_p;j')}[\eta;\vec{\lambda}'|-+\vdots\bar{\mathbf{M}}_p;j']W[\eta;\vec{\lambda}'|-+\vdots\bar{\mathbf{M}}_p]=0$$

for $0\le j'<j\le j_{max},\ p\le p_{max}$. (142)

In the following two subsections we discuss separately the X-Jacobi OPSs conventionally defined on the interval (-1,+1) and the RDC𝔖s of the R-Jacobi polynomials orthogonal on the infinite interval (+1,∞).

7.1. Infinite Net of Isospectral SLPs Solved by Simple **p-W𝔖s** *of Classical Jacobi Polynomials*

Let us set $\vec{\lambda}=-\lambda_{o;-},\lambda_{o;+}$, $\vec{\lambda}'=\vec{\lambda}_o$, $p_{max}=\lfloor\lambda_{o;-}\rfloor$, and $j_{max}=\infty$. This brings us to the net of the X-Jacobi OPSs composed of the p-WEOPs

$$\mathcal{P}_{\mathfrak{N}(\bar{\mathbf{M}}_p)+j}[\eta;\vec{\lambda}_o|-+\vdots\bar{\mathbf{M}}_p;j]:=\qquad(143)$$

$$\begin{vmatrix} P^{(\lambda_{o;+},-\lambda_{o;-})}_{m_1}(\eta) & P^{(\lambda_{o;+},-\lambda_{o;-})}_{m_2}(\eta) & \dots & \Theta^p_j[\eta]P^{(\lambda_{o;+},\lambda_{o;-})}_j(\eta) \\ \frac{d}{d\eta}P^{(\lambda_{o;+},-\lambda_{o;-})}_{m_1}(\eta) & \frac{d}{d\eta}P^{(\lambda_{o;+},-\lambda_{o;-})}_{m_2}(\eta) & \dots & P^{(1)}_{j+p-1}[\eta;\vec{\lambda}_o|-+] \\ & \dots & & \\ \frac{d^p}{d^p\eta}P^{(\lambda_{o;+},-\lambda_{o;-})}_{m_1}(\eta) & \frac{d^p}{d^p\eta}P^{(\lambda_{o;+},-\lambda_{o;-})}_{m_2}(\eta) & \dots & P^{(p)}_j[\eta;\vec{\lambda}_o|-+] \end{vmatrix}.$$

Note that the first p elements of the first row are represented by the R-Jacobi polynomials while the last element is the classical Jacobi polynomial multiplied by a constant. The weight function (143) takes the form:

$$W[\eta;-\lambda_{o;-},\lambda_{o;+}|-+\vdots\bar{\mathbf{M}}_p]:=\frac{\psi_0^2[\eta;-\lambda_{o;-}-p,\lambda_{o;+}+p]}{\left[W^{(\lambda_{o;+},-\lambda_{o;-})}_{\mathfrak{N}(\bar{\mathbf{M}}_p)}[\eta|-+\vdots\bar{\mathbf{M}}_p]\right]^2}\qquad(144)$$

for $-1<\eta<1\ (p\le\lfloor\lambda_{o;-}\rfloor)$,

where the Wronskian in the denominator is formed by orthogonal R-Jacobi polynomials and therefore is the subject of the general conjectures formulated in [58] for zeros of the Wronskians of orthogonal polynomials inside the normalization interval (real zeros larger than 1 in our case). Preposition 2 in Section 5 assures that the polynomial denominator of the fraction (144) remains finite inside the interval (-1,+1). This was the main rationale for us to consider only the seed polynomials with common pairs of Jacobi indexes.

On other hand, we could not pinpoint the similar proof for the polynomial denominator of the quasi-rational weight function (2.36) in [26]. From our point of view, the definition of the X-Jacobi OPS is incomplete until one can assure that the weight function in question does not have poles in the interval (-1,+1).

Another important new element of our approach is the proof that polynomials from each X-Jacobi OPS obey the Bochner-type ODE (121) with the regular singularities.. This was the central reason for restricting our analysis solely to q-RSs with the polynomial components represented by the simple p-W𝔍Ps.

In particular, based on Theorem 2.1 in [58] (summarizing Karlin and Szego's results [66]), we assert that any Wronskian of an even number of the R-Jacobi polynomials of sequential degrees may have only negative real zeros smaller than -1. This is one of rare cases when Preposition 2 above has been accurately confirmed.

7.2. Infinite Net of Isospectral SLPs Solved by **RDC𝔍s** *of R-Jacobi Polynomials*

An extension of Theorem 5 to the infinite interval (1,∞) is complicated by the mentioned constraint on the ratio of the logarithmic derivatives of two solutions as the prerequisite for the Sturm comparison theorem. We refer the reader to Appendix B in [21] for the rigorous arguments in support of Theorem 5 for the interval (1,∞).

Below w focus solely on the RDCTs using the infinitely many PFS of type $\underset{\sim}{\mathbf{a}}$ as the seed functions, i.e., by definition $\vec{\boldsymbol{\lambda}} = \vec{\boldsymbol{\lambda}}_o$ and $p_{max} = \infty$. The corresponding eigenfunctions of the prime SLE (24) solved under the DBCs on the interval (+1, ∞) are care formed by the R-Jacobi polynomials with the Jacobi indexes $\vec{\boldsymbol{\lambda}}' = -\boldsymbol{\lambda}_{o;-}, \boldsymbol{\lambda}_{o;+}$, , and their total number is equal to

$$n_{\underset{\sim}{c}} = j_{max} + 1 = \lceil \boldsymbol{\lambda}_{o;-} \rceil. \tag{145}$$

This brings us to the net of the finite EOP sequences composed of the p-PWs

$$\mathcal{P}_{\mathfrak{N}(\bar{\boldsymbol{M}}_p)+j}[\eta; -\lambda_{o;-}, \lambda_{o;+} \,|\, -+ \vdots \bar{\boldsymbol{M}}_p; j] := \tag{146}$$

$$\begin{vmatrix} P_{m_1}^{(\lambda_{o;+},\lambda_{o;-})}(\eta) & P_{m_2}^{(\lambda_{o;+},\lambda_{o;-})}(\eta) & \ldots & \Theta_1^p[\eta] P_j^{(\lambda_{o;+},-\lambda_{o;-})}(\eta) \\ \frac{d}{d\eta} P_{m_1}^{(\lambda_{o;+},\lambda_{o;-})}(\eta) & \frac{d}{d\eta} P_{m_2}^{(\lambda_{o;+},\lambda_{o;-})}(\eta) & \ldots & P_{j+p-1}^{(1)}[\eta; -\lambda_{o;-}, \lambda_{o;+} | -+] \\ & \ldots & & \\ \frac{d^p}{d^p\eta} P_{m_1}^{(\lambda_{o;+},\lambda_{o;-})}(\eta) & \frac{d^p}{d^p\eta} P_{m_2}^{(\lambda_{o;+},\lambda_{o;-})}(\eta) & \ldots & P_j^{(p)}[\eta; -\lambda_{o;-}, \lambda_{o;+} | -+] \end{vmatrix}.$$

This time the first p elements of the first row are represented by the classical Jacobi polynomials with positive indexes while the last element is the the R-Jacobi polynomial multiplied by a constant. The weight function (141) takes the form:

$$W[\eta; -\boldsymbol{\lambda}_{o;-}, \boldsymbol{\lambda}_{o;+} \,|\, -+ \vdots \bar{\boldsymbol{M}}_p] := \frac{\psi_0^2[\eta; -\boldsymbol{\lambda}_{o;-} - p, \boldsymbol{\lambda}_{o;+} + p]}{W_{\mathfrak{N}(\bar{\boldsymbol{M}}_p)}^{(\lambda_{o;+},\lambda_{o;-})}[\eta \,|\, -+ \vdots \bar{\boldsymbol{M}}_p]} \tag{147}$$

$$\text{for } -1 < \eta < 1,$$

where the Wronskian in the denominator is formed by the classical Jacobi polynomials with positive indexes.

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Again, based on Theorem 2.1 in [58], we conclude that any Wronskian of an even number of the classical Jacobi polynomials of sequential degrees may have only negative real zeros smaller than -1. However, this particular example is of less importance, compared with that on the finite integral of orthogonalization, because Theorem 5 assures that the JPW composed of the seed polynomials in question has no zeros larger than 1, which constitutes the question of fundamental significance for this study.

8. Discussion

Let us first point to the most essential element of our RSLP formalism – the advanced technique for selecting the sequences of the admissible RRZTs, using PFRs below the lowest eigenvalue as the q-RTFs. Each such sequence can be then re-interpreted as the admissible RDCT. In terms of [10] we suggested the systematic way for constructing a subfamily of the regular X-Bochner operators. To be more precise, we laid down the mathematical grounds for this innovation in Section 7 by converting the RCSLE (67) to its prime forms (129) and (130) on the intervals (-1,+1) and (+1,∞) accordingly and solving the resultant SLEs under the DBCs. The formulated SLPs allowed us to prove [27] that each RD$\mathfrak{T}$ of the PFS constitutes itself the PFS of the transformed SLE at the same energy.

As mentioned in Introduction, the RZT represents the generalization of the DT (in its original sense [13]) to the CSLEs in the same way as the factorization of the X-Jacobi differential operators [10] constitutes the natural extension of the conventional factorization technique in the framework of the SUSY quantum mechanics. The factorization chain of the rational operators analyzed in [10] is equivalent to a RDC$\mathfrak{T}$ of the JRef CSLE in our terms.

To illuminate advantages of our formalism developed here, compared to the PSLP accomplishments [9,10,15], let us formally re-formulate the results of subsection 7.1, based on the argumentation used in [9] to derive the explicit expressions for the Xm-Jacobi polynomials.

Let us start by introducing the Sturm-Liouville expression

$$\Big\{\frac{d}{d\eta}\,\mathcal{p}[\eta;\vec{\lambda}\,|-+\vdots\bar{\boldsymbol{M}}_p]\frac{d}{d\eta}-q_{\mathcal{p}}[\eta\,|-+\vdots\bar{\boldsymbol{M}}_p] \qquad (148)$$

$$+\varepsilon_j(\vec{\lambda})W[\eta;\vec{\lambda}\,|-+\vdots\bar{\boldsymbol{M}}_p]\Big\}\mathcal{P}_{\mathfrak{N}(\vec{\sigma}\vdots\bar{\boldsymbol{M}}_p)+j}[\eta;\vec{\lambda}\,|\,\vec{\sigma}\vdots\bar{\boldsymbol{M}}_p;j]=0$$

with the leading coefficient function related to the weight function (144) via the generic formula

$$\mathcal{p}[\eta;\vec{\lambda}\,|-+\vdots\bar{\boldsymbol{M}}_p]:=(1-\eta^2)W[\eta;\vec{\lambda}\,|-+\vdots\bar{\boldsymbol{M}}_p] \text{ for } -1<\eta<+1, \qquad (149)$$

assuming that the density function of the corresponding CSLE is given by (2). Note that we replaced the indexes $\mp\lambda_{o;\mp}$ of the weight function (144) for $\lambda_\mp$ in an attempt to expand its definition beyond the limits supported by Preposition 1.

Examination of the numerator of the fraction

$$\mathcal{p}[\eta;\vec{\lambda}\,|-+\vdots\bar{\boldsymbol{M}}_p]=\frac{(1-\eta^2)\psi_0^2[\eta;\lambda_- -p,\lambda_+ +p]}{\Big[W_{\mathfrak{N}(\bar{\boldsymbol{M}}_p)}^{(\lambda_{o;+},-\lambda_{o;-})}[\eta\,|-+\vdots\bar{\boldsymbol{M}}_p]\Big]^2} \qquad (150)$$

shows that it vanishes at ∓1

$$\lambda_\mp \mp p+1>0\ , \qquad (151)$$

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provided that the denominator remains finite at both singular ends. Preposition 1 assures that that the leading coefficient function vanishes under the latter constraint. As mentioned above, this was one of the reasons for restricting our analysis to the RefPFs (68).

Based on the stated observation, we conclude that the p-WPs forming the polynomial components of the q-RSs (139) under the constraints (151) obey the boundary conditions

$$\mathcal{W}\{\{\mathcal{P}_{\mathfrak{N}(\vec{\sigma}\vdots\bar{\boldsymbol{M}}_p;\,j)}[\eta;\vec{\lambda}'\,|-+\vdots\bar{\boldsymbol{M}}_p;\,j],\mathcal{P}_{\mathfrak{N}(\vec{\sigma}\vdots\bar{\boldsymbol{M}}_p;\,j')}[\eta;\vec{\lambda}'\,|-+\vdots\bar{\boldsymbol{M}}_p;0]\}\Big|_{\eta=\pm1}=0$$

$$\text{for } 0\le j'<j\le j_{max}\ (p\le p_{max}),\qquad (152)$$

where

$$\mathcal{W}_{\wp}\{P_j(\eta),P_{j'}(\eta)\}:=\wp[\eta;\vec{\lambda}\,|-+\vdots\bar{\boldsymbol{M}}_p]W\{P_j(\eta),P_{j'}(\eta)\}.\qquad (153)$$

Our next step would be to further narrow the range of the indexes $\lambda_{\mp}$, restrained by (151), in such a way that the denominator of the fraction (150) does not have zeros between -1 and +1. For the Xm-Jacobi OPS this was achieved in [9], using the Klein formula [64] for the numbers of zeros of a Jacobi polynomial in the interval (-1,+1). However, even the fact that we deal with the Wronskian of the mutually orthogonal R-Jacobi polynomials does not give any additional insight into how to handle this problem for p>1. In particular, Durán et al.'s conjectures dealing with the zeros of the Wronskians of orthogonal polynomials focus on the numbers of zeros inside the interval of orthogonalization, i.e., on the numbers of real zeros larger than 1 in our case. On the contrary, we are interested in the question how many of other zeros lie between -1 and +1. The striking feature of our approach is the use of the completely different technique to answer this question.

In summary, we have constructed the infinite net of the X-Jacobi DPSs composed of the simple p-WPs. Each X-DPS contains the finite EOP sequence formed by the RDC$\mathfrak{I}$ of the R-Jacobi polynomials using the classical Jacobi polynomials with the common positive indexes as seed polynomials. In addition, a finite subnet of these X-DPSs contains X-Jacobi OPSs, using the R-Jacobi polynomials with the common pair of indexes as the seed polynomials. The crucial point of our approach is that all the constructed EOP sequences obey the X-Bochner ODEs with polynomial coefficients. In the case of the X-OPSs there exists the one-to-one correspondence between these ODEs and regular X-Bochner operators in terms of [10]. On other hand, the finite EOP sequences are generally formed by X-orthogonal eigenpolynomials of irregular X-Bochner operators.

The net of the trigonometric ($|\eta|<1$) or radial ($\eta>1$) quantum-mechanical potentials exactly solvable in terms of the constructed infinite or accordingly finite EOP sequences can be obtained in following the prescriptions outlined by us in [21] for p=1.

The Liouville potentials quantized via the EOPs introduced in subsections 7.1 and 7.2 have the generic form:

$$V[\eta;-\lambda_{o;-},\lambda_{o;+}|\bar{\boldsymbol{M}}_p]=V_{t-PT}[\eta;\vec{\lambda}_o]\qquad (154)$$

$$+(1-\eta^2)\left\{I^o[\eta;\vec{\lambda}_o]-I^o[\eta;-\lambda_{o;-},\lambda_{o;+}|\bar{\boldsymbol{M}}_p]\right\}\ \text{for } -1<\eta<1$$

and

$$V[\eta;\vec{\lambda}_o|\bar{\boldsymbol{M}}_p]=V_{h-PT}[\eta;\vec{\lambda}_o]\qquad (155)$$

$$+(\eta^2-1)\left\{I^o[\eta;\vec{\lambda}_o]-I^o[\eta;\vec{\lambda}_o|\bar{\boldsymbol{M}}_p]\right\}\qquad \text{for } \eta>1$$

after being expressed in terms of the variables:

$$\eta(x)=\cos x\quad(-\pi<x<0)\qquad (156)$$

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and

$$\eta(r) = \cosh r \quad (0 < r < \infty) \qquad (157)$$

respectively, where the t-PT potential on the finite interval and the radial h-PT potential are parametrized as follows:

$$V_{t-PT}[\eta(x);\lambda_{o;+},\lambda_{o;-}] = \frac{\lambda_{o;+}^2 - 1/4}{4\sin^2 x/2} + \frac{\lambda_{o;-}^2 - 1/4}{4\cos^2 x/2} \quad (-\pi < x < 0), \qquad (158)$$

and

$$V_{h-PT}(r;\lambda_{o;+},\lambda_{o;-}] = \frac{\lambda_{o;+}^2 - 1/4}{4\sinh^2 r/2} + \frac{\lambda_{o;-}^2 - 1/4}{4\cosh^2 r/2} \quad (0 < r < \infty). \qquad (159)$$

As mentioned in Introduction, the rigorous mathematical studies [10,37] on the X-Jacobi and X-Laguerre OPSs made a few misleading references to the quantum-mechanical applications of the EOPs. To a certain extent this misinformation is traceable to the fact that the cited applications do not properly distinguish between the terms 'X-Jacobi DPS', 'X-Jacobi OPS', and 'finite EOP' sequences' (formed by the RD$\mathfrak{I}$s of the R-Jacobi polynomials), simply referring to the representatives of all the three manifolds as 'X-Jacobi polynomials'.

While the latter perplexity has already been detailed by us in [20,21], it seems useful to clarify again the sharp distinction between the TSIPs of groups A and B in Odake and Sasaki's [38] TSIP classification scheme. Namely, the prominent feature of the density function (2) is that the latter has only simple poles in the finite plane [24], which assured the energy-independence of the ExpDiffs for the poles of the JRef CSLE (1) at ∓ 1 [47]. Consequently, the coefficient function of the first derivative of the second-order ODE with polynomial coefficients turned independent of the degrees of the sought-for polynomial solutions, which allowed one to convert this equation into the eigenequation with rational coefficients [9,10,16].

The same comment is applied to the RDC$\mathfrak{I}$s of the three families of the Romanovski polynomials [29], namely, to the finite EOP sequences composed of the Romanovski-Bessel (R-Bessel) and Romanovski-Routh (R-Routh) polynomials analyzed by us in [67] and [68] respectively, as well as to the RDC$\mathfrak{I}$s of the R-Jacobi polynomials discussed in this paper. The associated Liouville potentials all belong to group A in Odake and Sasaki's [38] TSIP classification scheme and are indeed solved via the finite EOP sequences.

In the general case of the rational density function, allowing the solution of the JRef CSLE in terms of hypergeometric functions [40], the energy-dependent PF in (1) has second-order poles in the finite plane and as a result the associated Liouville potentials are quantized by the Jacobi polynomials with degree-dependent indexes. If the numerator of the given rational density function has no zeros at regular points of the JRef CSLE (or similarly of its confluent counterpart), then the associated Liouville potential turns into a TSIP of group B, with eigenfunctions expressible via the Jacobi (or respectively Laguerre) polynomials with at least one degree-dependent index, which have no direct relation to the theory of the EOPs.

To conclude, let us point to the crucial difference between the RDC$\mathfrak{I}$s of the R-Jacobi polynomials and those of the R-Bessel and R-Routh polynomials analyzed by us in [67] and [68] respectively. The common feature of the latter RDC nets is that each net is specified by a single series of Maya diagrams and as a result any finite EOP sequence allows the Wronskian representation [69]. On other hand, the complete net of the RDC$\mathfrak{I}$s of the R-Jacobi polynomials is specified by the two series of Maya diagrams, similar to the RDC$\mathfrak{I}$s of the classical Jacobi and classical Laguerre polynomials forming the X-Jacobi and X-Laguerre OPSs accordingly [25]. This implies that we managed to construct only a tiny manifold of the finite EOP sequences composed of the RDC$\mathfrak{I}$s of the R-Jacobi polynomials.

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We refer the reader to [25] for the scrupulous analysis of the equivalence relations between the various p-WPs . It should be however stressed that grouping of equivalent p-WPs together represents only a part of the problem. The next step would be to select the preferrable representation. For example, the RD$\mathfrak{I}$ of the h-PT potential with the TF $\underset{\sim}{\mathbf{b}},m$ seems easier to be dealt with, compared with the RDC$\mathfrak{I}$ of this potential with the m seed functions $\underset{\sim}{\mathbf{a}},k=1,2,\ldots,m$, though the final results will be absolutely the same.

The additional complication comes from the fact that one has to analyze the order of p-WP zeros at ∓ 1 to construct the appropriate X-Jacobi DPSs. And finally (assuming that the partitions selected in [26] are suitable) one has to require that the generalized Jacobi polynomials in the numerator of the weight function (2.36) in [26] do not have zeros inside the given interval of orthogonalization. The complexity of the outlined procedure helps to understand why our analysis was restricted merely to the relatively simple case of the seed Jacobi polynomials with the same pair of indexes.

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Abbreviations

ChExp	characteristic exponent
CSLE	canonical Sturm-Liouville equation
DBC	Dirichlet boundary condition
DPS	differential polynomial system
DCT	Darboux-Crum transformation
DC$\mathfrak{I}$	Darboux-Crum transform
DT	Darboux deformation
D$\mathfrak{I}$	Darboux transform
EOP	exceptional orthogonal polynomial
ExpDiff	exponent difference
GDT	generalized Darboux transformation
h-PT	hyperbolic Pöschl-Teller
JPW	Jacobi-polynomial Wronskian
JRef	Jacobi-reference
JS	Jacobi-seed
LC	limit circle
LDT	Liouville-Darboux transformation
LP	limit point
ODE	ordinary differential equation
OPS	orthogonal polynomial system
PD	polynomial determinant
PF	polynomial fraction
PFS	principal Frobenius solution
p-SLE	prime Sturm-Liouville equation
p-WP	pseudo-Wronskian polynomial

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p-WEOP	pseudo-Wronskian exceptional orthogonal polynomial
p-W$\mathfrak{T}$	pseudo-Wronskian transform
q-RS	quasi-rational solution
q-RTF	quasi-rational transformation function
RCSLE	rational canonical Sturm-Liouville equation
RDC	rational Darboux-Crum
RDCT	rational Darboux-Crum transformation
RDC$\mathfrak{T}$	rational Darboux-Crum transform
RDT	rational Darboux transformation
RD$\mathfrak{T}$	rational Darboux transform
restr-HRef	restrictive Heun-reference
R-Jacobi	Romanovski-Jacobi
R-Routh	Romanovski-Routh
RRZ$\mathfrak{T}$	rational Rudjak-Zakharov transform
W$\mathfrak{T}$	Wronskian transform

Appendix A. RZT of Generic CSLE

Let $\phi_\tau[\eta;\vec{\lambda}_o]$ be a nodeless solution of a CSLE

$$\left\{\frac{d^2}{d\eta^2}+I^o[\eta;\vec{\lambda}_o]+\varepsilon\,\rho[\eta]\right\}\Phi[\eta;\vec{\lambda}_o;\varepsilon]=0 \qquad (A1)$$

at the energy

$$\varepsilon=\varepsilon_\tau(\vec{\lambda}_o), \qquad (A2)$$

i.e.,

$$\left\{\frac{d^2}{d\eta^2}+I^o[\eta;\vec{\lambda}_o]+\varepsilon_\tau(\vec{\lambda}_o)\,\rho[\eta]\right\}\phi_\tau[\eta;\vec{\lambda}_o]=0. \qquad (A3)$$

We define the RZT of the given CSLE via the requirement that the function

$$*\phi_\tau[\eta;\vec{\lambda}_o]=\frac{\rho^{-1/2}[\eta]}{\phi_\tau[\eta;\vec{\lambda}_o]} \qquad (A4)$$

is the solution of the transformed CSLE:

$$\left\{\frac{d^2}{d\eta^2}+I^o[\eta;\vec{\lambda}_o\,|\,\tau]+\varepsilon\,\rho[\eta]\right\}\Phi[\eta;\vec{\lambda}_o;\varepsilon\,|\,\tau]=0 \qquad (A5)$$

at the same energy (A2), i.e.,

$$\left\{\frac{d^2}{d\eta^2}+I^o[\eta;\vec{\lambda}_o\,|\,\tau]+\varepsilon_\tau(\vec{\lambda}_o)\rho[\eta]\right\}*\phi_\tau[\eta;\vec{\lambda}_o]=0. \qquad (A6)$$

Representing both CSLEs (A3) and (A6) in the Riccati form:

$$I^o[\eta;\vec{\lambda}_o]=-ld^{\,2}\phi_\tau[\eta;\vec{\lambda}_o]-l\dot{d}\,\phi_\tau[\eta;\vec{\lambda}_o]-\varepsilon_\tau(\vec{\lambda}_o)\,\rho[\eta] \qquad (A7)$$

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and

$$I^{o}[\eta;\vec{\lambda}_{o}\,|\,\tau] := -ld^{\,2}{*}\phi_{\tau}[\eta;\vec{\lambda}_{o}] - \dot{ld}{*}\phi_{\tau}[\eta;\vec{\lambda}_{o}] - \varepsilon_{\tau}(\vec{\lambda}_{o})\rho[\eta], \qquad (A8)$$

subtracting one from another, and also taking into account that the logarithmic derivatives of the TF $\phi_{\tau}[\eta;\vec{\lambda}_{o}]$ and its reciprocal (A4) are related in the elementary fashion:

$$ld{*}\phi_{\tau}[\eta;\vec{\lambda}_{o}] = -ld\,\phi_{\tau}[\eta;\vec{\lambda}_{o}] - \tfrac{1}{2}\,ld\,\rho[\eta] \qquad (A9)$$

one finds [70]

$$I^{o}[\eta;\vec{\lambda}_{o}\,|\,\tau] = I^{o}[\eta;\vec{\lambda}_{o}] + 2\sqrt{\rho[\eta]}\,\frac{d}{d\eta}\frac{ld\,\phi_{\tau}[\eta;\vec{\lambda}_{o}]}{\sqrt{\rho[\eta]}} + \mathfrak{I}\{\rho[\eta]\}, \qquad (A10)$$

where the last summand represents the so-called [27] 'universal correction' defined via the generic formula

$$\mathfrak{I}\{f[\eta]\} := \tfrac{1}{2}\sqrt{f[\eta]}\,\frac{d}{d\eta}\frac{ld\,f[\eta]}{\sqrt{f[\eta]}}. \qquad (A11)$$

Appendix B. DCT of the Generic CSLE as a Sequence of RZTs

Let $\phi_{\tau_2}[\eta;\vec{\lambda}_{o}]$ be another solution of a CSLE (A1) at the energy $\varepsilon_{\tau_2}(\vec{\lambda}_{o})$. Then, as it was shown in [19], the CSLE (A6) with $\tau \equiv \tau_1$ has the solution

$$\phi_{\tau_2}[\eta;\vec{\lambda}_{o}\,|\,\tau_1] = \frac{W\{\phi_{\tau_{k=1,2}}[\eta;\vec{\lambda}_{o}]\}}{\rho^{1/2}[\eta]\,\phi_1[\eta;\vec{\lambda}_{o}]}. \qquad (A12)$$

Using this solution as the TF for the next RZT, we come to the CSLE

$$\left\{\frac{d^2}{d\eta^2} + I^{o}[\eta;\vec{\lambda}_{o}\,|\,\tau] + \varepsilon\,\rho[\eta]\right\}\Phi[\eta;\vec{\lambda}_{o};\varepsilon\,|\,\tau] = 0 \qquad (A13)$$

with the zero-energy free term

$$I^{o}[\eta;\vec{\lambda}_{o}|\tau_{k=1,2}] = I^{o}[\eta;\vec{\lambda}_{o}\,|\,\tau_1] \qquad (A14)$$

$$+2\sqrt{\rho[\eta]}\,\frac{d}{d\eta}\frac{ld\,\phi_{\tau_2}[\eta;\vec{\lambda}_{o}\,|\,\tau_1]}{\sqrt{\rho[\eta]}} + \mathfrak{I}\{\rho[\eta]\}.$$

Substituting (A12) into (A14), coupled with (A10) and (A11), then gives [69}

$$I^{o}[\eta;\vec{\lambda}_{o}|\tau_{k=1,2}] = I^{o}[\eta;\vec{\lambda}_{o}] + 2\sqrt{\rho[\eta]}\,\frac{d}{d\eta}\frac{ld\,W\{\phi_{\tau_{k=1,2}}[\eta;\vec{\lambda}_{o}]\}}{\sqrt{\rho[\eta]}}. \qquad (A15)$$

Let us now assume that the function [22,71]

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$$\phi_\tau[\eta;\vec{\lambda}_o\,|\,\tau_{k=1,\ldots,p}] = \frac{W\{\phi_{\tau_{k=1,\ldots,p}}[\eta;\vec{\lambda}_o], \phi_\tau[\eta;\vec{\lambda}_o]\}}{\rho^{p/2}[\eta]W\{\phi_{\tau_{k=1,\ldots,p}}[\eta;\vec{\lambda}_o]\}} \qquad (A16)$$

satisfies the CSLE

$$\left\{\frac{d^2}{d\eta^2} + I^0[\eta;\vec{\lambda}_o\,|\,\tau_{k=1,\ldots,p}] + \varepsilon\rho[\eta]\right\}\Phi[\eta;\vec{\lambda}_o;\varepsilon\,|\,\tau_{k=1,\ldots,p}] = 0 \qquad (A17)$$

with the zero-energy free term [22]

$$I^0[\eta;\vec{\lambda}_o|\tau_{k=1,\ldots,p}] = I^0[\eta;\vec{\lambda}_o] \qquad (A18)$$

$$+2\sqrt{\rho[\eta]}\frac{d}{d\eta}\frac{ld\,W\{\phi_{\tau_{k=1,\ldots,p}}[\eta;\vec{\lambda}_o]\}}{\sqrt{\rho[\eta]}} - p(p-2)\wp\{\rho[\eta]\}.$$

The RDT of the CSLE (A17) with the TF (A16) then results in the CSLE with the zero-energy free term defined by (A18) with p and $\tau_{k=1,\ldots,p}$ replaced for p+1 and $\tau_{k=1,\ldots,p+1}$ accordingly.

Theorem B. *The function (A16) is the solution of the CSLE (A17) at the energy* $\varepsilon_{\tau_p}(\vec{\lambda}_o)$.

Proof: Suppose that both functions $\phi_{\tau_p}[\eta;\vec{\lambda}_o\,|\,\tau_{k=1,\ldots,p-1}]$ and $\phi_{\tau_{p+1}}[\eta;\vec{\lambda}_o\,|\,\tau_{k=1,\ldots,p-1}]$ are solutions of the CSLE (A17) with p replaced by p-1. It is also assumed that the energies are equal to $\varepsilon_{\tau_p}(\vec{\lambda}_o)$ and $\varepsilon_{\tau_{p+1}}(\vec{\lambda}_o)$ accordingly. Then, by definition of the CSLE (17), the function

$$\phi_{\tau_{p+1}}[\eta;\vec{\lambda}_o\,|\,\tau_{k=1,\ldots,p}] \qquad (A19)$$

$$= \frac{W\{\phi_{\tau_p}[\eta;\vec{\lambda}_o\,|\,\tau_{k=1,\ldots,p-1}], \phi_{\tau_{p+1}}[\eta;\vec{\lambda}_o\,|\,\tau_{k=1,\ldots,p-1}]\}}{\rho^{1/2}[\eta]\phi_{\tau_p}[\eta;\vec{\lambda}_o\,|\,\tau_{k=1,\ldots,p-1}]}$$

must be its solution at the energy $\varepsilon_{\tau_{p+1}}(\vec{\lambda}_o)$. Replacing p and $\tau_{k=1,\ldots,p}$ in the right-hand side of (A16) for p-1 and $\tau_{k=1,\ldots,p-1}$ accordingly and then setting $\tau = \tau_p, \tau_{p+1}$ we can re-write (A19) as

$$\phi_{\tau_{p+1}}[\eta;\vec{\lambda}_o\,|\,\tau_{k=1,\ldots,p}] \qquad (A20)$$

$$= \frac{W\left\{W\{\phi_{\tau_{k=1,\ldots,p}}[\eta;\vec{\lambda}_o]\}, W\{\phi_{\tau_p}[\eta;\vec{\lambda}_o\,|\,\tau_{k=1,\ldots,p-1}], \phi_{\tau_{p+1}}[\eta;\vec{\lambda}_o]\}\right\}}{\rho^{1/2p}[\eta]W\{\phi_{\tau_{k=1,\ldots,p-1}}[\eta;\vec{\lambda}_o]\}}.$$

Choosing m = p-1, n = p+1, n-m =2 in the general Wronskian decomposition formula in [72] then gives:

$$W\{\phi_{\tau_{k=1,\ldots,p+1}}[\eta;\vec{\lambda}_o]\} \qquad (A21)$$

$$\frac{W\left\{W\{\phi_{\tau_{k=1,\ldots,p}}[\eta;\vec{\lambda}_o]\}, W\{\phi_{\tau_p}[\eta;\vec{\lambda}_o \mid \tau_{k=1,\ldots,p-1}], \phi_{\tau_{p+1}}[\eta;\vec{\lambda}_o]\}\right\}}{W\{\phi_{\tau_{k=1,\ldots,p-1}}[\eta;\vec{\lambda}_o]}$$

then brings us back to (A16) with $\tau = \tau_{p+1}$, which completes the proof. □

Here and in the other publications we refer to the CSLE (A17) with the zero-energy free term (A18) as 'Darboux-Crum transform' ($DC\mathfrak{T}$) of the CSLE (1) with the seed functions $\tau_{k=1,\ldots,p}$.

Appendix C. RDC Sequences of PFSs Near a 2nd-Order Pole with an Energy-Independent ExpDiff

Let $\Phi_{\mp}[\eta;\vec{\lambda};\varepsilon|\bar{M}_p]$ be the PFS of the RCSLE (63) near the pole at ∓ 1. Then the functions

$$\Psi_{-}[\eta;\vec{\lambda};\varepsilon|\bar{M}_p] = (1-\eta^2)^{-1/2}\,\Phi_{-}[\eta;\vec{\lambda};\varepsilon|\bar{M}_p] \quad \text{for } |\eta|<1 \tag{A22}$$

and

$$\Psi_{+}[\eta;\vec{\lambda};\varepsilon|\bar{M}_p] = \begin{cases} (1-\eta^2)^{-1/2}\,\Phi_{+}[\eta;\vec{\lambda};\varepsilon|\bar{M}_p] & \text{for } |\eta|<1, \\ (\eta-1)^{-1/2}\,\Phi_{+}[\eta;\vec{\lambda};-\varepsilon|\bar{M}_p] & \text{for } \eta>1 \end{cases} \tag{A23}$$

are the solutions of the prime SLEs (129) and accordingly (130), satisfying the DBCs at the corresponding singular endpoints:

$$\lim_{\eta\to\eta_{\mp}} \Psi_{\mp}[\eta;\vec{\lambda};\varepsilon|\bar{M}_p] = 0\,. \tag{A24}$$

Representing the $RRZ\mathfrak{T}s$ of the PFSs (A22) and (A23),

$$\Psi[\eta;\vec{\lambda};\varepsilon|\bar{M}_{p+1};\mp] = \frac{W\{\Psi[\eta;\vec{\lambda};\varepsilon_{m_{p+1}}(\vec{\lambda})|\bar{M}_p], \Psi_{\mp}[\eta;\vec{\lambda};\varepsilon|\bar{M}_p]\}}{\rho^{1/2}[\eta]\Psi[\eta;\vec{\lambda};\varepsilon_{m_{p+1}}(\vec{\lambda})|\bar{M}_p]}, \tag{A25}$$

as

$$\begin{aligned} \Psi[\eta;\vec{\lambda};\varepsilon|\bar{M}_{p+1};\mp] = {} & |1-\eta^2|^{1/2}\,\dot{\Psi}_{\mp}[\eta;\vec{\lambda};\varepsilon|\bar{M}_p] \\ & -|1-\eta^2|^{1/2}\, ld\,\Psi[\eta;\vec{\lambda};\varepsilon_{m_{p+1}}(\vec{\lambda})|\bar{M}_p]\Psi_{\mp}[\eta;\vec{\lambda};\varepsilon|\bar{M}_p] \end{aligned} \tag{A26}$$

shows that

$$\lim_{\eta\to\mp 1} \Psi[\eta;\vec{\lambda};\varepsilon|\bar{M}_{p+1};\mp] = 0 \tag{A27}$$

and therefore

$$\Psi[\eta;\vec{\lambda};\varepsilon|\bar{M}_{p+1};\mp] \equiv \Psi_{\mp}[\eta;\vec{\lambda};\varepsilon|\bar{M}_{p+1}] \tag{A28}$$

iff the ExpDiff for the corresponding pole of the RCSLE (67) lies within the LP range:

$$^{*}\lambda_{0;\mp}^{(p)} = |\lambda_{\mp} + p| > 1. \tag{A29}$$

Finally let us prove that the $RRZ\mathfrak{T}$ of the PFS $\Psi_{\infty}[\eta;\vec{\lambda};\underline{\varepsilon}|\bar{M}_p]$ of the prime SLE (130) near the pole at infinity vanishes in the limit $\eta\to\infty$. Taking into account that differentiating of the function η^{α} decreases the power exponent, we find that both summands in

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$$\Psi[\eta;\vec{\lambda};\underset{\sim}{\varepsilon}|\bar{M}_{p+1}]=(\eta^2-1)^{1/2}\,\dot{\Psi}_{\infty}[\eta;\vec{\lambda};\underset{\sim}{\varepsilon}|\bar{M}_p] \qquad (A30)$$

$$-(\eta^2-1)^{1/2}\,ld\,\Psi[\eta;\vec{\lambda};-\varepsilon_{m_{p+1}}(\vec{\lambda})|\bar{M}_p]\Psi_{\infty}[\eta;\vec{\lambda};\tilde{\varepsilon}|\bar{M}_p],$$

vanish at infinity which confirms that the q-RS (A30) is indeed the PFS of the transformed RCSLE near its pole at infinity.

Appendix D. Exact Solvability of the Prime SLEs (129) and (130) Under the DBCs.

Theorem 7. *All the Dirichlet problems for the prime SLEs (129) or alternatively (130), under the constraints (131) and (132) in both cases, have exactly the same discrete energy spectrum as the precursor prime SLE (24) solved under the DBCs on the intervals (-1,+1) or (+1,∞) accordingly.*

Proof: Let us first remind the reader that the theorem statement necessarily holds for p=1. For the prime SLE (130) it has been proven in [22] and one can reproduce very similar arguments for the finite interval. We shall come back to these arguments in the general case of $p>1$.

Re-writing (78) with $p=1$ as

$$\psi_{-+,\mathbf{m}_2}[\eta;-\lambda_{0;-},\lambda_{0;+}|\mathbf{m}_1]=\psi_0[\eta;-\lambda_{0;-}-1,\lambda_{0;+}+1]\times \qquad (A31)$$

$$\frac{W^{(-\lambda_{0;-},\lambda_{0;+-})}_{\mathbf{m}_1+\mathbf{m}_2-1}[\eta\,|\,\mathbf{m}_1,\mathbf{m}_2]}{P^{(-\lambda_{0;-},\lambda_{0;+})}_{\mathbf{m}_1}(\eta)},$$

we find that the power exponents of $\eta\pm1$ coincide with halves of the ExpDiffs

$$^{*}\lambda^{(p)}_{0;\mp}=|\lambda_{\mp}+p|=\lambda_{0;\mp}\mp p>0. \qquad (A32)$$

Keeping in mind that the exponent powers $(\lambda_{0;+}+1)/2$ are positive in both cases, we conclude that the listed solution vanishes in the limits $\eta\to1\mp$ and therefore represents the PFS of both prime SLEs (129) and (130) near this singular endpoint. We thus assert that the JPW (65) with the Jacobi indexes restricted by the conditions (131) may not have zeros inside the corresponding interval of orthogonalization.

Since each prime SLE (129) and (130) is exactly solvable for p=1 and $\bar{\mathbf{M}}_2$ is the admissible set of the polynomial seed solutions, we use the mathematical induction, assuming that the prime SLE (129) or (130) for the admissible set of the polynomial seed solutions, $\bar{\mathbf{M}}_p=\bar{\mathbf{M}}_{P-1}$, has exactly the same discrete energy spectrum as the prime SLE (24) with the leading coefficient function (28). First, one can repeat the above arguments to prove that the JPW (65) with $\bar{M}_p=\bar{\mathbf{M}}_P$ is nodeless inside the corresponding interval of orthogonalization and therefore $\bar{\mathbf{M}}_P$ is the admissible set of the polynomial seed solutions. Our next step is to prove that the given prime SLE with $\bar{\mathbf{M}}_p=\bar{\mathbf{M}}_P$ has exactly the same discrete energy spectrum as (24).

Suppose that the given prime SLE has another eigenfunction $\psi_{\mathbf{c},n}[\eta;\lambda'|\bar{\mathbf{M}}_P]$ at an energy E_n with the absolute value $|E_n|\neq\varepsilon_j(\vec{\lambda}')$ for any $j\leq$ jmax and therefore, by definition, it must obey the DBCs

$$\lim_{\eta\to\eta_{\mp}} \psi_{\mathbf{c},n}[\eta;\vec{\lambda}'\,|\,\bar{\boldsymbol{M}}_P] = 0 \cdot$$

(A33)

If $*\lambda_{0;-}^{(P)} > 1$ (which assures ExpDiffs $*\lambda_{0;\mp}^{(P-1)}$ lie within the LP range) then we can take advantage of the arguments presented in Appendix C to show that the RRZT with the TF

$$*\phi_{m_P}[\eta;\vec{\lambda}|\bar{\boldsymbol{M}}_{P-1}] = \frac{\rho^{-1/2}[\eta]}{\phi_{m_P}[\eta;\vec{\lambda}|\bar{\boldsymbol{M}}_{P-1}]} \qquad \text{(A34)}$$

converts the extraneous eigenfunction into the eigenfunction of the prime SLE (129) or (130) with $\bar{\boldsymbol{M}}_p = \bar{\boldsymbol{M}}_{P-1}$. However this conclusion contradicts the assumption that the prime SLE in question has exactly the same energy spectrum as the prime SLE (24). We thus assert that q-RSs (136) represent all possible eigenfunctions of the prime SLEs (129) and (130), which completes the proof of Theorem 7. □

The direct consequence of the proven theorem is that the p-WPs (99) with $\bar{M}_p = \bar{\boldsymbol{M}}_p$ and $\vec{\sigma} = -+$ have exactly j real zeros larger than 1.

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