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Taylor Tail Renormalization Theory Exact Shift Linearization, Orbit Rigidity, and Asymptotic Fingerprints of Analytic Power Series

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Abstract

Let $f(z) = \sum_{n\geq 0} a_n z^n$ be analytic at the origin, and assume that no Taylor coefficient vanishes. We study the *normalized Taylor tails* $T_n^f(w) := \sum_{k\geq 0} \frac{a_{n+k}}{a_n} w^k$, not as isolated remainders but as a discrete renormalization orbit on the space of normalized analytic germs. The governing map is the nonlinear operator $\mathcal{S}(F)(w) := \frac{F(w)-1}{wF'(0)}$, which acts as a coefficient shift followed by canonical normalization. The exact identity $T_{n+1}^f = \mathcal{S}(T_n^f)$ turns the Taylor coefficients of f into a dynamical system. We develop a self-contained theory of this dynamics. First, we prove that the nonlinearity of $\mathcal{S}$ is exactly linearized in ratio coordinates: the map $F \longmapsto \big((\mathcal{S}^n F)'(0)\big)_{n\geq 0}$ conjugates $\mathcal{S}$ to the one-sided shift on an explicit space of admissible ratio sequences. This yields complete reconstruction of the orbit and a realization theorem for all admissible orbits. Second, we classify the rigid orbit types: fixed points are exactly geometric series, periodic points are exactly rational functions with denominator $1 - \Lambda w^m$, and eventual periodicity is equivalent to a polynomial plus a rational tail. Third, we exhibit genuinely rich internal dynamics by constructing compact invariant subsystems on which $\mathcal{S}$ is conjugate to full shifts on finite alphabets. Fourth, on the asymptotic side, ratio limits force universal geometric profiles, while first- and second-order corrections to the coefficient ratios produce universal corrections to the tail orbit. In particular, dominant algebraic singularities leave a precise first asymptotic fingerprint on the renormalized tails. We also prove exact transport laws under differentiation and Hadamard products. The basic normalized tail object overlaps, up to an index shift, with the normalized remainders recently studied in the special-functions literature. The present contribution is different in focus: it isolates the renormalization operator itself, proves exact shift linearization and orbit realization, identifies symbolic invariant subsystems, and develops rigidity and asymptotic classification results for the resulting dynamical flow.

Keywords: analytic functions; Taylor series; normalized remainders; renormalization; symbolic dynamics; coefficient asymptotics

MSC: 30B10; 30B30; 37B10; 37B20; 05A16; 41A60

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1. Introduction

A classical analytic power series

$$f(z) = \sum_{n\geq 0} a_n z^n$$

may be studied from many familiar viewpoints: by its domain of analyticity, by its singularities, by the asymptotics of its coefficients, by its Padé approximants, or by coefficientwise operations such as differentiation and Hadamard products. The present paper isolates a different but very natural perspective: instead of viewing the series as a single global object, we examine its normalized tails after each Taylor order,

$$T_n^f(w) := \sum_{k\geq 0} \frac{a_{n+k}}{a_n} w^k, \qquad n \geq 0,$$

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and treat the sequence $(T_n^f)_{n\geq 0}$ as a dynamical orbit.

The relevance of such normalized tails is already suggested by classical coefficient asymptotics. Dominant singularities control coefficient growth, and ratio asymptotics often reflect poles, algebraic singularities, and related local structures [3,4,8,15]. A different line of work studies sections and tails of power series through the location and asymptotic distribution of their zeros [10]. More recently, Qi and collaborators have systematically studied *normalized remainders* (or normalized tails) for concrete special functions, proving monotonicity, convexity, logarithmic convexity, and inequalities for the exponential, Bernoulli, trigonometric, and related families [2,7,9,11,12,14,16]. Up to an index shift, the object used in that literature is precisely

$$\sum_{k\geq 0} \frac{a_{n+k}}{a_n} w^k,$$

which is our T_n^f.

The novelty pursued here is not the bare normalization itself, but the exact dynamical law satisfied by the whole tail family. If

$$\mathcal{S}(F)(w) := \frac{F(w)-1}{wF'(0)},$$

then every analytic series with nonzero Taylor coefficients satisfies

$$T_{n+1}^f = \mathcal{S}(T_n^f).$$

Thus the normalized tails form a renormalization orbit under a concrete nonlinear operator on the space of normalized analytic germs. Once this is recognized, several structural phenomena emerge. Some are rigid: fixed points are geometric, periodic points are rational, and eventual periodicity is equivalent to a polynomial plus a rational tail. Some are asymptotic: ratio limits force geometric universality, and corrections to the ratio asymptotics generate corrections to the tail profiles. Some are genuinely dynamical: after a change of coordinates, $\mathcal{S}$ becomes the left shift on the sequence of successive coefficient ratios, and this yields compact invariant subsets conjugate to full symbolic shifts.

The main contributions of the paper may be summarized as follows.

(i) We prove an *exact shift linearization*: the map

$$F \longmapsto \big((\mathcal{S}^n F)'(0)\big)_{n\geq 0}$$

is a bijection from normalized analytic germs to an explicit space of admissible ratio sequences and conjugates $\mathcal{S}$ to the one-sided shift.

(ii) We obtain an *orbit realization theorem*: every admissible ratio sequence produces a unique tail orbit, and every analytic series with nonzero coefficients is reconstructed from its initial scalar and its ratio cocycle.

(iii) We classify exact orbit types. Fixed points are exactly $(1-\rho w)^{-1}$; periodic points are exactly the rational germs $P(w)/(1-\Lambda w^m)$; eventual periodicity is equivalent to a polynomial plus a rational tail.

(iv) We show that the theory is not dynamically trivial: for any finite alphabet $A \subset \mathbb{C}^\times$, there is a compact invariant subset on which $\mathcal{S}$ is topologically conjugate to the full one-sided shift on A. In particular, $\mathcal{S}$ has periodic points of every period and positive topological entropy on such subsystems [6].

(v) We prove asymptotic rigidity theorems. If $a_{n+1}/a_n \to \rho$, then $T_n^f \to (1-\rho w)^{-1}$ locally uniformly for $\rho \neq 0$, and $T_n^f \to 1$ when $\rho = 0$. If the ratios admit first- or second-order asymptotic expansions, then the tail orbit has corresponding universal first- and second-order corrections.

(vi) We prove exact transport rules under differentiation and Hadamard products.

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The paper is organized as follows. Section 2 introduces the basic spaces, the normalized tails, and the renormalization operator. Section 3 proves exact shift linearization, reconstruction, orbit realization, and symbolic invariant subsystems. Section 4 contains the fixed/periodic/eventually periodic classification. Section 5 develops the universal limit theory and the first two asymptotic fingerprints. Section 6 records natural covariance properties, and Section 7 gives explicit model families.

2. Normalized Taylor Tails and the Renormalization Operator

Let $R \in (0, \infty]$, and write

$$D_R := \{z \in \mathbb{C} : |z| < R\}.$$

We work with analytic functions whose Taylor coefficients never vanish.

Definition 1. *For $R \in (0, \infty]$, let $\mathscr{A}_R^\times$ denote the set of all analytic functions*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{O}(D_R)$$

such that $a_n \neq 0$ for every $n \geq 0$.

Let $\mathscr{G}_R^\times$ denote the set of all analytic functions

$$F(w) = 1 + \sum_{n=1}^{\infty} c_n w^n \in \mathcal{O}(D_R)$$

such that $c_n \neq 0$ for every $n \geq 1$.

We also write

$$\mathscr{A}^\times := \bigcup_{R \in (0,\infty]} \mathscr{A}_R^\times, \qquad \mathscr{G}^\times := \bigcup_{R \in (0,\infty]} \mathscr{G}_R^\times.$$

If $f \in \mathscr{A}_R^\times$, define the *normalized Taylor tail of order n* by

$$T_n^f(w) := \sum_{k=0}^{\infty} \frac{a_{n+k}}{a_n} w^k. \tag{1}$$

When $n = 0$, this is simply

$$T_0^f(w) = \frac{f(w)}{a_0}.$$

The next statement shows that tail normalization preserves the radius of convergence exactly.

Proposition 1 (radius invariance)**.** *Let $f(z) = \sum_{n \geq 0} a_n z^n \in \mathscr{A}_R^\times$. Then for every $n \geq 0$, the power series T_n^f has radius of convergence exactly R. In particular, $T_n^f \in \mathscr{G}_R^\times$.*

Proof. By the Cauchy–Hadamard formula,

$$\frac{1}{R} = \limsup_{m \to \infty} |a_m|^{1/m}.$$

For fixed n, the coefficients of T_n^f are a_{n+k}/a_n, so its radius R_n satisfies

$$\frac{1}{R_n} = \limsup_{k \to \infty} \left| \frac{a_{n+k}}{a_n} \right|^{1/k} = \limsup_{k \to \infty} |a_{n+k}|^{1/k} |a_n|^{-1/k}.$$

Since $|a_n|^{-1/k} \to 1$ and $(n+k)/k \to 1$, we obtain

$$\limsup_{k\to\infty} |a_{n+k}|^{1/k} = \limsup_{k\to\infty} \left(|a_{n+k}|^{1/(n+k)}\right)^{(n+k)/k} = \frac{1}{R}.$$

Hence $R_n = R$. $\square$

We now introduce the renormalization operator that drives the tail dynamics.

Definition 2. *For $R \in (0, \infty]$, define*

$$\mathcal{S} : \mathscr{G}_R^\times \to \mathscr{G}_R^\times, \qquad \mathcal{S}(F)(w) := \frac{F(w) - 1}{wF'(0)}. \tag{2}$$

The next proposition makes the coefficient-shift nature of $\mathcal{S}$ explicit.

Proposition 2 (renormalized left shift). *Let $F(w) = 1 + \sum_{n\geq 1} c_n w^n \in \mathscr{G}_R^\times$. Then*

$$\mathcal{S}(F)(w) = \sum_{k=0}^{\infty} \frac{c_{k+1}}{c_1} w^k. \tag{3}$$

Consequently, for every $n \geq 0$,

$$\mathcal{S}^n(F)(w) = \sum_{k=0}^{\infty} \frac{c_{n+k}}{c_n} w^k. \tag{4}$$

Moreover, $\mathcal{S}$ is surjective but not injective. More precisely, for every $F \in \mathscr{G}_R^\times$, the full fiber over F is

$$\left\{G \in \mathscr{G}_R^\times : \mathcal{S}(G) = F\right\} = \left\{1 + \lambda w F(w) : \lambda \in \mathbb{C}^\times\right\}. \tag{5}$$

Proof. Since $F(w) - 1 = \sum_{n\geq 1} c_n w^n$ and $F'(0) = c_1 \neq 0$, we obtain

$$\mathcal{S}(F)(w) = \frac{1}{c_1} \sum_{n\geq 1} c_n w^{n-1} = \sum_{k\geq 0} \frac{c_{k+1}}{c_1} w^k,$$

which proves (3). Formula (4) follows by induction on n.

For the fiber description, let $G \in \mathscr{G}_R^\times$ satisfy $\mathcal{S}(G) = F$. Then

$$G(w) - 1 = wG'(0)F(w),$$

so writing $\lambda := G'(0) \in \mathbb{C}^\times$ yields $G(w) = 1 + \lambda w F(w)$. Conversely, any function of this form satisfies $\mathcal{S}(G) = F$. $\square$

The exact orbit law for normalized Taylor tails is now immediate.

Theorem 1 (exact renormalization law). *Let $f(z) = \sum_{n\geq 0} a_n z^n \in \mathscr{A}_R^\times$. Then for every $n \geq 0$,*

$$T_{n+1}^f = \mathcal{S}(T_n^f). \tag{6}$$

Equivalently,

$$T_n^f = \mathcal{S}^n(T_0^f) \qquad (n \geq 0). \tag{7}$$

In coefficient form,

$$T_n^f(w) = 1 + \frac{a_{n+1}}{a_n}\, w\, T_{n+1}^f(w). \tag{8}$$

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Proof. Write

$$T_n^f(w) = 1 + \sum_{k=1}^{\infty} \frac{a_{n+k}}{a_n} w^k = 1 + \frac{a_{n+1}}{a_n} w \sum_{j=0}^{\infty} \frac{a_{n+1+j}}{a_{n+1}} w^j = 1 + \frac{a_{n+1}}{a_n} w T_{n+1}^f(w),$$

which is (8). Since

$$(T_n^f)'(0) = \frac{a_{n+1}}{a_n},$$

we may solve for T_{n+1}^f to obtain

$$T_{n+1}^f(w) = \frac{T_n^f(w) - 1}{w (T_n^f)'(0)} = \mathcal{S}(T_n^f)(w).$$

Iterating gives (7). □

3. Ratio Coordinates, Exact Shift Linearization, and Orbit Realization

The operator $\mathcal{S}$ is nonlinear on the space of analytic germs, but it becomes an exact left shift after a change of coordinates.

Definition 3 (admissible ratio space). *Let $\mathscr{R}$ be the set of all sequences*

$$r = (r_n)_{n \geq 0} \in (\mathbb{C}^\times)^{\mathbb{N}_0}$$

such that

$$L(r) := \limsup_{n \to \infty} \left| \prod_{j=0}^{n-1} r_j \right|^{1/n} < \infty. \tag{9}$$

For $r \in \mathscr{R}$, define its associated radius by

$$R(r) := \begin{cases} L(r)^{-1}, & L(r) > 0, \\ \infty, & L(r) = 0. \end{cases}$$

Also define the left shift

$$\sigma : \mathscr{R} \to \mathscr{R}, \qquad \sigma(r_0, r_1, r_2, \ldots) = (r_1, r_2, r_3, \ldots).$$

If $r \in \mathscr{R}$ and we set

$$c_0 := 1, \qquad c_n := \prod_{j=0}^{n-1} r_j \qquad (n \geq 1),$$

then $L(r) = \limsup_{n \to \infty} |c_n|^{1/n}$. Thus $R(r)$ is exactly the Cauchy–Hadamard radius of the power series $1 + \sum_{n \geq 1} c_n w^n$. In other words, $\mathscr{R}$ is precisely the coefficient-ratio space for normalized analytic germs.

Theorem 2 (exact shift linearization). *Define*

$$\Phi : \mathscr{G}^\times \to \mathscr{R}, \qquad \Phi(F) := \big((\mathcal{S}^n F)'(0) \big)_{n \geq 0}.$$

Then Φ is a bijection with inverse

$$\Psi(r)(w) := 1 + \sum_{n=1}^{\infty} \Big(\prod_{j=0}^{n-1} r_j \Big) w^n, \qquad r \in \mathscr{R}. \tag{10}$$

Moreover,

$$\Phi \circ \mathcal{S} = \sigma \circ \Phi, \qquad \mathcal{S} \circ \Psi = \Psi \circ \sigma. \tag{11}$$

Thus $\mathcal{S}$ is exactly conjugate to the one-sided shift on $\mathscr{R}$.

Proof. Let $F(w) = 1 + \sum_{n\geq 1} c_n w^n \in \mathscr{G}^\times$. By (4),

$$\mathcal{S}^n(F)(w) = \sum_{k=0}^{\infty} \frac{c_{n+k}}{c_n} w^k,$$

so

$$(\mathcal{S}^n F)'(0) = \frac{c_{n+1}}{c_n} \qquad (n \geq 0). \tag{12}$$

Hence $\Phi(F) \in (\mathbb{C}^\times)^{\mathbb{N}_0}$, and its cumulative products satisfy

$$\prod_{j=0}^{n-1} (\mathcal{S}^j F)'(0) = \prod_{j=0}^{n-1} \frac{c_{j+1}}{c_j} = c_n.$$

Since F is analytic, $\limsup_{n\to\infty} |c_n|^{1/n} < \infty$, so $\Phi(F) \in \mathscr{R}$.

Conversely, let $r \in \mathscr{R}$, and define $c_0 := 1$, $c_n := \prod_{j=0}^{n-1} r_j$ for $n \geq 1$. By (9), the power series

$$1 + \sum_{n=1}^{\infty} c_n w^n$$

has radius of convergence $R(r) > 0$, and every coefficient c_n is nonzero. Thus $\Psi(r) \in \mathscr{G}^\times$.

Now let $F = 1 + \sum_{n\geq 1} c_n w^n \in \mathscr{G}^\times$. By (12),

$$\Psi(\Phi(F))(w) = 1 + \sum_{n=1}^{\infty} \Big(\prod_{j=0}^{n-1} \frac{c_{j+1}}{c_j} \Big) w^n = 1 + \sum_{n=1}^{\infty} c_n w^n = F.$$

Likewise, if $r \in \mathscr{R}$ and $c_n = \prod_{j=0}^{n-1} r_j$, then

$$\Phi(\Psi(r))_n = \frac{c_{n+1}}{c_n} = r_n,$$

so $\Phi(\Psi(r)) = r$. Therefore Φ and Ψ are inverse bijections.

Finally, for $F \in \mathscr{G}^\times$,

$$\Phi(\mathcal{S}(F))_n = (\mathcal{S}^{n+1} F)'(0) = \Phi(F)_{n+1} = \sigma(\Phi(F))_n,$$

which proves $\Phi \circ \mathcal{S} = \sigma \circ \Phi$. Since $\Psi = \Phi^{-1}$, the second identity in (11) follows. □

The conjugacy theorem gives an exact realization result for tail orbits.

Corollary 1 (orbit realization and coefficient reconstruction)**.** *Let $r = (r_n)_{n\geq 0} \in \mathscr{R}$ and let $a_0 \in \mathbb{C}^\times$. Define*

$$a_n := a_0 \prod_{j=0}^{n-1} r_j \qquad (n \geq 1),$$

and set

$$f_r(z) := \sum_{n=0}^{\infty} a_n z^n.$$

Then $f_r \in \mathscr{A}^\times$, and for every $n \geq 0$,

$$T_n^{f_r} = \Psi(\sigma^n r). \tag{13}$$

Conversely, every $f \in \mathscr{A}^\times$ has this form with

$$r_n = \frac{a_{n+1}}{a_n} = (T_n^f)'(0).$$

In particular,

$$a_n = a_0 \prod_{j=0}^{n-1} (T_j^f)'(0) \qquad (n \geq 1). \tag{14}$$

Proof. The coefficient sequence $(a_n/a_0)_{n \geq 0}$ is exactly the coefficient sequence of $\Psi(r)$, so $f_r = a_0 \Psi(r) \in \mathscr{A}^\times$. By Theorem 2,

$$T_n^{f_r} = \mathcal{S}^n(T_0^{f_r}) = \mathcal{S}^n\left(\frac{f_r}{a_0}\right) = \mathcal{S}^n(\Psi(r)) = \Psi(\sigma^n r).$$

The converse follows by taking $r_n = a_{n+1}/a_n$ and applying the already proved formula. □

A simple but important consequence is that the full tail orbit determines the analytic function up to a nonzero scalar.

Corollary 2 (completeness modulo scale). *Let $f, g \in \mathscr{A}^\times$. Then $T_n^f = T_n^g$ for all $n \geq 0$ if and only if there exists $C \in \mathbb{C}^\times$ such that $g = Cf$.*

Proof. If $g = Cf$, then $T_n^g = T_n^f$ by definition. Conversely, if all normalized tails agree, then in particular

$$\frac{f}{a_0} = T_0^f = T_0^g = \frac{g}{b_0},$$

where a_0 and b_0 are the constant terms of f and g. Hence $g = (b_0/a_0)f$. □

The shift linearization immediately identifies minimal periods.

Corollary 3 (minimal period equals ratio period). *Let $F \in \mathscr{G}^\times$, and let $m \geq 1$. Then F has minimal period m under $\mathcal{S}$ if and only if $\Phi(F)$ has minimal period m under the left shift.*

Proof. This is immediate from the conjugacy relation $\Phi \circ \mathcal{S} = \sigma \circ \Phi$. □

The shift conjugacy does more than classify rigid orbit types. It also transports genuine symbolic dynamics into the tail renormalization flow, producing compact invariant subsystems on which the dynamics is exactly a full shift. The next theorem makes this explicit.

Theorem 3 (symbolic invariant subsystems). *Let $A \subset \mathbb{C}^\times$ be a finite set with $q := |A| \geq 2$, and set*

$$M := \max_{\lambda \in A} |\lambda|.$$

Fix $r \in (0, 1/M)$, and let $\Sigma_A := A^{\mathbb{N}_0}$ with the product topology. Define

$$X_{A,r} := \{\Psi(\omega) : \ \omega \in \Sigma_A\} \subset H^\infty(D_r),$$

where $H^\infty(D_r)$ is equipped with the supremum norm on D_r. Then:

(i) *$X_{A,r}$ is compact and $\mathcal{S}$-invariant.*
(ii) *The map $\Psi : \Sigma_A \to X_{A,r}$ is a homeomorphism.*
(iii) *The restriction $\mathcal{S}|_{X_{A,r}}$ is topologically conjugate to the one-sided full shift $\sigma : \Sigma_A \to \Sigma_A$.*

In particular, $\mathcal{S}|_{X_{A,r}}$ has periodic points of every period and topological entropy $\log q$ [6].

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Proof. For $\omega = (\omega_n)_{n\geq 0} \in \Sigma_A$, let

$$c_n(\omega) := \prod_{j=0}^{n-1} \omega_j, \qquad c_0(\omega) := 1.$$

Since $|c_n(\omega)| \leq M^n$, the series

$$\Psi(\omega)(w) = 1 + \sum_{n=1}^{\infty} c_n(\omega) w^n$$

converges uniformly on D_r. Thus Ψ is well defined.

To prove continuity of Ψ, fix $\varepsilon > 0$. Choose N so large that

$$\sum_{n>N} (Mr)^n < \frac{\varepsilon}{2}.$$

If two sequences $\omega, \widetilde{\omega} \in \Sigma_A$ agree in their first N coordinates, then $c_n(\omega) = c_n(\widetilde{\omega})$ for $0 \leq n \leq N$, and therefore

$$\|\Psi(\omega) - \Psi(\widetilde{\omega})\|_{\infty, D_r} \leq \sum_{n>N} |c_n(\omega) - c_n(\widetilde{\omega})| r^n \leq 2 \sum_{n>N} (Mr)^n < \varepsilon.$$

Hence Ψ is continuous.

Now let $F_m \to F$ in $X_{A,r}$. For each fixed n, the coefficient $c_n(F_m)$ of w^n in F_m converges to the coefficient $c_n(F)$ of w^n in F, because coefficient functionals are continuous on $H^\infty(D_r)$. Since each $c_n(F_m)$ takes values in the finite set $\{\prod_{j=0}^{n-1} \omega_j : \omega_j \in A\}$, the convergence is eventually constant. Therefore, for each fixed n, the ratio

$$\frac{c_{n+1}(F_m)}{c_n(F_m)} \in A$$

is eventually constant and converges to $c_{n+1}(F)/c_n(F)$. This proves continuity of $\Phi = \Psi^{-1}$ on $X_{A,r}$. Hence Ψ is a homeomorphism from the compact space Σ_A onto $X_{A,r}$, and $X_{A,r}$ is compact.

Finally, invariance and conjugacy follow from Theorem 2:

$$\mathcal{S}(\Psi(\omega)) = \Psi(\sigma\omega) \qquad (\omega \in \Sigma_A).$$

Thus $\mathcal{S}(X_{A,r}) = X_{A,r}$, and Ψ conjugates σ to $\mathcal{S}|_{X_{A,r}}$. $\square$

Remark 1. *Theorem 3 shows that two apparently opposite features coexist in the theory. There is strong rigidity at the level of exact orbit types (fixed, periodic, eventually periodic), but there are also invariant subsystems with full symbolic dynamics. This coexistence becomes transparent only after the exact shift linearization of Theorem 2.*

4. Exact Rigidity of the Tail Dynamics

We now classify the algebraically rigid orbit types.

Theorem 4 (fixed points)**.** *Let $F \in \mathscr{G}^\times$. Then $\mathcal{S}(F) = F$ if and only if there exists $\rho \in \mathbb{C}^\times$ such that*

$$F(w) = \frac{1}{1 - \rho w}. \tag{15}$$

Proof. If $\mathcal{S}(F) = F$, then from the definition of $\mathcal{S}$,

$$F(w) = \frac{F(w) - 1}{wF'(0)}.$$

Writing $\rho := F'(0)$, we obtain

$$F(w) - 1 = \rho w F(w),$$

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hence $(1-\rho w)F(w)=1$, which is exactly (15). The converse is immediate by direct substitution. □

Periodic points admit a complete rational classification.

Theorem 5 (periodic points)**.** *Let $F(w)=1+\sum_{n\geq 1} c_n w^n \in \mathscr{G}^\times$, and let $m\geq 1$. The following are equivalent.*

(a) $\mathcal{S}^m(F)=F$.
(b) *The ratio sequence $\Phi(F)=(c_{n+1}/c_n)_{n\geq 0}$ is m-periodic.*
(c) *There exists $\Lambda\in\mathbb{C}^\times$ such that*
$$c_{m+k}=\Lambda c_k \qquad \text{for all } k\geq 0. \tag{16}$$
(d) *There exists $\Lambda\in\mathbb{C}^\times$ and a polynomial*
$$P(w)=1+c_1 w+\cdots+c_{m-1}w^{m-1}$$
of degree at most $m-1$ such that
$$F(w)=\frac{P(w)}{1-\Lambda w^m}. \tag{17}$$

In particular, every periodic point of $\mathcal{S}$ is rational.

Proof. The equivalence of (a) and (b) follows from Theorem 2. Assume (b). Then for every $k\geq 0$,
$$\frac{c_{m+k+1}}{c_{m+k}}=\frac{c_{k+1}}{c_k}.$$
Since $c_m\neq 0$, induction on k shows that
$$\frac{c_{m+k}}{c_k}=\frac{c_m}{c_0}=c_m \qquad (k\geq 0),$$
which is (16) with $\Lambda:=c_m$. Thus (b) implies (c).

Assume (c). Writing each index as $qm+j$ with $q\geq 0$ and $0\leq j<m$, induction on q gives
$$c_{qm+j}=\Lambda^q c_j.$$
Hence
$$F(w)=\sum_{j=0}^{m-1}\sum_{q=0}^{\infty} c_{qm+j}w^{qm+j}=\sum_{j=0}^{m-1} c_j w^j\sum_{q=0}^{\infty}(\Lambda w^m)^q=\frac{P(w)}{1-\Lambda w^m},$$
which is (d).

Finally, if (d) holds, then expanding the geometric denominator yields
$$F(w)=\sum_{j=0}^{m-1}\sum_{q=0}^{\infty} c_j\Lambda^q w^{qm+j},$$
so $c_{qm+j}=\Lambda^q c_j$. Therefore
$$\frac{c_{n+m+1}}{c_{n+m}}=\frac{c_{n+1}}{c_n} \qquad (n\geq 0),$$
which shows that $\Phi(F)$ is m-periodic. Thus (d) implies (b). □

Applying this to $T_0^f=f/a_0$ yields the classification for periodic tail flows.

Corollary 4 (periodic tail flows)**.** *Let $f(z)=\sum_{n\geq 0} a_n z^n\in\mathscr{A}^\times$, and let $m\geq 1$. The following are equivalent.*

(a) $T^f_{n+m}=T^f_n$ *for all* $n\geq 0$.
(b) *The ratio sequence $(a_{n+1}/a_n)_{n\geq 0}$ is m-periodic.*

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(c) *There exists* $\Lambda \in \mathbb{C}^\times$ *such that* $a_{m+k} = \Lambda a_k$ *for all* $k \geq 0$.
(d) *There exists* $\Lambda \in \mathbb{C}^\times$ *and a polynomial*

$$P(z) = a_0 + a_1 z + \cdots + a_{m-1} z^{m-1}$$

of degree at most $m-1$ *such that*

$$f(z) = \frac{P(z)}{1 - \Lambda z^m}. \tag{18}$$

Proof. Since $T_n^f = \mathcal{S}^n(T_0^f)$, condition (a) is equivalent to $\mathcal{S}^m(T_0^f) = T_0^f$. Apply Theorem 5 to $T_0^f = f/a_0$. □

Eventual periodicity also admits a complete description.

Theorem 6 (eventual periodicity). *Let* $F(w) = 1 + \sum_{n\geq 1} c_n w^n \in \mathscr{G}^\times$. *Fix integers* $N \geq 0$ *and* $m \geq 1$. *The following are equivalent.*

(a) $\mathcal{S}^{N+m}(F) = \mathcal{S}^N(F)$.
(b) *The ratio sequence* $\Phi(F) = (c_{n+1}/c_n)_{n\geq 0}$ *is eventually* m*-periodic after rank* N*, that is,*

$$\frac{c_{n+m+1}}{c_{n+m}} = \frac{c_{n+1}}{c_n} \qquad \text{for all } n \geq N. \tag{19}$$

(c) *There exist* $\Lambda \in \mathbb{C}^\times$ *and coefficients* $d_0, \ldots, d_{m-1} \in \mathbb{C}^\times$ *such that*

$$c_{N+qm+j} = d_j \Lambda^q \qquad (q \geq 0,\ 0 \leq j < m). \tag{20}$$

(d) *There exist a polynomial* Q *of degree at most* $N-1$*, a polynomial* P *of degree at most* $m-1$*, and* $\Lambda \in \mathbb{C}^\times$ *such that*

$$F(w) = Q(w) + w^N \frac{P(w)}{1 - \Lambda w^m}. \tag{21}$$

Proof. The equivalence of (a) and (b) follows from Theorem 2. Assume (b). Then the shifted germ $G := \mathcal{S}^N(F)$ has periodic ratio sequence of period m, hence Theorem 5 yields

$$G(w) = \frac{P_0(w)}{1 - \Lambda w^m}$$

for some polynomial P_0 of degree at most $m-1$. Since

$$G(w) = \sum_{k=0}^{\infty} \frac{c_{N+k}}{c_N} w^k,$$

this implies (20) and, after restoring the first N coefficients, (21). Thus (b) implies (c) and (d).

Conversely, (20) immediately implies that the tail starting at rank N has periodic ratio sequence of period m, which is (b). If (d) holds, expanding the rational tail shows that (20) holds, hence again (b) follows. □

Corollary 5 (eventually periodic tail flows for f). *Let* $f(z) = \sum_{n\geq 0} a_n z^n \in \mathscr{A}^\times$. *There exist integers* $N \geq 0$ *and* $m \geq 1$ *such that*

$$T_{n+m}^f = T_n^f \qquad \text{for all } n \geq N$$

if and only if

$$f(z) = Q(z) + z^N \frac{P(z)}{1 - \Lambda z^m} \tag{22}$$

for some polynomial Q*, some polynomial* P *of degree at most* $m-1$*, and some* $\Lambda \in \mathbb{C}^\times$.

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Proof. Apply Theorem 6 to $F = T_0^f = f/a_0$, then multiply by a_0. □

5. Universality and Asymptotic Fingerprints

We now turn from exact algebraic rigidity to asymptotic rigidity.

5.1. Geometric Universality from Ratio Asymptotics

Theorem 7 (ratio-limit universality). *Let $f(z) = \sum_{n\geq 0} a_n z^n \in \mathscr{A}^\times$, and write*

$$r_n := \frac{a_{n+1}}{a_n}.$$

(i) *If $r_n \to \rho \in \mathbb{C}^\times$, then*

$$T_n^f(w) \longrightarrow \frac{1}{1-\rho w} \tag{23}$$

locally uniformly on $D_{\min(R,1/|\rho|)}$, where R is the radius of convergence of f.

(ii) *If $r_n \to 0$, then*

$$T_n^f(w) \longrightarrow 1 \tag{24}$$

locally uniformly on D_R.

Proof. Fix $0 < r < \min(R, 1/|\rho|)$ in case $\rho \neq 0$, and $0 < r < R$ in case $\rho = 0$. For $|w| \leq r$, we have

$$T_n^f(w) = \sum_{k=0}^{\infty} \Big(\prod_{j=0}^{k-1} r_{n+j} \Big) w^k, \tag{25}$$

with the empty product equal to 1.

Assume first that $r_n \to \rho \neq 0$. Choose q such that $|\rho| r < q < 1$. For all sufficiently large n, we have $|r_m| r \leq q$ for every $m \geq n$. Thus for such n and all $|w| \leq r$,

$$\left| \Big(\prod_{j=0}^{k-1} r_{n+j} \Big) w^k \right| \leq q^k \qquad (k \geq 0).$$

For each fixed k,

$$\prod_{j=0}^{k-1} r_{n+j} \longrightarrow \rho^k \qquad (n \to \infty),$$

so dominated convergence applied to the series (25) yields

$$T_n^f(w) \longrightarrow \sum_{k=0}^{\infty} (\rho w)^k = \frac{1}{1-\rho w}$$

uniformly on $|w| \leq r$.

Now assume $r_n \to 0$. Define

$$M_n := \sup_{m \geq n} |r_m|,$$

so that $M_n \to 0$. For all sufficiently large n, we have $M_n r < 1$, and then (25) yields

$$\left| T_n^f(w) - 1 \right| \leq \sum_{k=1}^{\infty} (M_n r)^k = \frac{M_n r}{1 - M_n r} \longrightarrow 0 \qquad (n \to \infty)$$

uniformly on $|w| \leq r$. □

A useful consequence is that every nondegenerate locally uniform limit profile is necessarily geometric.

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Corollary 6 (limit rigidity)**.** *Let $f \in \mathscr{A}^\times$, and assume that for some $r > 0$, the sequence (T_n^f) converges locally uniformly on D_r to an analytic function T. If $T'(0) \neq 0$, then*

$$T(w) = \frac{1}{1 - T'(0)w} \qquad (|w| < r).$$

Proof. Local uniform convergence implies convergence of derivatives at the origin, so

$$\frac{a_{n+1}}{a_n} = (T_n^f)'(0) \longrightarrow T'(0) =: \rho \neq 0.$$

By Theorem 7, $T_n^f \to (1 - \rho w)^{-1}$ locally uniformly on a neighborhood of the origin. Uniqueness of locally uniform limits gives the claim. □

5.2. First- and Second-Order Asymptotic Fingerprints

The leading limit profile is geometric whenever the coefficient ratios converge to a nonzero limit. The next theorem shows that the first correction is also universal once the first correction to the ratios is known.

Theorem 8 (first-order tail fingerprint)**.** *Let $f(z) = \sum_{n \geq 0} a_n z^n \in \mathscr{A}^\times$. Assume that there exist $\rho \in \mathbb{C}^\times$, $\beta \in \mathbb{C}$, and $\eta > 0$ such that*

$$\frac{a_{n+1}}{a_n} = \rho\left(1 + \frac{\beta}{n} + O(n^{-1-\eta})\right) \qquad (n \to \infty). \tag{26}$$

Then for every fixed $s \in (0,1)$,

$$T_n^f(w) = \frac{1}{1 - \rho w} + \frac{\beta}{n}\frac{\rho w}{(1 - \rho w)^2} + o\left(\frac{1}{n}\right) \tag{27}$$

uniformly on the closed set $\{w : |\rho w| \leq s\}$.

Proof. Write

$$\frac{a_{n+1}}{a_n} = \rho\left(1 + \frac{\beta}{n} + \theta_n\right), \qquad \theta_n = O(n^{-1-\eta}).$$

Fix $s \in (0,1)$, and write $x := \rho w$, so $|x| \leq s$. Choose a constant $C > 0$ and set

$$K_n := \lfloor C \log n \rfloor.$$

We split the series defining T_n^f at K_n.

Step 1: uniform expansion for the head. For $0 \leq k \leq K_n$,

$$\frac{a_{n+k}}{a_n \rho^k} = \prod_{j=0}^{k-1}\left(1 + \frac{\beta}{n+j} + \theta_{n+j}\right). \tag{28}$$

Since $k \leq K_n = O(\log n)$, each factor is $1 + O(1/n)$, uniformly in j. Taking logarithms and using $\log(1+u) = u + O(u^2)$, we obtain

$$\log \frac{a_{n+k}}{a_n \rho^k} = \sum_{j=0}^{k-1}\left(\frac{\beta}{n+j} + \theta_{n+j} + O((n+j)^{-2})\right).$$

Now

$$\sum_{j=0}^{k-1} \frac{1}{n+j} = \frac{k}{n} + O\left(\frac{k^2}{n^2}\right),$$

and because $k = O(\log n)$, we have

$$\frac{k^2}{n^2} = o\left(\frac{1}{n}\right), \qquad \sum_{j=0}^{k-1} \theta_{n+j} = O\left(\frac{k}{n^{1+\eta}}\right) = o\left(\frac{1}{n}\right), \qquad \sum_{j=0}^{k-1} O((n+j)^{-2}) = o\left(\frac{1}{n}\right),$$

uniformly for $0 \le k \le K_n$. Hence

$$\log \frac{a_{n+k}}{a_n \rho^k} = \frac{\beta k}{n} + o\left(\frac{1}{n}\right)$$

uniformly in $k \le K_n$. Exponentiating gives

$$\frac{a_{n+k}}{a_n} = \rho^k \left(1 + \frac{\beta k}{n} + o\left(\frac{1}{n}\right)\right) \tag{29}$$

uniformly for $0 \le k \le K_n$.

Multiplying by w^k and summing, we obtain

$$\sum_{k=0}^{K_n} \frac{a_{n+k}}{a_n} w^k = \sum_{k=0}^{K_n} x^k + \frac{\beta}{n} \sum_{k=0}^{K_n} k x^k + o\left(\frac{1}{n}\right) \tag{30}$$

uniformly on $|x| \le s$, because $\sum_{k=0}^{K_n} |x|^k \le (1-s)^{-1}$.

Step 2: uniform control of the tail. Choose $\delta > 0$ such that $q := s(1+\delta) < 1$. By (26), for all large n and all $m \ge n$,

$$|r_m| := \left|\frac{a_{m+1}}{a_m}\right| \le |\rho|(1+\delta).$$

Therefore, for $|x| \le s$,

$$\left|\frac{a_{n+k}}{a_n} w^k\right| = \left|\prod_{j=0}^{k-1} r_{n+j} w\right| \le q^k.$$

Hence

$$\sup_{|x| \le s} \left|\sum_{k > K_n} \frac{a_{n+k}}{a_n} w^k\right| \le \sum_{k > K_n} q^k = O(q^{K_n}). \tag{31}$$

If C is chosen so large that $q^{K_n} = o(1/n)$, then the tail contribution is $o(1/n)$.

The same estimate shows that

$$\sum_{k > K_n} x^k = o\left(\frac{1}{n}\right), \qquad \frac{1}{n} \sum_{k > K_n} k x^k = o\left(\frac{1}{n}\right)$$

uniformly on $|x| \le s$. Therefore the truncated sums in (30) may be replaced by the full geometric series and its derivative:

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}, \qquad \sum_{k=0}^{\infty} k x^k = \frac{x}{(1-x)^2}.$$

Combining these identities with (30) and (31) yields

$$T_n^f(w) = \frac{1}{1-x} + \frac{\beta}{n} \frac{x}{(1-x)^2} + o\left(\frac{1}{n}\right), \qquad x = \rho w,$$

uniformly for $|x| \le s$, which is precisely (27). □

The next theorem keeps one more term and yields a second asymptotic fingerprint.

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Theorem 9 (second-order tail fingerprint). *Let $f(z) = \sum_{n\ge 0} a_n z^n \in \mathscr{A}^\times$. Assume that there exist $\rho \in \mathbb{C}^\times$, $\beta, \gamma \in \mathbb{C}$, and $\eta > 0$ such that*

$$\frac{a_{n+1}}{a_n} = \rho\left(1 + \frac{\beta}{n} + \frac{\gamma}{n^2} + O(n^{-2-\eta})\right) \qquad (n \to \infty). \tag{32}$$

Then for every fixed $s \in (0,1)$,

$$T_n^f(w) = \frac{1}{1-\rho w} + \frac{\beta}{n}\frac{\rho w}{(1-\rho w)^2} + \frac{1}{n^2}\left[\gamma\frac{\rho w}{(1-\rho w)^2} + \beta(\beta-1)\frac{(\rho w)^2}{(1-\rho w)^3}\right] + o\left(\frac{1}{n^2}\right) \tag{33}$$

uniformly on $\{w : |\rho w| \le s\}$.

Proof. Write

$$\frac{a_{n+1}}{a_n} = \rho\left(1 + \frac{\beta}{n} + \frac{\gamma}{n^2} + \theta_n\right), \qquad \theta_n = O(n^{-2-\eta}).$$

Fix $s \in (0,1)$, set $x := \rho w$, and choose

$$K_n := \lfloor C \log n \rfloor$$

with $C > 0$ to be chosen later.

Step 1: logarithmic expansion. For $0 \le k \le K_n$,

$$\frac{a_{n+k}}{a_n \rho^k} = \prod_{j=0}^{k-1}(1 + u_{n,j}), \qquad u_{n,j} := \frac{\beta}{n+j} + \frac{\gamma}{(n+j)^2} + \theta_{n+j}.$$

Because $u_{n,j} = O(n^{-1})$ uniformly for $0 \le j < K_n$, the expansion $\log(1+u) = u - u^2/2 + O(u^3)$ gives

$$\log(1 + u_{n,j}) = \frac{\beta}{n+j} + \frac{\gamma - \beta^2/2}{(n+j)^2} + \theta_{n+j} + O(n^{-3}),$$

uniformly in j. Summing from $j = 0$ to $k-1$, and using $k = O(\log n)$, we get

$$\log\frac{a_{n+k}}{a_n \rho^k} = \beta\sum_{j=0}^{k-1}\frac{1}{n+j} + \left(\gamma - \frac{\beta^2}{2}\right)\sum_{j=0}^{k-1}\frac{1}{(n+j)^2} + o\left(\frac{1}{n^2}\right) \tag{34}$$

uniformly for $0 \le k \le K_n$. The standard estimates

$$\sum_{j=0}^{k-1}\frac{1}{n+j} = \frac{k}{n} - \frac{k(k-1)}{2n^2} + O\left(\frac{k^3}{n^3}\right), \qquad \sum_{j=0}^{k-1}\frac{1}{(n+j)^2} = \frac{k}{n^2} + O\left(\frac{k^2}{n^3}\right)$$

therefore yield

$$\log\frac{a_{n+k}}{a_n \rho^k} = \frac{\beta k}{n} + \frac{1}{n^2}\left[-\frac{\beta}{2}k(k-1) + \left(\gamma - \frac{\beta^2}{2}\right)k\right] + o\left(\frac{1}{n^2}\right)$$

uniformly for $0 \le k \le K_n$. Exponentiating, with $e^{u+v} = 1 + u + v + u^2/2 + o(n^{-2})$ for $u = O(n^{-1})$ and $v = O(n^{-2})$, gives

$$\frac{a_{n+k}}{a_n} = \rho^k\left[1 + \frac{\beta k}{n} + \frac{1}{n^2}\left(\gamma k + \frac{\beta(\beta-1)}{2}k(k-1)\right) + o\left(\frac{1}{n^2}\right)\right] \tag{35}$$

uniformly for $0 \le k \le K_n$.

Step 2: tail domination. Choose $\delta > 0$ such that $q := s(1+\delta) < 1$. By (32), for all large enough n and all $m \ge n$,

$$\left|\frac{a_{m+1}}{a_m}\right| \le |\rho|(1+\delta).$$

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Hence, for $|x| \le s$,

$$\left|\frac{a_{n+k}}{a_n} w^k\right| = \left|\prod_{j=0}^{k-1} \frac{a_{n+j+1}}{a_{n+j}}\, w\right| \le q^k.$$

Therefore

$$\sup_{|x|\le s}\left|\sum_{k>K_n} \frac{a_{n+k}}{a_n} w^k\right| \le \sum_{k>K_n} q^k = O(q^{K_n}). \tag{36}$$

Taking C large enough makes $q^{K_n} = o(n^{-2})$. The same bound yields

$$\sum_{k>K_n} x^k = o\left(\frac{1}{n^2}\right), \qquad \frac{1}{n}\sum_{k>K_n} kx^k = o\left(\frac{1}{n^2}\right), \qquad \frac{1}{n^2}\sum_{k>K_n} k(k-1)x^k = o\left(\frac{1}{n^2}\right)$$

uniformly on $|x| \le s$.

Step 3: summation identities. Summing (35) over $0 \le k \le K_n$, we obtain

$$\sum_{k=0}^{K_n} \frac{a_{n+k}}{a_n} w^k = \sum_{k=0}^{K_n} x^k + \frac{\beta}{n}\sum_{k=0}^{K_n} kx^k + \frac{1}{n^2}\left[\gamma \sum_{k=0}^{K_n} kx^k + \frac{\beta(\beta-1)}{2}\sum_{k=0}^{K_n} k(k-1)x^k\right] + o\left(\frac{1}{n^2}\right),$$

uniformly on $|x| \le s$. By Step 2, the truncated sums can be replaced by the full identities

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}, \qquad \sum_{k=0}^{\infty} kx^k = \frac{x}{(1-x)^2}, \qquad \sum_{k=0}^{\infty} k(k-1)x^k = \frac{2x^2}{(1-x)^3}.$$

Substituting these formulas yields

$$T_n^f(w) = \frac{1}{1-x} + \frac{\beta}{n}\frac{x}{(1-x)^2} + \frac{1}{n^2}\left[\gamma\frac{x}{(1-x)^2} + \beta(\beta-1)\frac{x^2}{(1-x)^3}\right] + o\left(\frac{1}{n^2}\right)$$

uniformly on $|x| \le s$.

Conclusion. Since $x = \rho w$, this is exactly (33). □

The singular meaning of the first correction is classical and immediate.

Remark 2 (interpretation of the first fingerprint). *Suppose that f has a unique dominant algebraic singularity at $z = \zeta \ne 0$, with local form*

$$f(z) = g(z)(1 - z/\zeta)^{-\alpha},$$

where g is analytic and nonzero at ζ. Standard transfer theorems of singularity analysis imply

$$a_n \sim \frac{g(\zeta)}{\Gamma(\alpha)}\zeta^{-n} n^{\alpha-1},$$

and therefore

$$\frac{a_{n+1}}{a_n} = \zeta^{-1}\left(1 + \frac{\alpha-1}{n} + O(n^{-2})\right) \qquad (n \to \infty)$$

[3,4,8]. Hence Theorem 8 gives

$$T_n^f(w) = \frac{1}{1-\zeta^{-1}w} + \frac{\alpha-1}{n}\frac{\zeta^{-1}w}{(1-\zeta^{-1}w)^2} + o\left(\frac{1}{n}\right).$$

This is the first asymptotic fingerprint of the theory. The leading geometric profile identifies the dominant singularity location through $\rho = \zeta^{-1}$, while the n^{-1}-correction isolates the algebraic exponent α. In particular,

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two functions with the same dominant singularity location but different algebraic exponents already separate at first order in the tail flow. In this regime, the renormalized orbit is not merely convergent: its first deviation from the universal geometric attractor already records the singularity type.

6. Natural Covariance Properties

The tail flow behaves naturally under several standard coefficient operations.

Proposition 3 (scalar invariance and dilations). *Let* $f(z) = \sum_{n\geq 0} a_n z^n \in \mathscr{A}^\times$.

(i) *If* $c \in \mathbb{C}^\times$, *then* $T_n^{cf} = T_n^f$ *for every* $n \geq 0$.
(ii) *If* $\lambda \in \mathbb{C}^\times$ *and* $g(z) := f(\lambda z)$, *then*

$$T_n^g(w) = T_n^f(\lambda w) \qquad (n \geq 0). \tag{37}$$

Proof. The first statement is immediate from the definition. For the second, the coefficients of g are $b_n = a_n \lambda^n$, hence

$$T_n^g(w) = \sum_{k=0}^{\infty} \frac{b_{n+k}}{b_n} w^k = \sum_{k=0}^{\infty} \frac{a_{n+k}}{a_n} (\lambda w)^k = T_n^f(\lambda w).$$

□

Remark 3. *By Proposition 3, whenever a nonzero ratio limit* ρ *is present, one may rescale the variable by* $z \mapsto \rho^{-1} z$ *and reduce the universal geometric attractor to the normalized model* $(1-w)^{-1}$.

Differentiation admits an exact transport formula.

Proposition 4 (transport under differentiation). *Let* $f(z) = \sum_{n\geq 0} a_n z^n \in \mathscr{A}^\times$. *Then for every* $n \geq 0$,

$$T_n^{f'}(w) = T_{n+1}^f(w) + \frac{w}{n+1} (T_{n+1}^f)'(w). \tag{38}$$

Proof. Since

$$f'(z) = \sum_{n=0}^{\infty} (n+1) a_{n+1} z^n,$$

we have

$$T_n^{f'}(w) = \sum_{k=0}^{\infty} \frac{(n+k+1) a_{n+k+1}}{(n+1) a_{n+1}} w^k.$$

On the other hand,

$$T_{n+1}^f(w) = \sum_{k=0}^{\infty} \frac{a_{n+1+k}}{a_{n+1}} w^k,$$

so

$$\frac{w}{n+1} (T_{n+1}^f)'(w) = \sum_{k=1}^{\infty} \frac{k}{n+1} \frac{a_{n+1+k}}{a_{n+1}} w^k.$$

Adding the last two displays gives

$$\sum_{k=0}^{\infty} \left(1 + \frac{k}{n+1}\right) \frac{a_{n+1+k}}{a_{n+1}} w^k = \sum_{k=0}^{\infty} \frac{(n+k+1) a_{n+k+1}}{(n+1) a_{n+1}} w^k,$$

which is exactly (38). □

Finally, the theory is closed under Hadamard products.

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Proposition 5 (Hadamard covariance). *Let*

$$f(z) = \sum_{n\geq 0} a_n z^n, \qquad g(z) = \sum_{n\geq 0} b_n z^n$$

belong to $\mathscr{A}^\times$*, and define their Hadamard product*

$$(f \odot g)(z) := \sum_{n\geq 0} a_n b_n z^n.$$

Then for every $n \geq 0$*,*

$$T_n^{f\odot g} = T_n^f \odot T_n^g, \tag{39}$$

where the Hadamard product on the right-hand side is taken with respect to the variable w*.*

Proof. This is coefficientwise:

$$T_n^{f\odot g}(w) = \sum_{k=0}^{\infty} \frac{a_{n+k}b_{n+k}}{a_n b_n} w^k = \sum_{k=0}^{\infty} \left(\frac{a_{n+k}}{a_n}\right)\left(\frac{b_{n+k}}{b_n}\right) w^k = T_n^f \odot T_n^g(w).$$

□

7. Examples and Model Families

Example 1 (geometric series: the fixed-point model). *Let*

$$f(z) = \frac{1}{1-\rho z} = \sum_{n=0}^{\infty} \rho^n z^n, \qquad \rho \in \mathbb{C}^\times.$$

Then

$$\frac{a_{n+k}}{a_n} = \rho^k, \qquad T_n^f(w) = \sum_{k=0}^{\infty} (\rho w)^k = \frac{1}{1-\rho w}$$

for every $n \geq 0$*. Thus every normalized tail is identical, and the entire orbit is a fixed point of* $\mathcal{S}$*.*

Example 2 (exponential series: a degenerate limit class). *Let*

$$f(z) = e^{\rho z} = \sum_{n=0}^{\infty} \frac{\rho^n}{n!} z^n.$$

Then

$$\frac{a_{n+k}}{a_n} = \frac{\rho^k}{(n+1)(n+2)\cdots(n+k)} \qquad (k \geq 1),$$

so

$$T_n^f(w) = \sum_{k=0}^{\infty} \frac{(\rho w)^k}{(n+1)_k} = {}_1F_1(1; n+1; \rho w),$$

where $(n+1)_k$ *is the rising factorial. Since* $a_{n+1}/a_n = \rho/(n+1) \to 0$*, Theorem 7 gives*

$$T_n^f(w) \to 1$$

locally uniformly on $\mathbb{C}$*. The exponential function therefore belongs to the degenerate universality class* $\rho = 0$*, not to the nondegenerate fixed-point regime* $\rho \neq 0$*. The limit profile* 1 *is not a fixed point of* $\mathcal{S}$ *inside* $\mathscr{G}^\times$*, since it has vanishing derivative at the origin. Rather, the orbit approaches the boundary of the phase space, and* 1 *appears only as a degenerate boundary profile of the tail dynamics.*

Example 3 (algebraic model and the first two fingerprints). *Let*

$$f(z) = (1-\rho z)^{-\alpha} = \sum_{n=0}^{\infty} \frac{(\alpha)_n}{n!} \rho^n z^n, \qquad \rho \in \mathbb{C}^\times.$$

Then

$$\frac{a_{n+1}}{a_n} = \rho \frac{n+\alpha}{n+1} = \rho\Big(1 + \frac{\alpha-1}{n} + \frac{1-\alpha}{n^2} + O(n^{-3})\Big),$$

and

$$T_n^f(w) = \sum_{k=0}^{\infty} \frac{(n+\alpha)_k}{(n+1)_k} (\rho w)^k = {}_2F_1(1, n+\alpha; n+1; \rho w).$$

By Theorems 8 and 9, for every $s \in (0,1)$,

$$\begin{aligned} T_n^f(w) = {} & \frac{1}{1-\rho w} + \frac{\alpha-1}{n} \frac{\rho w}{(1-\rho w)^2} \\ & + \frac{1}{n^2}\left[(1-\alpha)\frac{\rho w}{(1-\rho w)^2} + (\alpha-1)(\alpha-2)\frac{(\rho w)^2}{(1-\rho w)^3}\right] + o\Big(\frac{1}{n^2}\Big) \end{aligned}$$

uniformly on $|\rho w| \le s$.

Example 4 (a genuine two-cycle). *Let*

$$F(w) = \frac{1+bw}{1-\Lambda w^2}, \qquad b, \Lambda \in \mathbb{C}^\times.$$

Its Taylor coefficients are

$$1,\ b,\ \Lambda,\ b\Lambda,\ \Lambda^2,\ b\Lambda^2, \ldots,$$

so by (3),

$$\mathcal{S}(F)(w) = \frac{1+(\Lambda/b)w}{1-\Lambda w^2}.$$

Applying $\mathcal{S}$ once more gives $\mathcal{S}^2(F) = F$. Hence F is periodic of period dividing 2. It is a genuine two-cycle exactly when $b^2 \neq \Lambda$; if $b^2 = \Lambda$, then

$$F(w) = \frac{1+bw}{1-b^2w^2} = \frac{1}{1-bw},$$

so the cycle collapses to the geometric fixed point.

Example 5 (binary symbolic dynamics inside $\mathcal{S}$). *Let $A = \{1,2\}$, and for $\omega = (\omega_n)_{n\ge 0} \in A^{\mathbb{N}_0}$ define*

$$F_\omega(w) := 1 + \sum_{n=1}^{\infty} \Big(\prod_{j=0}^{n-1} \omega_j\Big) w^n, \qquad |w| < \frac{1}{2}.$$

Then $F_\omega \in \mathcal{G}^\times_{1/2}$, and Theorem 2 gives

$$\mathcal{S}(F_\omega) = F_{\sigma\omega}.$$

Thus every binary sequence produces a valid orbit of renormalized tails. Periodic binary words give periodic rational germs; aperiodic binary sequences give aperiodic orbits. This is the simplest concrete manifestation of the symbolic dynamics from Theorem 3.

8. Discussion and Outlook

The operator $\mathcal{S}$ turns normalized Taylor tails into a discrete renormalization flow on analytic germs. The resulting theory has three layers.

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At the *structural* level, the dynamics is exactly linearized by ratio coordinates: $\mathcal{S}$ is conjugate to the left shift on admissible ratio sequences. This gives a complete orbit realization theorem and shows that the renormalized tails are not an ad hoc family of auxiliary objects but an exact dynamical encoding of the Taylor coefficients.

At the *rigidity* level, the theory is sharp: fixed points are geometric, periodic points are rational, and eventual periodicity is polynomial plus rational tail. These are exact equivalences, not heuristics.

At the *asymptotic* level, geometric profiles are universal whenever the ratios converge, and the first two corrections to the ratio asymptotics propagate into universal corrections of the tail orbit. In the algebraic model, the dominant singularity and the singular exponent appear directly in those corrections.

At the same time, the exact shift linearization shows that the theory is dynamically richer than the rigidity results alone might suggest: finite-alphabet ratio sequences produce compact invariant subsystems with full symbolic dynamics. Thus exact rigidity and dynamical complexity coexist naturally inside the same framework.

Several directions seem especially promising.

(1) **Higher-order fingerprints.** Theorem 9 suggests a full hierarchy in which successively finer ratio asymptotics produce successively finer universal corrections to the renormalized tails.

(2) **Rationality testing and approximation.** Since eventual periodicity is exactly rational-tail behavior, one may hope to connect finite orbit segments with effective rationality tests and Padé-type heuristics [1].

(3) **Restricted invariant classes.** On subsets defined by positivity, complete monotonicity, or special-function structure, one may seek finer dynamical invariants beyond the bare ratio sequence.

(4) **Multivariate extensions.** For a multivariate power series $\sum_\alpha a_\alpha z^\alpha$, one can define renormalized tails along rays or filtrations in multi-index space. The resulting dynamical geometry is likely far richer than in one variable.

The main lesson is simple but robust: an analytic power series can be studied not only through its full expansion, but through the orbit of its renormalized Taylor tails. That orbit is exact, computable, dynamically meaningful, rigid enough to recognize rational structure, and flexible enough to encode symbolic dynamics and asymptotic fingerprints of dominant singularities.

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