Article

Not peer-reviewed version

The Width of $f_0(980)$ in Isospin-Violating Decays

Yun Liang , Wencheng Yan * , Demin Li *

Posted Date: 15 May 2023

doi: 10.20944/preprints202305.1008.v1

Keywords: f0(980); isospin-violating; simultaneous fit; coupling constant

Article

The Width of $f_0(980)$ in Isospin-Violating Decays

Yun Liang [1], Wencheng Yan [1,*] and Demin Li [1,*]

[1] School of Physics and Microelectronics, Zhengzhou University, Zhengzhou 450001, P.R.China

* Correspondence: yanwc@zzu.edu.cn(W.Y.); lidm@zzu.edu.cn(D.L.)

Abstract: The scalar meson $f_0(980)$ has been a long-standing puzzle in light hadron physics. The mass and width of the $f_0(980)$ in normal decay processes are estimated to be $M = 990 \pm 20$ MeV$/c^2$ and $\Gamma = 40 - 100$ MeV, respectively. Theoretically, the internal structure of the $f_0(980)$ is proposed to be a conventional quark-antiquark meson, tetraquarks state, $K\overline{K}$ molecule, or quark-antiquark gluon hybrid. So far, explanations about the nature of $f_0(980)$ have been controversial for a long time. Recently, anomalously narrow width $f_0(980)$ were observed by the BESIII experiment in five independent isospin-violating decay channels. Based on the experimental data, we performed a simultaneous fit to $\pi\pi$ invariant mass distributions, and the mass and width in isospin-violating decays are determined to be $M = 990.0 \pm 0.4$ MeV$/c^2$ and $\Gamma = 11.4 \pm 1.1$ MeV, respectively. In addition, we use the parameterized Flatté formula to simultaneously fit the same $\pi\pi$ invariant mass distributions, the two coupling constants of $g_{f\pi\pi}$ and $g_{fK\overline{K}}$ are measured to be $g_{f\pi\pi} = 0.46 \pm 0.03$ and $g_{fK\overline{K}} = 1.24 \pm 0.32$, respectively. According to the joint confidence regions of $g_{f\pi\pi}$ and $g_{fK\overline{K}}$, we conclude that the experimental data tend to support the $K\overline{K}$ molecule model and the quark-antiquark ($q\bar{q}$) model, but tend not to support the tetraquarks ($q^2\bar{q}^2$) model and the quark-antiquark gluon ($q\bar{q}g$) hybrid model.

Keywords: $f_0(980)$; isospin-violating; simultaneous fit; coupling constant

1. Introduction

The scalar meson $f_0(980)(I^G J^{PC} = 0^+0^{++})$ was established experimentally 40 years ago; however, so far, its nature has been a long-standing puzzle. This scalar meson dominantly decays to $\pi\pi$ final states, but also decays to $K\bar{K}$ final states with a small branching fraction [1]. Resonant parameters of $f_0(980)$ were obtained via measuring decay channels, such as $\phi \to \pi^0\pi^0\gamma$ by SND [2] and KLOE [3], and $J/\psi \to \phi\pi^+\pi^-$ by BESII [4], or via analyzing scattering processes, such as $e^+e^- \to K^+K^-\pi^+\pi^-/K^+K^-\pi^0\pi^0$ by BABAR [5], $\pi\pi$ scattering data and K_{l4} decay data [6,7]. Based on those experimental measurements and data analysis, the mass and width of $f_0(980)$ estimated by the Particle Data Group (PDG) were $M = 990 \pm 20$ MeV$/c^2$ and $\Gamma = 40$ to 100 MeV until 2016 [8]. It should be noted that not all uncertainties come from experimental data. The shape of the $f_0(980)$ varies for different decay processes, and that is why the mass and width are quoted from their process-independent pole position at an unphysical sheet of the complex energy plane, defined as $\sqrt{s_{pole}} = M - i\Gamma/2$.

In 2012, the BESIII Collaboration first observed anomalously narrow widths of about 10 MeV of $f_0(980)$ via isospin-violating decays of $J/\psi \to \gamma\eta(1405), \eta(1405) \to \pi^0 f_0(980) \to \pi^0\pi^+\pi^-/\pi^0\pi^0\pi^0$ [9]. The isospin-violating ratio between $\eta(1405) \to \pi^0 f_0(980) \to \pi^0\pi^+\pi^-$ and $\eta(1405) \to \pi^0 a_0^0(980) \to \pi^0\eta\pi^0$ is up to $(17.9 \pm 0.42)\%$. Three years later, the BESIII Collaboration once again observed the similarly narrow widths of $f_0(980)$ via isospin-violating decays of $J/\psi \to \phi f_1(1285), f_1(1285) \to \pi^0 f_0(980) \to \pi^0\pi^+\pi^-/\pi^0\pi^0\pi^0$, in which the isospin-violating ratio is measured to be $(3.6 \pm 1.4)\%$ [10]. Since then the width of $f_0(980)$ in PDG has been updated to be Γ =10 to 100 MeV [1]. In 2018, the anomalously narrow width of $f_0(980)$ was further confirmed by the BESIII Collaboration via an isospin-violating decay of $\chi_{c1} \to \pi^0 f_0(980) \to \pi^0\pi^+\pi^-$ [11]. It is worth noting that the narrow-width $f_0(980)$ mesons are only produced in isospin-violating decays, which apparently disagrees with the normal width of $f_0(980)$ in isospin-conserving decays.

Theoretically, the internal structure of the $f_0(980)$ is not only considered as the conventional quark-antiquark, but also proposed to be tetraquarks [12], $K\overline{K}$ molecule [13], or quark-antiquark gluon hybrid [14]. However, explanations about the nature of $f_0(980)$ have been controversial to date. The most famous theoretical study about $f_0(980)$ is the $a_0^0(980)$-$f_0(980)$ mixing mechanism, which was first proposed in the late 1970s [15]. Because $a_0^0(980)$ and $f_0(980)$ both can decay into $K\overline{K}$, the charged and neutral kaon mass thresholds are different by about 8 MeV due to isospin-violating effects. The mixing amplitude between $a_0^0(980)$ and $f_0(980)$ is dominated by the unitary cuts of the intermediate two-kaon system and proportional to the phase-space difference between charged and neutral kaon systems. As a consequence, a narrow peak of about 8 MeV in width is predicted between the charged and neutral $K\bar{K}$ mass thresholds. The Feynman diagram of $a_0^0(980) \to f_0(980)$ mixing in the decay of $X \to \pi^0 a_0^0(980) \to \pi^0 f_0(980) \to 3\pi$ is shown in Figure 1(a), where X can be $\eta(1405)$, $f_1(1285)$ or χ_{c1}. The $a_0^0(980)$-$f_0(980)$ mixing mechanism has been investigated extensively for a long time, and many decay processes have been discussed [16–27]. There was no experimental results until the BESIII Collaboration reported $a_0^0(980)$-$f_0(980)$ mixing via the decays of $J/\psi \to \phi f_0(980) \to \phi a_0^0(980) \to \phi\eta\pi^0$ and $\chi_{c1} \to \pi^0 a_0^0(980) \to \pi^0 f_0(980) \to \pi^0\pi^+\pi^-$ [11,28]. The mixing intensity of $a_0^0(980) \to f_0(980)$, *i.e.* the isospin-violating ratio, is measure to be $0.40 \pm 0.07 \pm 0.14 \pm 0.07$, which is less than 1.0%. Obviously, the $a_0^0(980)$-$f_0(980)$ mixing mechanism can not completely describe the large isospin-violating ratio in the decays of $\eta(1405) \to \pi^0 f_0(980) \to \pi^0\pi^+\pi^-/\pi^0\pi^0\pi^0$ and $f_1(1285) \to \pi^0 f_0(980) \to \pi^0\pi^+\pi^-/\pi^0\pi^0\pi^0$.

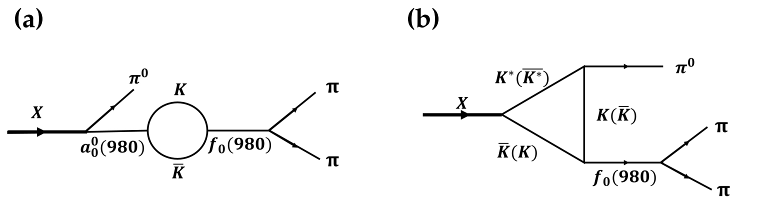

Figure 1. The Feynman diagram of the hadronic level: (**a**) The diagram of $a_0^0(980) \to f_0(980)$ mixing in the decay of $X \to \pi^0 a_0^0(980) \to \pi^0 f_0(980) \to 3\pi$. (**b**) The diagram of triangle singularity mechanism in the decay of $X \to \pi^0 f_0(980) \to 3\pi$. Here, X in the two diagrams can be $\eta(1405)$, $f_1(1285)$ or χ_{c1}.

Since the anomalously narrow widths of about 10 MeV of $f_0(980)$ was reported by the BESIII experiment [9], a novel scenario called triangle singularity mechanism was proposed to explain the internal behavior in isospin-violating decay processes [29]. Taking the decay $\eta(1405) \to \pi^0 f_0(980) \to 3\pi$ as an example, the intermediate $K\bar{K}^*$+c.c. pair can exchange an on-shell kaon, the three interaction vertices satisfy the energy-momentum conservation, the physical amplitude has a logarithmic triangle singularity, and the kinematic effects result in a narrow peak in the $\pi\pi$ invariant mass distributions. The Feynman diagram of the triangle singularity mechanism is shown in Figure 1(b). The triangle singularity mechanism can well explain the narrow width of the $f_0(980)$ and the large isospin-violating ratio in the decays of $\eta(1405) \to \pi^0 f_0(980) \to 3\pi$. The triangle singularity mechanism was continuously discussed more extensively and deeply [30–32], and applied to other decay processes [33–37].

In this article, firstly, we use the non-relativistic Breit-Wigner formula to simultaneously fit five $\pi\pi$ invariant mass distributions reported by the BESIII Collaboration, thereby obtaining the accurate mass and width of the $f_0(980)$ for isospin-violating decays in Section 2. Secondly, the energy-dependent Flatté formula as the shape of the $f_0(980)$ is used to simultaneously fit the $\pi\pi$ invariant mass distributions to determine the coupling constants of $f_0(980) \to K\overline{K}$ and $f_0(980) \to \pi\pi$, *i.e.*, $g_{fK\bar{K}}$ and $g_{f\pi\pi}$ in Section 3. Thirdly, we obtain the joint confidence region of the two coupling constants between $g_{fK\bar{K}}$ and $g_{f\pi\pi}$ as a way to provide quantitative constraints on different theoretical models of the $f_0(980)$ meson in Section 4. Finally, we summarize the whole article in Section 5.

2. Simultaneous fit to $\pi\pi$ invariant mass distributions in isospin-violating decays

So far, the BESIII Collaboration has reported anomalously narrow width of the $f_0(980)$ meson in five independent isospin-violating decay processes, they are $J/\psi \to \gamma\eta(1405), \eta(1405) \to \pi^0 f_0(980) \to \pi^0\pi^+\pi^-/\pi^0\pi^0\pi^0$ [9], $J/\psi \to \phi f_1(1285), f_1(1285) \to \pi^0 f_0(980) \to \pi^0\pi^+\pi^-/\pi^0\pi^0\pi^0$ [10], and $\psi(2S) \to \gamma\chi_{c1}, \chi_{c1} \to \pi^0 f_0(980) \to \pi^0\pi^+\pi^-$ [11], respectively. In order to accurately determine the mass and width of the $f_0(980)$ in isospin-violating decays, we perform a simultaneous fit to the $\pi\pi$ invariant mass spectra of the above five decay channels. In the simultaneous fit, the signal shape of the narrow $f_0(980)$ meson is described by the same non-relativistic Breit-Wigner function convolved with a Gaussian mass resolution of each decay channel. The mass resolution has an important influence on determining the width of the $f_0(980)$ meson, therefore the mass resolution is obtained in advance using Monte Carlo simulations, listed in Table 1. The backgrounds are represented by first-order or second-order Chebyshev polynomials, where the polynomial order is kept consistent with the fit in original papers published by BESIII.

The results of the simultaneous fit are illustrated in Figure 2. Figure 2 (a) and (b) are the fit results of the $\pi\pi$ invariant mass distributions in the decays of $\eta(1405) \to \pi^0 f_0(980)$, $f_0(980) \to \pi^0\pi^+\pi^-$ and $\pi^0\pi^0\pi^0$; Figure 2 (c) and (d) are the fit results of the $\pi\pi$ invariant mass spectra in the decays of $f_1(1285) \to \pi^0 f_0(980)$, $f_0(980) \to \pi^0\pi^+\pi^-$ and $\pi^0\pi^0\pi^0$; Figure 2 (e) is the fit result of the $\pi\pi$ invariant mass distribution in the decay of $\chi_{c1} \to \pi^0 f_0(980)$, $\chi_{c1} \to \pi^0\pi^+\pi^-$. The mass and width of the $f_0(980)$ meson obtained from the simultaneous fit, as well as the results reported by the BESIII Collaboration, are listed in Table 1. The mass and width from the simultaneous fit are $M = 990.0 \pm 0.4$ MeV$/c^2$ and $\Gamma = 11.4 \pm 1.1$ MeV, respectively. The fit results are consistent with the mass and width reported by the BESIII Collaboration for each channel, but the errors of the mass and width are improved remarkably. We can also obtain the joint confidence regions of the mass and width from the simultaneous fit, as shown in Figure 2 (e), in which the circles from inside to outside represent the confidence level of the two parameters from 1 to 6 standard deviations in order.

Table 1. The mass resolutions, the mass and width of $f_0(980)$ reported by the BESIII Collaboration. The bottom line is the results from the simultaneous fit.

Decay Channels	**Resolution (MeV)**	M **(MeV/c^2)**	Γ **(MeV)**
$\eta(1405) \to \pi^+\pi^-\pi^0$	3.3	989.9 ± 0.4	9.5 ± 1.1
$\eta(1405) \to \pi^0\pi^0\pi^0$	10.1	987.0 ± 1.4	4.6 ± 5.1
$f_1(1285) \to \pi^+\pi^-\pi^0$	3.5	989.0 ± 1.4	15.4 ± 4.9
$f_1(1285) \to \pi^0\pi^0\pi^0$	9.1	995.2 ± 4.9	15.5 ± 14.6
$\chi_{c1} \to \pi^+\pi^-\pi^0$	3.5	989.8 ± 1.4	10.0 ± 4.0
Simultaneous fit	—	$\mathbf{990.0 \pm 0.4}$	$\mathbf{11.4 \pm 1.1}$

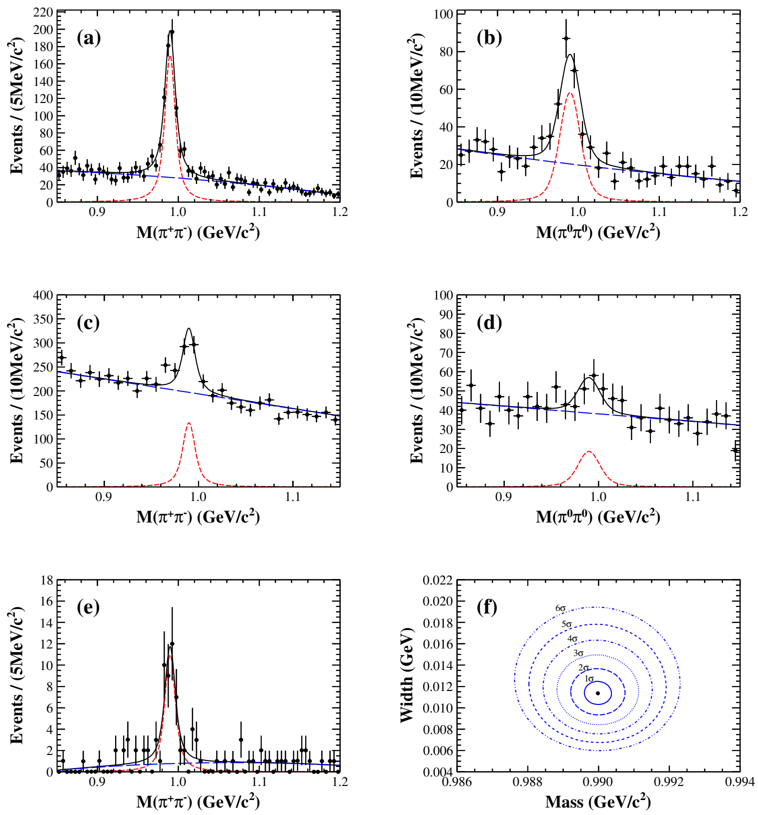

Figure 2. The simultaneous fit to the $\pi\pi$ invariant mass spectra of the decay channels of (**a**) $\eta(1405) \to \pi^0 f_0(980) \to \pi^0\pi^+\pi^-$, (**b**) $\eta(1405) \to \pi^0 f_0(980) \to \pi^0\pi^0\pi^0$; (**c**) $f_1(1285) \to \pi^0 f_0(980) \to \pi^0\pi^+\pi^-$, (**d**) $f_1(1285) \to \pi^0 f_0(980) \to \pi^0\pi^0\pi^0$; (**e**) $\chi_{c1} \to \pi^0 f_0(980) \to \pi^0\pi^+\pi^-$. The dots with error bars are the data reported by BESIII, the solid curves represent the total fit, the dashed curves represent the Breit-Wigner function of the $f_0(980)$ meson, and the long-dashed curves represent the background polynomials. (**f**) The joint confidence regions of the mass and width of the $f_0(980)$ meson from the simultaneous fit.

3. Determination of coupling constants of $g_{f\pi\pi}$ and $g_{fK\bar{K}}$

In general, the $f_0(980)$ meson can be descibed by the Flatté form of the propagator [29]:

$$G_f = \frac{1}{s - m_f^2 + i\sqrt{s}[\Gamma_{f\pi\pi}(s) + \Gamma_{fK\bar{K}}(s)]}, \tag{1}$$

where, s is the square of $\pi\pi$ invariant mass, m_f^2 is the square of the nominal mass of $f_0(980)$ in PDG [1], $\Gamma_{f\pi\pi}(s)$ and $\Gamma_{fK\bar{K}}(s)$ are energy-dependent partial widths of $f_0(980) \to \pi\pi$ and $f_0(980) \to K\bar{K}$, respectively. They are defined as:

$$\Gamma_{f\pi\pi}(s) = \frac{g_{f\pi\pi}^2}{16\pi\sqrt{s}}[\rho(\pi^0,\pi^0) + 2\rho(\pi^+,\pi^-)], \tag{2}$$

$$\Gamma_{fK\bar{K}}(s) = \frac{g_{fK\bar{K}}^2}{16\pi\sqrt{s}}[\rho(K^0,\bar{K}^0) + \rho(K^+,K^-)]. \tag{3}$$

where, $g_{f\pi\pi}$ and $g_{fK\bar{K}}$ are coupling constants of $f_0(980) \to K\bar{K}$ and $f_0(980) \to \pi\pi$, and $\rho(A,B) = \sqrt{(s-(m_A+m_B)^2)(s-(m_A-m_B)^2)}/2s$ is the momentum of the particle A or B in the center-of-mass frame of the two-body decay.

In order to obtain the coupling constants of $g_{f\pi\pi}$ and $g_{fK\overline{K}}$, a similar simultaneous fit to the $\pi\pi$ invariant mass spectra of the five decay channels is performed. In the new fit, only the shape of the $f_0(980)$ meson is replaced by the Flatté formula from the original Breit-Wigner function, and the remaining parts keep the same as the previous fit. The two coupling constants of $g_{f\pi\pi}$ and $g_{fK\overline{K}}$ from the simultaneous fit are determined to be $g_{f\pi\pi} = 0.46 \pm 0.03$ and $g_{fK\overline{K}} = 1.24 \pm 0.32$, respectively, which are listed in Table 2.

Table 2. The central mass and coupling constants, $g_{f\pi\pi}$ and $g_{fK\overline{K}}$, from the different theoretical predictions and the simultaneous fit.

Models	M (MeV/c^2)	$g_{f\pi\pi}$ (GeV)	$g_{fK\bar{K}}$ (GeV)
$q\bar{q}$	975	0.64	1.80
$q^2\bar{q}^2$	975	1.90	5.37
$K\bar{K}$	980	0.65	2.74
$q\bar{q}g$	975	1.54	1.70
Simultaneous fit	**990.0 $\pm$ 0.4**	**0.46 $\pm$ 0.03**	**1.24 $\pm$ 0.32**

4. Joint confidence regions of the coupling constants $g_{f\pi\pi}$ and $g_{fK\overline{K}}$

Theoretically, the $f_0(980)$ is not only considered as the conventional quark-antiquark ($q\bar{q}$) meson, but also the internal structures are proposed to be tetraquarks ($q^2\bar{q}^2$) state [12], $K\overline{K}$ molecule [13], or quark-antiquark gluon ($q\bar{q}g$) hybrid [14]. For those different theoretical models, the corresponding coupling constants, $g_{f\pi\pi}$ and $g_{fK\overline{K}}$, in the decays of $f_0(980) \to K\overline{K}$ and $f_0(980) \to \pi\pi$ have been predicted [27], which are also listed in Table 2 for comparison.

In order to distinguish the different theoretical models using the experimental data, we also draw the joint confidence regions of these two coupling constants from the simultaneous fit, shown in Figure 3. The dashed lines from inside to outside represent the confidence level of the two coupling constants from 1 to 5 standard deviations in order. The predicted positions from different theoretical models are also marked on the same plot, we find that only the prediction of the $K\overline{K}$ molecule model is located within the region of 5 standard deviations, while the predictions from other theoretical models are all outside the region of the five standard deviations, and the position of the quark-antiquark ($q\bar{q}$) model is very close to the boundary of the five standard deviations. The predicted positions of the tetraquarks ($q^2\bar{q}^2$) and the quark-antiquark gluon ($q\bar{q}g$) hybrid are far from the region of the five standard deviations. Therefore, the experimental data tend to support the $K\overline{K}$ molecule model and the quark-antiquark ($q\bar{q}$) model, but tend not to support the tetraquarks ($q^2\bar{q}^2$) model and the quark-antiquark gluon ($q\bar{q}g$) hybrid model.

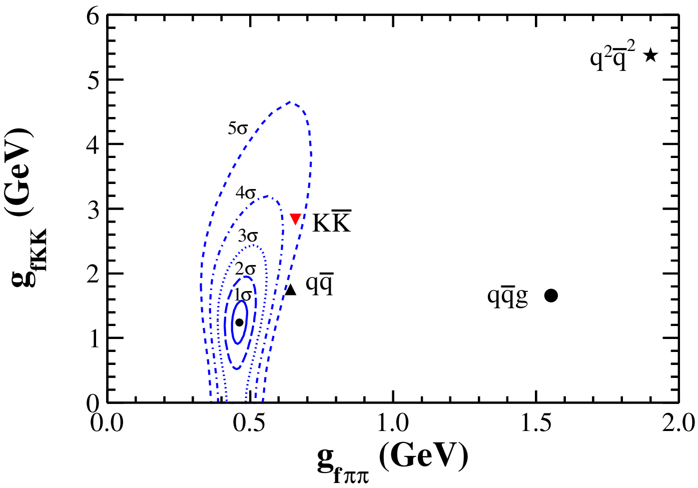

Figure 3. The joint confidence regions of the two coupling constants, $g_{f\pi\pi}$ and $g_{fK\bar{K}}$, from the simultaneous fit. The dashed lines from inside to outside represent the confidence level of the two coupling constants from one to five standard deviations in order. The predicted coupling constants from different theoretical models are also marked on the plot. The triangle represents the position of the traditional quark-antiquark model, the reversed triangle represents the position of the $K\bar{K}$ molecule model, and the round dot indicates the position of the quark-antiquark gluon hybrid, and the pentagon represents the position of the tetraquark model.

5. Summary

In summary, we perform a simultaneous fit to five $\pi\pi$ invariant mass distributions reported by the BESIII Collaboration, and determine the mass and width of the $f_0(980)$ meson in isospin-violating decay processes to be $M = 990.0 \pm 0.4$ MeV$/c^2$ and $\Gamma = 11.4 \pm 1.1$ MeV, respectively. The results are consistent with the mass and width reported by the BESIII experiment, but the errors of the mass and width are improved remarkably. We also use the parameterized Flatté formula to simultaneously fit the same $\pi\pi$ invariant mass distributions, and obtain the model-independent coupling constants of $g_{f\pi\pi}$ and $g_{fK\bar{K}}$ in the decays of $f_0(980) \to K\bar{K}$ and $f_0(980) \to \pi\pi$. Finally, we obtain the joint confidence regions of these two coupling constants from the simultaneous fit. We draw a conclusion that the experimental data tend to support the $K\bar{K}$ molecule model and the quark-antiquark ($q\bar{q}$) model, but tend not to support the tetraquarks ($q^2\bar{q}^2$) model and the quark-antiquark gluon ($q\bar{q}g$) hybrid model.

Author Contributions: Conceptualization, Y.L. and W.Y.; methodology, Y.L. and W.Y.; formal analysis, Y.L. and W.Y.; investigation, Y.L. and W.Y.; resources, Y.L. and W.Y.; data curation, Y.L. and W.Y.; writing—original draft preparation, W.Y.; writing—review and editing, W.Y. and D.L.; visualization, W.Y.; supervision, D.L.; project administration, W.Y. and D.L.; funding acquisition, W.Y. All authors have read and agreed to the published version of the manuscript.

Funding: This work is supported in part by National Natural Science Foundation of China (NSFC) under Contracts Nos. 11705006.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: We thank the BESIII Collaboration for publishing the data.

Conflicts of Interest: The authors declare no conflict of interest.

Abbreviations

The following abbreviations are used in this manuscript:

PDG	Particle Data Group
BESIII	Beijing Spectrometer III

References

1. R. L. Workman *et al.* (Particle Data Group). Review of Particle Physics. *Prog. Theor. Exp. Phys.* **2022**, *2022*, 083C01.
2. M. N. Achasov *et al.* (SND Collaboration). The $\phi(1020) \to \pi^0\pi^0\gamma$ decay. *Phys. Lett. B* **2000**, *485*, 349-356.
3. A. Aloisio *et al.* (KLOE Collaboration). Study of the decay $\phi \to \pi^0\pi^0\gamma$ with the KLOE detector. *Phys. Lett. B* **2002**, **537**, 21-27.
4. M. Ablikim *et al.* (BESIII Collaboration). Resonances in $J/\psi \to \phi\pi^+\pi^-$ and ϕK^+K^-. *Phys. Lett. B* **2005**, *607*, 243–253.
5. B. Aubert *et al.* (BABAR Collaboration). $e^+e^- \to K^+K^-\pi^+\pi^-$, $K^+K^-\pi^0\pi^0$ and $K^+K^-K^+K^-$ cross sections measured with initial-state radiation. *Phys. Rev. D* **2007**, *76*, 012008.
6. R. García-Martín, R. Kamiński, J. R. Peláez, J. Ruiz de Elvira and F. J. Ynduráin. Pion-pion scattering amplitude. IV. Improved analysis with once subtracted Roy-like equations up to 1100 MeV. *Phys. Rev. D* **2011**, *83*, 074004.
7. R. García-Martín, R. Kamiński, J. R. Peláez, and J. Ruiz de Elvira. Precise Determination of the $f_0(600)$ and $f_0(980)$ Pole Parameters from a Dispersive Data Analysis. *Phys. Rev. Lett.* **2011**, *107*, 072001.
8. K. A. Olive *et al.* (Particle Data Group). Review of Particle Physics. *Chin. Phys. C* **2014**, *38*, 090001.
9. M. Ablikim *et al.* (BESIII Collaboration). First Observation of $\eta(1405)$ Decays into $f_0(980)\pi^0$. *Phys. Rev. Lett.* **2012**, *108*, 182001.
10. M. Ablikim *et al.* (BESIII Collaboration). Observation of the isospin-violating decay $J/\psi \to \phi\pi^0 f_0(980)$. *Phys. Rev. D* **2015**, *92*, 012007.
11. M. Ablikim *et al.* (BESIII Collaboration). Observation of $a_0^0(980)-f_0(980)$ Mixing. *Phys. Rev. Lett.* **2018**, *121*, 022001.
12. J. Weinstein and N. Isgur. qq $\bar{q}$q system in a potential model. *Phys. Rev. D* **2015**, *27*, 588.
13. J. Weinstein and N. Isgur. $K\overline{K}$ molecules. *Phys. Rev. D* **1990**, *41*, 2236.
14. S. Ishida *et al.*, In Proceedings of the 6th International Conference on Hadron Spectroscopy, Manchester, United Kingdom, 10th-14th July 1995 (World Scientific, Singapore, 1995), p.454.
15. N. N. Achasov, S. A. Devyanin, and G. N. Shestakov. S^*-δ^0 mixing as a threshold phenomenon. *Phys. Lett. B* **1979**, *88*, 367-371.
16. B. Kerbikov and F. Tabakin. Mixing of the f_0 and a_0 scalar mesons in threshold photoproduction. *Phys. Rev. C* **2000**, *62*, 064601.
17. N. N. Achasov and G. N. Shestakov. Proposed Search for $a_0^0(980) - f_0(980)$ Mixing in Polarization Phenomena. *Phys. Rev. Lett.* **2004**, *92*, 182001.
18. N. N. Achasov and G. N. Shestakov. Manifestation of the $a_0^0(980) - f_0(980)$ mixing in the reaction $\pi^- p \to \eta\pi^0 n$ on a polarized target. *Phys. Rev. D* **2004**, *70*, 074015.
19. A. E. Kudryavtsev and V. E. Tarasov. On the possibility of observation of a_0-f_0 mixing in the $pn \to da_0$ reaction. *JETP Lett.* **2000**, *72*, 410.
20. A. E. Kudryavtsev, V. E. Tarasov, J. Haidenbauer, C. Hanhart, and J. Speth. Angular asymmetries in the reactions $\vec{p}p \to d\pi^+\eta$ and $\vec{p}n \to d\pi^0\eta$ and a_0-f_0 mixing. *Phys. At. Nucl.* **2003**, *66*, 1946.
21. A. E. Kudryavtsev, V. E. Tarasov, J. Haidenbauer, C. Hanhart, and J. Speth. Aspects of a_0-f_0 mixing in the reaction $\vec{p}\,\vec{n}$ da$_0$. *Phys. Rev. C* **2002**, *66*, 015207.
22. V. Y. Grishina, L. A. Kondratyuk, M. Büscher, W. Cassing, and H. Ströher. $a_0(980)$-$f_0(980)$ mixing and isospin violation in the reactions $pN \to da_0$, $pd \to^3 He/^3Ha_0$ and $dd \to^4 Hea_0$. *Phys. Lett. B* **2001**, *521*, 217-224.
23. F. E. Close and A. Kirk. Isospin breaking exposed in $f_0(980)$-$a_0(980)$ mixing. *Phys. Lett. B* **2000**, *489*, 24-28.
24. F. E. Close and A. Kirk. Large isospin mixing in ϕ radiative decay and the spatial size of the $f_0(980)$-$a_0(980)$ mesons. *Phys. Lett. B* **2001**, *515*, 13-16.
25. C. Hanhart, B. Kubis, and J. R. Pelaez. Investigation of a_0–f_0 mixing. *Phys. Rev. D* **2007**, *76*, 074028.

26. J. J. Wu, Q. Zhao and B. S. Zou. Possibility of measuring $a_0^0(980)$-$f_0(980)$ mixing from $J/\psi \to \phi a_0^0(980)$. *Phys. Rev. D* **2007**, *75*, 114012.
27. J. J. Wu and B. S. Zou. Study of $a_0^0(980)-f_0(980)$ mixing from $a_0^0(980) \to f_0(980)$ transition. *Phys. Rev. D* **2008**, *78*, 074017.
28. M. Ablikim *et al.* (BESIII Collaboration). Study of $a_0^0(980) - f_0(980)$ mixing. *Phys. Rev. D* **2011**, *83*, 032003.
29. J.-J. Wu, X.-H. Liu, Q. Zhao and B.-S. Zou. Puzzle of Anomalously Large Isospin Violations in $\eta(1405/1475) \to 3\pi$. *Phys. Rev. Lett.* **2012**, *108*, 081803.
30. F. Aceti, W. H. Liang, E. Oset, J. J. Wu and B. S. Zou. Isospin breaking and $f_0(980)$-$a_0(980)$ mixing in the $\eta(1405) \to \pi^0 f_0(980)$ reaction. *Phys. Rev. D* **2012**, *86*, 114007.
31. Meng-Chuan Du and Qiang Zhao. Internal particle width effects on the triangle singularity mechanism in the study of the $\eta(1405)$ and $\eta(1475)$ puzzle. *Phys. Rev. D* **2019**, *100*, 036005.
32. Meng-Chuan Du, Yin Cheng, and Qiang Zhao. Vertex corrections due to the triangle singularity mechanism in the light axial vector meson couplings to $K^*\overline{K}$+c.c.. *Phys. Rev. D* **2022**, *106*, 054019.
33. En Wang, Ju-Jun Xie, Wei-Hong Liang, Feng-Kun Guo, and Eulogio Oset. Role of a triangle singularity in the $\gamma p \to K^+\Lambda(1405)$ reaction. *Phys. Rev. C* **2017**, *95*, 015205.
34. Hao-Jie Jing, Shuntaro Sakai, Feng-Kun Guo and Bing-Song Zou. Triangle singularities in $J/\psi \to \eta\pi^0\phi$ and $\pi^0\pi^0\phi$. *Phys. Rev. D* **2019**, *100*, 114010.
35. Shuntaro Sakai, Eulogio Oset, and Feng-Kun Guo. Triangle singularity in the $B^- \to K^-\pi^0 X(3872)$ reaction and sensitivity to the $X(3872)$ mass. *Phys. Rev. D* **2020**, *101*, 054030.
36. Natsumi Ikeno, Raquel Molina, and Eulogio Oset. Triangle singularity mechanism for the $pp \to \pi^+ d$ fusion reaction. *Phys. Rev. C* **2021**, *104*, 014614.
37. A. Feijoo, R. Molina, L. R. Dai, Eulogio Oset. $\Lambda(1405)$ mediated triangle singularity in the $K^- d \to p\Sigma^-$ reaction. *Eur. Phys. J. C* **2022**, *82*, 1028.